Abh. Math. Semin. Univ. Hambg. (2008) 78: 91–98
DOI 10.1007/s12188-008-0003-y

Some natural equivalence relations in the Solovay model

Sy-David Friedman · Vladimir Kanovei

Received: 27 February 2008 / Published online: 31 May 2008

Abstract We obtain some non-reducibility results concerning some natural equivalence relations on reals in the Solovay model. The proofs use the existence of reals x which are minimal with respect to the cardinals in $L[x]$, in a certain sense.

Keywords Equivalence relations · Solovay model · Forcing · Constructibility

Mathematics Subject Classification (2000) 03E15 · 03E35 · 03E45

1 Introduction

The Borel reducibility of Borel and analytic equivalence relations is one of the key points of interest in modern descriptive set theory. Given a pair of equivalence relations E and F on Borel sets resp. X, Y (sets of reals or sets situated in any Polish space), E is said to be *Borel reducible* to F, symbolically $\mathsf{E} \le_{\text{BOR}} \mathsf{F}$, iff there exists a Borel map $\vartheta : X \to Y$ such that

$$x \mathrel{\mathsf{E}} x' \quad \Longleftrightarrow \quad \vartheta(x) \mathrel{\mathsf{F}} \vartheta(x')$$

for all $x, x' \in X$. Such a map ϑ obviously induces an injection from the quotient X/E to Y/F. Therefore the inequality $\mathsf{E} \le_{\text{BOR}} \mathsf{F}$ can be understood as the fact that the *Borel cardinality* of X/E is $\le$ that of Y/F. We refer to [7] for matters of original motivation and some basic results in this direction, and to [5] for a more modern exposition.

Communicated by B. Löwe.

S.-D. Friedman research was supported by Grant P 19375-N18 of the Austrian Science Fund (FWF).
V. Kanovei research was supported by Grants 06-01-00608 and 07-01-00445 of the Russian Foundation for Basic Research (RFBR).

S.-D. Friedman (✉)
Kurt Gödel Research Center for Mathematical Logic, Währinger Straße 25, 1090 Wien, Austria
e-mail: sdf@logic.univie.ac.at

V. Kanovei
Institute for Information Transmission Problems, Sector 1.1, GSP-4 Bol. Karetnyj Per. 19,
Moscow 101447, Russia

The structure of Borel cardinalities (that is, Borel equivalence relations under $\le_{\text{BOR}}$) is quite rich: in particular it embeds the structure of $\mathscr{P}(\omega)$ under inclusion modulo finite [9], and therefore embeds any partial order of size $\aleph_1$. Compare this to the structure of Borel cardinalities of pointsets in Polish spaces, which contains only finite cardinalities, $\aleph_0$, and the continuum $\mathfrak{c}$, and to the structure of true set theoretic cardinalities of pointsets and their quotients, which depends on the basic setup of the set theoretic universe.

This note belongs to a somewhat different branch of descriptive set theory whose broad description is real-ordinal definable (ROD for brevity) pointsets and relations in the Solovay model. (This model served as the background of several outstanding theorems in the early era of forcing. In particular Solovay [10] proved that in this model all ROD (including all projective) sets of reals are Lebesgue measurable and have the Baire property.)

Let $\le_{\text{ROD}}$ be the order of ROD reducibility, similar to $\le_{\text{BOR}}$ but with ROD maps ϑ. The $\le_{\text{ROD}}$ structure of ROD equivalence relations in the Solovay model has some striking similarities to the $\le_{\text{BOR}}$ structure of Borel and analytic equivalence relations. In particular the following dichotomy holds in the Solovay model, see [3]:

> if E is a ROD equivalence relation on the reals then either E admits a ROD reduction to equality on the set $2^{<\omega_1}$ of all countable transfinite dyadic sequences, or $\mathsf{E}_0 \le_{\text{BOR}} \mathsf{E}$,

where E_0 in this context can be identified with the Vitali equivalence relation on the real line. This can be compared with the Ulm-style dichotomy for analytic (that is, Σ^1_1) equivalence relations, proved under the hypothesis of sharps in [2] and under the hypothesis that the universe is a set generic extension of the constructible universe L in [4]:

> if E is an analytic equivalence relation then either E admits a Δ^1_2 (in the codes) reduction to the equality on $2^{<\omega_1}$, or $\mathsf{E}_0 \le_{\text{BOR}} \mathsf{E}$.

Another relevant result of [6] asserts that the $\le_{\text{ROD}}$-interval between E_0 and E_1 is empty in the Solovay model, similarly to the emptiness of the $\le_{\text{BOR}}$-interval between E_0 and E_1 by a classical result of [8].

These initial results lead us to a general problem of the structure of Borel, and, generally speaking, ROD equivalence relations under the ROD reducibility in the Solovay model. We consider, in the Solovay model, a series of OD (ordinal-definable) equivalence relations[1] Ω_n, $1 \le n < \omega$, where $x \,\Omega_n\, y$ iff $\omega_n^{L[x]} = \omega_n^{L[y]}$, and prove that they are pairwise $\le_{\text{ROD}}$-incomparable. Quite differently from the known irreducibility proofs in the theory of Borel reducibility, our proof involves some forcing coding systems, most notably a coding by a minimal real earlier developed in [1].

2 The main theorem

Let κ be inaccessible in L and consider $L[G]$, where G is generic for the gentle Lévy collapse P of κ to ω_1 (i.e., a condition in P is a finite function f from a subset of $\omega \times \kappa$ into κ such that $f(n, \alpha) < \alpha$ for each (n, α) in $\mathrm{Dom}(f)$). We refer to $M = L[G]$ as the *Solovay model*.[2] It was exactly the model where by [10] all ROD (including all projective) sets of

[1] Introduced by P. Kawa, who also conjectured their mutual $\le_{\text{ROD}}$-incomparability in the Solovay model, in a discussion with the second author of this paper in the course of a meeting at the University of Florida, Gainesville, May 2007.

[2] Sometimes the term "Solovay model" is used to refer not to M, but to the $L(\mathbb{R})$ of M. But as M and the $L(\mathbb{R})$ of M have the same notion of ROD-reducibility, this distinction is not relevant for the results of this paper.

reals are Lebesgue measurable. In M we consider the equivalence relations:

$$x\,\Omega_\xi\, y \quad \text{iff} \quad \omega_\xi^{L[x]} = \omega_\xi^{L[y]} \quad \text{for each } \xi,\ 0 < \xi < \kappa = \omega_1.$$

We make it clear that Ω_ξ are considered in this paper as equivalence relations on the reals (that is, on the Baire space ω^ω), although in principle they make sense for sets x, y of any kind.

Theorem 1 *In M, Ω_1 is not* ROD*-reducible to Ω_2.*

Proof For the sake of simplicity, we consider only the case of OD-reducibility. The general ROD case (that is, when a real parameter is added) is an easy relativisation. Thus we prove that Ω_1 is not reducible to Ω_2 via any OD function.

For the proof of this fact we need a lemma that involves a "cardinal-minimality" coding, and this is the key lemma in the proof. The lemma holds under the assumption of the countability of ω_4^L, therefore is true in the Solovay model.

Lemma 2 *Suppose that ω_4^L is countable. Then there is a real x such that $\omega_1^{L[x]} = \omega_2^L$, $\omega_2^{L[x]} = \omega_4^L$ but there is no real a in $L[x]$ such that $\omega_2^{L[a]} = \omega_3^L$.*

Proof Start with L as the ground model. First Lévy collapse ω_3^L to ω_2^L in the usual way, using conditions of size ω_1^L. As this forcing has only ω_4^L antichains in L by a simple cardinality argument, our hypothesis implies that a generic for this forcing exists in V. In this generic extension let A be a subset of ω_2^L which codes a wellordering of ω_2^L of length ω_3^L.

Now we introduce a forcing P in the new ground model $L[A]$ which adds the desired real x. This forcing bears some similarity to the forcing found in [1], Sect. 6.1. In $L[A]$, define a *tree* to be a set T of finite, increasing sequences of countable ordinals closed under initial segments with the property that if σ belongs to T then σ has uncountably many extensions in T. In addition, we require that whenever σ is a *splitting node* of T, i.e., an element of T such that $\sigma * \alpha$ belongs to T for more than one α, then in fact there are uncountably many such α's. The *nth splitting level* of T consists of those splitting nodes σ of T such that exactly n proper initial segments of σ are also splitting nodes of T.

Any such tree in $L[A]$ in fact belongs to L, as L and $L[A]$ have the same subsets of ω_1^L. The forcing P consists of those trees which code as much of A as possible, in the sense we next describe.

By induction on $i < \omega_2^L$ define the ordinal μ_i as follows: μ_i is the least ordinal μ greater than each μ_j, $j < i$, such that $\mathcal{A} = L_\mu[A \cap i]$ is admissible and has ω_1^L as its largest cardinal. We write $\mathcal{A}_i$ for $L_{\mu_i}[A \cap i]$. For each tree T we define $|T|$ to be the least i such that T belongs to $\mathcal{A}_i$ and call it the *rank of T*.

As ω_1^L is countable in V, any tree T has branches in V which are *cofinal in ω_1^L*, in the sense that the ordinals appearing in the branch are cofinal in ω_1^L. We say that the tree T *codes A at i* iff for each branch b through T in V which is cofinal in ω_1^L:

$$(*) \quad i \in A \quad \text{iff} \quad L_{\mu_i}[b] \text{ is admissible.}$$

Although this notion refers to branches through T in V, it is nonetheless expressible in the model $L[A]$, for the following reason: Suppose that $(*)$ were to fail for some b in V (where b is a branch through T which is cofinal in ω_1^L). Now let P be a forcing in $L[A]$ which forces that ω_1^L is countable. If G is P-generic over V, then $(*)$ fails for some b in

$V[G]$ and therefore by absoluteness, also for some b in $L[A][G]$ (as T and ω_1^L are countable in that model). So $(*)$ fails for some b in a set-generic extension of $L[A]$. Conversely, if $(*)$ fails for some b in a set-generic extension of $L[A]$, then it also fails for some b in a set-generic extension of V and therefore again by absoluteness, for some b in V. Thus instead of referring to branches in V we can equivalently refer to branches in a set-generic extension of $L[A]$, a quantifier expressible in the model $L[A]$.

Now let P consist of all trees T in $L[A]$ such that T codes A at i for each i less than $|T|$. Conditions in P are ordered by $T_0 \leq T_1$ iff T_0 is a subtree of T_1.

Sublemma 3 *Suppose that T belongs to P and $i < \omega_2^L$. Then T has an extension T^* such that $i \leq |T^*|$.*

Proof We prove this by induction on i. The case $i = 0$ is vacuous. Suppose that $i = j + 1$. By induction we may first extend T to have rank at least j and therefore can assume that $|T|$ equals j. Thus T belongs to $\mathcal{A}_j = L_{\mu_j}[A \cap j]$.

First suppose that j is an element of A. View T as a partial order which belongs to $\mathcal{A}_j$ and we will thin T to $T^* \in \mathcal{A}_i$ so that each branch b through T^* which is cofinal in ω_1^L is generic for the partial order T over $\mathcal{A}_j$. To achieve this, first note that if D_n, $n \in \omega$, are dense subsets of T in $\mathcal{A}_j$ and σ is any splitting node of T, we can thin $T(\sigma) = (T \text{ above } \sigma)$ to $T^*(\sigma)$ so that any branch through $T^*(\sigma)$ meets each D_n. The latter is done by thinning T below each $\sigma * \alpha$ to meet D_0, then thinning below each $\tau * \alpha$, where τ is an extension of σ on the next splitting level, to meet D_1, and so forth. Now using this, thin T to T^* as follows: List the dense subsets of T which belong to $\mathcal{A}_j$ as $\langle D_\alpha \mid \alpha < \omega_1^L \rangle$; such a list exists inside $\mathcal{A}_i$, as $\mathcal{A}_j$ has cardinality ω_1^L in $\mathcal{A}_i$. Now thin T below each $\sigma * \alpha$, where σ is on the 0th splitting level of T, to guarantee that any branch through $\sigma * \alpha$ meets each of the D_β, $\beta < \alpha$. Then thin below each node $\tau * \alpha$, where τ is on the first splitting level, to guarantee that any branch through $\tau * \alpha$ meets each of the D_β, $\beta < \alpha$, and so forth. The result is a tree T^* with the property that whenever the ordinal α appears on a branch b through T^*, b meets each D_β, $\beta < \alpha$. Thus whenever b is a branch through T^* which is cofinal in ω_1^L, b is generic for the partial order T over the model $\mathcal{A}_j$. As the enumeration of the D_α's was chosen in $\mathcal{A}_i$, it follows that T^* can also be chosen in $\mathcal{A}_i$, and therefore has rank i. And as any branch through T^* which is cofinal in ω_1^L is generic over $\mathcal{A}_j$ for the partial order $T \in \mathcal{A}_j$, it follows that $L_{\mu_j}[b]$ is admissible for any such branch b, as admissibility is preserved by set-forcing.

Now suppose that j does not belong to A. We wish to thin T to T^* so that any cofinal branch through T^* will destroy the admissibility of $\mathcal{A}_j$ (i.e., $L_{\mu_j}[b]$ will be inadmissible). Choose a subset B of ω_1^L in $\mathcal{A}_i$ such that B codes a wellordering of ω_1^L of length μ_j. This is possible as μ_j has cardinality ω_1^L in the model $\mathcal{A}_i$. Then $L_{\mu_j}[B]$ is inadmissible. For each $\alpha < \omega_1^L$, let $\beta_\alpha < \omega_1^L$ be the position of $B \cap \alpha$ in the canonical wellordering of L, and let C consist of these β_α's. Then C is unbounded in ω_1^L and $L_{\mu_j}[D]$ is inadmissible for any cofinal $D \subseteq C$, as from D we can easily recover B.

Now thin T to T^* as follows: Suppose that σ is on the 0th splitting level of T. List $\mathrm{Succ}_T(\sigma) = \{\alpha \mid \sigma * \alpha \in T\}$ in increasing order as $\langle \gamma_\alpha \mid \alpha < \omega_1^L \rangle$. Thin out T below σ by discarding the $\sigma * \gamma_\alpha$ for α not in C. Now repeat this for nodes σ that remain and are on the first splitting level, by saving only those $\sigma * \gamma$ which are "indexed" in C. After ω steps, the resulting tree T^* has the property that for any branch b:

> If σ is an initial segment of b which is a splitting node of T^*, then b extends $\sigma * \gamma$ where γ is "indexed" in C.

In particular, if b is a cofinal branch through T^*, then b determines a cofinal subset D of C, which in turn determines B, and therefore $L_{\mu_j}[b]$ is inadmissible, as desired.

Finally suppose that i is a limit ordinal. We may assume that $|T|$ is less than i. First suppose that i has L-cofinality ω and choose an ω-sequence $i_0 < i_1 < \cdots$ cofinal in i with $|T| < i_0$. Note that this sequence can be chosen in $\mathcal{A}_i$ as in this model i has cofinality either ω or ω_1^L and the latter cannot occur. Let σ be on the 0th splitting level of T. As T above any $\sigma * \alpha$ is a condition of rank at most that of T, we can apply induction to thin out T above each such node to a condition of rank i_0. Then for each remaining node σ on the first splitting level, thin out the tree above each $\sigma * \alpha$ to a condition of rank i_1. Continue in this way for ω steps and the result is a tree with the property that each cofinal branch b codes A at j for each j less than i. Moreover this construction can be carried out in $\mathcal{A}_i$, and therefore the resulting tree has rank i, as desired.

If i has L-cofinality ω_1^L then choose an ω_1^L-sequence $i_0 < i_1 < \cdots$ cofinal in i with $|T| < i_0$. Again we may assume that this sequence belongs to $\mathcal{A}_i$. Now thin out T in ω steps as in the case where i has L-cofinality ω, except when considering a node whose last component is the ordinal α, thin the tree above this node to have rank i_α. The result is a tree with the property that for any branch b and any ordinal α occurring on b, b codes A at j for each j less than i_α. It follows that for any cofinal branch b, b codes A at j for all j less than the supremum of the i_α's, namely i. □

Sublemma 4 *The forcing P collapses ω_1^L and preserves all other cardinals.*

Proof Clearly P collapses ω_1^L as the intersection of the trees in a generic produces an ω-sequence cofinal in ω_1^L. And as P has size ω_2^L in $L[A]$, it follows that cardinals greater than ω_2^L are preserved. So we need only check that ω_2^L is preserved. As ω_1^L is collapsed, it suffices to show that if T forces $\dot{f}$ to be a function from ω into ω_2^L, then some extension of T forces a bound on the range of $\dot{f}$. In $L[A]$ let $\langle M_n \mid n < \omega\rangle$ be a Σ_1-elementary chain of submodels of a large $H(\theta) = L_\theta[A]$ such that:

1. M_0 contains A, P, T, the name $\dot{f}$ and all countable ordinals as elements.
2. Each M_n has cardinality ω_1 and contains $\langle M_m \mid m < n\rangle$ as an element.
3. If M_ω is the union of the M_n's, then the sequence $\langle M_n \mid n \in \omega\rangle$ is definable over M_ω.

It is straightforward to obtain such a sequence, by taking the first ω-many Σ_1-elementary submodels of $L_\theta[A]$ which contain the parameters mentioned in 1 above. Note that if i_n denotes the intersection of M_n with ω_2^L, then the transitive collapse of M_n is an initial segment of $\mathcal{A}_{i_n}$ (as i_n is a cardinal in the former but not in the latter), which is in turn an initial segment of the transitive collapse of M_{n+1}. Also the transitive collapse of M_ω is an initial segment of $\mathcal{A}_{i_\omega}$, where i_ω is the supremum of the i_n's (as i_ω is a cardinal in the former but not in the latter).

Now thin T below each $\sigma * \alpha$, where σ is on the 0th splitting level of T, to a condition forcing a value of $\dot{f}(0)$. This can be done inside M_0. Thin T further in $\mathcal{A}_{i_0}$ so that the resulting T_0 is a condition of rank i_0 below each $\sigma * \alpha$, and therefore T_0 itself is a condition of rank i_0, belonging to the model $\mathcal{A}_{i_0}$. Then thin T_0 below each $\sigma * \alpha$, where σ is on the first splitting level of T_0, to a condition forcing a value of $\dot{f}(1)$. This can be done inside M_1. Thin further in $\mathcal{A}_{i_1}$ so that the resulting T_1 is a condition of rank i_1. The resulting sequence of T_n's can be chosen definably over M_ω and therefore belongs to $\mathcal{A}_{i_\omega}$. The intersection of the T_n's is therefore a condition forcing the range of $\dot{f}$ to be contained in the set of possible values of $\dot{f}(n)$ occurring in this construction. □

Sublemma 5 *Suppose that G is P-generic over $L[A]$. Let $f : \omega \to \omega_1^L$ be the unique infinite branch through all of the trees in G. Then the range of f is cofinal in ω_1^L and $L[A][G] = L[f]$.*

Proof The first conclusion is clear, as given any $\alpha < \omega_1^L$, any condition can be thinned so that any infinite branch includes an ordinal greater than α. It follows from the definition of the forcing and Sublemma 3 that f codes A at i for every i less than ω_2^L, and therefore $A \cap i$ can be inductively decoded in $L[f]$. So A belongs to $L[f]$. Finally, note that G consists precisely of those conditions T in $L[A]$ such that f is a branch through T, as if f is a branch through a condition T, then T must have uncountable intersection with each condition in G, else the range of f would be bounded in ω_1^L. □

Sublemma 6 *Suppose that a is a real in $L[f]$ and ω_1^L is countable in $L[a]$. Then f belongs to $L[a]$.*

Proof Suppose that T is a condition forcing $\dot{g}$ to be a cofinal function from ω into ω_1^L. We show that some extension of T forces that $\dot{f}$ belongs to $L[\dot{g}]$, where $\dot{f}$ is the canonical name for the cofinal function $f : \omega \to \omega_1^L$ added by G. Let σ be on the 0th splitting level of T and for each α such that $\sigma * \alpha$ belongs to T, thin T above $\sigma * \alpha$ to force a value of $\dot{g}(0)$. Then for each σ on the first splitting level of the resulting tree T_1, thin out above each $\sigma * \alpha$ in T_1 to force a value of $\dot{g}(1)$. Using an ω-sequence of Σ_1-elementary submodels as in the proof of Sublemma 4, we can ensure that after continuing this for ω steps, the result is a condition T^*, and moreover, the function that assigns to each node σ on the nth splitting level of T^* the value of $\dot{g}(n)$ forced by T^* below σ belongs to $\mathcal{A}_{|T^*|}$.

Now as T forces that $\dot{g}$ has range cofinal in ω_1^L, so does T^*, and therefore there are uncountably many values of $\dot{g}$ forced by T^* below its various splitting nodes σ. Therefore for some n_0, uncountably many values of $\dot{g}(n_0)$ are forced by T^* below nodes on the n_0th splitting level of T^*. Let X_0 be an uncountable subset of the n_0th splitting level so that if σ, τ are distinct elements of X_0, then T^* below σ and T^* below τ force distinct values of $\dot{g}(n_0)$. Thin out T^* by discarding nodes on the n_0th splitting level which do not belong to X_0. Now for each remaining node σ on the n_0th splitting level, we may choose n_1 and an uncountable X_1 consisting of nodes extending σ on the n_1-st splitting level so that if τ_0 and τ_1 are distinct nodes in X_1, then T^* below τ_0 and T^* below τ_1 force distinct values of $\dot{g}(n_1)$. Discard all nodes on the n_1-st splitting level that extend σ and do not belong to X_1. Continue this for ω steps and note that the resulting tree T^{**} still belongs to $\mathcal{A}_{|T^*|}$. As each node of T^{**} has uncountably many extensions in T^{**}, we may further thin T^{**} to a condition T^{***} in $\mathcal{A}_{|T^*|}$.

Now note that if G is P-generic and contains the condition T^{***}, then $f = \dot{f}^G$, the unique infinite branch through all of the conditions in G, can be recovered from $g = \dot{g}^G$, as any two distinct branches through T^{***} give rise to different versions of $\dot{g}$. So f belongs to $L[A, g]$. But as A is a subset of $\omega_2^L = \omega_1^{L[A,f]}$ with constructible proper initial segments, it then follows that forces f belongs to $L[g]$, as desired. □

Now come back to the proof of Lemma 2. Let f be as in Sublemma 5. First of all, there obviously exists a real x such that $L[x] = L[f]$. Further, all L-cardinals except for ω_1^L and ω_3^L are still cardinals in $L[x] = L[f] = L[A][G]$ by Sublemma 4 and the choice of A. It follows that $\omega_1^{L[x]} = \omega_2^L$ and $\omega_2^{L[x]} = \omega_4^L$. Now to finish the proof consider any real $a \in L[x]$ and prove that $\omega_2^{L[a]} \neq \omega_3^L$. There are two cases. If ω_1^L is countable in $L[a]$ then $f \in L[a]$

by Sublemma 6, hence $\omega_2^{L[a]} = \omega_4^L$. If $\omega_1^L = \omega_1^{L[a]}$ then $\omega_2^L = \omega_2^{L[a]}$ because ω_2^L remains a cardinal even in the bigger model $L[x]$. □

Now it does not take much to finish the proof of Theorem 1. (Recall that only the case of OD-reducibility is considered.) Suppose that Ω_1 were OD-reducible to Ω_2 via the OD function ϑ. Note that for each real z, $L[z]$ is closed under ϑ, as the fact that we are in the Solovay model implies that any real which is OD relative to z is contructible relative to z. Choose x as in Lemma 2; so $(\omega_1^{L[x]}, \omega_2^{L[x]}) = (\omega_2^L, \omega_4^L)$. Choose y a real arising from the usual Lévy collapse of ω_1^L to ω; then $(\omega_1^{L[y]}, \omega_2^{L[y]}) = (\omega_2^L, \omega_3^L)$. As $x\,\Omega_1\,y$ holds and ϑ reduces Ω_1 to Ω_2, it follows that $\vartheta(x)\,\Omega_2\,\vartheta(y)$ holds, i.e., that $\omega_2^{L[\vartheta(x)]} = \omega_2^{L[\vartheta(y)]}$. Now $\omega_2^{L[\vartheta(y)]}$ cannot be ω_2^L, else $\vartheta(y)\,\Omega_2\,0\,\Omega_2\,\vartheta(0)$ holds, which implies that $y\,\Omega_1\,0$ holds, contradicting $\omega_1^{L[y]} = \omega_2^L$. So $\omega_2^{L[\vartheta(y)]}$ must be ω_3^L. But by the choice of x, no real z in $L[x]$ satisfies $\omega_2^{L[z]} = \omega_3^L$, and in particular $\omega_2^{L[\vartheta(x)]}$ does not equal ω_3^L, contradicting $\vartheta(y)\,\Omega_2\,\vartheta(x)$. □

3 Generalization

Theorem 1 has the following straightforward generalisation:

Theorem 7 *In the Solovay model M, Ω_m is not* ROD*-reducible to Ω_n for any $0 < m < n < \omega$.*

Proof First suppose that m equals 1. Let $A \subseteq \omega_n^L$ code a Lévy collapse of ω_{n+1}^L to ω_n^L, code A by $B \subseteq \omega_2^L$ without collapsing cardinals, and finally code B by a real x as in the proof of Theorem 1, collapsing ω_1^L but preserving all other cardinals. Then for any real z in $L[x]$, either $\omega_1^{L[z]} = \omega_1^L$, in which case $\omega_n^{L[z]} = \omega_n^L$, or x belongs to $L[z]$, in which case $\omega_n^{L[z]} = \omega_{n+2}^L$. In particular, there is no real z in $L[x]$ such that $\omega_n^{L[z]} = \omega_{n+1}^L$.

Now let y code a Lévy collapse of ω_1^L to ω. Then x and y are Ω_1-equivalent. Suppose that ϑ were an OD-reduction of Ω_1 to Ω_n. Then we have $\omega_n^{\vartheta(x)} = \omega_n^{\vartheta(y)}$. Now $\omega_n^{\vartheta(y)}$ cannot be ω_n^L, else $\vartheta(y)\,\Omega_n\,\vartheta(0)$ and therefore $y\,\Omega_1\,0$, contradicting $\omega_1^{L[y]} = \omega_2^L$. So $\omega_n^{\vartheta(y)}$ equals ω_{n+1}^L. But this contradicts the fact that $\omega_n^{\vartheta(x)} = \omega_n^{\vartheta(y)}$ and no real z in $L[x]$, such as $\vartheta(x)$, can satisfy $\omega_n^{L[z]} = \omega_{n+1}^L$.

Now suppose that m is greater than 1. Then the proof is easier: Let $A \subseteq \omega_n^L$ code a Lévy collapse of ω_{n+1}^L to ω_n^L, let $B \subseteq \omega_{m-1}^L$ code both A and a Lévy collapse of ω_m^L to ω_{m-1}^L and then let $C \subseteq \omega_1^L$ code B. Now choose x to be a real coding C using ω-splitting trees, in analogy to the proof of Theorem 1, which used ω_1-splitting trees to code a subset of ω_2^L. Then $L[C]$ and $L[x]$ have the same cardinals and x has the property that for any real z in $L[x]$, either z belongs to L or x belongs to $L[z]$. In particular, for any real z in $L[x]$, $\omega_n^{L[z]}$ is either ω_n^L or ω_{n+2}^L.

Now let y code a Lévy collapse of ω_m^L to ω_{m-1}^L. Then x and y are Ω_m-equivalent. Suppose that ϑ were an OD-reduction of Ω_m to Ω_n. Then we have $\omega_n^{\vartheta(x)} = \omega_n^{\vartheta(y)}$. Now $\omega_n^{\vartheta(y)}$ cannot be ω_n^L, else $\vartheta(y)\,\Omega_n\,\vartheta(0)$ and therefore $y\,\Omega_m\,0$, contradicting $\omega_m^{L[y]} = \omega_{m+1}^L$. So $\omega_n^{\vartheta(y)}$ equals ω_{n+1}^L. But this contradicts the fact that $\omega_n^{\vartheta(x)} = \omega_n^{\vartheta(y)}$ and no real z in $L[x]$, such as $\vartheta(x)$, can satisfy $\omega_n^{L[z]} = \omega_{n+1}^L$. □

We finish with the easier result establishing irreducibility in the opposite direction:

Proposition 8 *In M, Ω_n is not* ROD*-reducible to Ω_m for $0 < m < n$.*

Proof Choose a real x such that $\omega_n^{L[x]} > \omega_n^L$ but $\omega_m^{L[x]} = \omega_m^L$. (Such a real is obtained by coding a collapse of ω_{n+1}^L to ω_n^L using almost disjoint coding; perfect-tree coding is not needed.) Then x is not Ω_n-equivalent to 0. If ϑ were an OD-reduction of Ω_n to Ω_m, then it follows that $\vartheta(x)$ is not Ω_m-equivalent to $\vartheta(0)$, which contradicts $\omega_m^{L[\vartheta(x)]} = \omega_m^L = \omega_m^{L[\vartheta(0)]}$. □

4 Questions

(1) In M, for any countable ordinal α define $x\,\Omega_\alpha\,y$ iff $\omega_\alpha^{L[x]} = \omega_\alpha^{L[y]}$. For which pairs α, β of countable ordinals is Ω_α OD-reducible to Ω_β in M? (Note, for example, that for limit ordinals ξ less than the least L-inaccessible, Ω_ξ and $\Omega_{\xi+1}$ are identical, as by Jensen's Covering Theorem, $\omega_{\xi+1}^{L[x]}$ is the least L-cardinal greater than $\omega_\xi^{L[x]}$ for any real x, which is uniquely determined by (and uniquely determines) $\omega_\xi^{L[x]}$.)

(2) Is there a real x such that $(\omega_1^{L[x]}, \omega_2^{L[x]}) = (\omega_2^L, \omega_4^L)$ and for each real y in $L[x]$, either $(\omega_1^{L[x]}, \omega_2^{L[y]})$ is (ω_2^L, ω_4^L) or y preserves cardinals over L?

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