where for a, $b \in \mathrm{At}(H)$ there exists a $c \in \mathrm{At}(H)$ such that $c \leq a+b$, $x = \langle a, \mathrm{At}(H) \setminus \{a, b, c\} \rangle$, $y = \langle b, \mathrm{At}(H) \setminus \{a, b, c\} \rangle$, and where for a, $b \in \mathrm{At}(H)$ there exists a $c \in \mathrm{At}(H)$ such that $a+b = a + c = b + c$ and $z = \langle 0, \mathrm{At}(H) \rangle$ in an arbitrary case. It is easily verified that $x \wedge y = x \wedge z$, but $x \wedge y \neq x \wedge (y + z)$. Consequently, the quasi-identity for cosemidistributivity is fulfilled only in the congruence lattices of the graphs K_n for $n \geq 1$ and $\overline{M(k, 0, m)}$ for $k \geq 0$ and $m \geq 0$.

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MODEL-COMPLETENESS OF A THEORY AND EVALUATION OF FORMULAS

V. A. Lyubetskii

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Macintyre in [1] considers special model-complete theories, which he defines by the condition that the reducing E-formula does not contain negations (he calls such theories positively model-complete). For example, one such is a field theory in which one can replace $x \neq 0$ by $\exists z(x \cdot z = 1)$. In the language of ring theory, this is essentially the only example. Probably, therefore, the author of [1], in deducing the basic theorem from [1] also in [2, p. 175], does this only for a field theory.

In [1, p. 88] it is said, "We would like to see an extension of the above-indicated method to noncommutative biregular rings." In [2] this problem is extended in the following manner: "Will there be a model-companion of a theory of certain natural Boolean extensions? ...a Boolean extension is a structure of sections of a sheaf over a Boolean space." A Boolean space is a complete disconnected compactum (a Stone space).

In this note, some answers are presented to the questions of Macintyre and a connection is established between some traditional questions of the theory of algebraic systems and Heyting-valued (nonstandard in the broad sense) analysis.

In [1], parallel with the language of ring theory, a general first-order language is also considered, including ring theory language, and also general sheaves (covering spaces)

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over Stone spaces X. Moreover, it is natural to impose on the atomic predicates of this wider language in one form or another, conditions making them similar to the predicate =, and that little depends on the choice of the sheaf. In this note, we restrict ourselves to ring theory language and, correspondingly, a Pierce sheaf for an arbitrary ring (associative and with 1). It is similar only in that our results also pertain to the mentioned generalization in [1] in the general languages and sheaves over X, including in sheaves not over a Stone space. They appear in such a form in another paper of the author.

Let us recall that any ring K is isomorphic to the ring (and it is identified further with it) of all global continuous sections of a sheaf $\mathcal{F}$ defined on the topology $\mathcal{T}$ of the Stone space X = X(K) of the Boolean algebra B = B(K) of all central idempotents of K (instead of B, $\mathcal{J}$ dc K is written). Moreover, the stalk (localization) K_p of $\mathcal{F}$ (respectively, K) over a point $p \in X$ (p is a prime ideal in B) is defined as $K/\overline{p}$, where $\overline{p} \neq p \cdot K$. The notation $\{K_p\} \models \varphi$ signifies the general validity of φ in all stalks (localizations) of K. Here and further, as desired, one can not mention the sheaves, speaking only of a ring K and the family of its residue class rings K_p; also one can not mention the topology $\mathcal{T}$ and the space X(K), operating only in terms relating to B(K).

First of all, let us once again formulate the problem. Let T be a theory in ring language, including the ring axioms and the axiom $0 \neq 1$, and having a model-companion T^*. Although it is not essential, it is convenient to consider that $T \subseteq T^*$. Let us form $\mathcal{K} = \{K \mid \{K_p\} \models T\}$ and $\mathcal{K}^* = \{K \mid \{K_p\} \models T^*, K \models \phi_1 \wedge \ldots \wedge \phi_m\}$, where $m \geq 0$ and the formulas $\phi_1, \ldots, \phi_m$ depend only on the class $\mathcal{K}$. Let us restrict ourselves here to the case when m = 3 and $\phi_1 = \forall k \exists e_0 \forall t \forall e \exists t_1 \exists t_2 (e_0^2 = e_0 \wedge e_0 t = t e_0 \wedge (e^2 = e \wedge e t_1 = t_1 e \wedge ek = 0 \Rightarrow e \leq e_0) \wedge (e^2 = e \wedge e t_2 = t_2 e \wedge e \leq e_0 \Rightarrow ek = 0))$ is the Horn formula, expressing the property of "normality" of K, $\phi_2 = \forall e \exists t \exists e_0 \forall t_1 (e^2 = e \wedge et = te \Rightarrow e_0{}^2 = e_0 \wedge e_0 t_1 = t_1 e_0 \wedge (e_0 = 0 \Rightarrow 0 = 1) \wedge (e_0 = e \Rightarrow 0 = 1) \wedge (e = 0 \vee e_0 \cdot e = e)$ is a formula expressing the lack of atomicity of K [or, what is the same, the lack of atomicity of B(K) or the absence of isolated points in X(K)]. This non-Horn formula can be replaced by a list of Horn formulas equivalent to it, since the class $\{K \mid K \models \phi_2\}$ is closed with respect to filtered products. Finally, $\phi_3 = \forall e \forall t (e^2 = e \Rightarrow et = te)$ is the Horn formula expressing commutativity of K. We show (Corollary "b" to Proposition 2) that each axiom of the class $\mathcal{K}^*$ of the form $\{K_p\} \models \varphi$, where $\varphi \in T^*$ is a Horn formula.

Therefore, the class $\mathcal{K}^*$ is axiomatizable and, what is more, Horn. For one condition on $\mathcal{K}^*$ it is model-complete (theorem 2); that is, its theorey Th $\mathcal{K}^*$ is model-complete. For one condition on the theory T^*, the class $\mathcal{K}$ is imbeddable in $\mathcal{K}^*$; that is, $\forall K \in \mathcal{K} \exists L \in \mathcal{K}^*$, $K \subseteq L$ (Theorem 3). Therefore, the class $\mathcal{K}^*$ is called a model-companion for $\mathcal{K}$. (Naturally, the class $\mathcal{K}^*$ is called a model-companion for $\mathcal{K}^*$, if these classes are mutually imbeddable and the class $\mathcal{K}^*$ is model-complete; that is $\forall K, L \in \mathcal{K}^*$, $K \subseteq L \Rightarrow K \preccurlyeq L$). Therefore, if the class $\mathcal{K}$ is axiomatizable, then Th $\mathcal{K}^*$ is a Horn model-companion for the theory Th $\mathcal{K}$ (Theorem 4). In this same theorem, for any (not necessarily axiomatizable) class K, there is indicated a dense Horn axiomatizable subclass, defined as the class of all models of T' (which is canonically determined according to an arbitrary initial theory

T), for which the class $\mathcal{K}^*$ is a model-companion; that is, the class $\mathcal{K}^*$ is a model-companion of the class Mod T' or, in other words, the theory Th$\mathcal{K}^*$ is a Horn model-companion for T'.

Let us provide two formulas: $\phi_4 = \forall e(e^2 = e \Rightarrow e = 0 \vee e = 1)$ is "without idempotents" and $\phi_5 = \forall e \exists t(e^2 = e \wedge et = te \Rightarrow e = 0 \vee e = 1)$ is "indivisible."

In the conclusion of the statement of the problem, let us remark that the indicated manner of giving the class $\mathcal{K}$ is based on the traditional approach to studying algebraic systems, in the framework of which the algebraic system is considered from the point of view of the family of all its localizations. From this point of view, such a method of giving a class is universal.

The condition mentioned above on $\mathcal{K}^*$ is such (the corresponding term, as also the other terms presented below, is conditional and pertains only to this note). Let us call the class of rings R, Boolean absolute if for any $K \in R$ we have $\forall L \in \mathcal{R}(K \subseteq L \Rightarrow \forall e_1 \in B(L)(e_1 \neq 0 \Rightarrow \exists p_1 \in e_1 (p_1 \cdot L) \cap K \subseteq (p_1 \cap K) \cdot K)$; that is, on a [dense in X(L)] set of points P_1, we have $\overline{P}_1 \cap K = \overline{p_1 \cap K}$. One can clarify the sense of this condition so. We intend to consider the extensions $K \subseteq L$, $K, L \in R$; that is, to consider in the language of the covering spaces, the surjective continuous function $f\colon X(L) \to X(K)$, $f(p_1) = p = p_1 \cap B = p_1 \cap K$ and family of homomorphisms $\overline{f}_{p_1}\colon K_p \to L_{p_1}$, $\overline{f}_{p_1}([k]_p) = [k]_{p_1}$. To the imbedding of K in L corresponds an imbedding of the covering spaces $(X(K), E(K)) \to (X(L), E(L))$ of the form $\langle f, \{\overline{f}_{p_1} | p_1 \in X(L)\}\rangle$. The kernel of the homomorphism $\overline{f}_{p_1}$ for an arbitrary point $p_1 \in X(L)$ equals $\{[k]_p | k \in K, k \in \overline{p}_1\}$ and our condition means precisely that the (dense in p_1) set of homomorphisms $\overline{f}_{p_1}$ consists of monomorphisms. In other words, in a Boolean absolute class, the imbedding of K in L induces imbeddings of K_p in L_{p_1} for a dense set of points p_1, where $p = f(p_1)$, $p_1 \in X(L)$.

For example, if R is a class of biregular rings; that is $\forall k \exists e \in B(K)$ $(\langle k\rangle = e \cdot K)$, then $R = \{K | \{K_p\} \models$ is a "prime ring"; moreover, K is an automatically normal ring and R_0 is a Boolean absolute class (here $\langle k\rangle$ is a principle ideal). While, as an appropriate dense set of points p_1, one can take all of X(L).

The second condition (on T*) mentioned above is that T* is a completely closed theory. Let us denote by $X_1(K)$ the set of all proper ideals in B(K). Let us note that $\overline{q} = q \cdot K$ is an ideal in K for any $q \in X_1(K)$ and one can let $K_q \rightleftharpoons K|\overline{q}$, $X(K) \subseteq X_1(K)$. Let us call the theory T closed if any model of it is imbedded in a model of it F, such that $\{F_p\} \models T$. Let us call a closed theory T completely (totally) closed, for which we have one more condition $F \models \phi_1 \wedge \phi_3$ (correspondingly for any model of it F, we have $\{F_q | q \in X_1(F)\} \models T$). For example, if $T \vdash \phi_4$, then T is completely closed and totally closed; if $T \vdash \phi_5$, then T is totally closed. Let us call a theory T normally (commutatively) closed if it is closed and the corresponding F has the property $F \models \phi_1$ (respectively, $F \models \phi_3$). For example, even the condition $T \vdash \phi_4$ is satisfied for the class R of strictly Rickart rings, which is given in the form $R = \{K | \{K_p\} \models \forall k, t(k \cdot t = 0 \Rightarrow k = 0 \vee t = 0),\ k \models \phi_1\}$. For another class R of abelian regular rings, which is given in the form $R = \{K | \{K_p\} \models$ "division ring"$\}$, we immediately have both conditions: this is a Boolean absolute class and $T \vdash \phi_4$. Let us recall that strict Rickartivity is characterized by the condition $\forall k \exists e \in B(K)$, $K^* = e \cdot K$, where k^* is a right annihilator of the element k of K; commutativity is property ϕ_3 and regularity is understood in the sense of von Neumann.

Let us recall the definition basic in a Heyting-valued analysis of a valuation [for the case of a ring language and a topology $\mathcal{T}$ in X(K): $[\![k=t]\!] = \cup\{e \in B(K) \mid e\cdot k = et\} = \{p \in X(K) \mid k(p) = t(p)\}$ {let us note that the normality of K is equivalent to the fact that $[\![k=t]\!] \in B(K), \forall k, t \in K$). Further, $[\![\varphi \vee \psi]\!] = [\![\varphi]\!] \cup [\![\psi]\!]$ (also for $\wedge$ and $\cap$), $[\![\neg \varphi]\!] = {}^\circ C [\![\varphi]\!]$, where °C is the interior of the complement $[\![\exists x \varphi]\!] = \cup\{[\![\varphi(k)]\!] \mid k \in K\}$, $[\![\forall x \varphi]\!] = {}^\circ\cap \{[\![\varphi(k)]\!] \mid k \in K\}$, where °∩ is the interior of the intersection. It is clear that $[\![\varphi(k_1, \ldots, k_n)]\!] \in \mathcal{T}$ (the parameters are always from K). Finally, different rings have different valuations. A valuation is similarly defined for the language ZF; moreover, the parameters run over $V^{\mathcal{T}}$. One can everywhere replace $\mathcal{T}$ by an arbitrary Heyting algebra Ω. In a reminder of Ω, one sometimes writes $[\![\cdot]\!]_\Omega$ and, in a reminder of K, one sometimes writes $[\![\cdot]\!]_K$. The connection of the languages of ring theory and ZF, and the valuations corresponding to them, is expressed next. For K, the object $K' \in V^{\mathcal{T}}$ is defined and the translation of the formulas φ of the ring theory language with parameters k_i from K to the formulas $\varphi^\vee$ of ZF with parameters P_{k_i} from K' (in the sense $[\![P_{k_i} \in K']\!] = 1$), for which $[\![\varphi(k_1, \ldots, k_n)]\!]_K = [\![\varphi^\vee(P_{k_1}, \ldots, P_{k_n})]\!]_{\mathcal{T}}$. Namely, P_k: $K \to \mathcal{T}$, $P_k(t) = [\![k=t]\!]$, where t runs over K; K': $\{P_k \mid k \in K\} \to \{1\}$, +': $\{\langle P_k, P_t, P_{k+t}\rangle^\Omega \mid k, t \in K\} \to \{1\}$ (it is similarly defined). Now the translation $\varphi \longmapsto \varphi^\vee$ is clear: $k \to P_k$, $+ \to +'$, $\cdot \to \cdot'$, $x \to (x \in K')$. For all the details of this, see [3] and also below in Proposition 5. Typically, one denotes P_k simply k and omits $\check{}$; that is, instead of $[\![\varphi^\vee(P_{k_1}, \ldots, P_{k_n})]\!]_{\mathcal{T}}$, one writes $[\![\varphi(k_1, \ldots, k_n)]\!]_{\mathcal{T}}$. Let us recall that $\hat{f}^\Omega = \{g \in V^\Omega \mid [g \in f]_\Omega = 1\}$, where 1 is the unit in Ω [3].

<u>Proposition 1.</u> If K is a normal ring and $\{K_p\} \models T$, where T is a model complete theory, then for any formula φ we have two properties: $[\![\varphi(k_1, \ldots, k_n)]\!]_K = \{p \in X(K) \mid K_p \models \varphi(k_1(p), \ldots, k_n(p))\}$ and $[\![\varphi(\bar{k})]\!]_K$ is a clopen set, where $k_1, \ldots, k_n \in K$.

<u>Proof.</u> For an atomic formula $[\![k=t]\!] = e_0$, where e_0 is from the definition of normality for the element $k - t$. For the connectives $\vee$, $\wedge$, $\neg$ everything is clear. For the connective $\exists$ let us observe $[\![\exists x \varphi]\!] = \{p \in X \mid K_p \models \exists x \varphi\}$. Using model-completeness for $\neg \exists x \varphi$, we obtain a reducing E-formula and by normality of the ring, the valuation $[\![\exists x \varphi]\!]$ is clopen. For $\forall$ let us observe $[\![\forall x \varphi]\!] \subseteq \{p \in X \mid K_p \models \forall x \varphi\} \subseteq \cap\{[\![\varphi(k)]\!] \mid k \in K\}$ and consider that $\{p \in X \mid K_p \models \exists x \neg \varphi\}$ is clopen.

<u>COROLLARY 1.</u> If T is a positive model-complete theory, then K is a normal ring (that is, the condition of normality in Proposition 1 can be omitted for such T).

This corollary is essentially noted in [1]. Without special assumptions about T, the equality from Proposition 1 is invalid even in the form $[\![\varphi]\!] = 1 \iff \{K_p\} \models \varphi$. Therefore, the valuation is an instrument not reducing to the consideration of the set $\{p \in X(K) \mid K_p \models \varphi\}$. However, we require the following:

COROLLARY 2. Let φ be in prenex normal form. If $[\![\varphi]\!]_K = 1$, then $\{K_p\} \models \varphi$.

Proof. For a quantifier-free φ let us verify the stronger: $p \in [\![\varphi]\!] \Rightarrow K_p \models \varphi$. By definition, $p \in [\![k \neq t]\!] \Rightarrow K_p \models k(p) = t(p)$ and $p \in [\![k = t]\!] \Rightarrow K_p \models k_p \neq t(p)$. For $\vee$, $\wedge$, it is obvious. For $\exists$ let us use accessibility (that is, in essence, the compactness of X), and for $\forall$, the flabbiness of a sheaf (that is, in essence, the epimorphicity of $K \to K_p$).

Remark. One can replace 1 by any clopen set. The normality of φ consists in the fact that negations are found only in the atomic formulas. If K is a normal ring, then for a quantifier-free formula, so that negations may not be available, we have $[\![\varphi]\!] = \{p \in X \mid K_p \models \varphi\}$.

Proposition 2. Let $\varphi(k_1, \ldots, k_n)$ be any formula in prenex disjunctive form in the ring language and $\varphi'(k_1, \ldots, k_n)$ be obtained from φ by transfer of the quantifier prefix Q, occurring in φ, and then by the assignment $\exists \bar{e}_s \forall t \forall e_0 \exists t_1 [e_1^2 = e_1 \wedge e_1 t = t e_1 \wedge \ldots \wedge \prod_s (1 - e_s) = 0 \wedge e_1 k_1 = e_1 t_1 \wedge \ldots \wedge (e_0^2 = e_0 \wedge e_0 t_1 - t_1 e_0 \wedge e_0 k_2 = e_0 t_2 \Rightarrow e_0 \leq 1 - e_1) \wedge \ldots]$. Then $[\![\varphi(\bar{k})]\!]_K = 1 \Leftrightarrow K \models \varphi'(\bar{k})$, where $\bar{k} \subseteq K$.

Proof. We consider that the quantifier-free part of φ is of the form $\bigvee_s \psi_s$. Then $\bigcup_s [\![\psi_s(\bar{k})]\!]_K = 1$ is equivalent to the existence of a set $\{e_s\} \subseteq B(K)$, such that $e_s \leqslant [\![\psi_s(\bar{k})]\!]_K$ and $\bigcup_s e_s = 1$. This is written by the above-indicated formula (where $k_1 = t_1$ is one of the equations, and $k_2 \neq t_2$ is one of the inequalities in ψ_1). If Q begins with $\exists$, then let us use accessibility in the clopen set. The case when Q begins with $\forall$ is obvious.

Remark. In Proposition 2 one can replace 1 by any clopen set and only accessibility is necessary from the sheaf and $\mathcal{F}$, in essence. Such a proposition is also true for other languages. The semantics of $\vee$ usually evokes the most difficulties — Proposition 2 reduces it to a Horn formula. Type φ' is easily described according to the type φ. The expressibility of this and a number of other predicates (see in particular Proposition 6 below) indicates that the forcing theory is expressed in an internal manner in the ring language, that gives the usual corollaries of this theory.

The transition $T \to T'$, where by definition $T' = \{\varphi' \mid \varphi \in T\}$ often is useful. The axioms of T' are easily written out according to the axioms of T.

COROLLARY. a) If an arbitrary theory is described with the help of any set of propositions of the form $[\![\varphi(\bar{k})]\!]_K = 1$, then it is axiomatizable and even Horn.

b) If $\mathcal{K} = \{K \mid \{K_p\} \models T, K \models \phi_1\}$, where T is model complete, then the class $\mathcal{K}$ is Horn-axiomatizable, namely $\mathcal{K} = \{K \mid K \models T', K \models \phi_1\}$. In particular, the class $\mathcal{K}^*$ is Horn-axiomatizable.

Proof. b) On one side, we use Proposition 1 and, in the converse, Corollary 2 to Proposition 1.

THEOREM 1. If $\mathcal{K} = \{K \mid \{K_p\} \models T, K \models \phi_1\}$, where T is an AE-theory (or in the absence of the condition $K \models \phi_1$, an AE-positive theory), then the class $\mathcal{K}$ is Horn-axiomatizable, namely $\mathcal{K} = \{K \mid K \models T', K \models \phi_1\}$.

Proof. It is known [3, p. 388] that under these conditions the relation $\{K_p\} \models \varphi$ is equivalent to $[\![\varphi]\!]_{\mathcal{T}} = 1$. Then, let us use Proposition 2. Hence, Corollary "b" to Proposition 2 is obtained, since the model-complete theory is an AE-theory.

Let us extend the theorem of Macintyre, which was spoken of at the beginning of the note, to the following:

THEOREM 2. If $\mathcal{K}^*$ is a Boolean absolute class, then it is model-complete.

LEMMA. For any $K \in \mathcal{K}^*$ and any primitive formula ψ with parameters from K, there exists a formula ψ_1 with those same parameters equivalent to $\neg\psi$ in each extension $L \supseteq K$, $L \in \mathcal{K}^*$, which has the form $\psi_1 = \exists\bar{k} \vee \forall e_0\, \psi_0$, where the block $\bar{k}$ runs over L, after which disjunction (concerning which, the sign $\vee$ speaks) can be encountered, and e_0 is a special variable in B(L) and ψ_0 is the conjunction of atomic formulas and formulas being the implication of two atomic formulas.

Proof. As in [1], let us form according to ψ the ψ_e-subformulas of ψ, containing all the equations from ψ and one inequality from ψ (one can consider that ψ contains among the inequalities also $0 \neq 1$). It was in essence proved in [1] that under the condition of normality and the lack of atomicity of L, there is satisfied (further, Proposition 1 is taken into account everywhere): $(4 \models \psi) \Leftrightarrow (\forall \ell ([\psi_\ell]_L \neq \emptyset \wedge \bigcup_\ell [\psi_\ell]_L = X)$. Therefore $(L \models \neg\psi) \Rightarrow (\exists \ell [\![\psi_\ell]\!]_L = 0) \vee (\bigcup_\ell [\![\psi_\ell]\!]_L < 1)$, where instead of ϕ and X, we write 0 and 1.

The first disjunctive term can be rewritten in an equivalent manner, sequentially so: $\exists \ell ([\![\neg\psi_\ell]\!] = 1)$, according to the model-completeness of the theory T*, there is an E-formula ψ_e' with a quantifier-free part $\bigvee_s \psi_{es}$, equivalent to $\neg\psi_e$ in the models for T*; that is, $\exists \ell ([\![\psi_\ell']\!] = 1)$, according to accessibility $[\![\cdot]\!]$ on the clopen set, we obtain $\exists \ell \exists \bar{k} (\bigcup_s [\![\psi_{\ell s}]\!] = 1)$, taking into account commutativity $\exists\bar{k}\exists\bar{e}_{\ell s} \bigvee_\ell \forall e_0 [e_{\ell 1}^2 = e_{\ell 1} \wedge \ldots \wedge \prod_s (1-e_{\ell s}) = 0 \wedge e_{\ell 1}\cdot k_1 = e_{\ell 1} t_1 \wedge \ldots \wedge (e_0 k_2 = e_0 t_2 \Rightarrow e_0 \leq 1 - e_{\ell t}) \wedge \ldots]$, where $k_1 = t_1$ is one of the equations, and $k_2 \neq t_2$ is one of the inequalities in ψ_{ℓ_1}.

Similarly, for the second disjunctive term: $\exists p \in X\, \forall \ell (p \notin [\![\psi_\ell]\!]$, $\exists p \in X\, (p \in [\![\bigwedge_\ell \neg\psi_\ell]\!])$, $[\![\bigvee_\ell \neg\psi_\ell]\!] > 0$, $\exists e \in B (e \neq 0 \wedge [\![\bigvee_\ell \neg\psi_e]\!] \geq e)$, according to the model-completeness of T*, there is an E-formula ψ_1 with a quantifier-free part $\bigvee_s \psi_{1s}$, for which $\exists B \in B\, (e \neq 0 \wedge [\![\psi_1]\!] \geq e)$, by accessibility $[\![\cdot]\!]$ on the clopen set, we obtain $\exists\bar{k}\exists e \in B (e \neq 0 \wedge \bigcup_s [\![\psi_{1s}]\!] \geq e)$, by commutativity $\exists\bar{k}\exists e \bigvee_s \forall e_0 [e^2 = e \wedge e \neq 0 \wedge e k_1 = e t_1 \wedge \ldots \wedge (e_0 k_2 = e_0 t_2 \Rightarrow e_0 \leq 1 - e) \wedge \ldots]$, where $k_1 = t_1$ and $k_2 \neq t_2$ from ψ_{11}.

Proof of Theorem 2. Let $K \subseteq L$, K, $L \in \mathcal{K}^*$. We want to transfer $\neg\psi$ from K to L. By Lemma 7 $\neg\psi$ in K and in L are equivalent to the same formula (with those same parameters) ψ_1. Therefore, it is sufficient to transfer ψ_1 from K to L; that is, to transfer $\forall e_0\psi_0$, more precisely, to transfer implication. This requires a strengthening, since B(L) can be wider than B(K); on account of commutativity $B(K) \subseteq B(L)$. It is given that $K \models \forall e_0 (e_0 k = 0 \Rightarrow e_0 \leq e)$, where k, $e \in K$. It is necessary to prove $L \models \forall e_0 (e_0 k = 0 \Rightarrow e_0 \leq e)$. Let us assume that $e_0 k = 0$, $e_0 \in B(L)$, and $e_0 \cap (1 - e) \neq 0$. By hypothesis, let us take $p_1 \in e_0 \cap (1 - e)$, for which $\bar{p}_1 \cap K \subseteq \bar{p}$, where $p = p_1 \cap K$. Then $(1 - e_0)\cdot k = k$ and $k = e_1 k$, where $e_1 \in p$. Therefore

$(1 - e_1) \notin p_1$ and $(1 - e_1)\cdot k = 0$, $(1 - e_1) \in B(K)$; that is, $(1 - e_1) \le e$, $e \in p_1$, and $(1 - e) \in p_1$. A contradiction.

Now let us consider the conditions ensuring the imbeddability of class $\mathscr{K}$ in class $\mathscr{K}^*$.

THEOREM 3. a) If T^* is normally closed, then $\mathscr{K}$ is imbedded in $\{K|\{K_p\} \models T^*, K \models \phi_1 \wedge \phi_2\}$.

b) If the theory T^* is completely closed, then $\mathscr{K}$ is imbedded in $\mathscr{K}^*$.

Proof. Let $K \in \mathscr{K}$ and, consequently, K is a subdirect product in $\prod_p K_p$, where all the K_p are models of T. By hypothesis, each K_p is imbedded in some F^p, where $F^p \models T^*$. As F^p let us choose exactly those models for T^*, regarding which it is spoken in the definition of a closed theory.

Let us denote by X_0 the Cantor discontinuum (a completely disconnected, separable, metrizable compactum without isolated points). In each F^p let us fix the discrete topology. We verify that all the rings $\overline{F^p} = C(X_0, F^p)$ belong to that class in which it is necessary to imbed K. These rings consist of all piecewise constant F^p-valued functions on X_0.

In correspondence with the corollary to Proposition 2, all the classes mentioned in the verified theorem are Horn, and that is why they are closed with respect to products. Therefore, $\prod_p \overline{F^p}$ belongs to the needed class and $K \rightarrowtail \prod_p K_p \twoheadrightarrow \prod_p F^p \rightarrowtail \prod_p \overline{F^p}$, that is required to prove.

Thus, let us consider the ring $\overline{F} = C(X_0, F)$, where $F \models T^*$ and F possesses the property from the definition of a closed theory. It is true that $B(\overline{F}) = C(X_0, B(F))$, and $\langle x_0, p_0\rangle = \{f \in B(\overline{F})|f(x_0) \in p_0\}$ is a prime ideal in $B(\overline{F})$ for any $x_0 \in X_0$ and $p_0 \in X(F)$. Any point from $X(\overline{F})$ has the same form, that is symbolically $X(\overline{F}) = X_0 \times X(F)$, since for $p \in X(\overline{F})$ there exists $x_0 \in X_0$, for which $p_0 = \{f(x_0)|f \in p\}$ does not contain a unit from B(F) (otherwise $\{\{x_0 \in X_0|f(x_0) = 1\}|f \in p\}$ is an open covering of X_0, and the subcovering leads to $f_1, \ldots, f_n \in p$, such that $f_1 \vee \ldots \vee f_r \in p$ and $f_1 \vee \ldots \vee f_n \equiv 1$ – a contradiction). Such a $p \subseteq \langle x_0, p_0\rangle$ is a prime ideal in B(F). Therefore, $p \subseteq \langle x_0, p_0\rangle$, due to the maximality of p this is possible only if $p = \langle x_0, p_0\rangle$.

Further, $\overline{\langle x_0, p_0\rangle} = \{f \in \overline{F}|f(x_0) \in \overline{p}_0\}$, where $\overline{p}_0 = p_0\cdot F$ and $\overline{\langle x_0, p_0\rangle} = \langle x_0, p_0\rangle\cdot F$. Therefore $(\overline{F})_{\langle x_0, p_0\rangle} = \overline{F}|\overline{\langle x_0, p_0\rangle} = F|\overline{p}_0 = F_{p_0}$, where p_0 runs over X(F). By hypothesis, for all F_{p_0} we have $F_{p_0} \models T^*$; hence $\{(\overline{F})_{\langle x_0, p_0\rangle}\} \models T^*$. Let us verify the normality of $\overline{F}$. If $f \in \overline{F}$, then let $e_0(x) = 1$, if $f(x) = 0$ and $e_0(x) = e_i$, where e_i is the element from F, corresponding, due to the normality of F, to an element $f(x) \neq 0$ from F; such e_0 satisfies the definition of normality for f in $\overline{F}$. Let us assume that f is an atom in $\overline{F}$. At least one "step" for f; for example, $f(x_0)$ is different from 0. This is a step over a clopen set containing at least two different points. Removing one of them along with its neighborhood, we obtain that f is not atomic. (The commutativity of $\overline{F}$ directly follows from the commutativity of F.)

Remark. Only two properties are actually used in Theorem 3 from the model-completeness of T^*: closure with respect to products and the model-imbeddability of T in T^*.

Joining Theorems 2 and 3, we obtain the following theorem. Let the axioms of T be written in prenex disjunctive form. Let us recall that $T' = \{\varphi' \mid \varphi \in T\}$ (φ' is explicitly constructed according to φ in accord with Proposition 2). This T' is a Horn theory. By Corollary 2 to Proposition 1, $[\![\varphi]\!] = 1 \Rightarrow \{K_p\} \models \varphi$; that is, any model of T' is contained in $\mathcal{K}$. Taking into account Proposition 1, we obtain

$$T' \subseteq Th\,\mathcal{K}^* \text{ и } \mathcal{K}^* \subseteq Mod(T') \subseteq \mathcal{K}.$$

THEOREM 4. If $\mathcal{K}^*$ is a Boolean absolute class and the theory T^* is completely closed, then Th $\mathcal{K}^*$ is a Horn model-companion for T and (in the case of an axiomatizable class $\mathcal{K}$) for the theory Th $\mathcal{K}$.

COROLLARY 1. If $\mathcal{K}$ is an axiomatizable subclass of the class of all biregular rings and T^* is abelian closed, then Th $\mathcal{K}^*$ is a Horn model-companion for Th $\mathcal{K}$.

COROLLARY 2. If $\mathcal{K}$ is an axiomatizable subclass of the class of abelian regular rings, then Th $\mathcal{K}^*$ is a Horn model-companion for Th $\mathcal{K}$.

In [1] the case is considered exactly when $\mathcal{K}$ is the class of all commutative regular rings (these are also the axioms for Th $\mathcal{K}$); such rings, of course, are abelian regular rings. Then it is clear that T is a field theory and T^* is the theory of algebraically closed fields. In order to write out the axioms for Th $\mathcal{K}^*$ according to the methodology indicated above, it is necessary to observe that commutativity and normality are ensured already by the "special axioms" $\{K_p\} \models T^*$. Therefore, ϕ_2 (lack of atomicity) remain and are still translated for the two axioms "field" and "algebraically closed." In correspondence with Proposition 2, let us form φ' and obtain, respectively, exactly "regular" and "algebraically closed." These are , in fact, the axioms for the theory Th $\mathcal{K}^*$ indicated in [1] and a number of other papers. In this manner, we indicate the general method of computing the corresponding axioms. Let us note that one can explicitly indicate in which class of formulas we fall in the transition $T \longmapsto T'$ for many T.

COROLLARY 3. If $\mathcal{K}$ is an axiomatizable subclass of the class of all strictly Rickart rings and $\mathcal{K}^*$ is a Boolean absolute class, then Th $\mathcal{K}^*$ is a Horn model companion for Th $\mathcal{K}$.

Proof. In this case, all the stalks are without divisors of zero and, consequently, without idempotents.

Remark. The class $\mathcal{L} = \{K_p \mid K \in \mathcal{K},\ K \models \phi_1 \wedge \phi_2 \wedge \phi_3,\ p \in X(K)\}$, where $T \vdash \phi_4$ is axiomatized by the theory T.

Proof. By the definition of the class $\mathcal{K}$ we have $K_p \models T$. If $F \models T$, then the ring $\overline{F} = C(X_0, F)$ has all stalks isomorphic to F; that is $\overline{F} \in \mathcal{K}$. Furthermore, $\overline{F}$ is normal atom-free and abelian. Therefore, $F \in \mathcal{L}$. (It is similarly true for the case $T \vdash \phi_5$.)

Let us present the imbeddability criterion of the class $\mathcal{K} = \{K|\{K_p\} \models T\}$ in some class $\mathcal{K}_1 = \{K|\{K_p\} \models T_1\}$ not assuming the model-completeness of T_1. Let us recall that the object K' is determined before Proposition 1. The crucial remark is that for a normal ring K the object $K' \in V^{\mathcal{B}}$ and, consequently, $K' \in (V^{\mathcal{B}} \cap V^J)$, where $\mathcal{B} = \mathcal{B}(K)$ is the Dedekind completion of B = B(K). One can identify such a $\mathcal{B}^{\bar{B}}$ with the algebra of regular open sets in X(K). These are sets $\mathcal{U} \in J(K)$ for which $\mathcal{U} = \alpha(\mathcal{U})$, where α is the operation consisting in computing the interior of the closure; that is $\alpha = \neg\neg$, where $\neg = {}^\circ C$ is the pseudocompletion. Therefore, one can judge the properties of K both from the point of view $[\![\cdot]\!]_B$ and from the point of view $[\![\cdot]\!]_{\mathcal{J}}$. In principle, the pair $\langle \mathcal{J}, \mathcal{B} \rangle$ corresponds to a Gödel negative translation of formulas. Let us present this in more detail.

Let two complete Heyting algebras Ω_1 and Ω_2 be comparable in the sense that $B \subseteq \Omega_1 \subseteq \Omega_2$ or $B \subseteq \Omega_2 \subseteq \Omega_1$. We will write $\Omega_1 \leq \Omega_2$, if $u \wedge_{\Omega_1} v = u \vee_{\Omega_2} v$, $u \wedge_{\Omega_1} v = u \wedge_{\Omega_2} v$, and $\neg_{\Omega_1} u = \neg_{\Omega_2} u$, $\bigvee_{\alpha}{}_{\Omega_1} u_\alpha \leq \bigvee_{\alpha}{}_{\Omega_2} u_\alpha, \bigwedge_{\alpha}{}_{\Omega_1} u_\alpha \leq \bigwedge_{\alpha}{}_{\Omega_2} u_\alpha$. And also write $\Omega_1 \leq \Omega_2$, if $\Omega_1 \leq \Omega_2$ and for $\neg$ the stronger $\neg_{\Omega_1} u = \neg_{\Omega_2} u$ is satisfied.

For example, $\mathcal{J}(K) \leq \mathcal{B}(K)$ [although $\mathcal{B}(K) \subseteq \mathcal{J}(K)$] and, furthermore, in this case $\bigwedge_{\alpha}{}_{\mathcal{J}} = \bigwedge_{\alpha}{}_{\mathcal{J}}$. Another meaningful example is such: it is well known that for any complete Heyting algebra Ω there is defined a complete Boolean algebra $\mathcal{B}(\Omega)$, for which $\Omega \subseteq \mathcal{B}(\Omega)$ and $\Omega \leq \mathcal{B}(\Omega)$; furthermore, in this case $\bigvee_{\alpha}{}_{\Omega} = \bigvee_{\alpha}{}_{\mathcal{B}}$ (In the Stone realization the algebras Ω are declared by open complements of compact-open sets.)

If $B \subseteq \Omega$ and Ω is a complete Heyting algebra, then let $[\![k = t]\!]_\Omega = \vee_\Omega \{e \in B | e \cdot k = e \cdot t\}$. This valuation is extended to the class of all formulas $\varphi(k_1, \ldots, k_n)$, where $k_1, \ldots, k_n \in K$, precisely the same as in the case $\Omega = \mathcal{J}(K)$.

<u>Proposition 3.</u> If $\Omega_1 \leq \Omega_2$ and K is a normal ring, and φ is in prenex normal form, then $[\![\varphi]\!]_{\Omega_1} \leq [\![\varphi]\!]_{\Omega_2}$. In particular, $([\![\varphi]\!]_{\Omega_1} = 1) \Rightarrow ([\![\varphi]\!]_{\Omega_2} = 1)$.

Proof by induction.

<u>COROLLARY.</u> If $\Omega_1 \leq \Omega_2$ and K is normal, then for φ that same assertion is satisfied in prenex form. In particular, $[\![\varphi]\!]_{J(K)} \leq [\![\varphi]\!]_{\mathcal{B}(K)}$, and for a quantifier-free φ we have $[\![\varphi]\!]_{J(K)} = [\![\varphi]\!]_{\mathcal{B}(K)}$.

<u>Proposition 4.</u> If T is an AE-theory and for a normal ring K, we have $\{K_p\} \models T$, then $[T]_{B(K)} = 1$

<u>Proof.</u> In accord with [3, p. 388] we obtain $[\![T]\!]_{\mathcal{J}} = 1$ (This transition for specific conditions on the sheaf is valid also for the more general theories in the role of T, that is discussed in the paper of the author mentioned in Sec. 2.) and we apply the corollary to Proposition 3.

Let us call the class $\mathcal{K}$ normal if for any $K \in \mathcal{K}$ there exists $L \in \mathcal{K}$, for which $K \subseteq L$, $L \models \phi_1$. Thus, we obtain the desired:

Proposition 5. Let the theory T be model-imbeddable in the theory T_1 (and this is derivable in ZFC). If T is an AE-theory and T_1 is a totally closed theory, and $\mathcal{K}$ is a normal class, then $\mathcal{K}$ is imbeddable in $\mathcal{K}_1$.

Proof. Let $K \in \mathcal{K}$ and by the normality of $\mathcal{K}$ one can consider that $K \models \phi_1$. Let $\bar{\mathcal{B}}$ be the Dedekind completion of B(K). By Proposition 4, $[\![T]\!]_{\bar{\mathcal{B}}} = 1$; that is, $[\![K' \models T]\!]_{\bar{\mathcal{B}}} = 1$. By hypothesis, $\forall x \exists f (x \models T \Rightarrow x \subseteq f \wedge f \models T_1)$ is derivable in ZFC. Hence, by accessibility in $V^{\bar{\mathcal{B}}}$ we obtain $[\![K' \subseteq f_1 \wedge f \models T_1]\!]_{\bar{\mathcal{B}}} = 1$, where $f \in V^{\bar{\mathcal{B}}}$. Let us denote $\hat{f}^{\bar{\mathcal{B}}} = L$. Then $K \subseteq L$ and we show that $L \in \mathcal{K}_1$ as is required to prove.

Actually $\bar{\mathcal{B}}$ is imbedded in B(L) according to the rule $b \mapsto b \cdot 1 + \neg b \cdot 0$. Therefore, one can consider that $\bar{\mathcal{B}} \subseteq B(L)$. Let us denote $L_{(p_0)} = L/\bar{p}_0$, where p_0 is a point of the Stone space $S(\bar{\mathcal{B}})$ of the Boolean algebra $\bar{\mathcal{B}}$. It is easy to verify that $L_{(p_0)}$ coincides with the factorization of L by the equivalence relation $(k \sim t) \Leftrightarrow ([\![k \neq t]\!]_{\bar{\mathcal{B}}} \notin p_0)$, and also that $([\![f \models \varphi(k_1, \ldots, k_n)]\!]_{\bar{\mathcal{B}}} = 1) \Leftrightarrow \{L_{(p_0)}\} \models \varphi([k_1]_{\bar{p}_0}, \ldots, [k_n]_{\bar{p}_0})$, where p_0 runs over $S(\bar{\mathcal{B}})$ and $k_1, \ldots, k_n \in L$ (by the induction on the length of φ). Thus, $\{L_{(p_0)}\} \models T_1$.

For any $p \in X(L)$ let us form $p_0 = p \cap \bar{\mathcal{B}} \subseteq \bar{\mathcal{B}}$. Moreover, $p_0 \in S(\bar{\mathcal{B}})$. It is clear that $\bar{p}_0$ is an ideal in L and $\bar{p}_0 \subseteq \bar{p}$, and $L_p = L/\bar{p} = (L/\bar{p}_0)(\bar{p}/\bar{p}_0) = L_{(p_0)}/\overline{p/\bar{p}_0}$. Let us remark that $q = p|\bar{p}_0$ has the properties: $q \subseteq B(L/\bar{p}_0)$ and q is closed with respect to $\wedge$ and does not contain $[1]_{\bar{p}_0}$: if $[1]_{\bar{p}_0} = [e]_{\bar{p}_0}$, where $e \in p$, then $1 - e = e_0 \cdot r$, $e_0 \in p_0$, and $1 = e + e_0 \cdot r = (e \vee e_0) \cdot 1$, $e \vee e_0 \in p$. A contradiction. Let us add to q all $[\ell]_{\bar{p}_0} \in B(L/\bar{p}_0)$ which $\leq [e]_{\bar{p}_0} \in q$; obtained in this manner is that q_1 is a proper ideal in $B(L/\bar{p}_0)$ and $\bar{q}_1 = \overline{p|\bar{p}_0}$. Thus, $L_p = L_{(p_0)}/\bar{q}_1$, where $q_1 \in X(L_{(p_0)})$, and by hypothesis, we obtain $L_p \models T_1$.

Let us call the class $\mathcal{K}$ atom-free if $\forall K \in \mathcal{K} \exists L \in \mathcal{K} (K \subseteq L \wedge L \models \phi_2)$. Let us call the class $\mathcal{K}$ completely normal if for any $K \in \mathcal{K}$, $K \models \phi_2$ there exists $L \in \mathcal{K}$, $K \subseteq L$, for which $L \models \phi_1 \wedge \phi_2$. Let us call the class $\mathcal{K}$ abelian if for any $K \in \mathcal{K}$, $K_1 \models \phi_1 \wedge \phi_2$ there exists $L \in \mathcal{K}$, $K \subseteq L$, for which $L \models \phi_1 \wedge \phi_2 \wedge \phi_3$.

COROLLARY. Let T be model-imbeddable in the theory T_1 (and this is deducible in ZFC). If T is an AE-theory and $T_1 \vdash \phi_5$, and $\mathcal{K}$ is an atom-free, completely normal class and the class $\mathcal{K}_1$ is abelian, then $\mathcal{K}$ is imbedded in $\{K \in \mathcal{K}_1 | K \models \phi_1 \wedge \phi_2 \wedge \phi_3 F\}$.

Remark. Considering the explicit form of the axioms φ' for $\mathcal{K}_1$ we obtain that it is necessary to transfer to the factor only implications of the form $e_0 \cdot k = 0 \Rightarrow e_0 \leq e$, that is done with the help of the technique of lifting idempotents. The commutativity of K with the help of the lifting of idempotents allows one to obtain the commutativity of L. Then, the condition of commutativity of the class $\mathcal{K}_1$ can be omitted and replaced in the definition of a totally closed theory $q \in X_1(L_{(p_0)})$ in $q \in X(L_{(p_0)})$.

In essence, the version of Proposition 5 is the following:

Proposition 6. a) If $T \vdash \psi$, then $(T' + \phi_{1\_} \vdash (\psi_\neg)^0$, where $\psi_\neg$ is obtained from $\neg\psi$ by a transition to disjunctive normal form, and the translation $\varphi \mapsto \varphi^0$ is defined below.

b) If $T \vdash \psi$, then $(T' + \phi_1) \vdash \psi+$, where $\varphi \mapsto \varphi^{+}$ is defined below.

To prove Proposition 6, we require the following lemmas:

LEMMA 1. If K is normal and $[\![\varphi(\bar{k})]\!]_{\mathcal{B}(K)} = 1$, then $[\![\varphi_{\neg}(\bar{k})]\!]_{\mathcal{J}(K)} = 0.$

Proof. If $[\![\psi_{\neg}]\!]_{\mathcal{J}} \neq 0$, then by the corollary to Proposition 3 $[\![\psi_1]\!]_{\mathcal{B}} \neq 0$. Hence $[\![\neg\psi]\!]_{\mathcal{B}} \neq 0$, $[\![\psi]\!]_{\mathcal{B}} < 1$. A contradiction.

LEMMA 2. Let K be normal and $\bar{k} \subseteq K$.

a) The predicate $[\![\varphi(\bar{k})]\!]_{J(k)} = 0$ is expressible in K; that is, equivalent to $K \models \varphi^{0}(\bar{k})$, where φ is in disjunctive normal form.

b) The predicate $[\![\psi_{\neg\neg}(\bar{k})]\!]_{J(k)} = 1$ is expressible in K; that is, equivalent to $K \models \psi^{\perp}(\bar{k})$, where $\psi_{\neg\neg}$ is obtained from ψ in prenex form by the addition of $\neg\neg$ before each $\exists$

Proof. a) Let us denote the quantifier-free part of φ by $\bigvee_{s} \psi_s$. It is clear that $[\![\bigvee_s \psi_s]\!]_{\mathcal{J}} = 0 \Leftrightarrow \forall s \forall e \forall e_0 \forall t_1 \exists t\, [\![e^2 = e \wedge et = te \wedge ek_1 = et_1 \wedge \ldots \wedge (e_0^2 = e_0 \wedge e_0 t_1 = t_1 e_0 \wedge e_0 k_2 = e_0 t_2 \Rightarrow e_0 \leq 1 - e) \wedge \ldots) \Rightarrow e \leq 0$. Thus, for a quantifier-free φ the translation φ^{0} is defined. For $\exists$ let $(\exists k \varphi)^{0} = \forall k \varphi^{0}$. For $\forall$ let $(\forall k \varphi)^{0} = \forall e \exists t, k\, (e^2 = e \wedge et = te \wedge (e \leq [\![\varphi(k)]\!]_{\mathcal{J}}) \Rightarrow e = 0)$. Here the predicate $e \leq [\![\varphi]\!]_{\mathcal{J}}$, in its turn, is revealed by induction (and, consequently, also expressible in K): $e \leq [\![\exists x \varphi]\!]_{\mathcal{J}} \Leftrightarrow \exists k (e \leq$ $\leq [\![\varphi(k)]\!]_{\mathcal{J}})$ and $e \leq [\![\forall x \varphi]\!]_{\mathcal{J}} \Leftrightarrow \forall k (e \leq [\![\varphi(k)]\!]_{\mathcal{J}})$, where $e \leq [\![\bigvee_s \psi_s]\!]_{\mathcal{J}}$ is determined similarly to Proposition 2 [instead of $\prod_s (1 - e_s) = 0$ let us write $\prod_s (1 - e_s) \leq 1 - e$]. It is clear that $[\![\exists k \varphi]\!]_{\mathcal{J}} = 0 \Leftrightarrow \forall k ([\![\varphi(k)]\!]_{\mathcal{J}} = 0) \Leftrightarrow \forall k \varphi^{0}(k)$ $[\![\forall k \varphi]\!]_{\mathcal{J}} = 0 \Leftrightarrow \forall e \exists t, k\, (e^2 = e \wedge et = te \wedge (e \leq [\![\varphi(k)]\!]_{\mathcal{J}}) \Rightarrow e = 0) \Leftrightarrow (\forall k \varphi)^{0}.$

b) For a quantifier-free ψ this predicate is expressed the same as $e \leq [\![\psi]\!]_{\mathcal{J}}$ in the previous point. Let us translate the quantifier prefix Q in the following manner. For the pair of connectives $\exists\forall$ in ψ we have: $[\![\neg\neg\exists x \forall y \psi_0]\!]_{\mathcal{J}} \geq e_0$, $e_0 \in B(K)$ is equivalent ($[\![\exists x \psi_0]\!]$ is an open set dense in e_0); further, let us transform $\forall e \exists t \exists e_1 \forall t_1 \exists k \forall k_1 (e^2 = e \wedge et = te \wedge e \neq 0 \wedge e \leq e_0 \Rightarrow e_1^2 = e_1 \wedge e_1 t_1 = t_1 e_1 \wedge e_1 \neq 0 \wedge e_1 \leq e \wedge e_1 \leq [\![\psi_0(k, k_1)]\!]_{\mathcal{J}}$, where $e_1 \leq [\![\psi_0(k, k_1)]\!]_{\mathcal{J}}$ is revealed by induction. For $\forall$ at the start of Q we have: $[\![\forall x \psi_0]\!]_{\mathcal{J}} \geq e \Leftrightarrow \forall k ([\![\psi_0]\!]_{\mathcal{J}} \geq e)$. For $\exists$ at the end of Q we have a particular case of the pair of connectives $\exists\forall$.

Proof of Proposition 6. a) In an arbitrary ring K let $K \models (T' + \phi_1)$. Then $[\![T]\!]_{J(K)} = 1$. By the corollary to Proposition 3 $[\![T]\!]_{\mathcal{B}(K)} = 1$. By hypothesis $[\![\psi]\!]_{\mathcal{B}(K)} = 1$ and Lemma 1 and 2 $[\![\psi_{\neg}]\!]_{J(K)} = 0$, $K \models (\psi_{\neg})^{0}$.

b) Let $K \models (T' + \phi_1)$. Then $[\![T]\!]_{J(K)} = 1$. Let us denote by $T_{\neg\neg}$ the Gödel negative translation of all the formulas from T (formulas from T in prenex form). Due to the normality of K, we obtain $[\![T_{\neg\neg}]\!]_J = 1$. Since $T_{\neg\neg}$ $\psi_{\neg\neg}$, then $[\![\psi_{\neg\neg}]\!]_J = 1$, where $\psi_{\neg\neg}$ is exactly as said in Lemma 2 "b". By this lemma $K \models \psi+$.

Remark. A similar proposition is also satisfied for other pairs in the role of $\langle J, \mathcal{B} \rangle$. The expressibility of the mentioned predicates occurs also for other languages. One can weaken the condition of normality of K; for example, replacing it by the condition $\{[\![k = 0]\!]_{J(K)} | k \in K\} \subseteq B(K)$.

Let us consider the question of the transfer of model-completeness to the "reverse side."

Proposition 7. If $\{K \mid \{K_p\} \models T_1,\ K \models (T_1)' + \phi_1 + \phi_2\} \subseteq \mathcal{K}_1 \subseteq \{K \mid \{K_p\} \models T_1\}$, where $T \vdash \phi_5$ and $T \subseteq T_1$, and $\mathcal{K}_1$ is a model-companion for $\mathcal{K}$ with the condition $\forall K \models T \forall L \in \mathcal{K}_1 \exists p_1 \in X(L)$ $(K \subseteq L \Rightarrow \bar{p}_1 \cap K = \overline{p_1 \cap K})$, then T_1 is a model-companion for T.

Proof. Let $F_1 \subseteq F_2$ be two models for T_1 and ψ be a primitive (over F_1) formula $F_1 \models \neg\psi$. The rings $\bar{F}_1 = C(X_0, F_1)$, $\bar{F}_2 = C(X_0, F_2)$ belong to $\mathcal{K}_1$ and $F_1 \subseteq F_2$. For the A-formula $\neg\psi$ (as also for any AF-formula) in the normal ring $[\![\neg\psi]\!]_{\bar{F}_1} = 1$ and, in correspondence with Proposition 2 for the translation φ', corresponding to $\neg\psi$, we have $\bar{F}_1 \models \varphi'$. Then $\bar{F}_2 \models \varphi'$ and by "uniformity" (not noted explicitly in Proposition 2) $[\![\neg\psi]\!]_{\bar{F}_2} = 1$; that is, $F_2 = (\bar{F}_2)_p \models \neg\psi$. Thus, T_1 is a model-complete theory.

Let F be a model of T. Then F is a stalk of the corresponding $\bar{F} \in \mathcal{K}$ and $\bar{F} \subseteq L \in \mathcal{K}_1$. In the language of covering spaces, this signifies that $\bar{f}_{p_1}$: $(\bar{F}_p) \to L_{p_1}$, where p_1 is any point from X(L). By hypothesis, one of the $\bar{f}_{p_1}$ is a monomorphism. Therefore, F is imbedded in L_{p_1} — one of the models for T_1.

COROLLARY. The class of all abelian regular rings does not have a model-companion $\mathcal{K}_1$ of the form expressed in Proposition 7, where T_1 includes the axioms of the division ring.

Proof. In this case, T is a theory of all division rings, which does not have a model-companion.

Remark. One can formulate conditions for which an arbitrary companion has the indicated form (and weaken the condition $T \vdash \phi_5$). Finally, the corollary pertains also to the subclasses, the localizations of the elements of which embrace the class of all division rings.

One can present many examples pertaining to Theorem 4 and its corollaries. For example, let T^* be a model-complete theory of a central algebra over an algebraically closed field. Let us for concreteness say that T^* is the theory of quaternions. Then, the corresponding class $\mathcal{K}^*$ is model-complete. In other words, the class of rings for which all the localizations are elementarily equivalent to the division ring of quaternions, is model-complete.

Let us call 1-primitive any such primitive formula in which there is no more than a single negation.

Proposition 8. If the theory T decides all the 1-primitive formulas, then $\mathrm{Th}\{K \mid \{K_p\} \models T,\ K \models \phi_2\}$ decides all the E-formulas.

Proof. Let K be an arbitrary element of this class. Since $K \models \phi_1$, then $\forall \ell \forall p (K_p \models \psi_\ell) \Rightarrow L \models \psi$ and everywhere $\exists \ell_0 \ \forall p (K_p \models \neg\psi_{\ell_0}) \Rightarrow L \models \neg\psi$. By hypothesis $T \vdash \psi_\ell$ for all ℓ or $\exists_{\ell_0}$ $(T \vdash \neg\psi_{\ell_0})$. Correspondingly $\forall \ell \forall p$ $(K_p \models \psi_\ell)$ or $\exists \ell_0 \forall p$ $(K_p \models \neg\psi_{\ell_0})$.

COROLLARY. If $\mathcal{K}^*$ is a Boolean absolute class and T^* decides all the 1-primitive formulas, then $\mathrm{Th}\,\mathcal{K}^*$ is a complete (and model-complete) Horn theory.

The basic results of the author's note were reported in a seminar under the direction of Prof. V. A. Smirnov in the spring of 1984.

Note Added in Proof. The condition of commutativity of ϕ_3 is, in essence, not used in the proofs, one can replace it by the condition of Boolean validity: $(K, L \in \mathcal{K}^* \wedge K \subseteq L) \Rightarrow B(K) \subseteq B(L)$; in particular, R_0 is any Boolean valid subclass.

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A QUESTION OF HIGMAN

A. S. Morozov

UDC 512.54.05

Our aim is to answer a question asked by Higman in a private conversation with Belegradek and Ershov.

The question runs as follows: "Is it true that any finitely generated group in which the equality problem is co-enumerable can be embedded in the group of all recursive permutations of the natural numbers?" A positive answer to this question seemed to be implied by the following results: first [1], any numerated set (δ, ν) such that the set $\{\langle x, y\rangle \mid \nu x \neq \nu y\}$ is enumerable is equivalent in the category of numerated sets to an object (δ', ν'), where ν' is a computable numeration of the family of general recursive functions δ'; second, it is obvious that any finitely generated group of recursive permutations of the natural numbers has a co-enumerable equality problem. Nevertheless, the general answer to the question turns out to be negative. That is the main result of this paper.

We go on to notation.

The permutation interchanging a and b and leaving all other elements fixed will be denoted by (a, b).

If $p_i = \{x \mid m_i \leq x \leq r_i\}$, i = 0, 1, are intervals in the set of natural numbers and k is a natural number, then the notation $k < p_i$, $p_0 < p_1$ will mean $k < m_i$, $r_0 < m_1$, respectively. By $|p_0 - p_1|$ we will mean the distance between the intervals — the number of elements k such that $p_0 < k < p_1$ if $p_0 < p_1$, or the number of elements such that $p_1 < k < p_0$ if $p_0 < p_1$, or the number of elements such that $p_1 < k < p_0$ if $p_1 < p_0$. If τ is a term, then we let $(\tau)_t{}^s$ denote the result of substituting s for t in τ.

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