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Mathematical Logic

A definable E_0 class containing no definable elements

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Abstract A generic extension $\mathbf{L}[x]$ by a real x is defined, in which the E_0-class of x is a lightface Π^1_2 (hence, ordinal-definable) set containing no ordinal-definable reals.

Keywords Jensen reals · Definable equivalence class · Definable elements

Mathematics Subject Classification 03E15 · 03E35

1 Introduction

It is known that the existence of a non-empty OD (ordinal-definable) set of reals X with no OD element is consistent with **ZFC**; the set of all non-constructible reals gives an example in many generic models including e.g. the Solovay model or the extension of $\mathbf{L}$, the constructible universe, by a Cohen real.

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> *Can such a set X be countable*? That is, is it consistent with **ZFC** that there is a countable OD (or outright definable by a precise set-theoretic formula) set of reals X containing no OD element?

This question was initiated and discussed at the *Mathoverflow* website[1] and at FOM.[2] In particular Ali Enayat (Footnote 2) conjectured that the problem can be solved by the finite-support countable product $\mathbb{P}^{<\omega}$ of Jensen's "minimal Π^1_2 real singleton forcing" $\mathbb{P}$ defined in [6] (see also Section 28A of [5]). Enayat proved that a symmetric part of the $\mathbb{P}^{<\omega}$-generic extension of $\mathbf{L}$ yields a model of **ZF** (not of **ZFC**!) in which the set of all reals $\mathbb{P}$-generic over $\mathbf{L}$ is a Dedekind-finite infinite OD set with no OD elements. In fact $\mathbb{P}^{<\omega}$-generic extensions of $\mathbf{L}$ and their symmetric submodels were considered in [1] (Theorem 3.3) with respect to some other questions.

Following the mentioned conjecture, we proved in [7] that indeed, in a $\mathbb{P}^{<\omega}$-generic extension of $\mathbf{L}$, the set of all reals $\mathbb{P}$-generic over $\mathbf{L}$ is a countable Π^1_2 set with no OD elements. The Π^1_2 definability is the best one can get in this context since it easily follows from the Π^1_1 uniformisation theorem that any non-empty Σ^1_2 set of reals definitely contains a Δ^1_2 element.

Jindra Zapletal[3] informed us that there is a totally different model of **ZFC** with an OD E_0-class[4] X containing no OD elements. The construction of such a model, not yet published, but described to us in a brief communication, involves a combination of several forcing notions and some modern ideas in descriptive set theory, like models of the form $\mathbf{V}[x]_{\mathsf{E}}$ for $\mathsf{E} = \mathsf{E}_0$, recently presented in [9]; it also does not look to yield X being analytically definable, let alone Π^1_2.

Theorem 1.1 *It is true in a suitable generic extension $\mathbf{L}[x]$ of $\mathbf{L}$, the constructible universe, by a real $x \in 2^\omega$ that the E_0-equivalence class $[x]_{\mathsf{E}_0}$ (hence a countable set) is Π^1_2, but it has no OD elements.*

The forcing $\mathbb{P}$ we use to prove the theorem is a clone of the abovementioned Jensen forcing, but defined on the base of the Silver forcing instead of the Sacks forcing. The crucial advantage of Silver's forcing here is that it leads to a Jensen-type forcing naturally closed under the 0-1 flip at any digit, so that the corresponding extension contains a Π^1_2 E_0-class of generic reals instead of a Π^1_2 generic singleton as in [6]. In fact a bigger family of E_0-*large* trees (perfect trees $T \subseteq 2^{<\omega}$ such that $\mathsf{E}_0 \restriction [T]$ is not smooth [8, Section 10.9]) would also work similarly to Silver trees.

It remains to note that a *finite* OD set of reals contains only OD reals by obvious reasons. On the other hand, by a result in [3] there can be two *sets* of reals X, Y such that the pair $\{X, Y\}$ is OD but neither X nor Y is OD.

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[1] A question about ordinal definable real numbers. *Mathoverflow*, March 09, 2010. http://mathoverflow.net/questions/17608.

[2] Ali Enayat. Ordinal definable numbers. FOM Jul 23, 2010. http://cs.nyu.edu/pipermail/fom/2010-July/014944.html.

[3] Personal communication, Jul 31/Aug 01, 2014.

[4] Recall that if $x, y \in \omega^\omega$ then $x \mathrel{\mathsf{E}_0} y$ iff $x(n) = y(n)$ for all but finite n.

Theorem 1.1 can be compared with some results obtained in the sharp hypothesis. In particular, Kechris and Woodin [10] proved that, in contrast to Theorem 1.1, assuming sharps, every countable non-empty Π^1_2 set X necessarily contains an OD element, basically, a real $x \in X$ characterised by $\mathbf{L}[x] \models \phi(\aleph_1, \ldots, \aleph_n)$ for some n and some formula ϕ. Sy Friedman [2, Theorem 4] demonstrated that, in this result, n cannot be taken one and the same for all countable Π^1_2 sets X, in particular, assuming sharps, there is a countable non-empty Π^1_2 set containing no Π^1_2 singletons.

2 Trees and Silver-type forcing

Let $2^{<\omega}$ be the set of all strings (finite sequences) of numbers $0, 1$. If $t \in 2^{<\omega}$ and $i = 0, 1$ then $t^\wedge k$ is the extension of t by k. If $s, t \in 2^{<\omega}$ then $s \subseteq t$ means that t extends s, while $s \subset t$ means proper extension. If $s \in 2^{<\omega}$ then $\mathtt{lh}(s)$ is the length of s, and $2^n = \{s \in 2^{<\omega} : \mathtt{lh}(s) = n\}$ (strings of length n). $\boldsymbol{\Lambda}$ is *the empty string*.

If $u \in 2^\omega$ then let $[u] = \{a \in 2^\omega : u \subset a\}$, the *Baire interval* in 2^ω; $[\boldsymbol{\Lambda}] = 2^\omega$.

Let any string $s \in 2^{<\omega}$ **act** on 2^ω so that $(s \cdot x)(k) = x(k) + s(k) \pmod 2$ whenever $k < \mathtt{lh}(s)$, and simply $(s \cdot x)(k) = x(k)$ otherwise.

If $X \subseteq 2^\omega$ and $s \in 2^{<\omega}$ then, as usual, let $s \cdot X = \{s \cdot x : x \in X\}$.

Similarly if $s \in 2^m$, $t \in 2^n$, $m \leq n$, then define $s \cdot t \in 2^n$ so that $(s \cdot t)(k) = t(k) + s(k) \pmod 2$ whenever $k < m$ and $(s \cdot t)(k) = t(k)$ whenever $m \leq k < n$. If $m > n$ then let $s \cdot t = (s \restriction n) \cdot t$. In both cases $\mathtt{lh}(s \cdot t) = \mathtt{lh}(t)$.

Let $s \cdot T = \{s \cdot t : t \in T\}$ for $T \subseteq 2^{<\omega}$.

If $T \subseteq 2^{<\omega}$ is a tree and $s \in T$ then put $T \restriction_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$.

Let **PT** be the set of all *perfect* trees $\varnothing \neq T \subseteq 2^{<\omega}$ (no endpoints, no isolated branches). If $T \in \mathbf{PT}$ then there is a largest string $s = \mathtt{stem}(T) \in T$ such that $T = T \restriction_s$ (the *stem* of T); we have $s^\wedge 1 \in T$ and $s^\wedge 0 \in T$ in this case. If $T \in \mathbf{PT}$ then

$$[T] = \{a \in 2^\omega : \forall n\, (a \restriction n \in T)\} \subseteq 2^\omega$$

is the perfect set of all *paths through* T.

Let **ST** be the set of all *Silver trees*. Thus $T \in \mathbf{ST}$ iff there is an infinite sequence of strings $u_k = u_k(T) \in 2^{<\omega}$ such that T consists of all strings of the form $s = u_1{}^\wedge i_1{}^\wedge u_2{}^\wedge i_2{}^\wedge u_3{}^\wedge i_3{}^\wedge \ldots {}^\wedge u_n{}^\wedge i_n$ and their substrings, where $n < \omega$ and $i_k = 0, 1$ whenever $1 \leq k \leq n$. (If $n = 0$ then $s = \boldsymbol{\Lambda}$.) Then $\mathtt{stem}(T) = u_1$, and $[T]$ consists of all infinite sequences $a = u_1{}^\wedge i_1{}^\wedge u_2{}^\wedge i_2{}^\wedge u_3{}^\wedge i_3{}^\wedge \cdots \in 2^\omega$, where $i_k = 0, 1, \forall k$.

Let a **Silver-type forcing** (**STF**) be any set $\mathbb{P} \subseteq \mathbf{ST}$ such that

(I) if $u \in T \in \mathbb{P}$ then $T \restriction_u \in \mathbb{P}$;
(II) if $T \in \mathbb{P}$ and $s \in 2^{<\omega}$ then $s \cdot T \in \mathbb{P}$.

Such a set $\mathbb{P}$ can be considered as a forcing notion (if $T \subseteq T'$ then T is a stronger condition), and then it adds a real in 2^ω.

Say that a **STF** $\mathbb{P}$ is *regular* iff it contains the full tree $2^{<\omega}$.

The set **ST** of all Silver trees is a regular **STF** by obvious reasons.

3 Splitting construction over a Silver-type forcing

Assume that $\mathbb{P} \subseteq \mathbf{ST}$ is a $\mathbf{STF}$ and $n < \omega$. The set $\mathbf{SS}_n(\mathbb{P})$ of *Silver splitting systems* of height $n = \mathtt{hgt}(\varphi)$ over $\mathbb{P}$ consists of all finite systems of trees, of the form $\varphi = \{T_s\}_{s \in 2^{\leqslant n}}$, satisfying the following requirements:

(1) each tree $T_s = \varphi(s)$ belongs to $\mathbb{P}$,—we let $r_s = \mathtt{stem}(T_s)$;
(2) if $s^\wedge i \in 2^{\leqslant n}$ $(i = 0, 1)$ then $T_{s^\wedge i} \subseteq T_s \upharpoonright_{r_s{}^\wedge i}$—it follows that $[T_{s^\wedge 0}] \cap [T_{s^\wedge 1}] = \varnothing$;
(3) there is an increasing sequence of numbers $h(0) < h(1) < \cdots < h(n)$ such that $\mathtt{lh}(r_s) = h(k)$ whenever $s \in 2^k$ and $k \le n$;
(4) if $m \le n$, $u, v \in 2^m$, and $t \in 2^{<\omega}$ then $r_u{}^\wedge t \in T_u \iff r_v{}^\wedge t \in T_v$—or equivalently $T_v = \sigma \cdot T_u$, where $\sigma = r_v \cdot r_u$.

Additionally, let $\mathbf{SS}_{-1}(\mathbb{P})$ consist of the *empty system* $\mathbb{\Lambda}$; $\mathtt{hgt}(\mathbb{\Lambda}) = -1$.

Put $\mathbf{SS}(\mathbb{P}) = \bigcup_{n \ge -1} \mathbf{SS}_n(\mathbb{P})$.

Say that a tree T *occurs in* $\varphi \in \mathbf{SS}(\mathbb{P})$ if $T = \varphi(s)$ for some $s \in 2^{\leqslant \mathtt{hgt}(\varphi)}$.

Lemma 3.1 *If* $\mathbb{P} \subseteq \mathbf{ST}$ *is a* ***STF****,* $\varphi = \{T_s\}_{s \in 2^{\leqslant n}} \in \mathbf{SS}_n(\mathbb{P})$ *and* $n = \mathtt{hgt}(\varphi) \ge 0$ *then the tree* $T = \bigcup_{s \in 2^n} T_s$ *belongs to* $\mathbf{ST}$ *(not necessarily to* $\mathbb{P}$*).*

Let φ, ψ be systems in $\mathbf{SS}(\mathbb{P})$. Say that

- φ *extends* ψ, symbolically $\psi \preccurlyeq \varphi$, if $n = \mathtt{hgt}(\psi) \le \mathtt{hgt}(\varphi)$ and $\psi(s) = \varphi(s)$ for all $s \in 2^{\leqslant n}$, and separately $\mathbb{\Lambda} \preccurlyeq \varphi$ for any $\varphi \in \mathbf{SS}(\mathbb{P})$;
- *properly extends* ψ, symbolically $\psi \prec \varphi$, if in addition $\mathtt{hgt}(\psi) < \mathtt{hgt}(\varphi)$;
- *refines* ψ, if $n = \mathtt{hgt}(\psi) = \mathtt{hgt}(\varphi)$, $\varphi(s) \subseteq \psi(s)$ for all $s \in 2^n$, and $\varphi(s) = \psi(s)$ for all $s \in 2^{<n}$.

In other words, the refinement allows to shrink trees in the top layer of the system, but does not change those in the lower layers.

Note that $\varphi = \mathbb{\Lambda}$ (the empty system) is the only one with $\mathtt{hgt}(\varphi) = -1$. To get a system φ with $\mathtt{hgt}(\varphi) = 0$ (and then $\mathtt{dom}\,\varphi = \{\boldsymbol{\Lambda}\}$) put $\varphi(\boldsymbol{\Lambda}) = T$, where $T \in \mathbf{ST}$. The following lemma leads to systems of bigger height.

Lemma 3.2 *Assume that* $\mathbb{P} \subseteq \mathbf{ST}$ *is a* ***STF*** *and* $\varphi = \{T_s\}_{s \in 2^{\leqslant n}} \in \mathbf{SS}(\mathbb{P})$.

(i) *If* $s_0 \in 2^n$*, and* $T \in \mathbf{ST}$*,* $T \subseteq T_{s_0}$*, then there is a system* $\varphi' = \{T'_s\}_{s \in 2^{\leqslant n}} \in \mathbf{SS}(\mathbb{P})$ *which refines* φ *and satisfies* $T'_{s_0} = T$.
(ii) *There is a system* $\varphi' = \{T'_s\}_{s \in 2^{\leqslant n+1}} \in \mathbf{SS}(\mathbb{P})$ *which properly extends* φ.
(iii) *If a system* ψ *properly extends* φ *and a system* ψ' *refines* ψ *then* ψ' *properly extends* φ *as well.*

Proof By definition all strings $r_s = \mathtt{stem}(T_s)$ with $s \in 2^n$ satisfy $\mathtt{lh}(r_s) = h$ for one and the same $h = h(n)$. To prove (i) put $T'_s = \{r_s{}^\wedge t : r_{s_0}{}^\wedge t \in T\} = (r_s \cdot r_{s_0}) \cdot T$ for all $s \in 2^n$, and still $T'_s = T_s$ for $s \in 2^{\leqslant n-1}$. The sets T'_s belong to $\mathbb{P}$ by (II) of Sect. 2.

(ii) Put $T'_{s^\wedge i} = T_s \upharpoonright_{r_s{}^\wedge i}$ for all $s \in 2^n$ and $i = 0, 1$, and still $T'_s = T_s$ for $s \in 2^{\leqslant n}$. The sets $T'_{s^\wedge i}$ belong to $\mathbb{P}$ by (I) of Sect. 2. □

By the lemma, if $\mathbb{P} \subseteq \mathbf{ST}$ is a **STF** then there are strictly $\prec$-increasing sequences $\{\varphi_n\}_{n<\omega}$ of systems $\varphi_n \in \mathbf{SS}_n(\mathbb{P})$. The limit system $\varphi = \bigcup_n \varphi_n = \{T_s\}_{s\in 2^{<\omega}}$ of such a sequence satisfies conditions (1)–(4) on the whole domain $2^{<\omega}$.

Lemma 3.3 (Fusion) *In this case, the tree* $T = \bigcap_n \bigcup_{s\in 2^n} T_s$ *is still a Silver tree in* **ST** (not necessarily in $\mathbb{P}$)*, and* $[T] = \bigcap_n \bigcup_{s\in 2^n}[T_s]$.

We define $\mathbf{SS}(\mathbb{P})^{<\omega}$, **the finite-support product** of countably many copies of $\mathbf{SS}(\mathbb{P})$, to be the set of all infinite sequences $\Phi = \{\Phi(k)\}_{k\in\omega}$, where each $\Phi(k)$ belongs to $\mathbf{SS}(\mathbb{P})$ and the set $|\Phi| = \{k : \Phi(k) \neq \mathbb{\Lambda}\}$ (the *support* of Φ) is finite. Sequences $\Phi \in \mathbf{SS}(\mathbb{P})^{<\omega}$ will be called *multisystems*.

Say that a tree T *occurs in* Φ if it occurs in some $\Phi(k)$, $k \in |\Phi|$.

Let Φ, Ψ be multisystems in $\mathbf{SS}(\mathbb{P})^{<\omega}$. We define that

- Φ *extends* Ψ, symbolically $\Psi \preccurlyeq \Phi$, if $\Psi(k) \preccurlyeq \Phi(k)$ (in $\mathbf{SS}(\mathbb{P})$) for all k—then obviously $|\Psi| \subseteq |\Phi|$;
- $\Psi \lll \Phi$, iff $|\Psi| \subseteq |\Phi|$ and $\Psi(k) \prec \Phi(k)$ properly for all $k \in |\Psi|$;
- Φ *refines* Ψ iff $\Phi(k)$ refines $\Psi(k)$ for all $k \in |\Psi|$.

Corollary 3.4 (of Lemma 3.2(ii)) *If* $\mathbb{P} \subseteq \mathbf{ST}$ *is a* ***STF*** *and* $\Psi \in \mathbf{SS}(\mathbb{P})^{<\omega}$ *then there is a multisystem* $\Phi \in \mathbf{SS}(\mathbb{P})^{<\omega}$ *such that* $\Psi \lll \Phi$.

4 Jensen's extension of a Silver-type forcing

Let $\mathbf{ZFC}'$ be the subtheory of **ZFC** including all axioms except for the power set axiom, plus the axiom saying that $\mathscr{P}(X)$ exists for all countable sets X. (Then ω_1 and continual sets like **PT** and **ST** exist as well.)

Definition 4.1 Let $\mathfrak{M}$ be a countable transitive model of $\mathbf{ZFC}'$. Suppose that $\mathbb{P} \in \mathfrak{M}$, $\mathbb{P} \subseteq \mathbf{ST}$ is a regular **STF**. Then the sets $\mathbf{SS}(\mathbb{P})$ and $\mathbf{SS}(\mathbb{P})^{<\omega}$ belong to $\mathfrak{M}$.

Let us consider any $\preccurlyeq$-increasing sequence $\Phi = \{\Phi^j\}_{j<\omega}$ of multisystems $\Phi^j = \{\Phi^j(k)\}_{k\in\omega} \in \mathbf{SS}(\mathbb{P})^{<\omega}$, *generic over* $\mathfrak{M}$ in the sense that it intersects every set $D \in \mathfrak{M}$, $D \subseteq \mathbf{SS}(\mathbb{P})^{<\omega}$, *dense*[5] in $\mathbf{SS}(\mathbb{P})^{<\omega}$. Then Φ intersects every set

$$D_k = \{\Phi \in \mathbf{SS}(\mathbb{P})^{<\omega} : \forall k' \le k\ (k \le \mathtt{hgt}(\Phi(k')))\}.$$

Hence if $k < \omega$ then the sequence $\{\Phi^j(k)\}_{j<\omega}$ of systems $\Phi^j(k) \in \mathbf{SS}(\mathbb{P})$ *eventually increases*: $\Phi^j(k) \prec \Phi^{j+1}(k)$ holds for infinitely many indices j (and $\Phi^j(k) = \Phi^{j+1}(k)$ for all other j). Thus there is a system of trees $\{\boldsymbol{T}^{\Phi}_k(s)\}_{k<\omega\wedge s\in 2^{<\omega}}$ in $\mathbb{P}$ such that $\Phi^j(k) = \{\boldsymbol{T}^{\Phi}_k(s)\}_{s\in 2^{\leqslant h(j,k)}}$ for all j,k, where $h(j,k) = \mathtt{hgt}(\Phi^j(k))$. Then

$$\boldsymbol{U}^{\Phi}_k = \bigcap_n \bigcup_{s\in 2^n} \boldsymbol{T}^{\Phi}_k(s) \quad \text{and} \quad \boldsymbol{U}^{\Phi}_k(s) = \bigcap_{n\ge \mathtt{lh}(s)} \bigcup_{t\in 2^n,\, s\subseteq t} \boldsymbol{T}^{\Phi}_k(t)$$

are trees in **ST** (not necessarily in $\mathbb{P}$) by Lemma 3.3 for each k and $s \in 2^{<\omega}$; and obviously $\boldsymbol{U}^{\Phi}_k = \boldsymbol{U}^{\Phi}_k(\boldsymbol{\Lambda})$. In fact $\boldsymbol{U}^{\Phi}_k(s) = \boldsymbol{U}^{\Phi}_k \cap \boldsymbol{T}^{\Phi}_k(s)$ by (2).

[5] Meaning that for any $\Psi \in \mathbf{SS}(\mathbb{P})^{<\omega}$ there is a multisystem $\Phi \in D$ such that $\Psi \preccurlyeq \Phi$.

Define a set of trees $\mathbb{U} = \{\sigma \cdot \boldsymbol{U}_k^{\Phi}(s) : k < \omega \wedge s \in 2^{<\omega} \wedge \sigma \in 2^{<\omega}\} \subseteq \mathbf{ST}$.

Lemma 4.2 *The set $\mathbb{U}$ is a **STF**. The union $\mathbb{P} \cup \mathbb{U}$ is a regular **STF**.*

Lemma 4.3 *The set $\mathbb{U}$ is dense in $\mathbb{U} \cup \mathbb{P}$.*

Proof Suppose that $T \in \mathbb{P}$. The set $D(T)$ of all multisystems $\varPhi$ in $\mathbf{SS}(\mathbb{P})^{<\omega}$, such that $\varPhi(k)(\boldsymbol{\Lambda}) = T$ for some $k \in |\varPhi|$, belongs to $\mathfrak{M}$ and obviously is dense in $\mathbf{SS}(\mathbb{P})^{<\omega}$. It follows that $\varPhi^j \in D(T)$ for some j, by the choice of Φ. Then $\boldsymbol{T}_k^{\Phi}(\boldsymbol{\Lambda}) = T$ for some k. However $\boldsymbol{U}_k^{\Phi}(\boldsymbol{\Lambda}) \subseteq \boldsymbol{T}_k^{\Phi}(\boldsymbol{\Lambda})$.

Lemma 4.4 *If a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$ is pre-dense in $\mathbb{P}$, and $U \in \mathbb{U}$, then $U \subseteq^{\mathtt{fin}} \bigcup D$, that is, there is a finite $D' \subseteq D$ with $U \subseteq \bigcup D'$.*

Moreover D remains pre-dense in $\mathbb{U} \cup \mathbb{P}$.

Proof Suppose that $U = \boldsymbol{U}_K^{\Phi}(s) \in \mathbb{U}$, $K < \omega$ and $s \in 2^{<\omega}$. (The general case, when $U = \sigma \cdot \boldsymbol{U}_K^{\Phi}(s)$ for some $\sigma \in 2^{<\omega}$, is easily reducible to the particular case $U = \boldsymbol{U}_K^{\Phi}(s)$ by substituting the set $\{\sigma \cdot T : T \in D\}$ for D.) Consider the set $\varDelta \in \mathfrak{M}$ of all multisystems $\varPhi \in \mathbf{SS}(\mathbb{P})^{<\omega}$ such that $K \in |\varPhi|$, $\mathtt{lh}(s) \le h = \mathtt{hgt}(\varPhi(K))$, and for each $t \in 2^h$ there is a tree $S_t \in D$ with $\varPhi(K)(t) \subseteq S_t$. The set $\varDelta$ is dense in $\mathbf{SS}(\mathbb{P})^{<\omega}$ by Lemma 3.2 and the pre-density of D. Therefore there is an index j such that $\varPhi^j$ belongs to $\varDelta$. Let this be witnessed by trees $S_t \in D$, $t \in 2^h$, where $\mathtt{lh}(s) \le h = \mathtt{hgt}(\varPhi^j(K))$, so that $\varPhi^j(K)(t) \subseteq S_t$. Then

$$U = \boldsymbol{U}_K^{\Phi}(s) \subseteq \boldsymbol{U}_K^{\Phi}(\boldsymbol{\Lambda}) \subseteq \bigcup\nolimits_{t \in 2^h} \varPhi^j(K)(t) \subseteq \bigcup\nolimits_{t \in 2^h} S_t \subseteq \bigcup D'$$

by construction, where $D' = \{S_t : t \in 2^h\} \subseteq D$ is finite.

To prove the pre-density of D, consider any string $t \in 2^h$ with $s \subset t$. Then $V = \boldsymbol{U}_K^{\Phi}(t) \in \mathbb{U}$ and $V \subseteq U$. On the other hand, $V \subseteq S_t \in D$. Thus the tree V witnesses that U is compatible with $S_t \in D$ in $\mathbb{U} \cup \mathbb{P}$, as required. □

Let $\textsc{Fin} = \{x \in 2^\omega : \text{the set } \{k : x(k) = 1\} \text{ is finite}\}$; a countable set.

Lemma 4.5 *If $U = \boldsymbol{U}_K^{\Phi}(s) \in \mathbb{U}$ then $[U] \cap \textsc{Fin} = \varnothing$.*

Proof If $n < \omega$ then the set $\varDelta_n \in \mathfrak{M}$ of all multisystems $\varPhi \in \mathbf{SS}(\mathbb{P})^{<\omega}$ such that $K \in |\varPhi|$, and for any $s \in 2^h$, where $h = \mathtt{hgt}(\varPhi(K))$, and each $x \in \varPhi(K)(s)$, there is $n' \ge n$ with $x(n') = 1$, is dense in $\mathbf{SS}(\mathbb{P})^{<\omega}$. □

5 Forcing a real to avoid a pre-dense set

We assume that $\mathbb{P}$ is a **STF** in this section.

Arguing in the conditions of Definition 4.1, the goal of Theorem 5.2 below will be to prove that, for any $\mathbb{P}$-name c of a real in 2^ω, it is explicitly forced by the extended forcing $\mathbb{P} \cup \mathbb{U}$ that c does not belong to sets $[U]$ where U is a tree in $\mathbb{U}$—unless c is a name of one of reals in the E_0-class of the generic real x itself.

Lemma 5.1 *If* $R \in \mathbb{P}$, $\varphi = \{T_s\}_{s \in 2^{\leqslant n}} \in \mathbf{SS}_n(\mathbb{P})$, *and* $f : 2^\omega \to 2^\omega$ *is continuous, then there exists a tree* $T \in \mathbb{P}$, $T \subseteq R$, *and a system* $\varphi' = \{T'_s\}_{s \in 2^{\leqslant n}} \in \mathbf{SS}_n(\mathbb{P})$ *which refines* φ, *such that* $[U] \cap (f\,"[T]) = \varnothing$, *where* $U = \bigcup_{s \in 2^n} T'_s$.

Proof Let $s \in 2^n$. There is a tree $T \in \mathbb{P}$, $T \subseteq R$, and a string $u \in 2^{<\omega}$ such that $\mathtt{lh}(u) > \mathtt{lh}((\mathtt{stem}(T_s)))$ and $f\,"[T] \subseteq [u]$. [6] There is a string $v \in T_s$ incomparable with u. Put $S = T_s \restriction_v$; then $[S] \cap f\,"[T] = \varnothing$. By Lemma 3.2(i), there is a system $\varphi' = \{T'_s\}_{s \in 2^{\leqslant n}} \in \mathbf{SS}(\mathbb{P})$ which refines φ and satisfies $T'_s = S$, hence $[T'_s] \cap f\,"[T] = \varnothing$. Iterating this construction for all $s \in 2^n$, we get a tree $T \subseteq R$ and a system $\varphi' = \{T'_s\}_{s \in 2^{\leqslant n}} \in \mathbf{SS}(\mathbb{P})$ which refines φ, such that $[T'_s] \cap f\,"[T] = \varnothing$ for each $s \in 2^n$.

Theorem 5.2 *In the assumptions of* Definition 4.1, *suppose that* $f : 2^\omega \to 2^\omega$ *is a continuous map coded*[7] *in* $\mathfrak{M}$, *and for every* $\sigma \in 2^{<\omega}$ *the set*

$$D_f(\sigma) = \{T \in \mathbb{P} : (\sigma \cdot [T]) \cap (f\,"[T]) = \varnothing\}$$

is dense in $\mathbb{P}$. *Assume that* $W \in \mathbb{P} \cup \mathbb{U}$, *and* $U \in \mathbb{U}$. *Then there exists a tree* $V \in \mathbb{U}$, $V \subseteq W$ *satisfying* $[U] \cap (f\,"[V]) = \varnothing$.

Proof By construction, $U = \rho \cdot \boldsymbol{U}^{\Phi}_K(s_0)$, where $K < \omega$ and $\rho, s_0 \in 2^{<\omega}$; we can assume that simply $s_0 = \boldsymbol{\Lambda}$, so that $U = \rho \cdot \boldsymbol{U}^{\Phi}_K$. Further, we can assume that $\rho = \boldsymbol{\Lambda}$, so that $U = \boldsymbol{U}^{\Phi}_K$. (Otherwise consider the map $f'(x) = \rho \cdot f(x)$.) By Lemma 4.3, we can assume that $W \in \mathbb{U}$, that is, $W = \eta \cdot \boldsymbol{U}^{\Phi}_L(t_0)$, where $\eta \in 2^{<\omega}$, $L < \omega$, and $t_0 \in 2^{<\omega}$. And we can assume that $\eta = \boldsymbol{\Lambda}$ (the empty string), that is, $W = \boldsymbol{U}^{\Phi}_L(t_0) \in \mathbb{U}$. (Otherwise consider the map $f'(x) = f(\eta \cdot x)$ and the tree $W' = \eta \cdot W = \boldsymbol{U}^{\Phi}_L(t_0)$.)

The indices K, L involved can be either equal or different.

There is a number J such that the multisystem $\Phi^J = \{\Phi^J(k)\}_{k \in \omega}$ satisfies $K, L \in |\Phi^J|$ and $\mathtt{hgt}(\Phi^J(L)) > \mathtt{lh}(t_0)$, so that the trees

$$S_0 = \Phi^J(K)(\boldsymbol{\Lambda}) = \boldsymbol{T}^{\Phi}_K(\boldsymbol{\Lambda}) \quad \text{and} \quad T_0 = \Phi^J(L)(t_0) = \boldsymbol{T}^{\Phi}_L(t_0)$$

in $\mathbb{P}$ are defined. Note that $U \subseteq S_0$ and $W \subseteq T_0$.

Consider the set $\mathscr{D}$ of all multisystems $\Phi = \{\Phi(k)\}_{k \in \omega} \in \mathbf{SS}(\mathbb{P})^{<\omega}$ such that

$$\mathtt{hgt}(\Phi^J(L)) < g = \mathtt{hgt}(\Phi(L)), \quad \mathtt{hgt}(\Phi^J(K)) < h = \mathtt{hgt}(\Phi(K)), \qquad (*)$$

and there is a tree $T \in \mathbb{P}$ satisfying the following:

(1) $[S] \cap (f\,"[T]) = \varnothing$, where $S = \bigcup_{s \in 2^h} \Phi(K)(s)$; and
(2) $T = \Phi(L)(t)$, where $t \in 2^g$ and $t_0 \subset t$.

[6] Recall that $[u] = \{a \in 2^\omega : u \subset a\}$ is the *Baire interval* in 2^ω.

[7] The *code* of a continuous $f : 2^\omega \to 2^\omega$ is the family of sets $C_t = \{u \in 2^{<\omega} : f\,"[u] \subseteq [t]\}$, $t \in 2^{<\omega}$.

Lemma 5.3 *$\mathscr{D}$ is dense in* $\mathbf{SS}(\mathbb{P})^{<\omega}$.

Proof Consider a multisystem $\Phi^* \in \mathbf{SS}(\mathbb{P})^{<\omega}$; the goal is to get a multisystem $\Phi' \in \mathscr{D}$ with $\Phi^* \preccurlyeq \Phi'$. By Corollary 3.4 there is an intermediate multisystem $\Phi = \{\Phi(k)\}_{k\in\omega} \in \mathbf{SS}(\mathbb{P})^{<\omega}$ satisfying $\Phi^* \lll \Phi$ and condition $(*)$ as above. Now if a multisystem $\Phi' \in \mathbf{SS}(\mathbb{P})^{<\omega}$ refines Φ then it satisfies $\Phi^* \lll \Phi'$ and $\Phi^* \preccurlyeq \Phi'$. Thus it suffices to find a multisystem $\Phi' \in \mathscr{D}$ which refines Φ.

Pick a string $t \in 2^g$ with $t_0 \subset t$ and let $R = \varphi_L(t)$; $R \subseteq T_0$ is a tree in $\mathbb{P}$.

Case 1 $K \neq L$. By Lemma 5.1, there exist: a tree $T \in \mathbb{P}$, $T \subseteq R$, and a splitting system $\varphi' = \{T'_s\}_{s\in 2^{\leqslant h}} \in \mathbf{SS}(\mathbb{P})$ which refines $\Phi(K)$, such that $[U] \cap (f\,"[T]) = \varnothing$, where $U = \bigcup_{s\in 2^h} T'_s$. Further, by Lemma 3.2(i), there is a system $\varphi'' = \{T''_s\}_{s\in 2^{\leqslant g}} \in \mathbf{SS}(\mathbb{P})$ which refines φ_L and satisfies $T''_t = T$. Define a multisystem $\Phi' \in \mathbf{SS}(\mathbb{P})^{<\omega}$ so that $\Phi'(K) = \varphi'$, $\Phi'(L) = \varphi''$, and $\Phi'(k) = \Phi(k)$ for all other values of k. Then Φ' belongs to $\mathscr{D}$ (witnessed by T and t) and is a refinement of Φ, as required.

Case 2 $L = K$, and hence $g = h$. The Case 1 argument does not work since we cannot independently shrink two different trees in the top level of $\Phi(K)$ because of (4) of Sect. 3. We'll make use of the density assumption of the theorem instead. By (3) of Sect. 3, let $H = \mathtt{lh}(\mathtt{stem}(\Phi(K)(s)))$ for any/all $s \in 2^h$.

By the density of sets $D_f(\sigma)$, there is a tree $T \in \mathbb{P}$, $T \subseteq R = \Phi(K)(t)$, which satisfies $(\sigma \cdot [T]) \cap (f\,"[T]) = \varnothing$ for all $\sigma \in 2^H$ (a finite set of σ s). As above, there is a system $\varphi' = \{T'_s\}_{s\in 2^{\leqslant h}} \in \mathbf{SS}(\mathbb{P})$ which refines $\Phi(K)$ and satisfies $T'_t = T$.

Still by (4) of Sect. 3, if $s \in 2^h$ then there is a string $\sigma \in 2^H$ such that $T'_s = \sigma \cdot T'_t = \sigma \cdot T$. Then by construction $[T'_s] \cap (f\,"[T]) = (\sigma \cdot [T]) \cap (f\,"[T]) = \varnothing$, so that we have $[T'_s] \cap (f\,"[T]) = \varnothing$ for all $s \in 2^h$. Therefore the multisystem $\Phi' \in \mathbf{SS}(\mathbb{P})^{<\omega}$ defined by $\Phi'(K) = \varphi'$ and $\Phi'(k) = \Phi(k)$ for all $k \neq K$, belongs to $\mathscr{D}$ (which is witnessed by T and t) and refines Φ, as required. □ (Lemma)

Come back to the proof of the theorem. The set $\mathscr{D}$ belongs to $\mathfrak{M}$ since f is coded in $\mathfrak{M}$. Therefore, by the lemma, there is an index $j \geq J$ such that the multisystem Φ^j belongs to $\mathscr{D}$. Let this be witnessed by a tree $T = \Phi^j(L)(t) \subseteq T_0 = \Phi^J(L)(t_0) = \boldsymbol{T}^{\Phi}_L(t_0)$, satisfying (1) above, where $t \in 2^g$, $g = \mathtt{hgt}(\Phi^j(L))$, and $t_0 \subset t$.

Consider the tree $V = \boldsymbol{U}^{\Phi}_L(t) \in \mathbb{U}$. Then both $V \subseteq W$ and $V \subseteq T \subseteq T_0$ by construction. Therefore $[S] \cap (f\,"[V]) = \varnothing$ by (1), where $S = \bigcup_{s\in 2^h} \Phi^j(K)(s) = \bigcup_{s\in 2^h} \boldsymbol{T}^{\Phi}_K(s)$, $h = \mathtt{hgt}(\Phi^j(K))$. Finally, $U = \boldsymbol{U}^{\Phi}_K \subseteq S$, so that $[U] \cap (f\,"[V]) = \varnothing$. □

6 Jensen's forcing

In this section, **we argue in $\mathbf{L}$, the constructible universe.**

Let $\leq_{\mathbf{L}}$ be the canonical wellordering of $\mathbf{L}$.

Definition 6.1 (in $\mathbf{L}$) Following Jensen [6, Section 3], define, by induction on $\xi < \omega_1$, a countable **STF** $\mathbb{U}_\xi$ (Sect. 2), as follows.

Let $\mathbb{U}_0$ consist of all trees $T_u = \{v : u \subseteq v \vee v \subset u\} \subseteq 2^{<\omega}$, where $u \in 2^{<\omega}$, including $2^{<\omega} = T_\Lambda$ itself. (Note that $[T_u] = [u] = \{x \in 2^\omega : u \subset x\}$.)

Suppose that $0 < \lambda < \omega_1$, and countable sets $\mathbb{U}_\xi \subseteq \mathbf{ST}$ are defined for $\xi < \lambda$. Let $\mathfrak{M}_\xi$ be the least model $\mathfrak{M}$ of $\mathbf{ZFC}'$ of the form $\mathbf{L}_\kappa$, $\kappa < \omega_1$, containing the sequence $\{\mathbb{U}_\xi\}_{\xi<\lambda}$ and such that $\lambda < \omega_1^{\mathfrak{M}}$ and all sets $\mathbb{U}_\xi$, $\xi < \lambda$, are countable in $\mathfrak{M}$. Then $\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{U}_\xi$ is a regular $\mathbf{STF}$ countable in $\mathfrak{M}$. Let $\Phi = \{\Phi^j\}_{j<\omega}$ be the $\le_{\mathbf{L}}$-least sequence of multisystems $\Phi^j \in \mathbf{SS}(\mathbb{P}_\lambda)^{<\omega}$, $\preccurlyeq$-increasing and generic over $\mathfrak{M}_\lambda$. Define $\mathbb{U}_\lambda = \mathbb{U}$ as in Definition 4.1. This completes the inductive step.

After the inductive construction is accomplished, let $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{U}_\xi$; then $\mathbb{P}$ is a regular $\mathbf{STF}$ since $\mathbb{U}_0 \subseteq \mathbb{P}$.

Proposition 6.2 (in $\mathbf{L}$) *The sequence $\{\mathbb{U}_\xi\}_{\xi<\omega_1}$ belongs to Δ_1^{HC}.*

Lemma 6.3 (in $\mathbf{L}$) *If a set $D \in \mathfrak{M}_\xi$, $D \subseteq \mathbb{P}_\xi$ is pre-dense in $\mathbb{P}_\xi$ then it remains pre-dense in $\mathbb{P}$. Therefore if $\xi < \omega_1$ then $\mathbb{U}_\xi$ is pre-dense in $\mathbb{P}$.*

Proof By induction on $\lambda \ge \xi$, if D is pre-dense in $\mathbb{P}_\lambda$ then it remains pre-dense in $\mathbb{P}_{\lambda+1} = \mathbb{P}_\lambda \cup \mathbb{U}_\lambda$ by Lemma 4.4. Limit steps are obvious. To prove the second part, note that $\mathbb{U}_\xi$ is dense in $\mathbb{P}_{\xi+1}$ by Lemma 4.3, and $\mathbb{U}_\xi \in \mathfrak{M}_{\xi+1}$. □

Lemma 6.4 (in $\mathbf{L}$) *If $X \subseteq \mathrm{HC} = \mathbf{L}_{\omega_1}$ then the set W_X of all ordinals $\xi < \omega_1$ such that $\langle \mathbf{L}_\xi\,;\, X \cap \mathbf{L}_\xi\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}\,;\, X\rangle$ and $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ is unbounded in ω_1. More generally, if $X_n \subseteq \mathrm{HC}$ for all n then the set W of all ordinals $\xi < \omega_1$, such that $\langle \mathbf{L}_\xi\,;\, \{X_n \cap \mathbf{L}_\xi\}_{n<\omega}\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}\,;\, \{X_n\}_{n<\omega}\rangle$ and $\{X_n \cap \mathbf{L}_\xi\}_{n<\omega} \in \mathfrak{M}_\xi$, is unbounded in ω_1.*

Proof Let $\xi_0 < \omega_1$. Let M be a countable elementary submodel of $\mathbf{L}_{\omega_2}$ containing ξ_0, ω_1, X, and such that $M \cap \mathbf{L}_{\omega_1}$ is transitive. Let $\phi : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ be the Mostowski collapse, and let $\xi = \phi(\omega_1)$. Then $\xi_0 < \xi < \lambda < \omega_1$ and $\phi(X) = X \cap \mathbf{L}_\xi$ by the choice of M. It follows that $\langle \mathbf{L}_\xi\,;\, X \cap \mathbf{L}_\xi\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}\,;\, X\rangle$. Moreover, ξ is uncountable in $\mathbf{L}_\lambda$, hence $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\xi$. We conclude that $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ as $X \cap \mathbf{L}_\xi \in \mathbf{L}_\lambda$ by construction.

The second claim does not differ much. □

Corollary 6.5 (compare to [6], Lemma 6) *The forcing $\mathbb{P}$ satisfies CCC in $\mathbf{L}$.*

Proof Suppose that $A \subseteq \mathbb{P}$ is a maximal antichain. By Lemma 6.4, there is an ordinal ξ such that $A' = A \cap \mathbb{P}_\xi$ is a maximal antichain in $\mathbb{P}_\xi$ and $A' \in \mathfrak{M}_\xi$. But then A' remains pre-dense, therefore, still a maximal antichain, in the whole set $\mathbb{P}$ by Lemma 6.3. It follows that $A = A'$ is countable. □

7 The model

We consider the set of trees $\mathbb{P} \in \mathbf{L}$ (Definition 6.1) as a forcing notion over $\mathbf{L}$.

Lemma 7.1 (compare to Lemma 7 in [6]) *A real $x \in 2^\omega$ is $\mathbb{P}$-generic over $\mathbf{L}$ iff $x \in Z = \bigcap_{\xi<\omega_1^{\mathbf{L}}} \bigcup_{U\in\mathbb{U}_\xi} [U]$.*

Proof If $\xi < \omega_1^{\mathbf{L}}$ then $\mathbb{U}_\xi$ is pre-dense in $\mathbb{P}$ by Lemma 6.3, therefore any real $x \in 2^\omega$ $\mathbb{P}$-generic over $\mathbf{L}$ belongs to $\bigcup_{U \in \mathbb{U}_\xi}[U]$.

To prove the converse, let $x \in Z$. Prove that x is $\mathbb{P}$-generic over $\mathbf{L}$. Consider a maximal antichain $A \subseteq \mathbb{P}$ in $\mathbf{L}$ and show that $x \in \bigcup_{T \in A}[T]$. Note that $A \subseteq \mathbb{P}_\xi$ for some $\xi < \omega_1^{\mathbf{L}}$ by Corollary 6.5. But then every tree $U \in \mathbb{U}_\xi$ satisfies $U \subseteq^{\mathtt{fin}} \bigcup A$ by Lemma 4.4, so that $\bigcup_{U \in \mathbb{U}_\xi}[U] \subseteq \bigcup_{T \in A}[T]$, and hence $x \in \bigcup_{T \in A}[T]$, as required. □

Corollary 7.2 (compare to Corollary 9 in [6]) *In any generic extension of* $\mathbf{L}$*, the set of all reals in* 2^ω $\mathbb{P}$*-generic over* $\mathbf{L}$ *is* Π_1^{HC} *and* Π_2^1*.*

Proof Use Lemma 7.1 and Proposition 6.2. □

Definition 7.3 From now on, let $G \subseteq \mathbb{P}$ be a set $\mathbb{P}$-generic over $\mathbf{L}$, so that $X = \bigcap_{T \in G}[T]$ is a singleton $X_G = \{x_G\}$, where $x_G \in 2^\omega$.

Compare the next lemma to Lemma 10 in [6]. While Jensen's forcing notion in [6] guarantees that there is a single generic real in the extension, the forcing notion $\mathbb{P}$ we use adds a whole E_0-class (a countable set) of generic reals!

Lemma 7.4 (in the assumptions of Definition 7.3) *If* $y \in \mathbf{L}[G] \cap 2^\omega$ *then* y *is a* $\mathbb{P}$*-generic real over* $\mathbf{L}$ *iff* $y \in [x_G]_{\mathsf{E}_0} = \{\sigma \cdot x_G : \sigma \in 2^{<\omega}\}$.

Proof The real x_G itself is $\mathbb{P}$-generic, of course. It follows that any real $y = \sigma \cdot x_G \in [x_G]_{\mathsf{E}_0}$ is $\mathbb{P}$-generic as well since the forcing $\mathbb{P}$ is by definition invariant under the action of any string $\sigma \in 2^{<\omega}$.

To prove the converse, *suppose towards the contrary* that there is a $\mathbb{P}$-real name $\mathbf{c} \in \mathbf{L}$ such that a tree $T \in \mathbb{P}$ $\mathbb{P}$-forces that $\mathbf{c}$ is $\mathbb{P}$-generic and $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}$ for all $\sigma \in 2^{<\omega}$, where $\dot{\boldsymbol{x}}$ is a canonical $\mathbb{P}$-name for the real x_G. To represent $\mathbf{c}$ as $f(x_G)$, where $f : 2^\omega \to 2^\omega$ is a continuous function coded in $\mathbf{L}$, we define $C_n^i = \{S \in \mathbb{P} : S$ forces that $\mathbf{c}(n) = i\}$ ($n < \omega$ and $i = 0, 1$), so that each set $C_n = C_n^0 \cup C_n^1$ is dense in $\mathbb{P}$. **Arguing in** $\mathbf{L}$, let $A_n \subseteq C_n$ be a maximal antichain; then each A_n is countable (in $\mathbf{L}$) by Corollary 6.5. Therefore there is a limit ordinal $\lambda < \omega_1^{\mathbf{L}}$ such that $\bigcup_n A_n \subseteq \mathbb{P}_\lambda$ and each A_n belongs to $\mathfrak{M}_\lambda$. We may assume that $T \in \mathbb{P}_\lambda$ (otherwise increase λ).

By construction and Lemma 4.3 there is a tree $R \in \mathbb{U}_\lambda$, $R \subseteq T$ (a stronger condition). By construction and Lemma 4.4 we have $R \subseteq^{\mathtt{fin}} \bigcup A_n$ for each n, meaning that there exists a finite set $A'_n \subseteq A_n$ with $R \subseteq \bigcup A'_n$. Let $g : [R] \to 2^\omega$ be defined so that $g(x)(n) = i$ iff $x \in [U]$ for some $U \in A'_n \cap C_n^i$. This is a continuous function on $[R]$, a compact set, coded in $\mathbf{L}$. One routinely checks that R $\mathbb{P}$-forces that $\mathbf{c} = g(\dot{\boldsymbol{x}})$.

Now, **still arguing in** $\mathbf{L}$, we are going to define a continuous extension $f : 2^\omega \to 2^\omega$ of g satisfying the following extra condition:
(*) if $x \in 2^\omega \smallsetminus [R]$ then $f(x) \in \textsc{Fin}$, that is, the set $\{k : f(x)(k) = 1\}$ is finite.

If $x \in [R]$ then put $f(x) = g(x)$. Now, suppose that $x \in 2^\omega \smallsetminus [R]$. Let m_x be the largest m with $x \restriction m \in R$. Let $s[x] = (x \restriction m_x)^\wedge(1 - x(m_x))$, so that $s[x] \in R$ but $s[x] \not\subset x$. Let y_x be the lexicographically leftmost element of the set $[R \restriction_{s[x]}] = \{y \in [R] : s[x] \subset y\}$. Finally let $f(x) = (g(y_x) \restriction m_x)^\wedge 0^\omega$ (the extension of $g(y_x)\restriction m_x \in 2^{<\omega}$ by infinitely many zeros). Then $\mathbf{1}_{f(x)}$ is finite. One easily proves that $f: 2^\omega \to 2^\omega$ a continuous extension of g satisfying (*) (and still coded in $\mathbf{L}$), and R forces $\mathbf{c} = g(\dot{\boldsymbol{x}}) = f(\dot{\boldsymbol{x}})$.

Once again we can assume that f is coded even in $\mathfrak{M}_\lambda$. (If not then replace λ by an appropriate ordinal $\lambda' > \lambda$ and replace R by a tree $R' \in \mathbb{U}_{\lambda'}$, $R' \subseteq R$.)

Let $\sigma \in 2^{<\omega}$. We claim that the set $D_f(\sigma)$ of all trees $S \in \mathbb{P}$ satisfying $(\sigma \cdot [S]) \cap (f\text{"}[S]) = \varnothing$, is dense in $\mathbb{P}$. Indeed let $U \in \mathbb{P}$. If $U \not\subseteq R$, then let $t \in U \setminus R$ and $S = U\restriction_t$; thus $S \in \mathbb{P}$, $S \subseteq U$, but $[S] \cap [R] = \varnothing$. We may assume that $S \in \mathbb{U}_\xi$ for some $\xi \geq 1$. Then $[S] \cap \textsc{Fin} = \varnothing$ by Lemma 4.5, hence $(\sigma \cdot [S]) \cap \textsc{Fin} = \varnothing$, but $f\text{"}[S] \subseteq \textsc{Fin}$ by construction, hence $(\sigma \cdot [S]) \cap (f\text{"}[S]) = \varnothing$ and $S \in D_f(\sigma)$. Now assume that $U \subseteq R$. Then U forces $\mathbf{c} = g(\dot{\boldsymbol{x}}) = f(\dot{\boldsymbol{x}})$, hence, forces $f(\dot{\boldsymbol{x}}) \neq \sigma \cdot \dot{\boldsymbol{x}}$. Then there are two incompatible strings $u, v \in 2^{<\omega}$ and a tree $S \in \mathbb{P}$, $S \subseteq U$, which forces $u \subset \sigma \cdot \dot{\boldsymbol{x}}$ and $v \subset f(\dot{\boldsymbol{x}})$. Then easily $S \subseteq [\sigma \cdot u] = \{\sigma \cdot x : u \subset x\}$ but $f\text{"}[S] \subseteq [v]$. It follows that $(\sigma \cdot [S]) \cap (f\text{"}[S]) = \varnothing$ (since u, v are incompatible), therefore $S \in D_f(\sigma)$.

By Lemma 6.4, there is a limit ordinal $\rho > \lambda$ such that each set $D'_f(\sigma) = D_f(\sigma) \cap \mathbb{P}_\rho$ is dense in $\mathbb{P}_\rho$. Applying Theorem 5.2 with $\mathbb{P} = \mathbb{P}_\rho$, $\mathbb{U} = \mathbb{U}_\rho$, and $\mathbb{P} \cup \mathbb{U} = \mathbb{P}_{\rho+1}$, we conclude that for each $U \in \mathbb{U}_\rho$ the set Q_U of all conditions $V \in \mathbb{P}_{\rho+1}$ satisfying $[U] \cap (f\text{"}[V]) \subseteq \textsc{Fin}$ is dense in $\mathbb{P}_{\rho+1}$. As obviously $Q_U \in \mathfrak{M}_{\rho+1}$, we further conclude that each Q_U is pre-dense in the whole forcing $\mathbb{P}$ by Lemma 6.3. This implies that R forces $\mathbf{c} = f(\dot{\boldsymbol{x}}) \notin \bigcup_{U \in \mathbb{U}_\rho} [U]$, hence, forces that $\mathbf{c}$ is not $\mathbb{P}$-generic, by Lemma 7.1. But this contradicts to the choice of T. $\square$

Lemma 7.5 (in the assumptions of Definition 7.3) *The real x_G is not OD in* $\mathbf{L}[G]$.

Proof Suppose towards the contrary that $T \in G$, $\vartheta(x)$ is a formula with ordinal parameters, and T $\mathbb{P}$-forces that x_G is the only $x \in 2^\omega$ satisfying $\vartheta(x)$. Let $s = \mathtt{stem}(T)$, so that both $s^\wedge 0$ and $s^\wedge 1$ belong to T. Then either $s^\wedge 0 \subset x_G$ or $s^\wedge 1 \subset x_G$; let, say, $s^\wedge 0 \subset x_G$. Let $n = \mathtt{lh}(s)$ and $\sigma = 0^n{}^\wedge 1$, so that all three strings $s^\wedge 0, s^\wedge 1, \sigma$ belong to 2^{n+1}, and $s^\wedge 1 = \sigma \cdot s^\wedge 0$. As the forcing $\mathbb{P}$ is invariant under the action of σ, the set $G' = \sigma \cdot G$ is $\mathbb{P}$-generic over $\mathbf{L}$, and $T' = \sigma \cdot T \in G'$. We conclude that it is true in $\mathbf{L}[G'] = \mathbf{L}[G]$ that the real $x' = x_{G'} = \sigma \cdot x_G$ is still the only x satisfying $\vartheta(x)$. However obviously $x' \neq x$! $\square$

Now, the set $X = [x_G]_{\mathsf{E}_0}$ is exactly the set of all $\mathbb{P}$-generic reals in the $\mathbb{P}$-generic model $\mathbf{L}[G] = \mathbf{L}[x_G]$ by Lemma 7.4, hence it belongs to Π^1_2 by Corollary 7.2, and it contains no OD elements by Lemma 7.5, as required. $\square$ (Theorem 1.1)

8 Concluding remarks

Theorem 1.1 also solves another related question asked at the *Mathoverflow* website[8]: namely, *is there an example of a set S definable in **ZFC** and provable in **ZFC** to be countably infinite, while at the same time, no set definable in **ZFC** can be proved in **ZFC** to be an element of S?* To define such an example, let (1) $S = [x]_{\mathsf{E}_0}$ provided

[8] A question about definable non-empty sets containing no definable elements. *Mathoverflow*, February 11, 2013, http://mathoverflow.net/questions/121484.

the set universe is equal to the class $\mathbf{L}[x]$ as in Theorem 1.1, and (2) simply $S = \omega$ otherwise. Suppose towards the contrary that **ZFC** proves that the real x, uniquely defined by a certain fixed formula, outright belongs to S. Then in particular this must be true in case (1), contrary to the definition of S via Theorem 1.1.

Speaking of Theorem 1.1, one may ask does it generalize to other typical equivalence relations. Here we immediately face a negative result.

Lemma 8.1 *Suppose that* E *is a Borel equivalence relation on* ω^ω*, smooth via a Borel map* $f : \omega^\omega \to 2^\omega$*, and* f *is* $\Delta^1_1(p)$*, where* $p \in \omega^\omega$ *is OD. If* $x \in \omega^\omega$ *and the equivalence class* $[x]_{\mathsf{E}}$ *is OD then* $[x]_{\mathsf{E}}$ *contains an OD element.*

Proof In our assumptions, $a = f(x)$ is OD as well. We conclude that $[x]_{\mathsf{E}} = f^{-1}(a)$ is a $\Delta^1_1(a, p)$ set. By the Kondo–Addison uniformization, it contains a $\Delta^1_2(a, p)$ element $y \in [x]_{\mathsf{E}}$. But $a, p \in$ OD, hence, y is OD as well. □

It follows that Theorem 1.1 fails for a smooth equivalence relation E defined in $\mathbf{L}$ instead of E_0. But it holds for Borel non-smooth *countable* relations (i.e., those with countable equivalence classes).

Lemma 8.2 *Suppose that, in the context of* Theorem 1.1*,* E *is a Borel non-smooth countable equivalence relation on* ω^ω *in* $\mathbf{L}$*. Then, in* $\mathbf{L}[x]$*, there is a real* y *such that the equivalence class* $[y]_{\mathsf{E}}$ *is OD but contains no OD elements.*

Proof By the Glimm – Effros dichotomy theorem [4], it is true in $\mathbf{L}$ that there is a $1-1$ Borel reduction $f : 2^\omega \to \omega^\omega$ of E_0 to E, so that $a \mathrel{\mathsf{E}_0} b$ iff $f(a) \mathrel{\mathsf{E}} f(b)$. Let f be $\Delta^1_1(p)$ in $\mathbf{L}$; $p \in \mathbf{L} \cap \omega^\omega$. By Shoenfield, it is true in $\mathbf{L}[x]$ that $f : 2^\omega \to \omega^\omega$ is still a $\Delta^1_1(p)$ reduction of E_0 to E.[9] Let $y = f(x)$. Then the E-class

$$[y]_{\mathsf{E}} = [\{f(x') : x' \in [x]_{\mathsf{E}_0}\}]_{\mathsf{E}}$$

of y is OD since such are $[x]_{\mathsf{E}_0}$ and f. Prove that $[y]_{\mathsf{E}}$ has no OD elements.

First of all, y itself cannot be OD as $x = f^{-1}(y)$ is not OD.

Further, it holds in $\mathbf{L}$ by the Feldman – Moore theorem (see, e.g. [8, 7.4.1]) that E is induced by a Borel action $\cdot$ of a countable group G on ω^ω, that is, $z \mathrel{\mathsf{E}} z'$ iff $\exists\, g \in G\,(z' = g \cdot z)$. Let $\cdot$ be $\Delta^1_1(r)$, where $r \in \omega^\omega$ in $\mathbf{L}$. Once again by Shoenfield, it holds in $\mathbf{L}[x]$ that $\cdot$ is a $\Delta^1_1(r)$ action of G that induces E. Now let $y' \in [y]_{\mathsf{E}}$ in $\mathbf{L}[x]$. Then there is an element $g \in G$ such that $y = g \cdot y'$. Therefore y is $\Delta^1_1(r, y')$. It follows that if y' is OD then so is y. But the latter is false by the above. □

Question 8.3 Is Lemma 8.2 still true for non-countable Borel equivalence relations E, beginning say with E_1?

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[9] We use the same symbol f to denote both $f \in \mathbf{L}$ itself, which is a Borel map in $\mathbf{L}$, and its natural extension in $\mathbf{V}$. The same for E_0 and E.

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