TRANSFER FROM DEDUCIBILITY IN THE CLASSICAL SET THEORY TO DEDUCIBILITY IN INTUITIONISTIC SET THEORY FOR THE LANGUAGE OF RINGS

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This article is a direct continuation of our previous publication [6] and is essentially a detailed study of one aspect of Theorem 1 cited in [6]. We show that for a large class of properties in the language of ring theory their deducibility in the classical Zermelo-Frankel set theory ZF implies their deducibility in intuitionistic set theories ZFI and ZFI' defined by Grayson in [1]. Furthermore, the derivation of the latter deducibility from the former one is done finitely (by means of a primitively recursive function), i.e., completely explicitly, and in such a way that the increase in the length of the second conclusion depends linearly on the length of the first conclusion.

Theories ZFI and (to a lesser degree) ZFI' have properties of disjunction and existence. Recall that the first of these properties means that if $ZFI \vdash \exists x \varphi(x)$, then a therm $t=\{y \mid \varphi(y)\}$ is defined explicitly in the language ZF and satisfies $ZFI \vdash \varphi(t)$ and $ZFI \vdash \exists! x(x=t)$. In the theory ZFI' this property means the same, but for certain restrictions on the form of the formula φ. Therefore, we can envision the following "programming" scheme: If $ZF \vdash \exists x \varphi(x)$ then $ZFI \vdash \exists x \varphi'(x)$ and therefore $ZFI \vdash \varphi'(t)$ where t is the corresponding therm. Here $\varphi' = \varphi$ and φ' is close in meaning to φ. If the formula $\exists x \varphi(x)$ is replaced by a formula $\forall \alpha \exists x \varphi(\alpha, x)$ then a similar scheme holds, where the therm t now depends on a parameter α. This scheme combines the expressive power of the language and theory ZF with a certain effectiveness (the possibility of constructing a program). Besides, apart from this circumstance, the topic of transferring from the classical logic to an intuitionistic one under which a formula φ does not acquire "meaningless" insertions (for example, connectives $\neg\neg$ and relativizations to the class of stable sets) has long been actively studied, beginning with [2]. In the latter publication, Novikov proved a theorem corresponding to this scheme and stated a conjecture that his theorem is "true for very broad conditions." We think that this conjecture in particular is shown to be true in this paper.

We also note a connection between our paper and Markov's principle. The theorem by Novikov mentioned above and other theorems on this topic can be considered to be special cases of a general semantic Markov principle.

Unless otherwise stated, throughout this paper the metamathematics used is deducibility in the theory ZFI; the other two possibilities are deducibility in the theory ZFI' and

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strictly finite metamathematics. Grayson made a similar assumption in [1].

In the sequel Ω is a fixed complete Heyting algebra. Grayson in his fundamental [1] defined a class V^{Ω} (all of whose functions are defined everywhere) and a valuation $[\![\cdot]\!]_{\Omega}$ for the language ZF with a family of parameters V^{Ω}. For more details on definitions and properties cited below (up to Proposition 1) see [1, 3].

The mentioned valuation satisfies the following properties:

$$[\![f\in g]\!]_{\Omega}=\bigcup\{g(h)\wedge[\![f=h]\!]_{\Omega}\mid h\in\mathcal{D}(g)\},$$
$$[\![f=g]\!]_{\Omega}=\bigcap\{[\![h\in f]\!]_{\Omega}\leftrightarrow[\![h\in g]\!]_{\Omega}\mid h\in\mathcal{D}(f)\cup\mathcal{D}(g)\}. \tag{1}$$

Let LEM be the scheme of axioms of the law of the excluded middle. This valuation has the following values on logical axioms of intuitionistic computation of predicates and on special axioms of the theory ZF. For axioms of capacity, pairing, union, power-set, infinity, separation, and $\mathcal{E}$-induction (which is equivalent to foundedness if the LEM axiom holds) it is equal to the unit 1 (in Ω). The same is true for the axiom of substitution and even the axiom of collection, but then the theory ZFI' has to be chosen as the metamathematics.

Let $Ord(x)$ be an ordinary formula stating " x is an ordinal." We define the rank of a set as $rk(x)\leftrightharpoons\bigcup\{rk(y)^{+}\mid y\in x\}$ and $V_{\alpha}\leftrightharpoons\bigcup\{\mathcal{P}(V_{\beta})\mid\beta\in\alpha\}$, $V\leftrightharpoons\bigcup\{V_{\alpha}\mid Ord(\alpha)\}$, where $\mathcal{P}(x)$ is the power set of the set x. We obtain

$$Ord(rk(x)),\ \forall x;\ \forall x\,(Ord(x)\Rightarrow rk(x)=x);\ \ \forall x\,(x\in V_{rk(x)^{+}});$$
$$\alpha\subseteq\beta\Rightarrow V_{\alpha}\subseteq V_{\beta};\ \ x\subseteq y\in V_{\alpha}\Rightarrow x\in V_{\alpha};\ \ V_{\alpha}\cap On=rk(V_{\alpha}). \tag{2}$$

Here α and β take values in On, on the class of all ordinals.

We define a mapping $(\cdot)^{\vee}\colon V\to V^{\Omega}$ by letting $x^{\vee}\in V^{\Omega}_{rk(x)}$ $x(f)\leftrightharpoons\bigcup\{[\![f=y]\!]\mid y\in x\}$, where

$$f\in V^{\Omega}_{<rk(x)}. \tag{3}$$

In the sequel we omit the index in the notation of this and subsequent valuations where the meaning is clear.

The following properties hold:

1) $[\![f=g]\!]\wedge[\![\varphi(f)]\!]\leq[\![\varphi(g)]\!]$, where φ is any formula with parameters;

2) $[\![\forall x\in f\ \varphi(x)]\!]=\bigcap\{f(x)\to[\![\varphi(x)]\!]\mid x\in\mathcal{D}(f)\}$, $[\![\exists x\in f\ \varphi(x)]\!]=\bigcup\{f(x)\cap[\![\varphi(x)]\!]\mid x\in\mathcal{D}(f)\}$ where φ is any formula with parameters.

Definition 1. A function $f\in V^{\Omega}$ is called extensional with respect to valuation $[\![\cdot]\!]_{\Omega}$ if

$$\forall x,y\in\mathcal{D}(f)\ (f(x)\cap[\![x=y]\!]\leq f(y)). \tag{4}$$

Proposition 1. If f is an extensional function then $\forall y\in\mathcal{D}(f)\,([\![y\in f]\!]=f(y))$.

The proof is obvious.

Definition 2. Let K be an associative ring with a unit. Let $\mathcal{B}(K)$ be a Boolean algebra of all central idempotents of the ring K, and $\mathcal{T}(K)$ a complete Heyting algebra of all simple ideals of $\mathcal{B}(K)$. The class $V^{\mathcal{T}(K)}$ is called a Heyting-valued universe corresponding to the ring K.

Let $A(K)$ be a zero-dimensional complete Heyting algebra of all J-operators on the algebra $\mathcal{T}(K)$ and $\mathscr{B}(K)$ a sublattice of $A(K)$ of all stable elements of $A(K)$. (An element A in u is called stable if $\neg\neg u = u$ where $\neg$ is computed in A.) This lattice $\mathscr{B}(K)$ is a complete Boolean algebra. All of this has been studied in greater detail in [4, 3]. The class $V^{\mathscr{B}(K)}$ is called a Boolean-valued universe corresponding to the ring K. In the above and subsequent notations we shall frequently omit the index K.

Recall that $\mathcal{T}$ canonically imbeds into $\mathscr{B}$ so therefore

$$V^{\mathcal{T}} \subseteq V^{\mathscr{B}}.$$

We define an auxiliary valuation:

$$[\![k=t]\!] \rightleftharpoons \{e\in \mathcal{B}(K) \mid e\cdot k = e\cdot t\} \in \mathcal{T}(K).$$

This estimate, like any other valuation, canonically extends to the set of all sentences of the corresponding language, in this case the set of all sentences in the language of rings with a parameter set K. One such extension of this valuation is obtained for values in $\mathcal{T}(K)$ (which we denote by $[\![\cdot]\!]_{\mathcal{T}}$), and another valuation is obtained for values in $\mathscr{B}(K)$ (which we denote by $[\![\cdot]\!]_{\mathscr{B}}$). Clearly, $[\![k=t]\!]_{\mathcal{T}} = [\![k=t]\!]_{\mathscr{B}}, \forall k,t\in K$ For more details see [3].

We note that

$$K\in V^{\mathcal{T}}_{rk(K)},\ K: V^{\mathcal{T}}_{<rk(K)} \to \mathcal{T} \text{ and } \forall k\in K\,(k\in V^{\mathcal{T}}_{rk(K)} \wedge rk(k)\in rk(K)),\ \{k \mid k\in K\}\subseteq V^{\mathcal{T}}_{<rk(K)}.$$

The same holds for $V^{\mathscr{B}(K)}$.

1) We define a function P_k by

$$P_k(\cdot): V^{\mathcal{T}}_{<rk(K)} \to \mathcal{T} \text{ } \kappa\ni\kappa\ P_k(f) \rightleftharpoons \cup\{[\![f=t]\!]_{\mathcal{T}} \cap [\![k=t]\!]_{\mathcal{T}} \mid t\in K\}.$$

Clearly, $\{P_k \mid k\in K\}\subseteq V^{\mathcal{T}}_{rk(K)}$. We define a function $K'_{\mathcal{T}}: V^{\mathcal{T}}_{<rk(K)}+ \to \mathcal{T}$ by $K'(f) \rightleftharpoons \cup\{[\![f=P_k]\!]_{\mathcal{T}} \mid k\in K\}$, where $f\in V^{\mathcal{T}}_{<rk(K)^+} = V^{\mathcal{T}}_{<rk(K)} \cup V'^{\mathcal{T}}_{rk(K)}$. Clearly, $K'\in V^{\mathcal{T}}_{rk(K)^+}$.

2) We similarly define a function $K'_{\mathscr{B}}$, replacing in the definition of $K'_{\mathcal{T}}$ the algebra $\mathcal{T}$ with algebra $\mathscr{B}$.

3) A ring K is called a $\mathcal{T}$-ring if $\forall k,t\in K\ ([\![k^{\vee}=t^{\vee}]\!]^{\mathcal{T}} \leq [\![k=t]\!]_{\mathcal{T}}$, and it is called a $\mathscr{B}$-ring if $\forall k,t\in K\ ([\![k^{\vee}=t^{\vee}]\!]_{\mathscr{B}}^{\mathcal{T}} \leq [\![k=t]\!]_{\mathscr{B}}$. It is easy to see a $\mathscr{B}$-ring is a $\mathcal{T}$-ring.

4) We define a function $+': V^{\mathcal{T}}_{<\alpha} \to \mathcal{T}$ as $+'(f) \rightleftharpoons \cup\{[\![f=\langle P_r, P_s, P_{r+s}\rangle]\!]_{\mathcal{T}} \mid r,s\in K\}$ where α is such that $\langle P_r, P_s, P_t\rangle \in V^{\mathcal{T}}_{\alpha}$ for all P_r, P_s, P_t in $V^{\mathcal{T}}_{rk(K)}$. We similarly define $+', -', \cdot'$ and $0' \rightleftharpoons P_0$, $1' \rightleftharpoons P_1$. All the dashes denote an index $\mathcal{T}$, since they are related to valuation $[\![\cdot]\!]_{\mathcal{T}}$. Replacing this valuation with $[\![\cdot]\!]_{\mathscr{B}}$ we obtain definitions of functions $+'_{\mathscr{B}}, -'_{\mathscr{B}}, \cdot$

$_{\mathcal{B}}', \mathcal{D}_{\mathcal{B}}', 1_{\mathcal{B}}'$. As before, we shall omit indices $\mathcal{T}$ and $\mathcal{B}$ if there is no danger of ambiguity.

Proposition 2. a) The functions P_k (for all $k \in K$) are extensional with respect to valuations $[\![\cdot]\!]_{\mathcal{T}}$ and $[\![\cdot]\!]_{\mathcal{B}}$.

b) A $\mathcal{T}$-ring K satisfies $[\![P_k = P_\ell]\!]_{\mathcal{T}} = [\![k=\ell]\!]$, $\forall k, \ell \in K$. A $\mathcal{B}$-ring K satisfies $[\![P_k = P_\ell]\!]_{\mathcal{B}} = [\![k=\ell]\!]$, $\forall k, \ell \in K$.

c) The following relations hold.

1) $[\![t_1 = r]\!] \cap [\![t_2 = s]\!] \leq [\![t_1 + t_2 = r+s]\!]$, $\forall t_1, r, t_2, s \in K$;

2) $[\![<P_r, P_s, P_{r+s}> \in +']\!]_{\mathcal{T}} = 1$, $\forall r, s \in K$. The same is true for valuation $[\![\cdot]\!]_{\mathcal{B}}$.

3) A $\mathcal{T}$-ring K satisfies the following: $[\![<P_r, P_s, P_\ell> \in +' \cap <P_r, P_s, P_t> \in +']\!]_{\mathcal{T}} \leq [\![P_\ell = P_t]\!]_{\mathcal{T}}$, $\forall r, s, \ell, t \in K$. The same holds for a $\mathcal{B}$-ring K with respect to valuation $[\![\cdot]\!]_{\mathcal{B}}$.

4) A $\mathcal{T}$-ring K satisfies $[\![+' : K'_{\mathcal{T}} x K'_{\mathcal{T}} \to K'_{\mathcal{T}}]\!]_{\mathcal{T}} = 1$. The same holds for a $\mathcal{B}$-ring K with respect to valuation $[\![\cdot]\!]_{\mathcal{B}}$.

Proof. a) We compute $P_k(f) \cap [\![f=g]\!]_{\mathcal{T}} = \cup\{[\![f=t]\!]_{\mathcal{T}} \cap [\![k=t]\!] \cap [\![f=g]\!]_{\mathcal{T}} \mid t \in K\} \leq \cup\{[\![g=t]\!]_{\mathcal{T}} \cap [\![k=t]\!] \mid t \in K\} = P_k(g)$. The valuation $[\![\cdot]\!]_{\mathcal{B}}$ is computed exactly the same way.

b) The existence property and Proposition 1 imply that the left-hand side can be computed as $\cap\{P_k(f) \leftrightarrow P_\ell(f) \mid f \in V^{\mathcal{T}}_{<rk(K)}\}$. We first check the inequality going one way: $[\![k=\ell]\!] \cap P_k(f) = \cup\{[\![f=t]\!]_{\mathcal{T}} \cap [\![k=t]\!] \cap [\![k=\ell]\!] \mid t \in K\} \leq \cup\{[\![f=t]\!]_{\mathcal{T}} \cap [\![\ell=t]\!] \mid t \in K\} = P_\ell(f)$. We obtain $[\![k=\ell]\!] \leq (P_k(f) \leftrightarrow P_\ell(f))$, $[\![k=\ell]\!] \leq [\![P_k = P_\ell]\!]_{\mathcal{T}}$.

We now check the inequality the other way: $[\![P_k = P_\ell]\!]_{\mathcal{T}} \leq \cap\{P_k(f) \to P_\ell(f) \mid f\}$ (letting f equal to k, we continue) $\leq P_k(k) \to P_\ell(k) = (\cup\{[\![k=t]\!]_{\mathcal{T}} \cap [\![k=t]\!] \mid t \in K\}) \to (\cup\{[\![k=t]\!]_{\mathcal{T}} \cap [\![\ell=t]\!] \mid t \in K\})$ (reducing the premise, leaving only the k-term, we continue) $\leq [\![([\![k=k]\!]_{\mathcal{T}} \cap [\![k=k]\!] = 1) \to (\dots)]\!] = \cup\{[\![k=t]\!]_{\mathcal{T}} \cap [\![\ell=t]\!] \mid t \in K\}$ (and using the assumption we obtain) $\leq \cup\{[\![k=t]\!] \cap [\![\ell=t]\!] \mid t \in K\} \leq [\![k=\ell]\!]$.

The valuation $[\![\cdot]\!]_{\mathcal{B}}$ is computed the same way.

c) 1) If e belongs to the left-hand side then $et_1 = er$, $et_2 = es$ so therefore $e(t_1+t_2) = e(r+s)$; i.e., e belongs to the right-hand side of the desired inclusion,

2) Clearly, we have $I = +'(<P_r, P_s, P_{r+s}>) \leq [\![<P_r, P_s, P_{r+s}> \in +']\!]_{\mathcal{T}}$;

3) The left-hand side (of this inequality) is computed as $\leq \bigcup_{f,g} \bigcup_{r1,s1,r2,s2} [\![f = <P_{r1}, P_{s1}, P_{r1+s1}> \cap f = <P_r, P_s, P_\ell> \cap g = <P_{r2}, P_{s2}, P_{r2+s2}> \cap g = <P_r, P_s, P_t>]\!] \leq \bigcup_{f,g,r1,s1,\dots} [\![P_{r1} = P_{r2} \cap P_{s1} = P_{s2} \cap P_{r1+s1} = P_\ell \cap P_{r2+s2} = P_t]\!]_{\mathcal{T}}$ using statement b) we continue] $\leq \bigcup_{r1,s1,r2,s2} [\![r1 = r2 \cap s1 = s2 \cap r1+s1 = \ell \cap r2+s2 = t]\!]$ (using relation 1 of this section we continue) $\leq \bigcup_{r1,s1,r2,s2} [\![r1+s1 = r2+s2]\!] \cap [\![r1+s1 = \ell \cap r2+s2 = t]\!] \leq \cup [\![\ell = t]\!]$ and once again using statement b) we obtain] $= [\![P_\ell = P_t]\!]_{\mathcal{T}}$;

4) We have to show that $[\![\forall f \in +'$ (where f is a triple of elements in $K']\!]_{\mathcal{T},\mathcal{B}} = 1$ and $[\![\forall x, y \in K' \exists f \in +' (P^1(f) = x \cap P^2(f) = y)]\!]_{\mathcal{T},\mathcal{B}} = [\![\forall f, g (P^1(f) = P^1(g) \cap P^2(f) = P^2(g) \Rightarrow P^3(f) = P^3(g))]\!]_{\mathcal{T},\mathcal{B}} = 1$, where P^i is the i-th term of the n-tuple. This is shown directly.

Definition 3. We define an interpretation of a formula φ of the language of rings in the language of a set theory ZF by induction on the construction of φ. The obtained interpretation will be denoted by $(\varphi)_K$. A formula $(\varphi)_K$ in the language ZF, for every free variable x in φ, contains a free variable P_x in addition to variables $K, +, -, \cdot, 0, 1$. Thus, an interpretation of a formula $\varphi \leftrightharpoons (x=y)$ is a formula $(\varphi)_K \leftrightharpoons (P_x = P_y)$. Note that the symbol = has different meanings on the left- and right-hand sides of the above equation; on the left it means identity, whereas on the right it means equicapacity. If $\varphi \leftrightharpoons (t_1+t_2 = y)$, then $(\varphi)_K \leftrightharpoons \exists u, v \in K\,[(t_1=u)_K \cap (t_2=v) \cap \langle u, v, P_y\rangle \in +']$. Here $(t_1=u)_K$ means that the formula $(t_1=u)$ in the language of rings is interpreted in the language ZF, the set-theoretical variable P_u corresponding to the ring variable u is replaced by a set-theoretical variable u, and $\langle\cdot,\cdot,\cdot\rangle$ denotes a triple. We similarly define interpretation of therms containing other operations. Propositional connectives are then transferred, and $\varphi \leftrightharpoons (\exists x\varphi(x))$ is interpreted as $\exists x \in K(\varphi)_K$. The case $\varphi \leftrightharpoons (\forall x\varphi)$ is defined similarly.

Proposition 3. Fix a ring K, and let $\bar{k} = \langle k_1, \ldots, k_n\rangle \in K$ and $P_{\bar{k}} \leftrightharpoons \langle P_{k1}, \ldots, P_{kn}\rangle$.

a) Let t be therm with free variables $\bar{x}$, y a variable in the language of rings, and $t(\bar{x}) = y$ a formula in the language of rings. Let $t(\bar{k}) = \ell$ be the corresponding sentence (where $\bar{k}, \ell \in K$). Then we have $[\![(t(\bar{P}_k) = P_\ell)_{K'}]\!]_{\mathcal{T}} = [\![t(\bar{k}) = \ell]\!]$ if K is a $\mathcal{T}$-ring, and $[\![t(\bar{P}_k) = P_\ell)_{K'}]\!]_{\mathcal{B}} = [\![t(\bar{k}) = \ell]\!]$ if K is a $\mathcal{B}$-ring.

b) Let $\varphi(\bar{x})$ be a formula with free variables $\bar{x}$ in the language of rings and $\varphi(\bar{k})$ the corresponding sentence $(\bar{k} \in K)$. Then $[\![(\varphi(\bar{P}_k))_{K'_{\mathcal{T}}}]\!]_{\mathcal{T}} = [\![\varphi(\bar{k})]\!]_{\mathcal{T}}$ if K is a $\mathcal{T}$-ring, and $[\![\varphi(\bar{P}_k))_{K'_{\mathcal{B}}}]\!]_{\mathcal{B}} = [\![\varphi(\bar{k})]\!]_{\mathcal{B}}$ if K is a $\mathcal{B}$-ring.

c) Under the same assumptions as in b) we have $[\![\langle K', +', -', \cdot', 0', 1'\rangle$ is a ring $]\!]_{\mathcal{T}} = [\![\langle K', +', -', \cdot', 0', 1'\rangle$ is a ring $]\!]_{\mathcal{B}} = 1$.

Proof. a) This is proven by induction on the length of the therm. The first step is contained in Proposition 2b).

We study the case appearing in Definition 3. We have

$$[\![((t_1+t_2)(\bar{P}_k) = P_\ell)_{K'}]\!]_{\mathcal{T}} = \cup\{K'(u) \cap K'(v) \cap [\![(t_1 = u)_{K'} \cap$$
$$\cap (t_2 = v)_{K'} \cap \langle u, v, P_\ell\rangle \in +']\!]_{\mathcal{T}} \mid u, v \in \mathcal{D}(K')\} \leq \cup\{[\![u = P_s \cap$$
$$v = P_r \cap (t_1 = P_s)_{K'} \cap (t_2 = P_r)_{K'} \cap \langle P_s, P_r, P_\ell\rangle \in +']\!]_{\mathcal{T}} \mid u, v \in V^{\mathcal{T}}_{<rk(K)}, s, r \in K\}$$

[using Proposition 2c) we continue] $\leq \cup\{[\![u = P_s]\!] \cap [\![v = P_r]\!] \cap [\![t_1 = P_s)_{K'}]\!] \cap [\![t_2 = P_r)_{K'}]\!] \cap [\![P_\ell = P_{s+r}]\!] \mid u, v;$ $s, r \in K\}$ (here $s+r$ is computed externally in the ring K, as is the case for valuations $[\![\cdot]\!]_{\mathcal{T},\mathcal{B}}) = \cup\{[\![u = P_s]\!]_{\mathcal{T}} \cap [\![v = P_r]\!]_{\mathcal{T}} \cap [\![t_1(\bar{k}) = s]\!] \cap [\![t_2(\bar{k}) = r]\!] \cap [\![\ell = s+r]\!] \mid u, v, s, r\}$ [again using Proposition 2c), we obtain] $\leq \cup\{[\![u = P_s]\!]_{\mathcal{T}} \cap [\![v = P_r]\!]_{\mathcal{T}} \cap [\![t_1(\bar{k}) + t_2(\bar{k}) = \ell]\!] \mid u, v, s, r\} \leq [\![(t_1+t_2)(\bar{k}) = \ell]\!]$.
We now show the opposite inequality $[\![t(\bar{k}) = \ell]\!]$ (where $t = t1 + t2$) $= [\![t_1^0 + t_2^0 = \ell]\!]$ (where $t_1^0 \leftrightharpoons t1(\bar{k})$ and $t_2^0 \leftrightharpoons t2(\bar{k})) \leq [\![P_{t_1^0+t_2^0} = P_\ell]\!]_{\mathcal{T}}$ (and in addition $[\![(t1(\bar{P}_k) = P_{t_1^0})_{K'}]\!]_{\mathcal{T}} = 1$, $[\![(t2(\bar{P}_k) = P_{t_2^0})_{K'}]\!]_{\mathcal{T}} = 1$ and $[\![\langle P_{t_1^0}, P_{t_2^0}, P_{t_1^0+t_2^0} \in']\!]_{\mathcal{T}} = 1$, so therefore) $\leq [\![\langle P_{t_1^0}, P_{t_2^0}, P_\ell\rangle \in +' \cap (t1(\bar{P}_k) = P_{t_1^0})_{K'} \cap (t2(\bar{P}_k) = P_{t_2^0})_{K'}]\!]_{\mathcal{T}}$ $\leq [\![((t1+t2)(\bar{P}_k) = P_\ell)_{K'}]\!]_{\mathcal{T}}$, since by letting $u \leftrightharpoons P_{t_1^0}$ and $P_{t_2^0}$ we obtain $K'(u) = K'(v) = 1$.

The case of the valuation $[\![\cdot]\!]_{\mathcal{B}}$ is treated in exactly the same way.

b) This is proven by induction on the length of the formula φ. For atomic formulas this has already been proven in a). For propositional connectives this equality is obvious. The case of quantifiers is dealt with as follows: $[\![(\exists x\varphi)_{K'}]\!]_{\mathcal{T}} = \bigcup\{K'(x) \cap [\![(\varphi(x,\bar{P}_k))_{K'}]\!]_{\mathcal{T}} \mid x\in\mathcal{D}(K')\} = \bigcup\{[\![x=P_t]\!]_{\mathcal{T}} \cap [\![\varphi(x,\bar{P}_k)]\!]_{\mathcal{T}} \mid x;\, t\in K\} = \bigcup\{[\![x=P_t]\!]_{\mathcal{T}} \cap [\![(\varphi(P_t,\bar{P}_k))_{K'}]\!]_{\mathcal{T}} \mid x,t\} = \bigcup\{[\![x=P_t]\!]_{\mathcal{T}} \cap [\![\varphi(t,\bar{k})]\!]_{\mathcal{T}} \mid x,t\} = \bigcup\{(\bigcup\{[\![x=P_t]\!]_{\mathcal{T}} \mid x\}) \cap [\![\varphi(t,\bar{k})]\!]_{\mathcal{T}} \mid t\} = [\![\exists\alpha\varphi(\alpha,\bar{k})]\!]_{\mathcal{T}}$. The quantifier $\forall$ is treated similarly: $[\![(\forall x\varphi)_{K'}]\!]_{\mathcal{T}} = \bigcup\{K'(x) \to [\![(\varphi(x,\bar{P}_k))_{K'}]\!] \mid x\in\mathcal{D}(K')\} = \bigcap\{K'(x) \to K'(x) \cap [\![(\varphi(x,\bar{P}_k))_{K'}]\!] \mid x\} = \bigcap\{K'(x) \to \bigcup\{[\![x=P_t]\!] \cap [\![(\varphi(x,\bar{P}_k))_{K'}]\!] \mid t\in K\} \mid x\} = \bigcap\{K'(x) \to \bigcup\{[\![x=P_t]\!] \cap (\varphi(P_t,\bar{P}_k))_{K'}]\!] \mid t\} \mid x\} = \bigcap\{K'(x) \to \bigcup\{[\![x=P_t]\!] \cap [\![\varphi(t,\bar{k})]\!] \mid t\} \mid x\} \leq \bigcap\bigcup\{[\![P_\ell = P_t]\!] \cap [\![\varphi(t,\bar{k})]\!] \mid t\} \mid \ell\in K\} \leq \bigcap\{[\![\varphi(\ell,\bar{k})]\!] \mid \ell\in K\} = [\![\forall x\varphi(x,\bar{k})]\!]$. To prove the converse inequality, it suffices to show that $\bigcap\{[\![\varphi(\ell,\bar{k})]\!] \mid \ell\in K\} \cap K'(x) \leq \bigcup\{[\![x=P_t]\!] \cap [\![\varphi(t,\bar{k})]\!] \mid t\in K\}$. The latter follows from an inequality $[\![x=P_r]\!] \cap [\![\varphi(r,\bar{k})]\!] \leq$ right-hand side.

The case of the valuation $[\![\cdot]\!]_{\mathcal{B}}$ is treated similarly.

Definition 4. 1) A normal $\mathcal{B}$-ring is called a (*)-ring. A formula "a K-normal $\mathcal{B}$-ring" in the language ZF is denoted by $*(K)$. A ring K is called normal if $(\Phi_1)_{K'}$, where Φ_1 is a formula in the language of rings that naturally expresses the following fact: $\forall k \exists e_0 \in B(K) \quad \forall e\in B(K)\ (e\cdot k=0 \Leftrightarrow e\leq e_0)$. The "usual" rings are normal.

2. A ring K is called strictly decidable (B-decidable) if its carrier K is strictly decidable (respectively, the set $B(K)$ is decidable). A set K is called strictly decidable if, for all $k,t\in K$ the heredity $\{k,t\}^+$ of a set $\{k,t\}$ (where $X^+ \rightleftharpoons \bigcup\{x^+ \mid x\in X\} \cup X$) satisfies the following property: $\forall a,b\in\{k,t\}^+ (a=b \cup a\neq b)$ where $a\neq b \rightleftharpoons \exists z((z\in a \cap z\notin b) \cup (z\notin a \cap z\in b))$. An arbitrary set X is called decidable if $\forall x, y\in X$, $(x=y \cup x\neq y)$; for the set $B(K)$ this is equivalent to a condition $\forall e\in B(K)\ (e=0 \cup e\neq 0)$. Clearly, in the classical sense , every ring is strictly decidable and B-decidable. On the other hand, in intuitionistic sense a ring with a countable carrier is strictly decidable, and every irreducible ring is B-decidable. A ring K is called irreducible if $(\Phi_3)_K$ where $\Phi_3 \rightleftharpoons \forall k(k^2=k \cap \forall t(kt=tk) \Rightarrow k=0 \cup k=1)$.

Proposition 4. a) If a ring K is B-decidable then it is a $\mathcal{T}$-ring.

b) If a ring K is strictly decidable (for example, has a countable carrier) then it is a $\mathcal{B}$-ring.

Proof. In both cases we have to show that $[\![k=t]\!] \leq [\![k=t]\!]$, where the left-hand side contains a valuation in either $\mathcal{T}$ or $\mathcal{B}$. To prove statement a) we have to show that $[\![k=t]\!]_{\mathcal{T}} \subseteq [\![k=t]\!]_{\mathcal{T}}$. Let $e\in[\![k=t]\!]_{\mathcal{T}}$, then $e=0 \cup e\neq 0$. If $e=0$ then e is contained in the right-hand side. Suppose that $e\neq 0$. By ε-induction on the first argument of the valuation we show that $(e\in[\![k=t]\!]_{\mathcal{T}}) \Rightarrow k=t$ (which immediately implies the desired inclusion). Suppose that $x\in k$. Then $e\in(k(x) \to [\![x\in t]\!]) = [\![x\in t]\!]$, since $x\in\mathcal{D}(k)$. Therefore, $e\in\bigcup\{t(g) \cap [\![x=g]\!] \mid g\in\mathcal{D}(t)\} = \bigcup\{[\![g=y]\!] \cap [\![x=g]\!] \mid g;\, y\in t\} \leq \bigcup\{[\![x=y]\!]_{\mathcal{T}} \mid y\in t\}$, $e=e_1+\ldots+e_n$, where $e_i\in[\![x=y_i]\!]$, $n\in N$. Induction on n shows that $e_i\neq 0$ for one of them. Therefore, $0\neq e_i\in[\![x=y_i]\!]$ and by the induction assumption we have $x=y_i\in t$. If $y\in t$, then we similarly obtain $e\in[\![y\in k]\!]$, $e=e_1+\ldots+e_n$ where $0\neq e_i\in[\![y=x_i]\!]$, $x_i\in k$. Descending in the first argument, we obtain $x_i=y$, $y\in k$.

Remark. a) If $e\in[\![k=t]\!]_{\mathcal{T}}\Rightarrow ek=et$, $\forall e\in\mathcal{B}(K)$, $\forall k,t\in K$ then K is a $\mathcal{T}$-ring.

b) First, $\mathcal{E}$-induction on the variable b shows that $a\notin b\triangleq[\![a\in b]\!]_{\mathcal{B}}=0$ and $a\#b\Rightarrow[\![a=b]\!]_{\mathcal{B}}=0$, $\forall a,b\in\{k,t\}^{+}$ where k, $t\in K$. If $a\notin b$ then $[\![a\in b]\!]_{\mathcal{B}}=\cup\{b(f)\cap[\![a=f]\!]\mid f\in\mathcal{D}(b)\}=\cup\{[\![f=y]\!]\cap[\![a=f]\!]\mid f;$ $y\in b\}\leq\cup\{[\![a=y]\!]\mid y\in b\}$ where $a\#y,\forall y\in b$. By the induction assumption the latter sum is equal to 0 and $[\![a\in b]\!]_{\mathcal{B}}=0$. If $a\#b$ then let, say, $\exists x(x\in a\cap x\notin b)$. We obtain $[\![a=b]\!]_{\mathcal{B}}\leq(a(x)\to[\![x\in b]\!])=[\![x\in b]\!]\leq\cup\{[\![x=y]\!]\mid y\in b\}$, where $x\#y$ $\forall y\in b$. By the induction assumption the latter sum is equal to 0.

Finally, either $k=t$ or $k\neq t$. In the first case $[\![k=t]\!]=1$. In the second case $[\![k=t]\!]_{\mathcal{B}}=0$.

Proposition 5. Let K be any $\mathcal{B}$-ring. Define $h: V_{\alpha}^{\mathcal{B}}\to\mathcal{B}$ where $V_{\alpha}^{\mathcal{B}}$ contains all pairs of elements of $V^{\mathcal{B}}_{<rk(K)^{+}}$ as $h(f)\leftrightharpoons\cup\{[\![f=<k,P_k>]\!]_{\mathcal{B}}\mid k\in K\}$. Then

a) $[\![h:K\to K'$ is an onto homomorphism $]\!]_{\mathcal{B}}=1$;

b) $[\![\{\ell\in K\mid h(\ell)=0\}=P_0]\!]_{\mathcal{B}}=1$; and

c) $[\![h:K/P_0\cong K']\!]_{\mathcal{B}}=1$.

Proof. a) We have $[\![h\subseteq K\times K']\!]_{\mathcal{B}}=1$ and $[\![\forall x\in K\ \exists y\in K'(<x,y>\in h)]\!]_{\mathcal{B}}=1$, $[\![\forall y\in K'\exists x\in K\ (<x,y>\in h)]\!]_{\mathcal{B}}=1$. Furthermore, $[\![\forall x,y\in h\ (P^1(x)=P^1(y)\Rightarrow P^2(x)=P^2(y))]\!]_{\mathcal{B}}=1$, since $[\![k=t]\!]_{\mathcal{B}}\leq[\![k=t]\!]_{\mathcal{B}}=[\![P_k=P_t]\!]_{\mathcal{B}}$. Finally, we show that the value of the following formula is equal to 1:

$$\begin{aligned}&\forall f,g,\ell\in h\ \forall x,y\in K^{V}[\![<x,y,P^1(f)>\in+\cap P^1(g)=x\ \cap\\&\cap P^1(\ell)=y)\Rightarrow<P^2(g),P^2(\ell),P^2(f)>\in+']\!],\ \text{since}\ [\![f=<k,P_k>]\!]\cap\\&\cap[\![g=<m,P_m>]\!]\cap[\![\ell=<n,P_n>]\!]\cap[\![x=r]\!]\cap[\![y=s]\!]\cap[\![<x,y,P^1(f)>\in+\\&\cap P^1(g)=x\cap P^1(\ell)=y]\!](\text{where } k,m,n,r,s\in K)\leq[\![<r,s,k>\in+\cap m=r\cap n=s)]\!]\leq\\&\leq[\![<m,n,k>\in+]\!]\leq[\![P_{m+n}=P_k]\!]\leq[\![<P_m,P_n,P_k>\in+']\!]\leq[\![P^2(g),P^2(\ell),P^2(f)\in+']\!].\end{aligned}$$

b) This statement is proven similarly.

c) This follows directly from previous results.

Definition 5. Let $\mathcal{X}(\cdot)$ be a formula in the language ZF (describing a family of sets $K<K,+,-,\cdot,0,1>$). The formula $\mathcal{X}(\cdot)$ is called absolute if $\forall K(\mathcal{X}(K)\Rightarrow K$ is a ring $\cap[\![\mathcal{X}(K/P_0)]\!]_{\mathcal{B}}=1)$.

Proposition 6. A usual formula in the language ZF describing the ring $\mathbb{Z}$ (or $\mathbb{Q}$ etc.) is an absolute formula.

Definition 6. An expression "$\bar{x}$-theory" T or "$\bar{x}$-formula" φ means that all formulas in T (respectively, formula φ) depend only on the free variables occurring in $\bar{x}$, where $\bar{x}=<x1,\ldots,xn>$, $n\geq 0$. For simplicity of formulations we assume that the theory T is recursively enumerable (even though this is not essential). Thus, a notation $K\models\ulcorner T\urcorner(\bar{k})$ is understood as $\forall n\in\omega K\models(\ulcorner T\urcorner(n)(\bar{k}))$, where $\ulcorner T$ is the code of a theory T $(\bar{k}=<k1,\ldots,kn>\in K,\ulcorner T\urcorner\in\mathbb{N})$. Since $\forall K(K\models\ulcorner\varphi\urcorner(\bar{k})\leftrightarrow\varphi(\bar{k}))_{K'}$ we can write $(T(\bar{k}))_{K}$. Another formulation of this question, using infinite disjunctions and conjunctions, is also possible. In this case we obtain

$[\![T(\bar{k})]\!] \leftrightharpoons \cap\{[\![\varphi(\bar{k})]\!] \mid \varphi \in T\}$ etc. Interestingly, this second path has already been noted in [1].

Definition 7. A formula φ in the language of rings is called a phi-formula if the premise of every implication appearing in φ satisfies the following two conditions: i) it does not contain the quantifier $\forall$; ii) it does not contain the quantifier $\exists$ in the domain of action of any implication. A theory consisting of phi-formulas is called a phi-theory. Given any theory T we can use certain natural methods to define a theory T^{φ} that consists of formulas composing T rewritten as phi-formulas. For example, an interesting case is Ar^{φ} where Ar is either Peano or complete arithmetic.

Definition 8. 1) A formula in the language of rings is called almost positive if it can be obtained from "blocks," i.e., formulas of the form $\forall \bar{t}(\varphi(\bar{t}) \Rightarrow \psi(\bar{t}))$, where φ is any Horn formula and ψ is any positive or an almost positive formula that has already been constructed, by imposing connectives $\exists, \forall, \cup, \cap$.

2) A sentence $\varphi(\bar{k})$ is called decidable with respect to a given ring K, where $\bar{k} \in K$ if $(\varphi(\bar{k}))_K \Rightarrow P[\varphi(\bar{k})]$, where the operator $P[\cdot]$ is defined by induction on the length of $\varphi(\bar{k})$ as follows. If φ is a positive formula then $P[\varphi] = T$. If this formula is a block then the value of P by definition is equal to $\exists \bar{t} \in K([\![\varphi(\bar{t})]\!] = 1 \cap \forall \bar{t} \in K((\psi(\bar{t}))_K \Rightarrow P[\psi(\bar{t})])$, furthermore, $P[\exists x\varphi] \leftrightharpoons \exists k \in K(\varphi(k))_K \cap P[\varphi(k)])$, $P[\forall x\varphi] \leftrightharpoons \forall k \in K(P[\varphi(k)])$, $P[\varphi \cap \psi] \leftrightharpoons P[\varphi] \cap P[\psi]$, $P[\varphi \cup \psi] \leftrightharpoons (P[\varphi] \cap (\varphi)_K) \cup (P[\psi] \cap (\psi)_K)$. A theory $T(\bar{k})$ is called decidable with respect to a given ring K if all its sentences are decidable with respect to this ring K.

Definition 9. Let $\varphi(\bar{k})$ be any sentence in the language of rings $(\bar{k} \in K)$. Define a translation $\varphi(\bar{k}) \longmapsto \varphi'(\bar{k}, e)$ (where φ' is also a formula in the language of rings and e is a special variable which assumes all values in $B(K)$] by induction on the length of φ. Let $(k=t)' \leftrightharpoons (e \cdot k = e \cdot t)$, $(\varphi \cap \psi)' \leftrightharpoons \varphi' \cap \psi'$, $(\exists x\varphi)' \leftrightharpoons \exists x\varphi'$, $(\forall x\varphi)' \leftrightharpoons \forall x\varphi'$ and the most important cases $(\varphi \Rightarrow \psi)' \leftrightharpoons \forall e0((e0 \leq e \cap \varphi'(e0)) \Rightarrow \psi'(e0))$ (here $e0$, $e1$, and $e2$ are special variables just like e), $(\varphi \cup \psi)' \leftrightharpoons \exists e1, e2((1-e1)\cdot(1-e2) = (1-e) \cap \varphi'(e1) \cap \psi'(e2))$. Let $\varphi'(\bar{k}) \leftrightharpoons \varphi'(\bar{k}, 1)$. If T is a theory then $T' \leftrightharpoons \{\varphi' \mid \varphi \in T\}$.

Proposition 7. Let K be any ring.

a) Condition $(\varphi'(\bar{k}, e))_K$ is equivalent to a condition $e \in [\![\varphi(\bar{k})]\!]_{\mathcal{T}(K)}$ for any formulas φ in the language of rings, parameters $\bar{k} \in K$ and $e \in B(K)$. In particular, $(\varphi'(\bar{k}))_K \leftrightarrow ([\![\varphi(\bar{k})]\!]_{\mathcal{T}(K)} = 1)$.

b) If $\varphi(\bar{k})$ is such that the premises of its implications contain positive formulas then $\neg\varphi'(\bar{k}, e)$ is a Horn formula.

Proof. a) This is proven by induction on the construction of the formula φ. For an atomic formula φ this is satisfied by the definition of the variation $[\![\cdot]\!]_{\mathcal{T}(K)}$. Cases $\varphi \cap \psi$, $\exists x\varphi$ (with the property of attainability of $[\![\cdot]\!]_{\mathcal{T}(K)}$ taken into account) and $\forall x\varphi$ are clear. A condition $((\varphi \cup \psi)'(e))_K$ is equivalent to $\exists e1, e2 \in B(K)$ $(e1 \cup e2 = e \cap e1 \in [\![\varphi]\!] \cap e2 \in [\![\psi]\!]) \Leftrightarrow e \in [\![\varphi \cup \psi]\!]$. A condition $((\varphi \Rightarrow \psi)'(e))_K$ is equivalent to a condition $\forall e0 \in B(K)$ $(e0 \leq e \Rightarrow (e0 \in [\![\varphi]\!] \Rightarrow e0 \in [\![\psi]\!])$, i.e., $\langle e \rangle \cap [\![\varphi]\!] \leq [\![\psi]\!]$.

b) We use induction on the construction of the formula φ to show that if φ is positive then φ' is also positive, and then prove the validity of the desired statement.

Proposition 8. Let K be a ring.

a) If $(\Phi_1)_K$ then $[\![\Phi_1]\!]_{\mathcal{T}(K)}=1$.

b) We have $[\![\Phi_3]\!]_{\mathcal{T}(K)}=1$.

c) If $(i)_K$ then $[\![i']\!]_{\mathcal{T}(K)}=1$ and $(\Phi_1)_K$ where pairs of symbols i . i' (i can take values 2, 3, 4, 5) denote the following pairs of properties in the language of rings: strictly biriccart $\longrightarrow$ prime ($i=2$) biregular $\longleftrightarrow$ quasisimple ($i=3$) strictly regular $\longleftrightarrow$ skew field ($i=4$) strictly riccart $\longleftrightarrow$ has no zero divisors ($i=5$) (for detailed definitions of these properties see [5, p. 389]).

Proof. a) Suppose that $(\Phi_1)_K$ is true. It suffices to show that $[\![\forall e[e^2=e \cap \forall t(et=te)\cap ek=0]\Leftrightarrow[e^2=e\cap\forall t(et=te)\cap e\leq e0]]\!]_{\mathcal{T}(K)}=1$ for every fixed k and $e0$ that corresponds to this k by the definition of normality. This valuation can be computed in both directions only for e such that the corresponding premise has value 1. In this case, if $[\![e^2=e\cap\forall t(et=te)\cap ek=0]\!]=1$, then $e\in\mathcal{B}(K)$, $ek=0$ and therefore $e\leq e0$. Conversely, if $[\![e^2=e\cap\forall t(et=te)\cap e\leq e0]\!]=1$ then $e\in\mathcal{B}(K)$, $e\leq e0$, and therefore $ek=0$.

b) We have to show that $[\![k^2=k]\!]\cap[\![\forall t(kt=tk)]\!]\leq[\![k=0]\!]\cup[\![k=1]\!]$ for all $k\in K$. Suppose that $e\in([\![k=k]\!]\cap\bigcap\{[\![kt=tk]\!]\mid t\in K\}$. Then $(ek)^2=ek$ and $ek\cdot t=t\cdot ek, \forall t\in K$, i.e., $ek\in\mathcal{B}(K)$. We obtain $ek\in[\![k=1]\!]$, since $ek^2=(ek)^2=ek$. On the other hand, $e(1-ek)\in[\![k=0]\!]$ since $e(1-ek)\cdot k=ek-e^2k^2=ek-(ek)^2=0$. Therefore, $ek\cup e(1-ek)\in[\![k=0\cup k=1]\!]$. Here $ek\cdot e(1-ek)=e(1-ek)\cdot ek=ek-e^2kek=0$. Therefore, $ek\cup e(1-ek)=ek+e(1-ek)=e$, i.e., $e\in[\![k=0]\!]\cup[\![k=1]\!]$.

c) The proof of this statement has been given in [5, pp. 388-393].

Proposition 9. Let K be any ring.

a) If φ is a positive sentence in the language of rings, then $(\varphi)_K$ implies $[\![\varphi]\!]_{\mathcal{T}(K)}=1$.

b) If ψ is a Horn sentence in the language of rings then $[\![\psi]\!]_{\mathcal{T}(K)}=1$ implies $(\psi)_K$.

c) Suppose that a theory $T(k)$ in the language of rings is decidable for a ring K ($\bar{k}\in K$). If $(T(\bar{k}))_K$, then $[\![T(\bar{k})]\!]_{\mathcal{T}(K)}=1$.

Proof. a) For atomic formulas this statement follows from the definition of the valuation. The rest of the proof proceeds using induction on the construction of the formula φ.

b) This is proven by induction on the length of the formula ψ. The cases of atomic formulas and connectives $\cap$, $\forall$ are obvious, and the case of the connective $\exists$ follows from the property of attainability of the valuation. The case $\varphi\Rightarrow\psi$ is studied as follows. Let $(\varphi)_K$. Then from a) we obtain $[\![\varphi]\!]=I$ and therefore $[\![\psi]\!]=1$, so by the induction hypothesis $(\psi)_K$.

c) We define the rank of an almost positive formula $\mathscr{X}$ in the language of rings. An almost positive formula such that all conclusions in its blocks contain positive formulas is defined to have rank 0. An almost positive formula such that all conclusions in its blocks contain almost positive formulas of rank up to and including n is defined to have rank $(n+1)$.

Now using induction on the rank of an almost positive formula $\mathscr{X}$ we obtain $((\mathscr{X})_K \cap P[\mathscr{X}]) \Rightarrow ([\![\mathscr{X}]\!]_{\mathcal{T}(K)} = 1)$. The desired assertion follows directly from this.

Suppose that $\mathscr{X}$ has rank 0. Let us consider induction on the construction of a sentence $\mathscr{X}$. Let $\mathscr{X}$ be a block. Assume that $(\mathscr{X})_K \cap P[\mathscr{X}]$. Taking into account $P[\mathscr{X}]$ we obtain $[\![\mathscr{X}]\!] = \{[\![\psi(t)]\!] \mid t \in K, [\![\varphi(t) = 1\}$ which is equal to 1, since every factor is equal to 1. Indeed, $[\![\varphi(t)]\!] = 1$ implies $(\varphi(t))_K$ and therefore $(\psi(t))_K$, $[\![\psi(t)]\!] = 1$, since ψ is a positive formula.

If $\mathscr{X} = \varphi \cap \psi$, then $(\varphi)_K \cap (\psi)_K$ and $P[\varphi] \cap P[\psi]$. If $\mathscr{X} = \varphi \cup \psi$, then $((\varphi)_K \cap P[\varphi]) \cup ((\psi)_K \cap P[\psi])$.

If $\mathscr{X} = \exists x \varphi$, then $\exists k \in K\, ((\varphi(k))_K \cap P[\varphi(k)])$. If $\mathscr{X} = \forall x \varphi$, then $\forall k \in K\, (\varphi(k))_K)$, $\forall k \in K\, (P[\varphi(k)])$.

Suppose that $\mathscr{X}$ has rank $n+1$. Let us once again use induction with respect to the construction of the sentence $\mathscr{X}$. If $\mathscr{X}$ is a block and $(\mathscr{X})_K$, $P[\mathscr{X}]$, then proceeding exactly as before we see that $\varphi(t)_K$ and using the second property in $P[\mathscr{X}]$ we obtain $P[\psi(t)]$. As before, we then apply the induction assumption on either the rank n or the construction of $\mathscr{X}$.

<u>Proposition 10.</u> Let K be a normal ring.

a) Every phi-sentence $\varphi(\bar{k})$ where $\bar{k} \in K$, satisfies $[\![\varphi(\bar{k})]\!]_{\mathcal{T}(K)} \leqslant [\![\varphi(\bar{k})]\!]_{\mathcal{B}(K)}$.

b) Every AE-sentence $\psi(\bar{k})$ where $\bar{k} \in K$, is such that if $[\![\psi(\bar{k})]\!]_{\mathcal{B}(K)} \geqslant a$, then $[\![\psi(\bar{k})]\!]_{\mathcal{B}(K)} \geqslant a$, $\forall a \in \mathcal{T}(K)$.

<u>Proof</u>. a) First, suppose that the formula φ does not contain the quantifier $\forall$ and does not contain the quantifier $\exists$ in the domain of action of a connective $\Rightarrow$. Using induction on the length of φ we shall show that if φ does not contain $\exists$, then $[\![\varphi]\!]_{\mathcal{T}} = [\![\varphi]\!]_{\mathcal{B}} \in \mathcal{B}(K)$ and if φ contains $\exists$, then $[\![\varphi]\!]_{\mathcal{T}} = [\![\varphi]\!]_{\mathcal{B}}$. An atomic formula φ satisfies the first assertion (because of the normality of the ring K). In the case $\varphi = (\varphi_1 \Rightarrow \varphi_2)$, we see that φ_1, φ_2 do not contain $\exists$ (by assumption), from which we obtain the first assertion. The case where $\varphi = \varphi_1 \cap \varphi_2$ or $\varphi_1 \cup \varphi_2$, or $\exists x \varphi_1$ is obvious.

Now suppose that φ is a phi-formula. Using induction we prove the desired inequality. If $\varphi = \varphi_1 \cap \varphi_2$ or $\varphi_1 \cup \varphi_2$, or $\exists x \varphi_1$, or $\forall x \varphi_1$, then the induction step is obvious. If $\varphi = (\varphi_1 \Rightarrow \varphi_2)$ then φ is the same as in the last paragraph. Therefore, it satisfies $[\![\varphi_1]\!]_{\mathcal{T}} = [\![\varphi_1]\!]_{\mathcal{B}}$ and $[\![\varphi_2]\!]_{\mathcal{T}} \leqslant [\![\varphi_2]\!]_{\mathcal{B}}$ simultaneously. From this we obtain the desired inequality.

b) We first note that for every normal ring K and every quantifier-free sentence $\varphi(\bar{k})$ we have the following

$$[\![\varphi(k)]\!]_{\mathcal{T}(K)} = [\![\varphi(k)]\!]_{\mathcal{B}(K)} \in B(K). \tag{11}$$

We prove this using induction on the length of φ. For an atomic formula this follows from the definition of valuation. Propositional operations on elements of $B(K)$ computed in $B(K)$ belong to $B(K)$ and coincide with the values of propositional operations in $\mathcal{T}$ and $\mathcal{B}$.

In case of $\mathcal{B}$-valuations sup is preserved in comparison to the $\mathcal{T}$-valuation. Therefore, we have

$$[\![\varphi]\!]_{\mathcal{T}} = [\![\varphi]\!]_{\mathcal{B}} \quad \text{for every } E\text{-formula } \varphi. \tag{12}$$

Thus, if $[\![\forall \bar{x}\varphi_1]\!]_{\mathcal{B}(K)} \geq a$, then $[\![\varphi_1(\bar{k})]\!]_{\mathcal{B}(K)} \geq a$, $\forall k$; $[\![\varphi_1(\bar{k})]\!]_{\mathcal{T}(K)} \geq a$, $[\![\forall \bar{x}\varphi]\!]_{\mathcal{T}(K)}$.

The following three theorems are purely finite.

In their statements T is any $\bar{x}$-theory in the language of rings, T_φ is a phi-theory, φ is a $\bar{x}$-formula in the language of rings, and ψ is a AE $\bar{x}$-formula. A notation of the form $[T(\bar{x}) \Rightarrow \varphi(\bar{x})]_K$ denotes $\forall \bar{k} \in K\ [T(\bar{k}) \Rightarrow \varphi(\bar{k})]_K$.

THEOREM 1. a) Suppose that $ZFI' \vdash \forall K\ [T(\bar{x}), \Phi_1, \Phi_3 \Rightarrow \varphi(\bar{x})]_K$. Then $ZFI' \vdash \forall K$ (K is a $\mathcal{T}$-ring $\Rightarrow [T'(\bar{x}), \Phi_1 \Rightarrow \varphi'(\bar{x})]_K$). The formula Φ_1 can be omitted simultaneously from the premise and the conclusion.

b) Suppose that $ZF \vdash \forall K\ [T_\varphi(\bar{x}) \Rightarrow \psi(\bar{x})]_K$. Then 1) $ZFI' \vdash \forall K\, (*(K) \Rightarrow [T_\varphi'(\bar{x}) \Rightarrow \psi'(\bar{x})]_K$ (recall that if ψ is Horn then $\psi' \Rightarrow \psi$) ; and 2) $ZFI' \vdash \forall K\, (*(K) \Rightarrow [T_\varphi(\bar{x}), \Phi_3 \Rightarrow \psi(\bar{x})]_K)$.

c) Suppose that $ZF \vdash \forall K\ [T_\varphi(\bar{x}), \Phi_3 \Rightarrow \psi(\bar{x})]_K$. Then 1) $ZFI' \vdash \forall K(*(K) \Rightarrow [T_\varphi'(\bar{x}) \Rightarrow (\neg\neg\psi)'(\bar{x})]_K$; and 2) $ZFI \vdash \forall K\,(*(K) \Rightarrow [T_\varphi(\bar{x}), \Phi_3 \Rightarrow \neg\neg\psi(\bar{x})]_K)$.

d) We can simultaneously add Φ_1 the premise of formula i' in a) and i to its conclusion, where i = 2, 3, 4, 5. We can simultaneously add i' to the premise of T_φ in b) and c), and i to its conclusion, where i = 3, 4, 5.

e) In all previous statements the length of the deduction in the conclusion depends linearly on the length of the deduction in the premise.

THEOREM 2. Suppose that T, T_φ, φ, ψ satisfy the conditions of Theorem 2. Then conclusions of Theorem 2 remain true if we replace both T in its premises and T' in its conclusions by T, adding to the conclusions the condition of decidability $T(\bar{x})$ for the ring K (or alternatively making a metamathematical assumption that $T(\bar{x})$ is a positive theory).

Proof of Theorems 1 and 2. a) Let K be a $\mathcal{T}$-ring and $(T'(\bar{k}), \Phi_1)_K$ where $\bar{k} \in K$ and Φ_1 could be omitted. Proposition 7a) implies that $[\![T(\bar{k})]\!]_{\mathcal{T}} = 1$. If $K \vDash \Phi_1$, then by Proposition 8a) we obtain $[\![\Phi_1]\!]_{\mathcal{T}} = 1$ and by Proposition b) we always have $[\![\Phi_3]\!]_{\mathcal{T}} = 1$. By Proposition 3b) we obtain $[\![(T(\bar{P}_k), \Phi_1, \Phi_3)_{K'_{\mathcal{T}}}]\!]_{\mathcal{T}} = 1$. A predicate $[\![\cdot]\!]_{\mathcal{T}} = 1$ is closed under ZFI'-deducibility, so therefore $[\![(\varphi(\bar{P}_k))_{K'_{\mathcal{T}}}]\!]_{\mathcal{T}} = 1$. Thus, $[\![\varphi(\bar{k})]\!]_{\mathcal{T}} = 1$ and $(\varphi'(\bar{k}))_K$.

In the case $(i)_K$ of statement d), using Proposition 8c)we obtain $\| i' \|_{\mathcal{T}}$ and the proceed as before.

If T satisfies the conditions of Theorem 2 then by Proposition 9 we obtain $[\![T(\bar{k})]\!]_{\mathcal{T}} = 1$ and then proceed as before.

We note that one could write a number of logical laws (for example, ones related to the second infinite law of distributivity) for which the value of a valuation $[\![\cdot]\!]_{\mathcal{T}}$ is equal to 1; all of such laws can be eliminated in a way similar to Φ_3.

b) Let K be a (*)-ring and $T'_{\varphi}(\bar{k})_K$ Proceeding as in the proof of statement a), we obtain $[\![T(\bar{k})]\!]_{\mathcal{T}} = 1$. By Proposition 10a) we have $[\![T(\bar{k})]\!]_{\mathcal{B}} = 1$. By Proposition 3b) we have $[\![(T(\bar{P}_k))_{K'_{\mathcal{B}}}]\!]_{\mathcal{B}} = 1$. A predicate $[\![\cdot]\!]_{\mathcal{B}} = 1$ is closed with respect to ZF-deducibility, so therefore $[\![(\psi(\bar{P}_k))_{K'_{\mathcal{B}}}]\!]_{\mathcal{B}} = 1$. This in turn implies that $[\![\psi(\bar{k})]\!]_{\mathcal{B}} = 1$ and by Proposition 10b) we have $[\![\psi(\bar{k})]\!]_{\mathcal{T}} = 1$, i.e., $(\psi'(\bar{k}))_K$. In the second part we additionally have $(\varphi_3)_K$. We now prove that

$$([\![\varphi(\bar{k})]\!]_{\mathcal{T}} = 1) \iff ((\varphi(\bar{k}))_K) \tag{6}$$

for all sentences $\varphi(\bar{k})$ of the language of rings, and from this the desired result directly follows.

We prove it by induction on the construction of φ. For atomic formulas (6) follows from the definition of valuation. For conjunction it is obvious. We prove the case of disjunction as follows. Let $[\![\varphi]\!] \cup [\![\psi]\!] = \bar{I}$, then either $I = e1 \cup e2$, $e1 \in [\![\varphi]\!]$, $e2 \in [\![\psi]\!]$, $ei = 0$ or $ei = 1$. Let us consider the case $e1 = 0$ and $e2 = 0$. We obtain a contradiction. The case of implication is proven as follows. Let $\varphi \Rightarrow \psi$ Since either $[\![\varphi]\!] = 0$ or $[\![\varphi]\!] = 1$ we have $[\![\varphi]\!] \leq [\![\psi]\!]$. The case of the existence quantifier is treated as follows. Let $[\![\exists x \varphi]\!] = 1$. Then the property of attainability implies that $[\![\varphi(k)]\!] = 1$ for some $k \in K$ and hence $(\exists x \varphi)_K$. The case of the quantifier of universality is obvious.

Concerning d), if $(i)_K$ then by Proposition 8c) we have $[\![i']\!]_{\mathcal{T}} = 1$ and i' is a phi-formula. Therefore, $[\![i']\!]_{\mathcal{B}} = 1$ and then we proceed as before.

c) Here we supplement the proof of b) by the fact that the classical proof uses a sentence Φ_3 as an axiom. This is not related to any special role of the sentence Φ_3 itself (for example, Φ_3 could be replaced with Φ_1), but it illustrates the possibility of using something like Φ_3 which is not a phi-formula. Exactly as in b) we obtain $[\![\Phi_3 \Rightarrow \psi]\!]_{\mathcal{B}} = 1$. Let $\Phi'_3 \leftrightharpoons \forall k \exists t (kt = tk \cap k^2 = k \Rightarrow k = 0 \cup k = 1)$. Classically, Φ'_3 is equivalent to Φ_3, so therefore, $[\![\Phi'_3]\!]_{\mathcal{B}} \leq [\![\psi]\!]_{\mathcal{B}}$. However, Φ'_3 is a phi-formula, so therefore $[\![\Phi'_3]\!]_{\mathcal{T}} \leq [\![\psi]\!]_{\mathcal{T}}$ [using Propositions 10a) and b)]. Intuitionistically, we have $\Phi_3^- \Rightarrow \Phi_3'^-$ where $(\cdot)^-$ is the usual Gödel negative translation. Using the normality of the ring K we obtain $[\![\Phi_3^-]\!] = [\![\Phi_3]\!]_{\mathcal{T}} \leq [\![\neg\neg\Phi'_3]\!]_{\mathcal{T}} \leq [\![\neg\neg\psi]\!]_{\mathcal{T}}$, i.e., $\neg\neg[\![\psi]\!]_{\mathcal{T}} = \bar{I}$. By Proposition 7 the latter is equivalent to $((\neg\neg\psi)')_K$, whereas it is also equivalent to the following condition:

$$[\forall e0 \leq e\ (\psi'(e0) \Rightarrow e0 = 0)] \Rightarrow e = 0, \quad \forall e. \tag{7}$$

Recall that a letter e (perhaps with some indices) denotes an element in $\mathcal{B}(K)$.

In the framework of classical metamathematics condition (7) is equivalent to a condition $\forall e \neq 0 \exists e0 (\neq 0, \leq e)(\psi'(e0))$.

The above equivalence is easy to prove. Suppose that (7) holds. Then $[\ldots] \leftrightarrow (e = 0)$. Therefore, $\langle e \rangle \bar{d} [\![\psi]\!]_{\mathcal{T}} \leftrightarrow (e = 0)$ where $a d b \leftrightharpoons \forall x \in a (x \in b \Rightarrow x = 0)$, since the left-hand side is equivalent to $[\ldots]$. Thus, $\neg\neg[\![\psi]\!]_{\mathcal{T}} = \{\langle e \rangle \mid \langle e \rangle \bar{d} [\![\psi]\!]_{\mathcal{T}}\} = 0.\ \neg\neg[\![\psi]\!]_{\mathcal{T}} = 1$. Suppose that $\neg\neg[\![\psi]\!]_{\mathcal{T}} = 1$ and the premise in (7) is true. Then $\langle e \rangle \bar{d} [\![\psi]\!]_{\mathcal{T}}$. Therefore, $\langle e \rangle \leq \neg[\![\psi]\!]_{\mathcal{T}}, \langle 1 - e \rangle \geq \neg\neg[\![\psi]\!]_{\mathcal{T}} = 1,\ 1 - e = 1,\ e = 0$.

The second assertion requires an opening up of condition (7). Let $e = I$. Then we have $\neg\forall e0 (\psi'(e0) \Rightarrow e0 = 0).\ \neg(\psi'(1) \Rightarrow 1 = 0), \neg\neg\psi'$ and using equivalence (6) we obtain $\neg\neg\psi$.

d) This statement has already been proven in the context of other statements.

e) This directly follows from the previous proof.

Remark. A translation $\varphi \mapsto \varphi'$ [and consequently conditions of the form (7)] has a direct set-theoretical meaning. We first clarify it using the example of a quantifier-free formula φ of the form $\varphi \leftrightharpoons \bigcup\{(k_{1i} = t_{1i} \cap \ldots \cap r_{1i} \neq s_{1i} \cap \ldots) \mid i = 1, \ldots, n\}$ where the multiple dots denote other equations and other inequalities, respectively, in the i-th disjunctive term, and k_{1i}, t_{1i}, r_{1i}, s_{1i} are both polynomials and the values of these polynomials for given parameters. Then $(\varphi')_K$ is equivalent to the following condition: There exists a decomposition of the ring K as a direct sum $K = \oplus\{e_i \cdot K \mid i = 1, \ldots, n\}$ where $e_i \in \mathcal{B}(K)$, such that $e_i \cdot k_{1i} = e_i \cdot t_{1i}$ (for all equations contained in the i-th term) and $e_i \cdot r_{1i} \neq e_i \cdot s_{1i}$ (for all inequalities contained in the i-th term), where $e \cdot r \neq e \cdot s \leftrightharpoons \forall \varepsilon 0 (\neq 0, \leq \varepsilon)(e0 \cdot r \neq e0 \cdot s)$. If $\psi \leftrightharpoons Q\varphi$, where Q is a quantifier prefix, then $\psi' = Q\varphi'$.

THEOREM 3. Conclusions of Theorems 1 and 2 remain true if in their statement $\forall K \ldots$ is replaced with $\forall K (\mathscr{X}(K) \Rightarrow \ldots)$, where $\mathscr{X}(\cdot)$ is any absolute formula.

Proof. We need only to add the following argument to the previous proof. We know that $[\![T(\bar{P}_K) \ldots)_{K'_{\mathcal{B}}}]\!]_{\mathcal{B}} = 1$ and furthermore $[\![\mathscr{X}(K'_{\mathcal{B}})]\!]_{\mathcal{B}} = 1$ since by Proposition 5 we have $[\![K'_{\mathcal{B}} \cong K / \mathcal{P}_0]\!]_{\mathcal{B}} = 1$. (The case $K'_{\sim}$ is formulated and studied similarly.)

In particular, we note that with this method many difficult proofs in the theory ZF are automatically transformed into intuitionistic proofs of same assertions. For example, this is true for the known positive solution of Hilbert's seventeenth problem.

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