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Linearization of definable order relations ☆

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Abstract

We prove that if $\preccurlyeq$ is an analytic partial order then either $\preccurlyeq$ can be extended to a $\boldsymbol{\Delta}^1_2$ linear order similar to an antichain in $2^{<\omega_1}$, ordered lexicographically, or a certain Borel partial order $\preccurlyeq_0$ embeds in $\preccurlyeq$. Similar linearization results are presented, for κ-bi-Souslin partial orders and real-ordinal definable orders in the Solovay model. A corollary for analytic equivalence relations says that any (lightface) Σ^1_1 equivalence relation E, such that E_0 does not embed in E, is fully determined by intersections with E-invariant Borel sets coded in L.

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Notation. A binary relation $\preccurlyeq$ on a set X is a *partial quasi-order*, or PQO, on X, iff $x \preccurlyeq y \wedge y \preccurlyeq z \Rightarrow x \preccurlyeq z$, and $x \preccurlyeq x$ for any $x \in X$. In this case, $\approx$ is the associated equivalence, i.e., $x \approx y$ iff $x \preccurlyeq y \wedge y \preccurlyeq x$.

If $x \approx y \Rightarrow x = y$ for any x, y then $\preccurlyeq$ is a *partial order*, or PO. If in addition $x \preccurlyeq y \vee y \preccurlyeq x$ for all $x, y \in X$ then $\preccurlyeq$ is a *linear* order (LO).

Let $\preccurlyeq$ and $\preccurlyeq'$ be PQOs on resp. X and X'. A map $h : X \to X'$ will be called *half-order preserving*, or h.o.p., iff $x \preccurlyeq y \Rightarrow h(x) \preccurlyeq' h(y)$. Finally, a *linearization* is any h.o.p. map $h : \langle X; \preccurlyeq \rangle \to \langle X'; \preccurlyeq' \rangle$, where $\preccurlyeq'$ is a LO, satisfying $x \approx y \Leftrightarrow h(x) = h(y)$.

By $\preccurlyeq_{\mathrm{lex}}$ we denote the lexicographical order (where applicable).

$\mathcal{N} = \omega^\omega$ is the *Baire space*.

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0. Introduction

It is a simple application of Zorn's lemma that any partial order can be extended to a linear order on the same domain. More generally, any partial quasi-order admits a linearization.

A much more difficult problem is to provide a descriptive characterization of the linear order in the assumption that one has such for the given PQO. For instance, not every Borel PQO is *Borel linearizable*. Indeed, recall that E_0 is an equivalence relation on 2^ω defined as follows: $a\,\mathsf{E}_0\,b$ iff $a(k)=b(k)$ for all but finite k. Then E_0, considered as a PQO, is not Borel linearizable, as it is known that there is no Borel map h defined on 2^ω and satisfying $a\,\mathsf{E}_0\,b \Leftrightarrow h(a)=h(b)$ (see [2]).

This example can be converted to a partial order. Define the *anti-lexicographical* PO $\leqslant_0$ on 2^ω as follows: $a \leqslant_0 b$ iff either $a=b$ or there is $m\in\omega$ such that $a(m)<b(m)$ and $a(k)=b(k)$ for all $k>m$.[2] Clearly $a \leqslant_0 b$ implies $a\,\mathsf{E}_0\,b$, and $\leqslant_0$ linearly orders each E_0-equivalence class similarly to the integers $\mathbb{Z}$, except for the class of $\omega\times\{0\}$ (ordered as ω) and the class of $\omega\times\{1\}$ (ordered as the inverted ω). Finally, a simple argument (see [11]) shows that $\leqslant_0$ is not Borel linearizable.

There are Borel-non-linearizable Borel orders of various nature, e.g., the PQO: $a\preccurlyeq b$ iff $a(k)\leqslant b(k)$ for all but finite k on 2^ω, or the dominance relation on ω^ω. However, by the next theorem, proved in [11], the relation $\leqslant_0$ is actually a *minimal* Borel-non-linearizable Borel order.[3]

Theorem 1 (Kanovei [11]). *Suppose that $\preccurlyeq$ is a Borel* PQO *on $\mathcal{N}=\omega^\omega$. Then exactly one of the following two conditions is satisfied*:

(I^{B}) *$\preccurlyeq$ is Borel linearizable – moreover, in this case there are an ordinal $\alpha<\omega_1$ and a Borel linearization $h:\langle\mathcal{N};\preccurlyeq\rangle\to\langle 2^\alpha;\leqslant_{\mathrm{lex}}\rangle$*;

(II^{B}) *there exists a continuous h.o.p. 1–1 map $F:\langle 2^\omega;\leqslant_0\rangle\to\langle\mathcal{N};\preccurlyeq\rangle$ such that $a \not\!\mathsf{E}_0\, b\Rightarrow F(a)\not\preccurlyeq F(b)$.*

Condition (II^{B}) could be, informally, commented upon as follows: $\preccurlyeq$ admits at least $\mathcal{N}/\mathsf{E}_0$-many mutually incomparable chains of length $\mathbb{Z}$ or less, assuming that quantities are measured in a Borel way.

To see that Theorem 1 fails already for *analytic* orders, let $\mathtt{WO}=\{x\in\mathcal{N}: x \text{ codes an ordinal}\}$; for $x\in\mathtt{WO}$ let $|x|$ be the ordinal coded by x. Define a Σ^1_1 PQO: $x\preccurlyeq y$ iff either $y\notin\mathtt{WO}$, or $x,y\in\mathtt{WO}$ and $|x|\leqslant|y|$. Then $\preccurlyeq$ does not satisfy (I^{B}) even with a non-Borel map h since orders $\langle 2^\alpha;\leqslant_{\mathrm{lex}}\rangle$, $\alpha<\omega_1$, do not admit strictly increasing ω_1-chains. Requirement (II^{B}) also cannot be fulfilled via analytic maps F by a restriction theorem.

[2] If one enlarges $\leqslant_0$ so that, in addition, $a\leqslant_0 b$ whenever $a,b\in 2^\omega$ are such that $a(k)=1$ and $b(k)=0$ for all but finite k, then the enlarged relation can be induced by a Borel action of $\mathbb{Z}$ on 2^ω, such that $a\leqslant_0 b$ iff $a=zb$ for some $z\in\mathbb{Z}$, $z>0$.

[3] Compare with the "Glimm–Effros" theorem of Harrington et al. [2].

1. Main results

The aim of this paper is to prove similar dichotomical linearization theorems for some *non-Borel* partial orders. This will include analytic and bi-κ-Souslin (for cardinals $\kappa > \omega_1$) orders, and definable orders in the Solovay model.

1.1. Analytic partial orders

Following ideas of Hjorth and Kechris [7], we involve longer linear orders, $2^{<\omega_1}$ and 2^{ω_1}, to match the nature of analytic PQOS. A set $A \subseteq 2^{<\omega_1}$, consisting of pairwise $\subseteq$-incomparable elements, will be called an *antichain*.

Theorem 2. *Suppose that* $\preccurlyeq$ *is a* $\boldsymbol{\Sigma}^1_1$ PQO *on* $\mathcal{N}$. *Then at least one of the following two conditions*, (I^A) *or* (II^A), *is satisfied*:

(I^A) *There is a linearization* $h : \langle \mathcal{N}; \preccurlyeq \rangle \to \langle 2^{\omega_1}; \leqslant_{\mathrm{lex}} \rangle$ *such that for any* $\gamma < \omega_1$ *the map* $h_\gamma(x) = h(x)(\gamma)$ *is Borel. In addition in each of the two following cases* [4] *there is an antichain* $A \subseteq 2^{<\omega_1}$ *and a* $\boldsymbol{\Delta}^1_2$ *in the codes linearization* $h : \langle \mathcal{N}; \preccurlyeq \rangle \to \langle A; \leqslant_{\mathrm{lex}} \rangle$:
  (a) *for any* x, *the* $\approx$*-class* $[x]_\approx = \{y : y \approx x\}$ *of* x *is Borel*; [5]
  (b) *the universe is a set-generic extension of a class* $\mathrm{L}[z_0]$, $z_0 \in \mathcal{N}$.

(II^A) *As* (II^B) *of Theorem* 1.

Note that (I^A) and (II^A) here are compatible for instance in the assumption $\mathrm{V} = \mathrm{L}$. There may exist reasonable sufficient conditions (like: all $\boldsymbol{\Delta}^1_2$ sets are Lebesgue measurable) for (I^A) and (II^A) to be incompatible.

In the "additional" part of (I^A), the linearization $h : \langle \mathcal{N}; \preccurlyeq \rangle \to \langle A; \leqslant_{\mathrm{lex}} \rangle$ will be in fact slightly better than just $\boldsymbol{\Delta}^1_2$: indeed, for any ordinal $\lambda < \omega_1$, the set of all values $h(x)$ which belong to 2^λ is a Borel set in 2^λ and the partial map $\{\langle x, h(x) \rangle : h(x) \in 2^\lambda\}$ is an analytic subset of $\mathcal{N} \times 2^\lambda$.

1.2. Applications for analytic equivalence relations

Theorem 2 applies for analytic equivalence relations viewed as a particular case of PQOS.

Corollary 3. *Let* E *be a* $\boldsymbol{\Sigma}^1_1$ *equivalence relation on* $\mathcal{N}$. *Then at least one of the following two conditions*, (I^E) *or* (II^E), *is satisfied*:

[4] An obvious parallel with the "Ulm classification" theorem in [7] allows us to conjecture that the additional assertion is also true in the assumption of the existence of "sharps", or an even weaker assumption in [1]. However the most interesting problem is to prove the additional assertion in **ZFC**.

[5] This applies, e.g., when $\preccurlyeq$ is a PO. Recall that $x \approx y$ iff $x \preccurlyeq y \wedge y \preccurlyeq x$.

(I^{E}) *There is a map* h: $\mathcal{N} \to 2^{\omega_1}$ *such that* $x \mathsf{E} y \Rightarrow h(x) = h(y)$ *and for any* $\gamma < \omega_1$ *the map* $h_\gamma(x) = h(x)(\gamma)$ *is Borel. In addition in each of the two following cases there is an antichain* $A \subseteq 2^{<\omega_1}$ *and a* $\mathbf{\Delta}^1_2$ *in the codes map* $h : \mathcal{N} \to A$ *such that* $x \mathsf{E} y \Leftrightarrow h(x) = h(y)$:
(a) *for any* x, *the* E *class* $[x]_{\mathsf{E}} = \{y\colon y \mathsf{E} x\}$ *of* x *is Borel*;
(b) *the universe is a set generic extension of a class* $\mathrm{L}[z_0]$, $z_0 \in \mathcal{N}$.

(II^{E}) *There exists a continuous* 1–1 *function* F: $2^\omega \to \mathcal{N}$ *such that we have* $a \mathsf{E}_0 b \Leftrightarrow F(a) \mathsf{E} F(b)$. [6]

Note that the maps h_γ in (I^{A}) and (I^{E}) will be Borel in certain effective sense, i.e., they will have Borel codes [7] in $\mathrm{L}[z]$ provided $\preccurlyeq$ is $\Sigma^1_1[z]$. This implies the following corollary: [8]

Corollary 4. *Assume that* E *is a* $\Sigma^1_1[z]$ *equivalence relation,* $z \in \mathcal{N}$, *and* (II^{E}) *of Corollary* 3 *fails. Then* $x \mathsf{E} y$ *iff we have* $x \in X \Leftrightarrow y \in X$ *for every* E*-invariant Borel set* $X \subseteq \mathcal{N}$ *with a Borel code in* $\mathrm{L}[z]$.

Corollary 3, with (I^{E}) in the additional form, has been obtained by Hjorth and Kechris [7] in the subcase (a) (as well as in the assumption of existence of sharps), by Friedman and Velickovic [1] in a hypothesis dealing with weakly compact cardinals, and by Kanovei [10] in the subcase (b). It is not clear whether the reduction, given in [7], satisfies the requirement that all maps h_γ are Borel, as in (I^{E}), and whether it leads to Corollary 4.

1.3. Bi-Souslin partial orders

Theorem 1 admits a generalization on bi-κ-Souslin order relations, which follows a few known patterns (see [14, 5]) in its appeal to a kind of Cohen-generic stability requirement.

Recall that if T is a tree on $\omega \times \omega \times \lambda$, λ being an ordinal, then

$$[T] = \{\langle x, y, f\rangle \in \mathcal{N}^2 \times \lambda^\omega : \forall m\ T(x \restriction m, y \restriction m, f \restriction m)\}$$

and $\mathfrak{p}[T] = \{\langle x, y\rangle : \exists f [T](x, y, f)\}$, which is a λ-*Souslin* set. Recall that a set is $(\lambda + 1)$-*Borel* if it belongs to the smallest algebra of sets containing all open sets and closed under $\complement$, the complement, and unions of $\leqslant \lambda$ sets. *Coding* of $(\lambda + 1)$-Borel sets will be introduced in Section 2.2.

Theorem 5. *Suppose that* $\omega \leqslant \lambda$, *and* T *and* S *are trees on* $\omega \times \omega \times \lambda$ *such that the sets* $\preccurlyeq_T = \mathfrak{p}[T]$ *and* $\preccurlyeq_S = \complement\mathfrak{p}[S]$ *are* PQOS *on* $\mathcal{N}$ *and* $\preccurlyeq_T \subseteq \preccurlyeq_S$. *Assume that* $\complement\mathfrak{p}[S]$ *remains*

[6] A map F as in (II^{E}) is called a (continuous) *embedding* of E_0 in E. A map h as in (I^{E}) is called a *reduction* of E to the equality.

[7] That is, λ-Borel codes for an ordinal $\lambda < \omega_1$, not necessarily countable in $\mathrm{L}[z]$.

[8] Hjorth and Kechris told the author in April 1997 that they had known the result.

a PQO *in Cohen generic extensions of the universe. Then at least one of the following two conditions is satisfied*:

(I*) *There is* $\alpha<(\lambda^+)^{L[S,T]}$ *and a* $(\lambda+1)$-*Borel, coded in* $\mathrm{L}[S,T]$, *h.o.p. map* $h:\langle\mathscr{N};\preccurlyeq_T\rangle\to\langle 2^\alpha;\leqslant_{\mathrm{lex}}\rangle$ *such that* $h(x)=h(y)\Rightarrow x\approx_S y$.

(II*) *There exists a continuous* 1–1 *h.o.p. map* $F:\langle 2^\omega;\leqslant_0\rangle\to\langle\mathscr{N};\preccurlyeq_T\rangle$ *such that* $a\not\mathsf{E}_0 b\Rightarrow F(a)\not\preccurlyeq_S F(b)$.

Note that if $\preccurlyeq_T=\preccurlyeq_S$ then $\preccurlyeq=\preccurlyeq_T=\preccurlyeq_S$ is a bi-λ-Souslin PQO, and conditions (I*) and (II*) take the form:

(I$^\lambda$) *There are an ordinal* $\alpha<(\lambda^+)^{\mathrm{L}[S,T]}$ *and a* $(\lambda+1)$-*Borel, coded in* $\mathrm{L}[S,T]$, *linearization* $h:\langle\mathscr{N};\preccurlyeq\rangle\to\langle 2^\alpha;\leqslant_{\mathrm{lex}}\rangle$.

(II) *As* (II$^{\mathrm{B}}$) *of Theorem* 1.

The case $\lambda=\omega$ in Theorem 5 obviously implies Theorem 1. (The `Cohen`-stability condition follows in this case from the Shoenfield absoluteness.)

The case $\lambda=\omega_1$ includes, in particular, $\mathbf{\Delta}^1_2$ relations, the $\mathbf{\Pi}^1_2$ side of which is `Cohen`-stable. It is not known whether the stability condition can be dropped in the $\mathbf{\Delta}^1_2$ case. It can be expected, however, that the `Cohen`-stability cannot be dropped in the bi-ω_1-Souslin case, as an example, given by Shelah [14], shows for a related theorem.

1.4. Definable partial orders in the Solovay model

The next theorem describes the state of affairs in the Solovay model.[9] Let OD mean *ordinal definable* and ROD mean *real-ordinal definable*.

Theorem 6 (In the Solovay model). *Suppose that* $\preccurlyeq$ *is a* ROD PQO *on* $\mathscr{N}$. *Then exactly one of the following two conditions is satisfied*:

(I$^{\mathrm{s}}$) $\preccurlyeq$ *is* ROD *linearizable. Moreover in this case there are an antichain* $A\subseteq 2^{<\omega_1}$ *and a* ROD *linearization* h: $\langle\mathscr{N};\preccurlyeq\rangle\to\langle A;\leqslant_{\mathrm{lex}}\rangle$.

(II) *As* (II$^{\mathrm{B}}$) *of Theorem* 1.

One of the crucial steps in the proof of this theorem will be to show that, in the Solovay model for **ZFC**, obtained by the collapse of a constructible model up to an inaccessible cardinal Ω, any set $X\subseteq 2^\vartheta$, $\vartheta<\Omega^+$, which is a ROD image of the reals, is lexicographically ROD order isomorphic to an antichain $A(X)\subseteq 2^{<\Omega}$.

[9] The *Solovay model* is a generic extension of a constructible model (as defined by Solovay [15]) where all projective sets of reals are Lebesgue measurable.

1.5. Organization of the proofs

The main part of the paper consists of the proof of Theorem 5. After preliminaries in Section 2, [10] mainly devoted to a coding system for Borel sets, we introduce the dichotomy in Section 2.4. Then the proof of Theorem 5 naturally develops itself in Sections 3 and 4. The principal technical scheme goes back to the papers of Harrington and Shelah [4], Shelah [14], and especially Hjorth [5], containing theorems on bi-κ-Souslin equivalence and order relations. However our version of the technique is free of any use of model theory, including admissible sets. On the other hand, we apply an effective version of a classical separation theorem, proved in [9].

Two issues can be underlined. First, the most important properties of Borel codes will be associated with the behaviour of the coded sets in a collapse generic extension of the universe. Second, the splitting construction, that leads to (II*) of Theorem 5, applies another forcing argument (in fact a countable subforcing, to make use of the Cohen-stability requirement).

We also exploit several technical achievements made in the study of the Borel orders [3, 11, 13] by means of the Gandy–Harrington topology (the topology generated by Σ^1_1 sets). The two technical schemes, the one we use and the one based on the Gandy–Harrington topology, involve different kinds of "effective" sets in the forcing, but have many common points in the construction of the proofs (like a similar definition of the "regular" and "singular" cases, a similar construction of splitting systems, etc.), although differ in many details. As a matter of fact the Gandy–Harrington topology technique provides a short direct proof of Theorem 1 (see [11]), but it has problems with the analytic case as it does not capture the proper type of effectiveness.

Theorem 2 (Section 5) will require a reflection argument saying that any analytic PQO has uncountably many indices for "upper" Borel approximations which are PQOS, together with a delicate reasoning in the additional case, in Section 5.4. We will also show that (II$^{\mathrm{A}}$) of Theorem 2 is essentially a Σ^1_2, therefore Shoenfield-absolute, statement (Section 5.1, via an argument due to Hjorth and Kechris [7]). Our approach will be to cut the "long" (of length ω_1) invariants, given by the general part of Theorem 2, and then transform "shorter" (of countable length) invariants so that they form an antichain.

Finally, Theorem 6 is proved in Section 6. We observe that, in the Solovay model, all ROD sets are ω_1-Souslin; hence Theorem 5 can be applied (the Cohen-stability condition is easily verifiable). Thus it suffices to convert a linearization given by Theorem 5 (in the case $\preccurlyeq_T = \preccurlyeq_S$) to the form required by Theorem 6.

[10] The material of Section 2 and, to some extent, 3, has predecessors in [4, 5], Friedman and Velickovic [1], but the remarkable brevity of those notes forced the author to present self-contained proofs of some key results, for instance Theorem 9, instead of references of the form: "using the idea outlined in … one proves …".

2. Preliminaries

This section begins the part of the paper devoted to the proof of Theorem 5. Starting the proof, we fix an ordinal $\lambda \geqslant \omega$ and trees $T, S \subseteq (\omega \times \omega \times \lambda)^{<\omega}$. Suppose that $\preccurlyeq_T = \mathfrak{p}[T]$ and $\preccurlyeq_S = \complement \mathfrak{p}[S]$ are PQOs on $\mathcal{N}$, which satisfy $\preccurlyeq_T \subseteq \preccurlyeq_S$. Define the relation $x \approx_S y$ iff $x \preccurlyeq_S y \wedge y \preccurlyeq_S x$, and $x \approx_T y$ similarly.

2.1. Extending the universe and improving the orders

By the possible uncountability of λ in V, the basic set universe where Theorem 5 is being proved, the properties of the orders $\preccurlyeq_T$ and $\preccurlyeq_S$ in V are somewhat occasional. Extending V properly, we can reveal more fundamental properties of the relations. Let $\lambda^\star = (\lambda^{++})^{\mathrm{L}[S,T]}$ Let $\mathrm{V}^\star$ be a $\lambda^\star$-collapse extension of V. In particular, $\lambda^\star$ is a countable ordinal in $\mathrm{V}^\star$.

We now encounter a problem. Unless the ordinal λ is countable (so that the Shoenfield absoluteness theorem can be applied), the relations $\preccurlyeq_T$ and $\preccurlyeq_S$ may not remain partial orders in $\mathrm{V}^\star$. However this can be fixed, to some extent. First of all, $\preccurlyeq_T \subseteq \preccurlyeq_S$ in $\mathrm{V}^\star$ by some other sort of absoluteness. We also know that $\preccurlyeq_S$ is a PQO in Cohen-generic extensions of V. In addition, we can assume that

– In $\mathrm{V}^\star$, $\preccurlyeq_T$ is a PQO on $\mathcal{N}$ while $\preccurlyeq_S$ is a binary relation satisfying $x \preccurlyeq_S x$ and the implication $x' \preccurlyeq_T x \preccurlyeq_S y \preccurlyeq_T y' \Rightarrow x' \preccurlyeq_S y'$.

To justify the $\preccurlyeq_T$-part of the assumption, we simply replace T by another tree $T' \in \mathrm{L}[S,T]$, $T' \subseteq (\omega \times \omega \times \lambda)^{<\omega}$, such that

$$\langle x, y\rangle \in \mathfrak{p}[T'] \Leftrightarrow \exists n \exists x_0 \cdots \exists x_n \ (x = x_0 \preccurlyeq_T x_1 \preccurlyeq_T \cdots \preccurlyeq_T x_n = y).$$

(Note that $\mathfrak{p}[T'] = \mathfrak{p}[T]$ in V.) Then, to justify the $\preccurlyeq_S$-part of the assumption, change S to a tree $S' \in \mathrm{L}[S,T]$, $S' \subseteq (\omega \times \omega \times \lambda)^{<\omega}$, such that

$$\langle x, y\rangle \notin \mathfrak{p}[S'] \Leftrightarrow \forall x' \forall y' (x' \preccurlyeq_{T'} x \wedge y \preccurlyeq_{T'} y' \Rightarrow x' \preccurlyeq_S y').$$

2.2. Coding Borel sets

The coding system we use is based on an infinitary language. This is equivalent to the ordinary coding using λ-branching wellfounded trees. However infinitary formulas bring some technical advantage.

We let $\mathscr{L}_{\lambda+1,0} \in \mathrm{L}[S,T]$ be the infinitary language defined as follows:

(i) Atomic formulas are of the form $\dot{x}(k) = l$ and $\dot{f}(k) = \alpha$, where $\dot{x}$ is a constant symbol for an indefinite element of $\mathcal{N} = \omega^\omega$, $\dot{f}$ is a constant symbol for an indefinite element of the set λ^ω, while $k, l \in \omega$ and $\alpha < \lambda$.

(ii) Non-atomic formulas are composed, in $\mathrm{L}[S,T]$, by conjunctions and disjunctions of $\mathrm{L}[S,T]$-size $\leqslant \lambda$, together with the ordinary connectives $\wedge$, $\vee$, $\neg$, but any formula contains only finitely many constant symbols of types $\dot{x}$ and $\dot{f}$ mentioned in (i) (that is, for indefinite elements of $\mathscr{N}$ and λ^ω).

(Quantifiers are not allowed.) By definition, the set of all $\mathscr{L}_{\lambda+1,0}$-formulas belongs to $\mathrm{L}[S,T]$ and has cardinality $(\lambda^+)^{\mathrm{L}[S,T]} < \lambda^\star$ in $\mathrm{L}[S,T]$.

For instance $[T](\dot{x},\dot{y},\dot{f})$ is a $\mathscr{L}_{\lambda+1,0}$-formula

$$\bigwedge_{m\in\omega}\ \bigvee_{(s,t,\psi)\in T_m}\ \bigwedge_{k<m}(\dot{x}(k)=s(k)\wedge\dot{y}(k)=t(k)\wedge\dot{f}(k)=\psi(k)),$$

where $T_m=\{\langle s,t,\psi\rangle\in T:\mathtt{dom}\,s=\mathtt{dom}\,t=\mathtt{dom}\,\psi=m\}$; we shall denote it by $\dot{x}\preccurlyeq_{T,\dot{f}}\dot{y}$. Similarly, the formula $\neg[S](\dot{x},\dot{y},\dot{f})$ will be denoted by $\dot{x}\preccurlyeq_{S,\dot{f}}\dot{y}$. The formulas like $\dot{x}\not\preccurlyeq_{S,\dot{f}}\dot{y}$, $\dot{x}\approx_{S,\dot{f}}\dot{y}$, etc., are derivatives. Then

$$x\preccurlyeq_T y \;\Leftrightarrow\; \exists f\in\lambda^\omega\; x\preccurlyeq_{T,f}y \quad\text{and}\quad x\preccurlyeq_S y \;\Leftrightarrow\; \forall f\in\lambda^\omega\; x\preccurlyeq_{S,f}y.$$

Formulas in $\mathscr{L}_{\lambda+1,0}$ code $(\lambda+1)$-Borel subsets of spaces $\mathscr{N}^m\times(\lambda^\omega)^n$: for a formula, say, $\varphi(\dot{x},\dot{f})$ we put $[\![\varphi]\!]=\{\langle x,f\rangle\in\mathscr{N}\times\lambda^\omega:\varphi(x,f)\}$ and define $[\![\varphi]\!]^\star$ similarly, but in $\mathrm{V}^\star$, so that, in particular, $[\![\varphi]\!]=[\![\varphi]\!]^\star\cap\mathrm{V}$.

We may view φ as a code, in $\mathrm{L}[S,T]$, for the sets $[\![\varphi]\!]$ and $[\![\varphi]\!]^\star$.

2.3. Consistency and separation

A $\mathscr{L}_{\lambda+1,0}$-*theory* (or simply: theory) will be any set Φ of $\mathscr{L}_{\lambda+1,0}$-formulas, such that the list of all constants of type (i) occurring in at least one formula in Φ, is finite. We shall write: a theory $\Phi(\dot{x}_1,\ldots,\dot{f}_n)$, to mean that all constants of type (i) which occur in Φ are included in the list $\dot{x}_1,\ldots,\dot{f}_n$.

The following is the principal concept.

Definition 7. A formula or a theory of $\mathscr{L}_{\lambda+1,0}$ is $\star$-*consistent* if it has a model in $\mathrm{V}^\star$, i.e., becomes true after one suitably substitutes, in $\mathrm{V}^\star$, its constants of type (i) by elements of $\mathscr{N}$ and λ^ω. Notions like: *to be* $\star$-*consistent with* and *to* $\star$-*imply* will have the same meaning.

Remark 8. Let $\Phi\in\mathrm{L}[S,T]$ be a $\mathscr{L}_{\lambda+1,0}$-formula or $\mathscr{L}_{\lambda+1,0}$-theory. Then, for Φ "to be $\star$-consistent" is describable in $\mathrm{L}[S,T]$. Indeed, it easily follows from the Shoenfield absoluteness theorem by ordinary forcing arguments that Φ is $\star$-consistent iff it is true in $\mathrm{L}[S,T]$ that "Φ has a model in a $\lambda^\star$-collapse generic extension of the universe".

Theorem 9. *A $\mathscr{L}_{\lambda+1,0}$-theory $\Phi\in\mathrm{L}[S,T]$ is $\star$-consistent iff every its subtheory $\Psi\in\mathrm{L}[S,T]$ of cardinality $\leqslant\lambda$ in $\mathrm{L}[S,T]$ is $\star$-consistent. Therefore, a theory Φ $\star$-implies a formula φ iff φ is $\star$-implied by a subtheory $\Psi\in\mathrm{L}[S,T]$ of Φ of cardinality $\leqslant\lambda$ in $\mathrm{L}[S,T]$.*

Proof. Prove the essential direction from right to left. This is an application of the downward Lowenheim–Skolem argument. Suppose, for brevity, that $\dot{x}$ is the only constant which occurs in Φ.

The set $\mathbf{P}$ of all theories $\Pi(\dot{x}) \in \mathrm{L}[S,T]$ of cardinality $\leqslant \lambda$ in $\mathrm{L}[S,T]$, which are $\star$-consistent with every subtheory $\Psi \subseteq \Phi$, $\Psi \in \mathrm{L}[S,T]$, of cardinality $\leqslant \lambda$ in $\mathrm{L}[S,T]$, belongs to $\mathrm{L}[S,T]$ and has cardinality $< \lambda^\star$ in $\mathrm{L}[S,T]$ by the above. Let us view $\mathbf{P}$ as a forcing notion (bigger theories are stronger). By the choice of $\mathrm{V}^\star$, there is a $\mathbf{P}$-generic over $\mathrm{L}[S,T]$ set $G \subseteq \mathbf{P}$ in $\mathrm{V}^\star$. Prove that the theory $\Gamma(\dot{x}) = \bigcup G$ has a model in $\mathrm{V}^\star$. (Then Φ is $\star$-consistent, being a subtheory of $\Gamma(\dot{x})$.)

Let D_n be the set of all theories $\Pi(\dot{x}) \in \mathbf{P}$ which contain $\dot{x}(n) = k$ for some k. Then D_n belongs to $\mathrm{L}[S,T]$. Note that D_n is dense in $\mathbf{P}$.

(Let $\Pi(\dot{x}) \in \mathbf{P}$. Assume on the contrary that $\Pi_k = \Pi \cup \{\dot{x}(n) = k\} \notin \mathbf{P}$ for all k. Then there is, in $\mathrm{L}[S,T]$, a sequence of subtheories Ψ_k of Φ of $\mathrm{L}[S,T]$-cardinality $\leqslant \lambda$, such that Π_k is $\star$-inconsistent with Ψ_k for any k. The theory $\Psi = \bigcup_k \Psi_k$ has also $\mathrm{L}[S,T]$-cardinality $\leqslant \lambda$, and is $\star$-inconsistent with $\Pi \cup \{\dot{x}(n) = k\colon\ k \in \omega\}$, hence $\star$-inconsistent with Π, a contradiction with the choice of Π).

It follows that $G \cap D_n \neq \emptyset$ for any n, so that there is (unique) $x \in \mathcal{N}$ in $\mathrm{V}^\star$ such that $x(n) = k$ iff $\dot{x}(n) = k$ belongs to Γ. It remains to prove that, for any $\mathscr{L}_{\lambda+1,0}$-formula $\varphi(\dot{x})$, $\varphi(x)$ is true (for this x) iff $\varphi(\dot{x}) \in \Gamma$.

This holds for elementary formulas $\dot{x}(n) = k$ by the above. The induction step $\neg$ is easy, so we concentrate on the induction step $\bigvee$. Assume that φ is a formula of the form $\bigvee_{\alpha<\lambda} \varphi_\alpha(\dot{x})$. Consider the set $D \in \mathrm{L}[S,T]$ of all theories $\Pi(\dot{x}) \in \mathbf{P}$ such that *either* both φ and some of φ_α belong to Π, *or* $\neg\varphi$ and all of $\neg\varphi_\alpha$ belong to Π. Then D is dense in $\mathbf{P}$.

(Indeed suppose that $\Pi(\dot{x}) \in \mathbf{P}$. If $\Pi_\alpha = \Pi \cup \{\varphi_\alpha(\dot{x})\}$ belongs to $\mathbf{P}$ for some α then easily $\Pi' = \Pi \cup \{\varphi(\dot{x})\} \cup \{\varphi_\alpha(\dot{x})\}$ belongs to $\mathbf{P}$, hence to D. Otherwise for any $\alpha < \lambda$ the theory Π_α is $\star$-inconsistent with a subtheory Ψ_α of Φ of $\mathrm{L}[S,T]$-cardinality $\leqslant \lambda$. Then each theory Π_α is $\star$-inconsistent with the union $\Psi = \bigcup_{\alpha<\lambda} \Psi_\alpha$. It easily follows that then $\Psi \cup \Pi$ $\star$-implies each of $\neg\varphi_\alpha$, so that the theory $\Pi' = \Pi \cup \{\neg\varphi(\dot{x})\} \cup \{\neg\varphi_\alpha(\dot{x})\colon\ \alpha < \lambda\}$ belongs to D.)

It follows that there is a theory $\Pi(\dot{x}) \in D \cap G$. In the *either* case, φ and one of φ_α belong to Γ and, by the induction hypothesis, $\varphi_\alpha(x)$ is true for this α, hence $\varphi(x)$ is true. The *or* case is analogous. □

The next theorem belongs to the type of *separation* theorems.

Theorem 10. *Assume that* $\Phi(\dot{x}, \dot{y}, \dot{f}, \ldots)$, $\Psi(\dot{x}, \dot{y}', \dot{f}', \ldots)$ *are* $\mathscr{L}_{\lambda+1,0}$-*theories in* $\mathrm{L}[S,T]$, *having* $\dot{x}$ *as the only common constant in the* (*finite*) *lists of constants. Assume that the theory* $\Phi(\dot{x}, \dot{y}, \dot{f}, \ldots) \cup \Psi(\dot{x}, \dot{y}', \dot{f}', \ldots)$ *is* $\star$-*inconsistent. Then there is a* $\mathscr{L}_{\lambda+1,0}$-*formula* $\pi(\dot{x})$ *which* $\star$-*separates* Φ *from* Ψ *in the sense that* $\Phi(\dot{x}, \ldots)$ $\star$-*implies* $\pi(\dot{x})$ *while* $\Psi(\dot{x}, \ldots)$ $\star$-*implies* $\neg\pi(\dot{x})$.

Proof. We can assume, by Theorem 9, that Φ and Ψ consist of single $\mathscr{L}_{\lambda+1,0}$-formulas, resp. $\varphi(\dot{x}, \dot{y}, \dot{f}, \ldots)$ and $\psi(\dot{x}, \dot{y}', \dot{f}', \ldots)$. Let, for the sake of simplicity, φ be $\varphi(\dot{x}, \dot{y})$ and

ψ be $\psi(\dot{x}, \dot{f})$. Consider, in $\mathrm{V}^\star$, the sets

$$[\![\varphi]\!]^\star = \{\langle x, y\rangle \in \mathcal{N}^2 : \varphi(x, y)\} \quad \text{and} \quad [\![\psi]\!]^\star = \{\langle x, f\rangle \in \mathcal{N} \times \lambda^\omega : \psi(x, f)\}.$$

The projections $X = \{x : \exists y [\![\varphi]\!]^\star(x, y)\}$ and $Y = \{x : \exists f [\![\psi]\!]^\star(x, f)\}$ are disjoint (by the inconsistency assumption) $\mathbf{\Sigma}^1_1$ sets, hence by a classical separation theorem they are separable by a Borel set. Moreover, as we proved in [9] (Theorem 7), in this case a separating set is coded in $\mathrm{L}[S, T]$ so that it has the form $B = [\![\pi]\!]^\star$ for some $\mathscr{L}_{\lambda+1,0}$-formula $\pi(\dot{x})$. $\square$

2.4. Borel h.o.p. maps and the dichotomy

To introduce the dichotomy, we have to extend the language $\mathscr{L}_{\lambda+1,0}$ by Borel functions mapping $\mathcal{N}$ in a set of the form 2^α, $\alpha < (\lambda^+)^{\mathrm{L}[S,T]}$. Let a *function code* be a sequence of the form $\vec{\varphi} = \{\varphi_\gamma(\dot{x})\}_{\gamma<\alpha}$, where $\alpha \in \mathtt{Ord}$ and each φ_γ is a $\mathscr{L}_{\lambda+1,0}$-formula. Such a sequence defines a function $h_{\vec{\varphi}}$: $\mathcal{N} \to 2^\alpha$ so that $h_{\vec{\varphi}}(x)(\gamma) = 1$ iff $\varphi_\gamma(x)$.

Let H_α be the set of all maps $h_{\vec{\varphi}}$: $\langle \mathcal{N}; \preccurlyeq_T\rangle \to \langle 2^\alpha; \leqslant_{\mathrm{lex}}\rangle$, where $\vec{\varphi} \in \mathrm{L}[S, T]$ is a function code, which are h.o.p. in $\mathrm{V}^\star$. Define $\mathrm{H} = \bigcup_{\alpha<(\lambda^+)^{\mathrm{L}[S,T]}} \mathrm{H}_\alpha$. Thus every function in H is a $\lambda + 1$-Borel, coded in $\mathrm{L}[S, T]$, map $\mathcal{N} \to$ some 2^α, $\alpha < (\lambda^+)^{\mathrm{L}[S,T]}$, satisfying $x \preccurlyeq_T y \Rightarrow h(x) \leqslant_{\mathrm{lex}} h(y)$ in $\mathrm{V}^\star$ (then also in V). As a matter of fact, functions in H will be used only via equalities of the form $h(x) = h(y)$ where $h = h_{\vec{\varphi}} \in \mathrm{H}_\alpha$ for some $\alpha < (\lambda^+)^{\mathrm{L}[S,T]}$, viewed as shorthand for the formula $\bigwedge_{\gamma<\alpha}(\varphi_\gamma(x) \Leftrightarrow \varphi_\gamma(y))$.

Let $\dot{x} \equiv_{\mathrm{H}} \dot{y}$ be the theory which contains all formulas $h(\dot{x}) = h(\dot{y})$, where $h \in \mathrm{H}$. Thus $\equiv_{\mathrm{H}}$ defines an equivalence relation extending $\approx_T$.

We have two cases. [11]

Case 1: *The theory* $\dot{x} \equiv_{\mathrm{H}} \dot{y}$ $\star$-*implies* $\dot{x} \approx_{S,\dot{f}} \dot{y}$. Then, by Theorem 9, there is a set $\mathrm{H}' = \{h_\alpha : \alpha < \lambda\} \subseteq \mathrm{H}$, such that the subtheory $\dot{x} \equiv_{\mathrm{H}'} \dot{y}$ already $\star$-implies $\dot{x} \approx_{S,\dot{f}} \dot{y}$ and $\mathrm{L}[S, T]$ contains a sequence of function codes for functions h_α. Let $h(x)$ be the concatenation of all $h_\alpha(x)$, $\alpha < \lambda$. Then $h \in \mathrm{H}$ and $h(\dot{x}) = h(\dot{y})$ already $\star$-implies $\dot{x} \approx_{S,\dot{f}} \dot{y}$, so that h satisfies (I^*) of Theorem 5.

Case 2: *The theory* $(\dot{x} \equiv_{\mathrm{H}} \dot{y}) \cup \{\dot{x} \not\approx_{S,\dot{f}} \dot{y}\}$ *is* $\star$-*consistent*. Assuming this, we work towards (II^*) of Theorem 5. To define a required map F, we shall apply a sophisticated splitting construction based on some forcing ideas. The next section introduces the forcing notions involved. (The assumption of Case 2 will be applied only to show that the forcing notions are non-empty.) Section 4 defines the embedding and ends the proof of Theorem 5.

[11] There is a point of dissatisfaction in the distribution on the two cases we use. It would be more natural to define Case 1 as that $\dot{x} \leqslant_{\mathrm{H}} \dot{y}$ $\star$-implies $\dot{x} \preccurlyeq_S \dot{y}$, where $\dot{x} \leqslant_{\mathrm{H}} \dot{y}$ is the theory $\{h(\dot{x}) \leqslant_{\mathrm{lex}} h(\dot{y})\colon h \in \mathrm{H}\}$, which would improve ($\mathrm{I}^*$) of Theorem 5 to the existence of a h.o.p. map satisfying $h(x) \leqslant h(y) \Rightarrow x \preccurlyeq_S y$. However then the arguments for Case 2, especially the key lemmas in the next section, do not go through.

2.5. Hulls

By $\mathscr{F}(\dot{x})$ we shall denote the collection of all $\mathscr{L}_{\lambda+1,0}$-formulas $\varphi(\dot{x})$ with the only constant $\dot{x}$; clearly $\mathscr{F}(\dot{x}) \in \mathrm{L}[S,T]$. For a theory $\Phi(\dot{x},\dot{y},\ldots)$, let $\mathscr{F}_{\dot{x}}[\Phi(\dot{x},\dot{y},\ldots)]$ be the set of all formulas $\varphi(\dot{x}) \in \mathscr{F}(\dot{x})$ which are $\star$-implied by $\Phi(\dot{x},\dot{y},\ldots)$.

Lemma 11. *Let* $\mathrm{R}(\dot{x}) \subseteq \mathscr{F}(\dot{x})$ *and* $\Pi(\dot{x},\ldots)$ *be* $\mathscr{L}_{\lambda+1,0}$*-theories in* $\mathrm{L}[S,T]$. *Assume that* $\mathscr{F}_{\dot{x}}[\Pi(\dot{x},\ldots)] \subseteq \mathrm{R}(\dot{x})$. *Then the theory* $\Pi'(\dot{x},\ldots) = \Pi(\dot{x},\ldots) \cup \mathrm{R}(\dot{x})$ *satisfies* $\mathscr{F}_{\dot{x}}[\Pi'(\dot{x},\ldots)] = \mathscr{F}_{\dot{x}}[\mathrm{R}(\dot{x})]$.

Proof. Show that $\mathscr{F}_{\dot{x}}[\Pi'(\dot{x},\ldots)] \subseteq \mathscr{F}_{\dot{x}}[\mathrm{R}(\dot{x})]$ (the non-trivial direction). Consider a formula $\psi(\dot{x}) \in \mathscr{F}_{\dot{x}}[\Pi'(\dot{x},\ldots)]$. By Theorem 9 there is a subtheory $\mathrm{R}'(\dot{x}) \in \mathrm{L}[S,T]$ of $\mathrm{R}(\dot{x})$, of cardinality $\leqslant \lambda$ in $\mathrm{L}[S,T]$, such that $\Pi(\dot{x},\ldots) \cup \mathrm{R}'(\dot{x})$ $\star$-implies $\psi(\dot{x})$. We conclude that the formula $(\bigwedge \mathrm{R}'(\dot{x})) \Rightarrow \psi(\dot{x})$ belongs to $\mathscr{F}_{\dot{x}}[\Pi(\dot{x},\ldots)]$, thus to $\mathrm{R}(\dot{x})$, which guarantees $\psi(\dot{x}) \in \mathscr{F}_{\dot{x}}[\mathrm{R}(\dot{x})]$. $\square$

Let $\mathscr{H}(\dot{x})$ be the set of all formulas $\eta(\dot{x}) \in \mathscr{F}(\dot{x})$ such that the theory $\dot{x} \equiv_{\mathrm{H}} \dot{x}'$ $\star$-implies $(\eta(\dot{x}) \wedge \dot{x}' \preccurlyeq_{T,f} \dot{x}) \Rightarrow \eta(\dot{x}')$.

Note that $\mathscr{H}(\dot{x}) \in \mathrm{L}[S,T]$, see Remark 8.

Lemma 12. *Suppose that* $\eta(\dot{x}) \in \mathscr{H}(\dot{x})$. *Then there exists a function* $h \in \mathrm{H}_{\alpha+1}$, *for some* $\alpha < (\lambda^+)^{\mathrm{L}[S,T]}$, *such that, in* $\mathrm{V}^\star$, $\eta(x) \Leftrightarrow h(x)(\alpha) = 0$. *Therefore, the theory* $\dot{x} \equiv_{\mathrm{H}} \dot{y}$ $\star$*-implies* $\eta(\dot{x}) \Leftrightarrow \eta(\dot{y})$.

Proof. By definition there exists a function $g \in H_\alpha$ for some $\alpha < (\lambda^+)^{\mathrm{L}[S,T]}$ satisfying $(g(x) = g(y) \wedge \eta(x) \wedge x' \preccurlyeq_T x) \Rightarrow \eta(x')$ in $\mathrm{V}^\star$. Define $h(x) = g(x)^\wedge 0$ whenever $\eta(x)$, and $h(x) = g(x)^\wedge 1$ otherwise. $\square$

For a theory $\Phi(\dot{x},\dot{y},\ldots)$, let $\mathscr{H}_{\dot{x}}[\Phi(\dot{x},\dot{y},\ldots)] = \mathscr{F}_{\dot{x}}[\Phi(\dot{x},\dot{y},\ldots)] \cap \mathscr{H}(\dot{x})$.

3. The forcing

Let $\Xi(\dot{x})$ be the set of all formulas $\xi(\dot{x}) \in \mathscr{F}(\dot{x})$ which are $\star$-implied by the theory $\dot{x} \equiv_{\mathrm{H}} \dot{y} \cup \{\dot{x} \not\approx_{S,f} \dot{y}\}$. Note that $\Xi(\dot{x}) \in \mathrm{L}[S,T]$ and $\Xi(\dot{x})$ is $\star$-consistent (because the theory $\dot{x} \equiv_{\mathrm{H}} \dot{y} \cup \{\dot{x} \not\approx_{S,f} \dot{y}\}$ is such).

Let $\mathbb{P}$ be the set of all $\star$-consistent theories $\Pi(\dot{x}) \in \mathrm{L}[S,T]$ such that $\Xi(\dot{x}) \subseteq \Pi(\dot{x})$. The set $\mathbb{P}$ belongs to $\mathrm{L}[S,T]$, see Remark 8 above, so we can view it as a forcing notion over $\mathrm{L}[S,T]$ (assuming that Π is stronger than Π' when $\Pi' \subseteq \Pi$). Note that $\mathbb{P} \neq \emptyset$: for instance $\Xi \in \mathbb{P}$.

As for the action of this forcing, one easily proves (using the proof of Theorem 9 above) that any $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ produces a real $x = x_G$ such that $\pi(x_G)$ holds for any formula $\pi(\dot{x}) \in \bigcup G$.

3.1. Key lemmas

The lemmas proved below assert that different theories are $\star$-consistent and satisfy some other requirements. Note that the $\star$-consistency part of the lemmas is based on the assumption of Case 2.

We argue in the extended universe $\mathrm{V}^\star$ *in this subsection.* Note that $\preccurlyeq_T$ is a PQO, but $\preccurlyeq_S$ is not necessarily a PQO in $\mathrm{V}^\star$, see Section 2.1.

Lemma 13. *Let $\Pi(\dot x)$ be a theory in $\mathbb{P}$. Then the theory*

$$\Phi_\Pi(\dot x, \dot y, \dot f) =_{\mathrm{df}} \Pi(\dot x) \cup \Pi(\dot y) \cup \dot x \equiv_{\mathrm{H}} \dot y \cup \{\dot x \not\preccurlyeq_{S,\dot f} \dot y\}$$

is $\star$-consistent. Moreover, it satisfies the equalities $\mathscr{F}_{\dot x}[\Phi_\Pi(\dot x, \dot y, \dot f)] = \mathscr{F}_{\dot x}[\Pi(\dot x)]$ and $\mathscr{F}_{\dot y}[\Phi_\Pi(\dot x, \dot y, \dot f)] = \mathscr{F}_{\dot y}[\Pi(\dot y)]$.

Proof. Let us first prove the $\star$-consistency. Otherwise by Theorem 9 there exists a formula $\pi(\dot x) \in \mathscr{F}_{\dot x}[\Pi(\dot x)]$ and a function $h \in \mathrm{H}$ such that the formula $\pi(\dot x) \wedge \pi(\dot y) \wedge h(\dot x) = h(\dot y) \wedge \dot x \not\preccurlyeq_{S,\dot f} \dot y$ is $\star$-inconsistent. The plan will be to find functions $h', h'' \in \mathrm{H}$ such that the formulas

$$\pi(\dot x) \wedge h'(\dot x) = h'(\dot y) \wedge \dot y \not\preccurlyeq_{S,\dot f} \dot x \quad \text{and} \quad \pi(\dot x) \wedge h''(\dot x) = h''(\dot y) \wedge \dot x \not\preccurlyeq_{S,\dot f} \dot y$$

are $\star$-inconsistent: then the formula $\neg\pi(\dot x)$ belongs to $\Xi(\dot x)$, which is a contradiction because $\Xi(\dot x) \subseteq \Pi(\dot x)$.

To define h' (the case of h'' is similar), it suffices to get a formula $\psi(\dot x) \in \mathscr{H}(\dot x)$ such that $X = [\![\pi]\!]^\star \subseteq U = [\![\psi]\!]^\star$ and, for all $x \in X$ and $u \in U$, $h(x) = h(u)$ implies $u \preccurlyeq_S x$. (Indeed, let, by Lemma 12, $f \in \mathrm{H}_{\alpha+1}$ satisfy $\psi(x) \Leftrightarrow f(x)(\alpha) = 0$. Let $h' \in \mathrm{H}$ be a concatenation of h and f, so that $h'(x) = h'(y)$ implies $h(x) = h(y)$ and $f(x) = f(y)$. Now, the formula $\pi(x) \wedge h'(x) = h'(u)$ implies both $\psi(x)$ and $\psi(x) \Leftrightarrow \psi(u)$, thus implies $\psi(u)$, and finally implies $u \preccurlyeq_S x$ by the choice of ψ.)

Let $Z = \{z\colon \forall x \in X (h(z) = h(x) \Rightarrow z \preccurlyeq_S x)\}$. Then $X \subseteq Z$ by the $\star$-inconsistency assumption above.

Define a sequence of sets $X = X_0 \subseteq U_0 \subseteq X_1 \subseteq U_1 \subseteq \cdots \subseteq Z$ and formulas $\pi_n(\dot x) \in \mathscr{F}$ so that $U_n = \{u\colon \exists x \in X_n (h(x) = h(u) \wedge u \preccurlyeq_T x)\}$, $X_n = [\![\pi_n(\dot x)]\!]^\star$, and the sequence of formulas φ_n belongs to $\mathrm{L}[S, T]$.

Now, the $\mathscr{L}_{\lambda+1,0}$ formula $\psi(\dot x) = \bigvee_n \varphi_n(\dot x)$ clearly defines the set $[\![\psi]\!]^\star = \bigcup_n X_n = \bigcup_n U_n$. Furthermore ψ belongs to $\mathscr{H}(\dot x)$, as any of the sets U_n satisfies $(u \in U_n \wedge h(u) = h(u') \wedge u' \preccurlyeq_T u) \Rightarrow u' \in U_n$ by the construction. Finally, take $x \in X$ and $u \in U$, suppose that $h(x) = h(u)$, and prove $u \preccurlyeq_S x$. Note that $u \in U_n$ for some n, therefore we have $y \in X_n$ such that $h(y) = h(u)$ and $u \preccurlyeq_T y$. Then $y \in Z$ and $h(y) = h(x)$, therefore $y \preccurlyeq_S x$. It follows that $u \preccurlyeq_S x$. (We used the properties of $\preccurlyeq_T$ and $\preccurlyeq_S$ in $\mathrm{V}^\star$, see Section 2.1.) Thus ψ is a required formula.

It remains to carry out the construction of X_n, U_n, π_n.

Suppose that a set $X_n = [\![\pi_n(\dot x)]\!]^\star \subseteq Z$ has been defined. Define U_n by the equality above. Then $X_n \subseteq U_n$, and $U_n \subseteq Z$. (Assume that $u \in U_n$, so $u \preccurlyeq_T x$ for some $x \in X_n$

satisfying $h(x)=h(u)$. Take any $x'\in X$ such that $h(x')=h(u)$ and prove $u\preccurlyeq_S x'$. First of all $h(x)=h(x')$ hence $x\preccurlyeq_S x'$ because $x\in X_n\subseteq Z$. Now $u\preccurlyeq_S x'$ as $u\preccurlyeq_T x$.)

Theorem 10 yields a formula $\beta(\dot x)\in\mathscr{F}(\dot x)$ such that the set $B=[\![\beta]\!]^\star$ satisfies $U_n\subseteq B\subseteq Z$. Take $X_{n+1}=B$ and $\pi_{n+1}=\beta$.

As the choice of the formulas π_n can be forced in $\mathrm{L}[S,T]$, the sequence of formulas can be chosen in $\mathrm{L}[S,T]$. This ends the proof of the consistency.

The equality $\mathscr{F}_{\dot x}[\Phi_\Pi(\dot x,\dot y,\dot f)]=\mathscr{F}_{\dot x}[\Pi(\dot x)]$ does not cause much trouble. Indeed suppose that $\pi(\dot x)\in\mathscr{F}(\dot x)\setminus\mathscr{F}_{\dot x}[\Pi(\dot x)]$ (the non-trivial direction). Then the theory $\Pi'(\dot x)=\Pi(\dot x)\cup\{\neg\pi(\dot x)\}$ is $\star$-consistent, hence belongs to $\mathbb{P}$. It follows from the already proved part of the lemma that $\Phi_\Pi(\dot x,\dot y,\dot f)\cup\{\neg\pi(\dot x)\}$ is $\star$-consistent as well, hence $\pi(\dot x)\notin\mathscr{F}_{\dot x}[\Phi_\Pi(\dot x,\dot y,\dot f)]$. $\square$

Lemma 14. *Assume that* $\Pi(\dot x)$, $\mathrm{R}(\dot x)$ *belong to* $\mathbb{P}$, *and* $\mathscr{H}_{\dot x}[\Pi(\dot x)]=\mathscr{H}_{\dot x}[\mathrm{R}(\dot x)]$. *Then* $\Psi_{\Pi\mathrm{R}}(\dot x,\dot y,\dot f)=_{\mathrm{df}}\Pi(\dot x)\cup\mathrm{R}(\dot y)\cup\dot x\equiv_{\mathrm{H}}\dot y\cup\{\dot x\preccurlyeq_{T,\dot f}\dot y\}$ *is a* $\star$-*consistent theory satisfying* $\mathscr{F}_{\dot x}[\Psi_{\Pi\mathrm{R}}(\dot x,\dot y,\dot f)]=\mathscr{F}_{\dot x}[\Pi(\dot x)]$ *and* $\mathscr{F}_{\dot y}[\Psi_{\Pi\mathrm{R}}(\dot x,\dot y,\dot f)]=\mathscr{F}_{\dot y}[\mathrm{R}(\dot y)]$.

Proof. As in the previous lemma, it suffices to prove the $\star$-consistency. Suppose otherwise. Then there exist formulas $\pi(\dot x)\in\mathscr{F}_{\dot x}[\Pi(\dot x)]$ and $\rho(\dot y)\in\mathscr{F}_{\dot y}[\mathrm{R}(\dot y)]$, and a function $h\in\mathrm{H}$ such that the formula $\pi(\dot x)\wedge\rho(\dot y)\wedge h(\dot x)=h(\dot y)\wedge\dot x\preccurlyeq_{T,\dot f}\dot y$ is $\star$-inconsistent. In other words we have, in $\mathrm{V}^\star$, $x\not\preccurlyeq_T y$ whenever $x\in X=[\![\pi]\!]^\star$ and $y\in Y=[\![\rho]\!]^\star$ satisfy $h(x)=h(y)$.

Define $Z=\{z\colon \forall\, y\in Y\,(h(y)=h(z)\Rightarrow z\not\preccurlyeq_T y)\}$, so that $X\subseteq Z$ but $Y\cap Z=\emptyset$. The same iterated procedure as in the proof of Lemma 13, but with $U_n=\{u\colon \exists\, x\in X_n\ (h(x)=h(u)\wedge x\preccurlyeq_T u)\}$, results in a formula $\psi(\dot x)\in\mathscr{H}(\dot x)$ such that the set $U=\complement[\![\psi]\!]^\star$ satisfies $X\subseteq U\subseteq Z$. But this contradicts the assumption $\mathscr{H}_{\dot x}[\Pi(\dot x)]=\mathscr{H}_{\dot x}[\mathrm{R}(\dot x)]$. $\square$

Corollary 15. *Suppose that* $\Pi(\dot x)$, $\mathrm{R}(\dot x)$ *belong to* $\mathbb{P}$ *and* $\mathscr{H}_{\dot x}[\Pi(\dot x)]=\mathscr{H}_{\dot x}[\mathrm{R}(\dot x)]$. *Then* $\mathrm{U}(\dot x,\dot y,\dot f)=_{\mathrm{df}}\Pi(\dot x)\cup\mathrm{R}(\dot y)\cup\dot x\equiv_{\mathrm{H}}\dot y\cup\{\dot x\not\preccurlyeq_{S,\dot f}\dot y\}$ *is a* $\star$-*consistent theory satisfying* $\mathscr{F}_{\dot x}[\mathrm{U}(\dot x,\dot y,\dot f)]=\mathscr{F}_{\dot x}[\Pi(\dot x)]$ *and* $\mathscr{F}_{\dot y}[\mathrm{U}(\dot x,\dot y,\dot f)]=\mathscr{F}_{\dot y}[\mathrm{R}(\dot y)]$.

(This is a generalization of Lemma 13.)

Proof. It suffices, as above, to prove the $\star$-consistency. Suppose otherwise. Then the theory $\Phi_\Pi(\dot x,\dot z,\dot f)\cup\Psi_{\mathrm{R}\Pi}(\dot y,\dot z,\dot g)$ is $\star$-inconsistent as well. (Otherwise we have, in $\mathrm{V}^\star$, reals x, y, z satisfying $\Pi(x)$, $\Pi(z)$, $x\equiv_{\mathrm{H}}z$, $\mathrm{R}(y)$, and $y\equiv_{\mathrm{H}}z$, hence $x\equiv_{\mathrm{H}}z$, and, in addition, $x\not\preccurlyeq_S z$ and $y\preccurlyeq_T z$, hence $x\not\preccurlyeq_S y$.) Theorem 10 yields a formula $\pi(\dot z)\in\mathscr{F}(\dot z)$ $\star$-implied by $\Phi_\Pi(\dot x,\dot z,\dot f)$ but $\star$-inconsistent with $\Psi_{\mathrm{R}\Pi}(\dot y,\dot z,\dot g)$, which is a contradiction as $\mathscr{F}_{\dot z}[\Phi_\Pi(\dot x,\dot z,\dot f)]=\mathscr{F}_{\dot z}[\Pi(\dot z)]=\mathscr{F}_{\dot z}[\Psi_{\mathrm{R}\Pi}(\dot y,\dot z,\dot g)]$ by Lemmas 13 and 14. $\square$

Corollary 16. *Suppose that* $\Pi(\dot x)$, $\mathrm{R}(\dot x)$ *belong to* $\mathbb{P}$ *and* $\mathscr{H}_{\dot x}[\Pi(\dot x)]=\mathscr{H}_{\dot x}[\mathrm{R}(\dot x)]$. *Then there are theories* $\Pi'(\dot x)$, $\mathrm{R}'(\dot x)\in\mathbb{P}$ *such that* $\Pi\subseteq\Pi'$, $\mathrm{R}\subseteq\mathrm{R}'$, *still* $\mathscr{H}_{\dot x}[\Pi'(\dot x)]=\mathscr{H}_{\dot x}[\mathrm{R}'(\dot x)]$, *and* $\Pi'(\dot x)\cup\mathrm{R}'(\dot x)$ *is* $\star$-*inconsistent.*

Proof. The theory $\Phi(\dot{x},\dot{y}) =_{\mathrm{df}} \Pi(\dot{x}) \cup \mathrm{R}(\dot{x}) \cup (\dot{x} \equiv_{\mathrm{H}} \dot{y}) \cup \{\dot{x} \neq \dot{y}\}$ is $\star$-consistent by the previous corollary. It easily follows that there exist natural numbers m and $k \neq k'$ such that the theory $\Phi'(\dot{x},\dot{y}) =_{\mathrm{df}} \Phi(\dot{x},\dot{y}) \cup \{\dot{x}(m)=k\} \cup \{\dot{y}(m)=k'\}$ is still $\star$-consistent. Set $\Pi'(\dot{x}) = \mathscr{H}_{\dot{x}}[\Phi'(\dot{x},\dot{y})]$ and $\mathrm{R}'(\dot{y}) = \mathscr{H}_{\dot{y}}[\Phi'(\dot{x},\dot{y})]$. We have $\mathscr{H}_{\dot{x}}[\Pi'(\dot{x})] = \mathscr{H}_{\dot{x}}[\mathrm{R}'(\dot{x})]$, because the theory Φ' still includes $\dot{x} \equiv_{\mathrm{H}} \dot{y}$. $\square$

3.2. Countable subforcing

It will be a key moment below that we consider, for a moment, a generic extension of the universe V, in which $\preccurlyeq_S$ has to remain a PQO. The forcing notion involved should be a derivate of $\mathbb{P}$, most likely uncountable in V: therefore, the `Cohen` stability condition cannot be applied directly. Fortunately, there is a trick which allows to settle the problem. We shall replace $\mathbb{P}$ by a good enough $\mathrm{L}[S,T]$-countable subset $\mathbb{P}^\circ \subseteq \mathbb{P}$, so that some basic properties of $\mathbb{P}$ will be preserved.

To define $\mathbb{P}^\circ$, note that by definition $\mathbb{P} \in \mathfrak{M} = \mathrm{L}_{\lambda^\star}[S,T]$. Let us consider an $\mathrm{L}[S,T]$-countable elementary submodel $\mathfrak{M}^\circ \in \mathrm{L}[S,T]$ of $\mathfrak{M}$, containing λ, T, S – then containing $\mathbb{P}$, $\mathscr{F}(\dot{x})$, $\mathscr{H}(\dot{x})$, Ξ, H as well. Define $\mathbb{P}^\circ = \mathbb{P} \cap \mathfrak{M}^\circ$.

Thus $\mathbb{P}^\circ$ belongs to $\mathrm{L}_{\lambda^\star}[S,T]$, is countable in $\mathrm{L}[S,T]$, and is "enough" elementarily equivalent subset of $\mathbb{P}$. We can consider $\mathbb{P}^\circ$ as a forcing over $\mathrm{L}[S,T]$. However we shall also use a weaker form of genericity. Let $\mathfrak{D}$ be the set (it belongs to and is countable in $\mathrm{L}[S,T]$) of all sets $D \subseteq \mathfrak{M}$ which are definable in $\mathfrak{M}$ by an $\in$-formula containing λ, S, T, H, $\mathfrak{M}^\circ$, and all elements of $\mathfrak{M}^\circ$, as parameters. (Then $\mathfrak{M}^\circ$ and $\mathbb{P}^\circ$ belong to $\mathfrak{D}$.)

Lemma 17. *Let $G \subseteq \mathbb{P}^\circ$ be $\mathbb{P}^\circ$-generic over $\mathfrak{D}$* (i.e., G intersects every dense subset $D \in \mathfrak{D}$ of $\mathbb{P}^\circ$). *Then there is a real $x = x_G \in \mathrm{L}[S,T,G]$ such that, for any $\mathscr{L}_{\lambda+1,0}$-formula $\varphi(\dot{x}) \in \mathfrak{M}^\circ$, $\varphi(x)$ is true iff $\varphi(\dot{x}) \in \bigcup G$.*

Proof. Let us follow the proof of Theorem 9. First, we observe that for any n the set D_n of all theories $\Pi(\dot{x}) \in \mathbb{P}$, containing $\dot{x}(n)=k$ for some (clearly unique) k, is dense in $\mathbb{P}$. Moreover, $D_n \in \mathrm{L}[S,T]$ (see Remark 8) and a careful execution of the argument used in Remark 8 shows that in fact $D_n \in \mathfrak{M}$ and D_n is $\in$-definable in $\mathfrak{M}$ using S and T as parameters, so $D_n \in \mathfrak{M}^\circ$ and the set $D_n^\circ = D_n \cap \mathbb{P}^\circ \in \mathfrak{D}$ is dense in $\mathbb{P}^\circ$.

It follows that D_n° intersects G for all n; hence, there is a real $x \in \mathrm{L}[S,T,G]$ such that $x(n)=k$ whenever $\dot{x}(n)=k$ belongs to $\Gamma = \bigcup G$.

We continue to argue by induction on the complexity of π. The negation step is obvious, so let us concentrate on the step $\bigvee_{\gamma<\lambda}$. Consider a formula $\varphi \in \mathfrak{M}^\circ$ of the form $\bigvee_{\gamma<\lambda} \varphi_\gamma(\dot{x})$. The dense in $\mathbb{P}$ set $D \subseteq \mathbb{P}$, defined as in the proof of Theorem 9, then belongs to $\mathfrak{M}^\circ$, too, and the intersection $D^\circ = D \cap \mathfrak{M}^\circ$ is dense in $\mathbb{P}^\circ$. Moreover, $D^\circ \in \mathfrak{D}$. It follows that there is a theory $\Pi(\dot{x}) \in D^\circ \cap G$. If Π is of the *or* type (see the proof of Theorem 9), then clearly $\varphi(\dot{x})$ cannot belong to Γ and, by the induction hypothesis (for formulas φ_γ), $\varphi(x)$ is false. Suppose that Π is of the *either* type, so that it contains $\varphi(\dot{x})$ and a formula $\varphi_\gamma(\dot{x})$, $\gamma<\lambda$. Since both Π and φ belong to $\mathfrak{M}^\circ$,

Π even contains a formula $\varphi_\gamma(\dot{x})$ which belongs to $\mathfrak{M}^\circ$. Now apply the induction hypothesis for this φ_γ. $\square$

3.3. Two-dimensional modifications

We define here several forcing notions, based on $\mathbb{P}^\circ$, which force pairs of $\mathbb{P}^\circ$-generic reals x, y, in particular, such pairs which satisfy $x \preccurlyeq_T y$ or $x \not\preccurlyeq_S y$. First of all, introduce "full", $\mathbb{P}$-oriented versions.

- Let $\mathbb{P}_{(2)}$ be the set of all $\star$-consistent theories $\Delta(\dot{x},\dot{y}) \in \mathrm{L}[S,T]$ such that $\Xi(\dot{x}) \cup \Xi(\dot{y}) \subseteq \Delta(\dot{x},\dot{y})$.
- Let ${}^T\mathbb{P}_{(2)}(\dot{x},\dot{y},\dot{f})$ be the set of all $\star$-consistent theories $\mathscr{T}(\dot{x},\dot{y},\dot{f})$ of the form $\Delta(\dot{x},\dot{y}) \cup \mathrm{F} \cup \dot{x} \equiv_{\mathrm{H}} \dot{y} \cup \{\dot{x} \preccurlyeq_{T,\dot{f}} \dot{y}\}$, where $\Delta \in \mathbb{P}_{(2)}$ and F is a finite collection of formulas $\dot{f}(k) = \alpha$ (where $k \in \omega$ and $\alpha < \lambda$).
- Let ${}^S\mathbb{P}_{(2)}(\dot{x},\dot{y},\dot{f})$ be the collection of all $\star$-consistent theories $\Sigma(\dot{x},\dot{y},\dot{f})$ of the form $\Delta(\dot{x},\dot{y}) \cup \mathrm{F} \cup \dot{x} \equiv_{\mathrm{H}} \dot{y} \cup \{\dot{x} \not\preccurlyeq_{S,\dot{f}} \dot{y}\}$, where F and Δ are as in the definition of ${}^T\mathbb{P}_{(2)}$.

For instance, the theory $\Xi(\dot{x}) \cup \Xi(\dot{y}) \cup \dot{x} \equiv_{\mathrm{H}} \dot{y} \cup \{\dot{x} \preccurlyeq_{T,\dot{f}} \dot{y}\}$ (which is $\star$-consistent by Lemma 14) belongs to ${}^T\mathbb{P}_{(2)}$ while $\Xi(\dot{x}) \cup \Xi(\dot{y}) \cup \dot{x} \equiv_{\mathrm{H}} \dot{y} \cup \{\dot{x} \not\preccurlyeq_{S,\dot{f}} \dot{y}\}$ ($\star$-consistent by Lemma 13) belongs to ${}^S\mathbb{P}_{(2)}$, so that the collections are non-empty. Note also that $\mathbb{P}_{(2)}$, ${}^T\mathbb{P}_{(2)}$ and ${}^S\mathbb{P}_{(2)}$ belong to $\mathrm{L}[S,T]$, as above, and even to $\mathfrak{M}^\circ$. Now, define the $\mathbb{P}^\circ$-versions.

- ${}^T\mathbb{P}^\circ_{(2)} = {}^T\mathbb{P}_{(2)} \cap \mathfrak{M}^\circ$ and ${}^S\mathbb{P}^\circ_{(2)} = {}^S\mathbb{P}_{(2)} \cap \mathfrak{M}^\circ$.

It follows from the above that ${}^T\mathbb{P}^\circ_{(2)}$ and ${}^S\mathbb{P}^\circ_{(2)}$ are non-empty and belong to $\mathfrak{D}$. Moreover they are countable in $\mathrm{L}[S,T]$.

Lemma 18. *Let $G \subseteq {}^T\mathbb{P}^\circ_{(2)}$ be ${}^T\mathbb{P}^\circ_{(2)}$-generic over $\mathfrak{D}$. There is a unique triple $\langle x,y,f\rangle \in \mathrm{L}[S,T,G] \cap (\mathscr{N}^2 \times \lambda^\omega)$ such that $\tau(x,y,f)$ holds for any formula $\tau(\dot{x},\dot{y},\dot{f})$ in $(\bigcup G) \cap \mathfrak{M}^\circ$. In particular we have $x \preccurlyeq_T y$.*

Lemma 19. *Let $G \subseteq {}^S\mathbb{P}^\circ_{(2)}$ be ${}^S\mathbb{P}^\circ_{(2)}$ generic over $\mathfrak{D}$. There is a unique triple $\langle x,y,f\rangle \in \mathrm{L}[S,T,G] \cap (\mathscr{N}^2 \times \lambda^\omega)$ such that $\sigma(x,y,f)$ holds for any formula $\sigma(\dot{x},\dot{y},\dot{f})$ in $(\bigcup G) \cap \mathfrak{M}^\circ$. In particular we have $x \not\preccurlyeq_S y$.*

Proof. Proofs are analogous to the proof of Lemma 17. The additional parts are motivated by the fact that formula $\dot{x} \preccurlyeq_{T,\dot{f}} \dot{y}$ belongs to any $\mathscr{T} \in {}^T\mathbb{P}^\circ_{(2)}$ while formula $\dot{x} \not\preccurlyeq_{S,\dot{f}} \dot{y}$ belongs to any theory $\Sigma \in {}^S\mathbb{P}^\circ_{(2)}$. $\square$

3.4. The product forcing

The forcing notion ${}^S\mathbb{P}^\circ_{(2)}$ executes too a tight control over generic reals. Fortunately, generic $\preccurlyeq_S$-incomparable pairs can be obtained by another forcing, which connects the components in a much looser way, so that it is rather a kind of product forcing, with the factors equal to $\mathbb{P}^\circ$.

Let $\mathbb{P} \times_{\mathrm{H}} \mathbb{P}$ be the set of all theories $\Upsilon(\dot{x},\dot{z})$ of the form $\Pi(\dot{x}) \cup \mathrm{R}(\dot{z})$, where Π and R belong to $\mathbb{P}$ and satisfy $\mathscr{H}_{\dot{x}}[\Pi(\dot{x})] = \mathscr{H}_{\dot{x}}[\mathrm{R}(\dot{x})]$. The set $\mathbb{P} \times_{\mathrm{H}} \mathbb{P}$ is non-empty and belongs to $\mathfrak{M}^\circ$. As above, the set $\mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ = (\mathbb{P} \times_{\mathrm{H}} \mathbb{P}) \cap \mathfrak{M}^\circ$ belongs to $\mathfrak{D}$ and is non-empty.

Theorem 20. *Let $G \subseteq \mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$ be $\mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$-generic over $\mathfrak{D}$. There is a unique pair $\langle x,z\rangle \in \mathrm{L}[S,T,G] \cap \mathscr{N}^2$ such that $\pi(x,z)$ holds for any formula $\pi(\dot{x},\dot{z}) \in (\bigcup G) \cap \mathfrak{M}^\circ$. Those reals x and z are $\preccurlyeq_S$-incomparable.*

Pairs $\langle x,z\rangle$ as in the theorem will be denoted by $\langle x_G, z_G\rangle$.

Proof. Let us concentrate on the proof that x_G and z_G are $\preccurlyeq_S$-incomparable; the rest is analogous to the above.

Paradoxically, the proof appeals to generic extensions of V, the true universe. (Genericity over $\mathfrak{D}$ does not seem to work.) We first prove the theorem in the case when $G \subseteq \mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$ *is* $\mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$*-generic over* V. Assume on the contrary that $x_G \preccurlyeq_S z_G$ is $\mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$-forced over V by a "condition" $\Upsilon_0(\dot{x},\dot{z}) = \Pi_0(\dot{x}) \cup \mathrm{R}_0(\dot{z}) \in \mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$, where Π_0 and R_0 belong to $\mathbb{P}^\circ$ and satisfy $\mathscr{H}_{\dot{x}}[\Pi_0(\dot{x})] = \mathscr{H}_{\dot{x}}[\mathrm{R}_0(\dot{x})]$.

We are going to define a generic "rectangle" of reals x, z, x', z', such that the following is forced: $x \preccurlyeq_S z$ and $x' \preccurlyeq_S z'$ – by the contrary assumption, $z \preccurlyeq_T x'$– by Lemma 18, and $x \not\preccurlyeq_S z'$– by Lemma 19, leading to a contradiction. To reach this goal, apply the forcing $\mathbb{P}^\circ_{(4)}$ which consists of forcing conditions of the following general form:

$$\mathrm{Q} = \langle\, \Upsilon(\dot{x},\dot{z}), \mathscr{T}(\dot{z},\dot{x}',\dot{f}), \Upsilon'(\dot{x}',\dot{z}'), \Sigma(\dot{x},\dot{z}',\dot{f})\rangle\,,$$

where the theories $\Upsilon(\dot{x},\dot{z}) = \Pi(\dot{x}) \cup \mathrm{R}(\dot{z})$ and $\Upsilon'(\dot{x}',\dot{z}') = \Pi'(\dot{x}') \cup \mathrm{R}'(\dot{z}')$ belong to $\mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$, $\mathscr{T}$ belongs to ${}^T\mathbb{P}^\circ_{(2)}$, Σ belongs to ${}^S\mathbb{P}^\circ_{(2)}$, and we have

$$\Pi(\dot{x}) = \mathscr{F}_{\dot{x}}[\Sigma(\dot{x},\dot{z}',\dot{f}], \quad \mathrm{R}(\dot{z}) = \mathscr{F}_{\dot{z}}[\mathscr{T}(\dot{z},\dot{x}',\dot{f})],$$
$$\Pi'(\dot{x}') = \mathscr{F}_{\dot{x}'}[\mathscr{T}(\dot{z},\dot{x}',\dot{f})], \quad \mathrm{R}'(\dot{z}') = \mathscr{F}_{\dot{z}'}[\Sigma(\dot{x},\dot{z}',\dot{f})].$$

Let us order $\mathbb{P}^\circ_{(4)}$ componentwise: Q_1 is stronger than Q_2 iff $\Upsilon_2(\dot{x},\dot{z}) \subseteq \Upsilon_1(\dot{x},\dot{z})$, $\mathscr{T}_2(\dot{z},\dot{x}',\dot{f}) \subseteq \mathscr{T}_1(\dot{z},\dot{x}',\dot{f})$, $\Upsilon'_2 \subseteq \Upsilon'_1$, and $\Sigma_2 \subseteq \Sigma_1$.

To get a condition in $\mathbb{P}^\circ_{(4)}$, we start with the given theory $\Upsilon_0(\dot{x},\dot{z}) = \Pi_0(\dot{x}) \cup \mathrm{R}_0(\dot{z}) \in \mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$. By definition $\mathscr{H}_{\dot{x}}[\Pi_0(\dot{x})] = \mathscr{H}_{\dot{x}}[\mathrm{R}_0(\dot{x})]$. It can be supposed that $\mathscr{F}_{\dot{x}}[\Pi_0(\dot{x})] = \Pi_0(\dot{x})$ (otherwise we can replace $\Pi_0(\dot{x})$ by $\mathscr{F}_{\dot{x}}[\Pi_0(\dot{x})]$) and $\mathscr{F}_{\dot{z}}[\mathrm{R}_0(\dot{z})] = \mathrm{R}_0(\dot{z})$.

The theory $\Sigma_0(\dot{x},\dot{z}',\dot{f}) =_{\mathrm{df}} \Pi_0(\dot{x}) \cup \mathrm{R}_0(\dot{z}') \cup \dot{x} \equiv_{\mathrm{H}} \dot{z}' \cup \{\dot{x} \not\preccurlyeq_{S,\dot{f}} \dot{z}'\}$ belongs to ${}^S\mathbb{P}^\circ_{(2)}(\dot{x},\dot{z}')$ and satisfies $\mathscr{F}_{\dot{x}}[\Sigma_0(\dot{x},\dot{z}',\dot{f})] = \Pi_0(\dot{x})$ and $\mathscr{F}_{\dot{z}'}[\Sigma_0(\dot{x},\dot{z}',\dot{f})] = \mathrm{R}_0(\dot{z}')$ by Corollary 15. Similarly, by Lemma 14, the theory

$$\mathscr{T}_0(\dot{z},\dot{x}',\dot{f}) =_{\mathrm{df}} \mathrm{R}_0(\dot{z}) \cup \Pi_0(\dot{x}') \cup \dot{z} \equiv_{\mathrm{H}} \dot{x}' \cup \{\dot{z} \preccurlyeq_{T,\dot{f}} \dot{x}'\}$$

belongs to ${}^T\mathbb{P}^\circ_{(2)}$ and satisfies $\mathscr{F}_{\dot z}[\mathscr{T}_0(\dot z,\dot x',\dot f)]=\mathrm{R}_0(\dot z)$ and $\mathscr{F}_{\dot x'}[\mathscr{T}_0(\dot z,\dot x',\dot f)]=\Pi_0(\dot x')$. Now $\mathrm{Q}_0=\langle \Upsilon_0(\dot x,\dot z), \mathscr{T}_0(\dot z,\dot x',\dot f), \Upsilon_0(\dot x',\dot z'), \Sigma_0(\dot x,\dot z',\dot f)\rangle$ belongs to $\mathbb{P}^\circ_{(4)}$.

Assertion. *Suppose that* $\mathrm{Q}=\langle \Upsilon,\mathscr{T},\Upsilon',\Sigma\rangle\in\mathbb{P}^\circ_{(4)}$, $\Upsilon_1(\dot x,\dot z)\in\mathbb{P}^\circ\times_{\mathrm{H}}\mathbb{P}^\circ$, *and* $\Upsilon(\dot x,\dot z)\subseteq\Upsilon_1(\dot x,\dot z)$. *Then there is a condition* $\mathrm{Q}_1=\langle \Upsilon_1,\mathscr{T}_1,\Upsilon_1',\Sigma_1\rangle\in\mathbb{P}^\circ_{(4)}$ *stronger than* Q (i.e., $\mathscr{T}\subseteq\mathscr{T}_1$, $\Upsilon'\subseteq\Upsilon_1'$, *and* $\Sigma\subseteq\Sigma_1$).

The same is true when we strengthen any of the other three components.

Proof. By definition $\Upsilon_1(\dot x,\dot z)=\Pi_1(\dot x)\cup\mathrm{R}_1(\dot z)$ where Π_1 and R_1 belong to $\mathbb{P}^\circ$ and $\mathscr{H}_{\dot x}[\Pi_1(\dot x)]=\mathscr{H}_{\dot x}[\mathrm{R}_1(\dot x)]$.

Let $\mathscr{T}_1(\dot z,\dot x',\dot f)=\mathscr{T}(\dot z,\dot x',\dot f)\cup\mathrm{R}_1(\dot z)$. By Lemma 11, we have $\mathscr{H}_{\dot z}[\mathscr{T}_1(\dot z,\dot x',\dot f)]=\mathscr{H}_{\dot z}[\mathrm{R}_1(\dot z)]$. Let $\Pi_1'(\dot x')=\mathscr{F}_{\dot x'}[\mathscr{T}_1(\dot z,\dot x',\dot f)]$. Now $\mathscr{H}_{\dot z}[\Pi_1'(\dot z)]=\mathscr{H}_{\dot z}[\mathrm{R}_1(\dot z)]$ by Lemma 12 because even $\mathscr{T}(\dot z,\dot x',\dot f)$ includes $\dot z\equiv_{\mathrm{H}}\dot x'$ by the definition of ${}^T\mathbb{P}^\circ_{(2)}$. Hence, $\mathscr{H}_{\dot x}[\Pi_1'(\dot x)]=\mathscr{H}_{\dot x}[\Pi_1(\dot x)]$ by the above.

We shall define $\mathrm{R}_1'(\dot z')$ using the other side of the rectangle. Let $\Sigma_1(\dot x,\dot z',\dot f)=\Sigma(\dot x,\dot z',\dot f)\cup\Pi_1(\dot x)$ and $\mathrm{R}_1'(\dot z')=\mathscr{F}_{\dot z'}[\Sigma_1(\dot x,\dot z',\dot f)]$. Then $\mathscr{H}_{\dot x}[\mathrm{R}_1'(\dot x)]=\mathscr{H}_{\dot x}[\Pi_1(\dot x)]$ by Lemmas 11 and 12 as above. Thus in particular $\mathscr{H}_{\dot x'}[\mathrm{R}_1'(\dot x')]=\mathscr{H}_{\dot x'}[\Pi_1'(\dot x')]$, so that the theory $\Upsilon_1'(\dot x',\dot z')=\Pi_1'(\dot x')\cup\mathrm{R}_1'(\dot z')$ belongs to $\mathbb{P}^\circ\times_{\mathrm{H}}\mathbb{P}^\circ$, closing the diagram. Clearly $\Upsilon\subseteq\Upsilon_1$. It easily follows from the construction that $\mathrm{Q}_1=\langle \Upsilon_1,\mathscr{T}_1,\Upsilon_1',\Sigma_1\rangle\in\mathbb{P}^\circ_{(4)}$ is as required. $\square$

We continue the proof of Theorem 20. Consider a $\mathbb{P}^\circ_{(4)}$-generic extension $\mathrm{V}[\mathbb{G}]$ by a generic set $\mathbb{G}\subseteq\mathbb{P}^\circ_{(4)}$ containing the condition Q_0 defined above. It easily follows from the assertion just proved that $\mathbb{G}$ results in a "rectangle" of reals x, z, x', $z'\in\mathrm{V}[\mathbb{G}]$ such that

(1) The pair $\langle z,x'\rangle$ is ${}^T\mathbb{P}^\circ_{(2)}$-generic over V, hence we have $z\preccurlyeq_T x'$ in $\mathrm{V}[\mathbb{G}]$ by Lemma 18.
(2) The pair $\langle x,z'\rangle$ is ${}^S\mathbb{P}^\circ_{(2)}$-generic over V, hence we have $x\not\preccurlyeq_S y$ in $\mathrm{V}[\mathbb{G}]$ by Lemma 19.
(3) The pairs $\langle x,z\rangle$ and $\langle x',z'\rangle$ are $\mathbb{P}^\circ\times_{\mathrm{H}}\mathbb{P}^\circ$-generic over V, and moreover, the corresponding generic subsets of $\mathbb{P}^\circ\times_{\mathrm{H}}\mathbb{P}^\circ$ contain the "condition" $\Upsilon_0(\dot x,\dot z)$ fixed above; hence we have $x\preccurlyeq_S z$ and $x'\preccurlyeq_S z'$ in $\mathrm{V}[\mathbb{G}]$ by the choice of Υ_0.

This is a contradiction because, first, $\preccurlyeq_T\subseteq\preccurlyeq_S$ in $\mathrm{V}[\mathbb{G}]$ (see Section 2.1), and second, since the forcing notion $\mathbb{P}^\circ_{(4)}$ is countable, $\mathrm{V}[\mathbb{G}]$ is a Cohen-generic extension, therefore $\preccurlyeq_S$ remains a PQO in $\mathrm{V}[\mathbb{G}]$. This ends the proof of the theorem in the case when G is $\mathbb{P}^\circ\times_{\mathrm{H}}\mathbb{P}^\circ$-generic over V.

It follows, by ordinary forcing arguments, that the theorem is also true in the case when $G\subseteq\mathbb{P}^\circ\times_{\mathrm{H}}\mathbb{P}^\circ$ is $\mathbb{P}^\circ\times_{\mathrm{H}}\mathbb{P}^\circ$-generic over $\mathrm{L}[S,T]$.

Thus $\mathbb{P}^\circ\times_{\mathrm{H}}\mathbb{P}^\circ$ forces, over $\mathrm{L}[S,T]$, the existence of a function $f\in\lambda^\omega$ such that $\langle x_G,z_G,f\rangle\in[S]$. Let $\mathbf{f}\in\mathrm{L}[S,T]$ be a term for such a function, so that $\mathbb{P}^\circ\times_{\mathrm{H}}\mathbb{P}^\circ$ forces $[S](x_G,z_G,\mathbf{f})$. Basically $\mathbf{f}\subseteq(\mathbb{P}^\circ\times_{\mathrm{H}}\mathbb{P}^\circ)\times\lambda^{<\omega}$, hence $\mathbf{f}\in\mathfrak{M}=\mathrm{L}_{\lambda^\star}[S,T]$. Now $\mathbf{f}$ can be assumed to be definable in $\mathfrak{M}$, hence a member of $\mathfrak{D}$. (Take the $<_{\mathrm{L}[S,T]}$-least such a term.) Then, for any n, the set D_n of all theories $\Upsilon \in \mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$, which force $u \subset x_G$, $v \subset z_G$, $w \subset \mathbf{f}$ for some $u, v \in \omega^n$ and $w \in \lambda^n$ (note that then $\langle u, v, w\rangle \in S$), belongs to $\mathfrak{D}$, and, by the choice of $\mathbf{f}$, is dense in $\mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$.

Let us now carry out the general case in the theorem, i.e., G is assumed to be $\mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$-generic over $\mathfrak{D}$. Then G intersects each of the sets D_n by the above. Therefore, G gives rise to a triple $\langle x_G, z_G, f_G\rangle \in [S]$ by the definition of D_n. It follows that $\langle x_G, z_G\rangle \in \mathfrak{p}[S]$, as required. $\square$

4. The embedding

We are going to define (in the assumption of Case 2 of Section 2.4) a continuous 1–1 map $F: 2^\omega \to \mathcal{N}$ satisfying (II*) of Theorem 5. Our strategy will be to carry out a splitting construction, based on Theorem 20 and Lemmas 17 and 18. The construction is carried out in V, the basic universe: note that the forcing notions $\mathbb{P}^\circ$, ${}^T\mathbb{P}^\circ_{(2)}$, and $\mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$, which Theorem 20 and Lemmas 17, 18 deal with, belong to V and are countable in V (even in $\mathrm{L}[S,T]$), and so is the set $\mathfrak{D}$, containing all relevant dense sets.

The construction of the embedding will accomplish the proof of Theorem 5.

4.1. Generic splitting family of theories

Let a *crucial pair* be any (ordered) pair $\langle u, v\rangle$ such that $u, v \in 2^m$ for some m and $u = 1^{k\wedge}0^\wedge w$, $v = 0^{k\wedge}1^\wedge w$ for some $k < m$ and $w \in 2^{m-k-1}$.

Let $\{D(n): n \in \omega\}$, $\{D^2(n): n \in \omega\}, \{D_2(n): n \in \omega\}$ be enumerations (in V) of the collections of all dense (we mean: *open dense*) subsets of resp. $\mathbb{P}^\circ$, $\mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$, ${}^T\mathbb{P}^\circ_{(2)}$, which belong to $\mathfrak{D}$. It can be assumed that each dense set has infinitely many indices in the relevant enumeration.

Our plan will be to define a family of theories $\Pi_u(\dot x) \in \mathbb{P}^\circ$ (where $u \in 2^{<\omega}$) and $\mathcal{T}_{uv}(\dot x, \dot y, \dot f) \in {}^T\mathbb{P}^\circ_{(2)}$ (where $\langle u, v\rangle$ is a crucial pair in some 2^n) satisfying the following conditions, for all $u \in 2^{<\omega}$ and $i = 0, 1$:

(i) $\Pi_u \in D(n)$ whenever $u \in 2^n$; $\Pi_u(\dot x) \subseteq \Pi_{u^\wedge i}(\dot x)$;

(ii) if $\langle u, v\rangle$ is a crucial pair in 2^n then $\mathcal{T}_{uv}(\dot x, \dot y, \dot f) \in D_2(n)$; in addition, for any $i = 0, 1$, we have $\mathcal{T}_{uv}(\dot x, \dot y, \dot f) \subseteq \mathcal{T}_{u^\wedge i, v^\wedge i}(\dot x, \dot y, \dot f)$;

(iii) if $u, v \in 2^n$ and $u(n-1) \neq v(n-1)$ then $\Pi_u(\dot x) \cup \Pi_v(\dot z) \in \mathbb{P}^\circ \times_{\mathrm{H}} \mathbb{P}^\circ$, moreover, $\in D^2(n)$, and the theory $\Pi_u(\dot x) \cup \Pi_v(\dot x)$ is $\star$-inconsistent;

(iv) $\mathcal{F}_{\dot x}[\mathcal{T}_{uv}(\dot x, \dot y, \dot f)] = \Pi_u(\dot x)$ and $\mathcal{F}_{\dot y}[\mathcal{T}_{uv}(\dot x, \dot y, \dot f)] = \Pi_v(\dot y)$ – then in particular $\mathcal{F}_{\dot x}[\Pi_u(\dot x)] = \Pi_u(\dot x)$ for all u.

Remark 21. Since theories in ${}^T\mathbb{P}^\circ_{(2)}$ contain $\dot x \equiv_{\mathrm{H}} \dot y$, it follows from (iv) by Lemma 12 that $\mathcal{H}_{\dot x}[\Pi_u(\dot x)] = \mathcal{H}_{\dot x}[\Pi_v(\dot x)]$ for all crucial pairs $\langle u, v\rangle$. Therefore, $\mathcal{H}_{\dot x}[\Pi_u(\dot x)] = \mathcal{H}_{\dot x}[\Pi_v(\dot x)]$ for all $u, v \in 2^n$ and $n \in \omega$ as any two tuples $u, v \in 2^n$ are connected by a (unique) chain of crucial pairs.

4.2. Construction of theories

To define Π_Λ (where Λ is the empty sequence, the only member of 2^0) we begin with the theory $\Xi(\dot{x})$, see Section 3. As clearly $\Xi(\dot{x}) \in \mathbb{P}^\circ$, there is a theory $\Pi(\dot{x}) \in D(0)$ including $\Xi(\dot{x})$. Let $\Pi_\Lambda(\dot{x}) = \Pi(\dot{x})$.

Suppose that the construction has been completed up to a level n, and extend it to the next level.

To begin with, set $\Pi_{s^\wedge i}(\dot{x}) = \Pi_s(\dot{x})$ for all $s \in 2^n$ and $i = 0, 1$, and define $\mathcal{T}_{s^\wedge i, t^\wedge i}(\dot{x}, \dot{y}, \dot{f}) = \mathcal{T}_{st}(\dot{x}, \dot{y}, \dot{f})$ whenever $i = 0, 1$ and $\langle s, t\rangle$ is crucial in 2^n. For the "initial" pair $\langle 1^{n\wedge}0, 0^{n\wedge}1\rangle$, let $\mathcal{T}_{1^{n\wedge}0, 0^{n\wedge}1}$ be the theory

$$\Pi_{1^n}(\dot{x}) \cup \Pi_{0^n}(\dot{y}) \cup (\dot{x} \equiv_{\mathrm{H}} \dot{y}) \cup \{\dot{x} \preccurlyeq_{T,\dot{f}} \dot{y}\}.$$

Then, by Lemma 14, $\mathcal{T}_{1^{n\wedge}0, 0^{n\wedge}1} \in {}^T\mathbb{P}^\circ_{(2)}$ and $\mathcal{F}_{\dot{x}}[\mathcal{T}_{1^{n\wedge}0, 0^{n\wedge}1}(\dot{x}, \dot{y}, \dot{f})] = \Pi_{1^{n\wedge}0}(\dot{x})$, $\mathcal{F}_{\dot{y}}[\mathcal{T}_{1^{n\wedge}0, 0^{n\wedge}1}(\dot{x}, \dot{y}, \dot{f})] = \Pi_{0^{n\wedge}1}(\dot{y})$.

This ends the definition of "initial values" at the $(n+1)$th level. The plan is to gradually strengthen the theories in order to fulfill the requirements.

Step 1: We take care of item (i). Consider an arbitrary $u_0 = s_0{}^\wedge i \in 2^{n+1}$. As the set $D(n)$ is dense, there is a theory $\Pi'(\dot{x}) \in D(n)$ including $\Pi_{u_0}(\dot{x})$. We can assume that $\Pi'(\dot{x}) = \mathcal{F}_{\dot{x}}[\Pi'(\dot{x})]$: for otherwise change $\Pi'(\dot{x})$ to $\mathcal{F}_{\dot{x}}[\Pi'(\dot{x})]$.

The intention is to take $\Pi'(\dot{x})$ as the "new" Π_{u_0}. But this change has to be expanded through the net of crucial pairs, in order to preserve (iv). (Fortunately, the tree of all crucial pairs in 2^{n+1} is a chain.)

Thus put $\Pi'_{u_0}(\dot{x}) = \Pi'(\dot{x})$. Suppose that $\Pi'_u(\dot{x})$ has been defined, includes Π_u, the older version, and satisfies $\mathcal{F}_{\dot{x}}[\Pi'_u(\dot{x})] = \Pi'_u(\dot{x})$ – for some $u \in 2^{n+1}$ which is connected by a crucial pair with a not yet encountered $v \in 2^{n+1}$. Define $\mathcal{T}'_{uv}(\dot{x}, \dot{y}, \dot{f})$ to be $\Pi'_u(\dot{x}) \cup \mathcal{T}_{uv}(\dot{x}, \dot{y}, \dot{f})$ and $\Pi'_v(\dot{y})$ to be $\mathcal{F}_{\dot{y}}[\mathcal{T}'_{uv}(\dot{x}, \dot{y}, \dot{f})]$. Note that the theory $\Pi'_v(\dot{y})$ includes $\Pi_v(\dot{y})$ because (iv) is assumed for the old theories Π_u, Π_v, $\mathcal{T}_{uv}$. Note also that (iv) holds for the new theories Π'_u, Π'_v, $\mathcal{T}'_{uv}$: indeed, the equality $\mathcal{F}_{\dot{x}}[\mathcal{T}'_{uv}(\dot{x}, \dot{y}, \dot{f})] = \Pi'_u(\dot{x})$ follows from Lemma 11.

The construction describes how the change from Π_{u_0} to Π'_{u_0} spreads through the chain of crucial pairs in 2^{n+1}, resulting in a system of new theories, Π'_u and $\mathcal{T}'_{uv}$, which satisfy (i) for the particular $u_0 \in 2^{n+1}$.

Let us iterate this construction consecutively for all $u_0 \in 2^{n+1}$, getting finally a system of theories satisfying (i) (fully) (and (iv)), which we shall denote by Π_u and $\mathcal{T}_{uv}$ from now on.

Step 2: We take care of item (iii). Let us fix a pair of u_0 and v_0 in 2^{n+1}, such that $u_0(n) = 0$ and $v_0(n) = 1$. It follows from the density of $D^2(n)$ that there is a theory $\Pi'_{u_0}(\dot{x}) \cup \Pi'_{v_0}(\dot{y}) \in D^2(n)$ which includes $\Pi_{u_0}(\dot{x}) \cup \Pi_{v_0}(\dot{y})$. We may assume that $\Pi'_{u_0}(\dot{x}) = \mathcal{F}_{\dot{x}}[\Pi'_{u_0}(\dot{x})]$ and $\Pi'_{v_0}(\dot{y}) = \mathcal{F}_{\dot{y}}[\Pi'_{v_0}(\dot{y})]$. We can also assume, by Corollary 16, that $\Pi'_{u_0}(\dot{x}) \cup \Pi'_{v_0}(\dot{x})$ is $\star$-inconsistent.

Let us spread the change from Π_{u_0} to Π'_{u_0} and from Π_{v_0} to Π'_{v_0} through the chain of crucial pairs in 2^{n+1} until the two waves of spreading meet each other at the pair

$\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$. This leads to a system of theories Π'_u and $\mathscr{T}'_{uv}$ which satisfy (iii) for the particular pair $\langle u_0, v_0\rangle$ and still satisfy (iv) with the exception of the "meeting" crucial pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$ (for which basically $\mathscr{T}'_{1^n{}^\wedge 0, 0^n{}^\wedge 1}$ is not yet defined for this step).

Take notice that the construction of Step 1 leaves $\mathscr{T}_{1^n{}^\wedge 0, 0^n{}^\wedge 1}$ in the form $\Pi_{1^n{}^\wedge 0}(\dot x)\cup \Pi_{0^n{}^\wedge 1}(\dot y)\cup(\dot x \equiv_{\mathrm H} \dot y)\cup\{\dot x \leqslant_{T,\dot f} \dot y\}$ (where $\Pi_{1^n{}^\wedge 0}$ and $\Pi_{0^n{}^\wedge 1}$ are the "versions" at the end of Step 1. We now have new $\star$-consistent theories, $\Pi'_{1^n{}^\wedge 0}$ and $\Pi'_{0^n{}^\wedge 1}$, including resp. $\Pi_{1^n{}^\wedge 0}$ and $\Pi_{0^n{}^\wedge 1}$ and satisfying the equality $\mathscr{H}_{\dot x}[\Pi'_{1^n{}^\wedge 0}(\dot x)] = \mathscr{H}_{\dot x}[\Pi'_{0^n{}^\wedge 1}(\dot x)]$. (See Remark 21; recall that $\mathscr{H}_{\dot x}[\Pi'_{u_0}] = \mathscr{H}_{\dot x}[\Pi'_{v_0}]$ for the initial pair simply because $\Pi'_{u_0}(\dot x)\cup \Pi'_{v_0}(\dot y)\in \mathbb{P}^\circ \times_{\mathrm H} \mathbb{P}^\circ$.) We observe that the theory $\Pi'_{1^n{}^\wedge 0}(\dot x)\cup \Pi'_{0^n{}^\wedge 1}(\dot y)\cup (\dot x \equiv_{\mathrm H} \dot y)\cup\{\dot x \leqslant_{T,\dot f} \dot y\}$ taken as $\mathscr{T}'_{1^n{}^\wedge 0, 0^n{}^\wedge 1}$ belongs to ${}^T\mathbb{P}^\circ_{(2)}$ and satisfies (iv) for the pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$ by Lemma 14. This ends the consideration of the pair $\langle u_0, v_0\rangle$.

Applying this construction consecutively for all pairs of $u_0\in P_0$ and $v_0\in P_1$ (including the pair $\langle 1^n{}^\wedge 0, 0^n{}^\wedge 1\rangle$) we finally get a system of theories satisfying (i), (iii), and (iv), which will be denoted still by Π_u and $\mathscr{T}_{uv}$.

Step 3: We finally take care of requirement (ii). Consider a particular crucial pair $\langle u_0, v_0\rangle$ in 2^{n+1}. By the density of $D_2(n)$, there is a theory $\mathscr{T}'_{u_0,v_0}(\dot x, \dot y, \dot f)$ in $D_2(n)$ including $\mathscr{T}_{u_0,v_0}(\dot x, \dot y, \dot f)$.

Let us define $\Pi'_{u_0}(\dot x) = \mathscr{F}_{\dot x}[\mathscr{T}'_{u_0,v_0}(\dot x, \dot y, \dot f)]$ and $\Pi'_{v_0}(\dot y) = \mathscr{F}_{\dot y}[\mathscr{T}'_{u_0,v_0}(\dot x, \dot y, \dot f)]$ and spread this change through the chain of crucial pairs in 2^{n+1}. (Note that $\mathscr{H}_{\dot x}[\Pi'_{u_0}(\dot x)] = \mathscr{H}_{\dot x}[\Pi'_{v_0}(\dot x)]$, because theories in ${}^T\mathbb{P}^\circ_{(2)}$ include $\dot x \equiv_{\mathrm H} \dot y$. This implies $\mathscr{H}_{\dot x}[\Pi'_u(\dot x)] = \mathscr{H}_{\dot x}[\Pi'_v(\dot x)]$ for all $u, v\in 2^{n+1}$, after the spreading.)

Executing this construction for all crucial pairs in 2^{n+1}, we finally end the construction of a system of theories satisfying (i)–(iv).

4.3. Ending the proof of Theorem 5

Thus, we have seen that the assumption of Case 2 of Section 2.4 implies the existence of a family of theories Π_u and $\mathscr{T}_{uv}$ satisfying (i)–(iv). To prove Theorem 5, it remains to show that the existence of such a system provides a continuous map F which witnesses (II*) of Theorem 5.

To prove this note that Lemma 17 and (i) imply that for any $a\in 2^\omega$ there is a unique real, denoted by $F(a)$, satisfying every formula in $\bigcup_{n\in\omega} \Pi_{a\restriction n}(\dot x)$, and the map F is continuous. Moreover F is 1–1 by the $\star$-inconsistency in (iii).

Suppose that $a, b\in 2^\omega$ and $a \not\mathsf{E}_0\, b$, so that $a(n)\neq b(n)$ for infinitely many n. It follows then from (iii) and Theorem 20 that $F(a) \not\leqslant_S F(b)$.

Let us check that F satisfies (II*) of Theorem 5. Suppose that $a, b\in 2^\omega$ are $\leqslant_0$-neighbours, i.e., $a = 1^k{}^\wedge 0^\wedge c$ and $b = 0^k{}^\wedge 1^\wedge c$ for some $k\in\omega$ and $c\in 2^\omega$. Then $\langle a\restriction n, b\restriction n\rangle$ is a crucial pair for all $n > k$. Therefore, by (ii) and Lemma 18, there is a unique triple $\langle x, y, f\rangle \in \mathscr{N}^2\times \lambda^\omega$ which satisfies every formula in $\bigcup_{n\in\omega}\mathscr{T}_{a\restriction, n, b\restriction n}(\dot x, \dot y, \dot f)$, and also satisfies $x\leqslant_T y$. On the other hand, we have $x = F(a)$, $y = F(b)$ by (iv), so that $F(a)\leqslant_T F(b)$. $\square$ (Theorem 5)

5. Analytic order relations

This section proves Theorem 2.

We start with a technical absoluteness lemma, necessary because some parts of the proof will appeal to generic extensions of the universe. (In fact, there is another way to the same goal, but at the cost of complications of another kind.)

5.1. Why embedding $\leqslant_0$ is absolute

The proof of the following lemma involves an idea communicated to the author by Hjorth, with a reference to Hjorth and Kechris [7], Section 3, where the idea is realized in terms of category.

Lemma 22. *If $p \in \mathscr{N}$ and $\preccurlyeq$ is a $\Sigma^1_1(p)$* PQO *then* (II^{A}) *of Theorem* 2 *is equivalent to a $\Sigma^1_2(p)$ statement, uniformly in p.*

Proof. The goal does not seem easy: at the first look the statement is Σ^1_3. To improve this to Σ^1_2, we use *Borel approximations* of $\preccurlyeq$.

Recall that $\mathtt{WO} = \{z \in \mathscr{N} : z$ codes an ordinal$\}$; for $z \in \mathtt{WO}$ let $|z|$ be the ordinal coded by z, and $\mathtt{WO}_\nu = \{z \in \mathtt{WO} : |z| = \nu\}$.

Being a $\mathbf{\Sigma}^1_1$ subset of $\mathscr{N}^2$, the relation $\preccurlyeq$ classically has the form $\preccurlyeq = \bigcup_{\nu<\omega_1} \preccurlyeq_\nu$ where $\langle \preccurlyeq_\nu : \nu < \omega_1 \rangle$ is an increasing sequence of Borel subsets of $\preccurlyeq$. Moreover, there is a Π^1_1 formula $\pi(z,x,y)$ (containing p as a parameter) such that we have $x \preccurlyeq_\nu y \Leftrightarrow \pi(z,x,y)$ whenever $z \in \mathtt{WO}_\nu$. (There also exists a $\mathbf{\Sigma}^1_1$ formula with the same property, which we do not nead here.)

The following statement is clearly $\Sigma^1_2(p)$ (use formula π):

(II') *There is a continuous* 1–1 *map $F' : 2^\omega \to \mathscr{N}$ and a countable ordinal ν such that*:
  (a) *$a \leqslant_0 b$ implies $F'(a) \preccurlyeq_\nu F'(b)$*;
  (b) *$a \not\leqslant_0 b$ implies $F'(a) \not\preccurlyeq F'(b)$.*

Thus it remains to prove that (II^{A}) of Theorem 2 is equivalent to (II'). The hard part in the equivalence is to prove that (II^{A}) implies (II'). To prove this direction we consider a κ-collapse generic extension V^+ of V, the universe of all sets, where κ is $2^{\aleph_0}$ in V. As (II^{A}) is $\Sigma^1_3(p)$ while (II') is $\Sigma^1_2(p)$, it suffices to prove that (II^{A}) of Theorem 2 implies (II') *in* V^+.

We can enumerate in V^+ by natural numbers all dense subsets of $2^{<\omega}$ and $2^{<\omega} \times 2^{<\omega}$ (the Cohen forcing and its square) which belong to V. This allows to define in V^+ infinite sequences $\langle u_n : n \in \omega \rangle$ and $\langle v_n : n \in \omega \rangle$ such that $u_n, v_n \in 2^{l(n)}$ for some $l(n)$ for all n, and for any n:

(1°) if $u, v \in 2^l$ where $l = n + \sum_{m=0}^{n-1} l(n)$ then the pairs $\langle u^\wedge u_n, v^\wedge v_n \rangle$ and $\langle u^\wedge v_n, v^\wedge u_n \rangle$ belong to the n-th dense subset of $2^{<\omega} \times 2^{<\omega}$.

Define in V^+, for each $a \in 2^\omega$, $G(a) = w_0{}^\wedge w_1{}^\wedge w_2{}^\wedge \ldots$, where $w_n = u_n{}^\wedge 0$ whenever $a(n) = 0$ and $w_n = v_n{}^\wedge 1$ whenever $a(n) = 1$. Then G is continuous and 1–1, therefore the map $F'(a) = F(G(a))$ is continuous and 1–1 as well. (Here F is a map which witnesses (II^{A}) of Theorem 2 in V^+.) Prove that F' witnesses (II').

Suppose that $a, b \in 2^\omega$ and $a \leqslant_0 b$. Then by (1°) both $a' = G(a)$ and $b' = G(b)$ are $\mathtt{Cohen}$-generic over V and $a' \leqslant_0 b'$, hence $F(a') \preccurlyeq F(b')$ (by the choice of F), even in $\mathrm{V}[a', b']$, which implies $F(a') \preccurlyeq_\nu F(b')$ for an ordinal $\nu < \omega_1^{\mathrm{V}}$. (Note that $\mathrm{V}[a'] = \mathrm{V}[a', b']$ is a $\mathtt{Cohen}$-generic extension of V, hence $\omega_1^{\mathrm{V}} = \omega_1^{\mathrm{V}[a', b']}$.) Since the difference between a' and b' is finite (because $a' \leqslant_0 b'$), the latter statement is a property of a', hence it is $\mathtt{Cohen}$-forced over V. It follows, by the $\mathtt{ccc}$ property of the Cohen forcing, that there exists an ordinal $\nu < \omega_1^{\mathrm{V}}$ such that $F'(a) \preccurlyeq_\nu F'(b)$ whenever $a, b \in 2^\omega$ in V^+ satisfy $a \leqslant_0 b$.

Suppose that $a, b \in 2^\omega$ and $a \not\leqslant_0 b$. Then by definition $\langle G(a), G(b) \rangle$ is $\mathtt{Cohen}^2$-generic over V, in particular $a' = G(a)$ and $b' = G(b)$ satisfy $a' \not\leqslant_0 b'$, therefore $F(a') \not\preccurlyeq F(b')$ by the choice of F. $\square$

5.2. *Setup for the analytic case*

Consider an analytic PQO $\preccurlyeq$ on $\mathscr{N}$. We shall w.l.o.g. assume that $\preccurlyeq$ is Σ^1_1, so that $\preccurlyeq = \preccurlyeq_T = \mathfrak{p}[T]$, where T is a recursive tree in $(\omega \times \omega \times \omega)^{<\omega}$. We shall also suppose that

(2°) $\preccurlyeq_T$ does *not* satisfy (II^{A}) of Theorem 2.

The aim is to prove that then $\preccurlyeq_T$ satisfies (I^A) of Theorem 2.

Since $\preccurlyeq_T$ is Σ^1_1, there is a continuous, coded in L, map $C : \mathscr{N}^2 \to \mathscr{N}$ such that $x \preccurlyeq_T y \Leftrightarrow C(x, y) \notin \mathtt{WO}$. It follows that $\preccurlyeq_T = \bigcap_{\nu < \omega_1} \preccurlyeq^\nu_T$, where

$$\preccurlyeq^\nu_T = \complement \{\langle x, y \rangle : C(x, y) \in \mathtt{WO}_\nu\}$$

(see the notation above). It is known that each $\preccurlyeq^\nu_T$ is an ω_1-Borel set, coded in L, and $\preccurlyeq^\mu_T \subseteq \preccurlyeq^\nu_T$ provided $\nu < \mu$. In addition we have the following:

Boundedness principle: If $\preccurlyeq_T \subseteq X$, where $X \subseteq \mathscr{N}^2$ is a $\mathbf{\Pi}^1_1$ set, then there is an ordinal $\nu < \omega_1$ such that $\preccurlyeq^\nu_T \subseteq X$.

We have to make some changes in the definition of sets H_α in Section 2.4. Now, a *function code of type* α, λ will be a sequence of the form $\vec{\varphi} = \{\varphi_\beta(\dot{x})\}_{\beta < \alpha}$, where each φ_γ is a *constructible* $\mathscr{L}_{\lambda+1,0}$-formula. Such a sequence defines a function $h_{\vec{\varphi}}$: $\mathscr{N} \to 2^\alpha$ just as in Section 2.4. Let $\vec{\varphi}_\gamma$ be the γth, in the sense of Gödel's well ordering of L, function code $\vec{\varphi} \in \mathrm{L}$ (of any type). Note that $\{\vec{\varphi}_\gamma : \gamma < \omega_1\}$ is exactly the set of all function codes of types α, λ for $\alpha, \lambda < \omega_1$. We shall assume that each $\vec{\varphi}_\gamma$ is of type α, λ for some $\alpha, \lambda \leqslant \gamma$.

Now, for any $\gamma \in \mathtt{Ord}$, define $h_\gamma = h_{\vec{\varphi}_\gamma}$, whenever $h_{\vec{\varphi}_\gamma}$ is a h.o.p. map $\langle \mathscr{N}; \preccurlyeq_T \rangle \to \langle 2^\alpha; \leqslant_{\mathrm{lex}} \rangle$ *in the* γ*-collapse extension of* L, and $h_\gamma(x) = \Lambda$ (the empty sequence) otherwise. By the Shoenfield absoluteness theorem, if $\gamma < \omega_1$ then $h_\gamma = h_{\vec{\varphi}_\gamma}$ iff $h_{\vec{\varphi}_\gamma}$ is h.o.p.

in the universe V, therefore $\mathrm{H}^* = \{h_\gamma\colon \gamma<\omega_1\}$ is the set of all ω_1-Borel, coded in L, maps from $\mathscr{N}$ to some $2^\alpha, \alpha<\omega_1$, satisfying $x \preccurlyeq_T y \Rightarrow h(x) \leqslant_{\mathrm{lex}} h(y)$.

Suppose that M is a transitive model of $\mathbf{ZFC}^-$ (minus Power Sets), and $\gamma<\omega_1$ belongs to M. Then $\vec{\varphi}_\gamma \in (\mathrm{L})^M$. One can determine, *within* M, whether $h_\gamma = h_{\vec{\varphi}_\gamma}$, for instance, checking whether the γ-collapse forcing forces, in $(\mathrm{L})^M$, that $h_{\vec{\varphi}_\gamma}$ is h.o.p. (in the collapse extension of $(\mathrm{L})^M$). It follows that there is a Σ_1 formula $\Phi(\cdot,\cdot,\cdot)$ such that

(3°) If M is a transitive model of $\mathbf{ZFC}^-$ (minus Power Sets), $x \in \mathscr{N} \cap M$, and $\lambda \in M$, $\lambda<\omega_1$, then $u = \{h_\alpha(x)\}_{\alpha<\lambda} \in M$ and u is the only member of M such that $\Phi(x,\lambda,u)$ holds in M.

Define the concatenation (which is here a proper class, of course):

$$\mathbf{h}(x) = h_0(x)^\wedge h_1(x)^\wedge \ldots {}^\wedge h_\alpha(x)^\wedge \ldots \quad (\alpha \in \mathtt{Ord}).$$

5.3. *The general case of analytic relations*

We first prove that $\preccurlyeq_T$ satisfies the general part of $(\mathrm{I}^{\mathrm{A}})$ (leaving aside the additional statement) of Theorem 2, via the map $\mathbf{h}^{\omega_1}(x) = \mathbf{h}(x) \restriction \omega_1$, i.e., we show that $\mathbf{h}^{\omega_1}\colon \langle \mathscr{N}; \preccurlyeq_T\rangle \to \langle 2^{\omega_1}; \leqslant_{\mathrm{lex}}\rangle$ is a linearization. First of all $\mathbf{h}^{\omega_1}$ is a h.o.p. map from $\langle \mathscr{N}; \preccurlyeq_T\rangle$ to $\langle 2^{\omega_1}; \leqslant_{\mathrm{lex}}\rangle$ because each h_α is h.o.p. by definition. Thus it remains to prove that $\mathbf{h}^{\omega_1}(x) = \mathbf{h}^{\omega_1}(y) \Rightarrow x \approx_T y$.

This involves a *reflection lemma* for analytic PQOs.

The sets $\preccurlyeq_T^\nu$ above are not necessarily PQOs. However:

Lemma 23. *Assume that* $B \subseteq \mathscr{N}^2$ *is a Borel set and* $\preccurlyeq_T \subseteq B$. *Then there is* $\nu<\omega_1$ *such that* $\preccurlyeq_T^\nu \subseteq B$ *and* $\preccurlyeq_T^\nu$ *is a* PQO. [12]

Proof. First prove a weaker statement: *there is* $\mu<\omega_1$ *such that*

$$x \preccurlyeq_T^\mu y \Rightarrow \forall x'\, \forall y'\, (x' \preccurlyeq_T x \wedge y \preccurlyeq_T y' \Rightarrow x' \preccurlyeq_T^\mu y').$$

Note that by the boundedness there exists an ordinal $\mu_0<\omega_1$ such that $\preccurlyeq_T^{\mu_0} \subseteq B$. Suppose that an ordinal $\mu_n \geqslant \mu_0$ has been defined. Let us define $Z(x,y)$ iff $\forall x' \forall y'\, (x' \preccurlyeq_T x \wedge y \preccurlyeq_T y' \Rightarrow x' \preccurlyeq_T^{\mu_n} y')$, so that Z is a $\mathbf{\Pi}^1_1$ relation and $\preccurlyeq_T \subseteq Z \subseteq \preccurlyeq_T^{\mu_n}$. Using the boundedness principle again, we get an ordinal $\mu_{n+1} \geqslant \mu_n$ satisfying $\preccurlyeq_T \subseteq \preccurlyeq_T^{\mu_{n+1}} \subseteq Z$, so that by definition

$$x \preccurlyeq_T^{\mu_{n+1}} y \Rightarrow \forall x'\, \forall y'\, (x' \preccurlyeq_T x \wedge y \preccurlyeq_T y' \Rightarrow x' \preccurlyeq_T^{\mu_n} y').$$

It remains to define $\mu = \mathtt{sup}_n\, \mu_n$.

Starting the proof of the lemma, we choose ν_0 so that $\preccurlyeq_T^{\nu_0} \subseteq B$ and

(4°) $x \preccurlyeq_T^{\nu_0} y \Rightarrow \forall x'\, \forall y'\, (x' \preccurlyeq_T x \wedge y \preccurlyeq_T y' \Rightarrow x' \preccurlyeq_T^{\nu_0} y')$.

[12] This lemma belongs to the category of reflection principles very useful in the study of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$ sets, see, e.g., [3, Section 1] for more general formulations. We present a rather short proof, to make the exposition self-contained.

Suppose that an ordinal $\nu_n \geqslant \nu_0$ satisfying (4°) has been defined. Put $Z(x, y)$ iff $x \preccurlyeq_T^{\nu_n} y \wedge \forall z(y \preccurlyeq_T^{\nu_n} z \Rightarrow x \preccurlyeq_T^{\nu_n} z)$, so that Z is a $\mathbf{\Pi}_1^1$ relation and $\preccurlyeq_T \subseteq Z \subseteq \preccurlyeq_T^{\nu_n}$. As above there is an ordinal $\nu_{n+1} \geqslant \nu_n$ satisfying $\preccurlyeq_T \subseteq \preccurlyeq_T^{\nu_{n+1}} \subseteq C$, so that by definition

$$x \preccurlyeq_T^{\nu_{n+1}} y \Rightarrow \forall z(y \preccurlyeq_T^{\nu_n} z \Rightarrow x \preccurlyeq_T^{\nu_n} z).$$

Take $\nu = \mathtt{sup}_n \nu_n$. □

Now assume $x \not\approx_T y$ and prove that $\mathbf{h}^{\omega_1}(x) \neq \mathbf{h}^{\omega_1}(y)$. By the lemma there is an ordinal $\nu < \omega_1$ such that $x \not\approx_T^{\nu} y$ and $\preccurlyeq_T^{\nu}$ is a PQO. As $\preccurlyeq_T^{\nu}$ is equal to $\complement\{\langle x, y\rangle : C(x, y) \in \mathtt{WO} \wedge |C(x, y)| \leqslant \nu\}$ by the above, it easily follows that there exists a tree $S = S_\nu \in \mathrm{L}$, $S \subseteq (\omega \times \omega \times \nu)^{<\omega}$, such that $\preccurlyeq_T^{\nu} = \complement \mathfrak{p}[S]$. Apply Theorem 5 for the relations $\preccurlyeq_T = \mathfrak{p}[T] \subseteq \preccurlyeq_T^{\nu} = \complement \mathfrak{p}[S]$. (As ν is countable, the $\mathtt{Cohen}$-generic stability requirement is satisfied.)

We observe that (II*) of Theorem 5 fails by assumption (2°). Therefore (I*) of Theorem 5 holds, so that there exists an ordinal $\alpha < \omega_1$ such that the map h_α satisfies $h_\alpha(x) = h_\alpha(y) \Rightarrow x \approx_T^{\nu} y$. It follows that $h_\alpha(x) \neq h_\alpha(y)$ by the choice of ν, hence $\mathbf{h}^{\omega_1}(x) \neq \mathbf{h}^{\omega_1}(y)$, as required.

5.4. Special cases: reduction to countable sequences

Consider the "additional" part in (I^{A}) of Theorem 2. We still assume that the Σ_1^1 PQO $\preccurlyeq_T = \mathfrak{p}[T]$ does *not* satisfy (II^{A}) of Theorem 2, but either it satisfies (a) in the "additional" part of Theorem 2 or we have (b) of Theorem 2. In the second case, we shall assume the following: there exists a real z_0 such that each real x in the universe V belongs to a set [13] generic extension of $\mathrm{L}[z_0]$. [14] For the sake of brevity, we shall actually drop z_0, i.e., suppose that *each real x in the universe* V *belongs to a set generic extension of* L: the general case does not differ. The aim is to find an antichain $A \subseteq 2^{<\omega_1}$ and a Δ_1^{HC} linearization $\langle \mathcal{N}; \preccurlyeq_T\rangle \to \langle A; \leqslant_{\mathrm{lex}}\rangle$.

Lemma 24. *Let $x \in \mathcal{N}$. There is an ordinal $\lambda < \omega_1$ such that the following is true for $u = \mathbf{h}(x) \upharpoonright \lambda$:*
(a) $\mathrm{L}_\lambda[u]$ *models* $\mathbf{ZFC}^-$ (that is, $\mathbf{ZFC}$ minus Power Sets); [15]
(b) *there is a set generic extension of* $\mathrm{L}_\lambda[u]$ *which contains a real x' satisfying* $\mathbf{h}(x') \upharpoonright \lambda = u$; [16]
(c) *if y, z are reals in a set generic extension M of* $\mathrm{L}_\lambda[u]$, *satisfying* $\mathbf{h}(y) \upharpoonright \lambda = \mathbf{h}(z) \upharpoonright \lambda$, *then* $y \approx_T z$;

[13] It is not clear to what extent *class* forcing universes can accomodate the reasoning below, in particular the proofs of Lemmas 24 and 27.
[14] The extensions can be different for different reals x. Moreover, the extensions can be Boolean valued extensions of $\mathrm{L}[z_0]$ rather than factual classes in the universe.
[15] By $\mathrm{L}_\lambda[u]$ we understand the result of the Gödel construction of length λ arranged so that only the restriction $u \upharpoonright \gamma$ is available at each step $\gamma < \lambda$. Note that $u \notin \mathrm{L}_\lambda[u]$. In this case, u can be used as an extra class parameter in the $\mathbf{ZFC}^-$ schemata.
[16] Why (b), a consequence of (d), is included will be explained below.

(d) *there is a set generic extension of* $\mathrm{L}_\lambda[u]$ *which contains a real* x' *satisfying* $x' \approx_T x$.

Proof. Consider case (b) of Theorem 2. There is a cardinal κ such that x belongs to a set generic extension of L_{κ^+}, where κ^+ is taken in the sense of L. Note that L_{κ^+} models $\mathbf{ZFC}^-$. There is an ordinal $\lambda < \omega_1$ such that $\langle \mathrm{L}_\lambda; x, \in \rangle$ is $\in$-isomorphic to a countable elementary submodel of $\langle \mathrm{L}_{\kappa^+}; x, \in \rangle$ – hence L_λ models $\mathbf{ZFC}^-$ – and x belongs to a set generic extension of L_λ. Note that $u = \mathbf{h}(x) \upharpoonright \lambda$ is a definable class in $\mathrm{L}_\lambda[x]$ by (3°) above. It is known (see Lemma 4.4 in Solovay [15] or Lemma 5 in [10], as particular cases) that x belongs to a set generic extension of $\mathrm{L}_\lambda[u]$. Take $x' = x$.

Consider case (a) of Theorem 2. As $[x]_{\approx_T}$ is Borel, there is, by Lemma 23, an ordinal $\nu < \omega_1$ such that $x \approx_T y \Leftrightarrow x \approx_T^\nu y$ and $\preccurlyeq_T^\nu$ is a PQO. Therefore (see the end of Section 5.3) there is an ordinal $\alpha < \omega_1$ satisfying $h_\alpha(x) = h_\alpha(y) \Leftrightarrow x \approx_T y$ (for any y). Let $\lambda_0 = \sum_{\gamma \leqslant \alpha} \rho_\gamma$, where $\rho_\gamma < \omega_1$ satisfies $\mathtt{ran}\, h_\gamma \subseteq 2^{\rho_\gamma}$. Then, for all y, $x \approx_T y \Leftrightarrow \mathbf{h}(x) \upharpoonright \lambda_0 = \mathbf{h}(y) \upharpoonright \lambda_0$. There is an ordinal λ, $\lambda_0 < \lambda < \omega_1$ such that $\langle \mathrm{L}_\lambda; x, \in \rangle$ is $\in$-isomorphic to a countable elementary submodel of $\langle \mathrm{L}_{\kappa^+}; x, \in \rangle$, and λ_0 is countable in L_λ. Prove (b) for this case. Let $u = \mathbf{h}(x) \upharpoonright \lambda$. We show that even $\mathrm{L}_\lambda[u]$ itself contains a real y satisfying $y \approx_T x$. Indeed, the statement "there is a real y such that $\mathbf{h}(y) \upharpoonright \lambda_0 = u \upharpoonright \lambda_0$" is Σ^1_1 provided λ_0 is countable. It follows that it is absolute for $\mathrm{L}_\lambda[u]$.

Prove (c). As (II^A) of Theorem 2 is essentially a Σ^1_2 sentence (by Lemma 22), false in V by the assumption (2°), it is false in $\mathrm{L}_{\kappa^+}[x]$, as well as in any set generic extension of $\mathrm{L}_{\kappa^+}[x]$. Therefore, by the already proved, in Section 5.3, part of Theorem 2, if reals y, z belong to a set generic extension of $\mathrm{L}_{\kappa^+}[\mathbf{h}(x) \upharpoonright \kappa^+]$, then $\mathbf{h}(y) \upharpoonright \kappa^+ = \mathbf{h}(z) \upharpoonright \kappa^+$ implies $y \approx_T z$. By the choice of λ, this remains true for λ instead of κ^+. □

The following lemma explains why (b) is considered.

Lemma 25. *In the presence of* (a)–(c), *condition* (d) *is equivalent to*

(d′) *for any real* x' *in a set generic extension of* $\mathrm{L}_\lambda[u]$, *if* $\mathbf{h}(x') \upharpoonright \lambda = u$ *then* $x' \approx_T x$.

Proof. In the presence of (b), (d′) ⇒ (d) is clear. Prove the opposite implication. Let x' witness that (d) is true, so that x' belongs to a P′-generic extension $\mathrm{L}_\lambda[u, G']$ of $\mathrm{L}_\lambda[u]$, and $x' \approx_T x$. Then $\mathbf{h}(x') \upharpoonright \lambda = u$. Consider another real x'' in a P″-generic extension $\mathrm{L}_\lambda[u, G'']$ of $\mathrm{L}_\lambda[u]$, satisfying $\mathbf{h}(x'') \upharpoonright \lambda = u$. Here $P', P'' \in \mathrm{L}_\lambda[u]$ are forcing notions. We have to prove that $x' \approx_T x''$. We can assume that the equalities $\mathbf{h}(x') \upharpoonright \lambda = u$ and $\mathbf{h}(x'') \upharpoonright \lambda = u$ are forced by resp. P' and P''. Consider a set $G \subseteq P'$, which is generic over both $\mathrm{L}_\lambda[u, G']$ and $\mathrm{L}_\lambda[u, G'']$. Let y be obtained from G as x' from G'. Then $\mathbf{h}(y) \upharpoonright \lambda = u$ and y, x' belong to $\mathrm{L}_\lambda[u, G', G]$, a set generic extension of $\mathrm{L}_\lambda[u]$, so that $y \approx_T x'$ by (c). Similarly $y \approx_T x''$. □

Remark 26. The possibility of getting a real y, satisfying $\mathbf{h}(y) \upharpoonright \lambda = u$ and "compatible" with each of x', x'' over $\mathrm{L}_\lambda[u]$, is the key point of the proof. To carry out the reasoning, we shall define λ_x below via a real x' in a generic rather than arbitrary extension of $\mathrm{L}_\lambda[\mathbf{h}(x)\restriction\lambda]$ (see (b) of Lemma 24) and, therefore, to suppose that every real belongs to a generic extension of L. (Unless the $\approx_T$-classes are Borel.) To eliminate this assumption, it would be sufficient to prove that

* $*$ for any real x there is a real y in a set generic extension of L which preserves "true" ω_1 and satisfies $\mathbf{h}(y)\restriction\omega_1=\mathbf{h}(x)\restriction\omega_1$.

Possibly, $0^\#$ can lead to a counterexample. It would be interesting to prove that the negation of $(*)$ implies the existence of $0^\#$. $\square$

For any $x\in\mathcal{N}$, let λ_x be the least ordinal $\lambda<\omega_1$ satisfying the requirements of the lemma. We put $\mathfrak{h}(x)=\mathbf{h}(x)\restriction\lambda_x$.

Corollary 27. *We have*: $\mathfrak{h}(x)=\mathfrak{h}(y)$ *iff* $x\approx_T y$.

Proof. Suppose that $\mathfrak{h}(x)=\mathfrak{h}(y)=u\in 2^\lambda$ (so that $\lambda=\lambda_x=\lambda_y$) and prove $x\approx_T y$ (the non-trivial direction). Let y' witness that $\mathfrak{h}(y)=u$, in other words, y' belongs to a set generic extension of $\mathrm{L}_\lambda[u]$, and $y\approx_T y'$. Then $\mathbf{h}(y')=\mathbf{h}(y)=u$, so that, by Lemma 25, $y'\approx_T x$, as required. $\square$

Proposition 28. *The map* $\mathfrak{h}$ *and the set* $R=\mathtt{ran}\,\mathfrak{h}$ *are* Δ_1^{HC}.

Proof. First consider the map $x\mapsto\lambda_x$. Note that each of conditions (a)–(c) of Lemma 24 can be forced within $\mathrm{L}_\lambda[u]$, therefore (a)–(c) are Δ_1^{HC}, while (d) and (d$'$) are resp. Σ_1^{HC} and Π_1^{HC}. It follows that $x\mapsto\lambda_x$ is a Δ_1^{HC} map, because the function $x,\lambda\mapsto\mathbf{h}(x)\restriction\lambda$ is Δ_1^{HC} via the formula Φ of (3°). Therefore $\mathbf{h}$ is Δ_1^{HC} as well. Consider the set R.

We assert that $u\in 2^\lambda$ belongs to R iff: $\mathrm{L}_\lambda[u]$ satisfies (a)–(c) of Lemma 24, and there is a set generic extension of $\mathrm{L}_\lambda[u]$, where

(5°) there exists a real x' such that $u=\mathbf{h}(x')\restriction\lambda$ and, for any ordinal $\lambda'<\lambda$, if $\mathrm{L}_{\lambda'}[u\restriction\lambda']$ satisfies requirements (a)–(c) of Lemma 24 then there is a forcing notion $P\in\mathrm{L}_{\lambda'}[u\restriction\lambda']$ which forces (in $\mathrm{L}_\lambda[u]$) a real y in $\mathrm{L}_{\lambda'}[u\restriction\lambda',G]$ satisfying $\mathbf{h}(y)\restriction\lambda'=u\restriction\lambda'$ but $y\not\approx_T x'$.

This obviously implies that R is Δ_1^{HC}, so it remains to prove the assertion.

Let $u=\mathfrak{h}(x)=\mathbf{h}(x)\restriction\lambda_x$. By definition there is a real $x'\approx_T x$ in a set generic extension of $\mathrm{L}_\lambda[u]$ (where $\lambda=\lambda_x$). Then $\mathbf{h}(x')\restriction\lambda=u$. Consider any $\lambda'<\lambda$ such that (a)–(c) hold for $u'=u\restriction\lambda'$. By (b) assumed, there is a forcing notion $P\in\mathrm{L}_{\lambda'}[u']$ which forces (in $\mathrm{L}_{\lambda'}[u']$) a real y such that $\mathbf{h}(y)\restriction\lambda'=u'$. Since $\lambda'<\lambda=\lambda_x$, any such y satisfies $y\not\approx_T x'$. It follows that P provides (5°) for λ'.

Conversely, suppose that (5°) holds in a set generic extension of $\mathrm{L}_\lambda[u]$. Then (d) is also here (take x' as x). It remains to prove the minimality. Consider any $\lambda'<\lambda$ such that (a)–(c) hold for $u'=u\restriction\lambda'$. Then, by (5°), there is a real z in a set generic extension of $\mathrm{L}_{\lambda'}[u']$ such that $\mathbf{h}(z)\restriction\lambda'=u'$ but $z\not\approx_T x'$. It follows that (d$'$) does not hold for x' and λ', so that $\lambda=\lambda_x$ is really minimal. $\square$

5.5. Reduction to an antichain

Thus we have defined a Δ_1^{HC} map $\mathfrak{h}:\mathcal{N}\to 2^{<\omega_1}$, such that, for any real x, we have $x\approx_T y \Leftrightarrow \mathfrak{h}(x)=\mathfrak{h}(y)$ and $\mathfrak{h}(x)=\mathbf{h}(x)\restriction\lambda_x$ for some $\lambda_x<\omega_1$. However the range $\mathtt{ran}\,\mathfrak{h}$ may not be an antichain in $2^{<\omega_1}$. To fix this problem, we define a new Δ_1^{HC} map $\mathfrak{h}'$: $\mathcal{N}\to 3^{<\omega_1}$, as follows.

For any x, we put $\mathtt{dom}\,\mathfrak{h}'(x)=\lambda_x+1$, $\mathfrak{h}'(x)(\gamma+1)=\mathfrak{h}(x)(\gamma)$ for all $\gamma<\lambda_x$, and $\mathfrak{h}'(x)(\lambda_x+1)=1$. The values of $\mathfrak{h}'(x)(\gamma)$ for limit ordinals γ need more care. Let $\gamma<\lambda_x$ be a limit ordinal. Then:

(i) if there is no real y such that $\mathfrak{h}(x)\restriction\gamma=\mathfrak{h}(y)$ then $\mathfrak{h}'(x)(\gamma)=1$;
(ii) if $\mathfrak{h}(x)\restriction\gamma=\mathfrak{h}(y)$ for a real y, and $x\preccurlyeq_T y$, then set $\mathfrak{h}'(x)(\gamma)=0$;
(iii) if $\mathfrak{h}(x)\restriction\gamma=\mathfrak{h}(y)$ for a real y, and $x\not\preccurlyeq_T y$, then set $\mathfrak{h}'(x)(\gamma)=2$.

There is no controversy here because $\mathfrak{h}(y')=\mathfrak{h}(y'')$ implies $y'\approx_T y''$.

Lemma 29. *$A'=\mathtt{ran}\,\mathfrak{h}'$ is a Δ_1^{HC} antichain in $3^{<\omega_1}$, and $\mathfrak{h}'$ is a Δ_1^{HC} linearization $\langle\mathcal{N};\preccurlyeq_T\rangle\to\langle A';\leqslant_{\mathrm{lex}}\rangle$.*

Proof. By definition, $\mathfrak{h}(x)\neq\mathfrak{h}(y)$ implies $\mathfrak{h}'(x)\neq\mathfrak{h}'(y)$ and, moreover, that $\mathfrak{h}'(x)$ and $\mathfrak{h}'(y)$ are $\subseteq$-incomparable in $2^{<\omega_1}$, so that $A'=\mathtt{ran}\,\mathfrak{h}'$ is an antichain in $2^{<\omega_1}$, and $x\approx_T y\Leftrightarrow\mathfrak{h}'(x)=\mathfrak{h}'(y)$. It remains to show that $\mathfrak{h}'(x)<_{\mathrm{lex}}\mathfrak{h}'(y)$ implies $y\not\preccurlyeq_T x$. Let γ be the least ordinal such that $\mathfrak{h}'(x)(\gamma)<\mathfrak{h}'(y)(\gamma)$. If $\gamma=\xi+1$ then obviously $\mathfrak{h}(x)\restriction\xi=\mathfrak{h}(y)\restriction\xi$ but $\mathfrak{h}(x)(\xi)<\mathfrak{h}(y)(\xi)$, so that $\mathbf{h}(x)<_{\mathrm{lex}}\mathbf{h}(y)$ and $y\not\preccurlyeq_T x$.

Thus let γ be a limit ordinal. Then $\mathfrak{h}(x)\restriction\gamma=\mathfrak{h}(y)\restriction\gamma$. If actually $\gamma=\lambda_y$, so that $\mathfrak{h}(x)\restriction\gamma=\mathfrak{h}(y)$, then $\mathfrak{h}'(y)(\gamma)=1$ while $\mathfrak{h}'(x)(\gamma)$ is computed by (ii) or (iii), therefore, in fact, by (ii) because $\mathfrak{h}'(x)(\gamma)<\mathfrak{h}'(y)(\gamma)$. In other words, $x\preccurlyeq_T y$. However $x\approx_T y$ is impossible since $\mathfrak{h}'(x)\neq\mathfrak{h}'(y)$. Thus $y\not\preccurlyeq_T x$. The case $\gamma=\lambda_x$ is considered similarly.

It remains to handle the case when $\gamma<\mathtt{min}\{\lambda_x,\ \lambda_y\}$. There must be a real z such that $\mathfrak{h}(x)\restriction\gamma=\mathfrak{h}(y)\restriction\gamma=\mathfrak{h}(z)$, because otherwise we would have $\mathfrak{h}'(x)(\gamma)=\mathfrak{h}'(y)(\gamma)=1$ by (i). Now $\mathfrak{h}'(x)(\gamma)$ has to be computed by (ii), so that $x\preccurlyeq_T z$, while $\mathfrak{h}'(y)(\gamma)$ must be computed by (iii), so that $y\not\preccurlyeq_T z$, which is incompatible with $y\preccurlyeq_T x$, as required.

Note that the binary relation: "$u=\mathfrak{h}(x)$ for a real $x\preccurlyeq_T y$" is Δ_1^{HC} by Proposition 28: the formula "$u\in R$ and $\forall x(u=\mathfrak{h}(x)\Rightarrow x\preccurlyeq_T y)$" gives a (less trivial) Π_1 expression. This observation shows that $\mathfrak{h}'$ is Δ_1^{HC}. To prove that $R'=\mathtt{ran}\,\mathfrak{h}'$ is Δ_1^{HC}, note that each $u'\in R'$ is obtained from a unique $u\in R$ by the procedure described above (to get u throw away all limit terms of u'), and the connection between u and u' is Δ_1^{HC}. $\square$

The lemma ends the proof of Theorem 2. (Improvement to $2^{<\omega_1}$ is easy.)

$\square$ (Theorem 2)

5.6. More on the nature of invariants

Following Hjorth and Kechris [7], let us call *invariants* the values which functions like $\mathfrak{h}$ or $\mathfrak{h}'$ take. We know that both $R=\mathtt{ran}\,\mathfrak{h}$ and $R'=\mathtt{ran}\,\mathfrak{h}'$ are Δ_1^{HC} sets, assuming, as above, that $\preccurlyeq_T$ is a lightface Σ^1_1 relation. (If actually $\preccurlyeq_T$ is $\mathbf{\Sigma}^1_1$, with some $p \in \mathcal{N}$ as the only real parameter, then R and R' become $\mathbf{\Delta}^{\mathrm{HC}}_1$, with the same parameter.) However, there exist some extra properties, which make the system of invariants somewhat better than "just" Δ^{HC}_1 or, generally, $\mathbf{\Delta}^{\mathrm{HC}}_1$.

Indeed, it follows from our proof of Proposition 28 that R is Δ^{HC}_1 just because there is a formula φ (which expresses certain forcing phenomena) such that $u \in 2^\lambda$ belongs to R iff φ is true in $\mathrm{L}_\lambda[u]$. It follows that, for any particular $\lambda < \omega_1$, the set $R_\lambda = R \cap 2^\lambda$ is a Borel subset of 2^λ while the partial map

$$F_\lambda = \{\langle x, u\rangle : u = \mathfrak{h}(x) \in R_\lambda\}$$

is $\mathbf{\Sigma}^1_1$. (The existential quantifier expresses the existence of a real x' in a set generic extension of $\mathrm{L}_\lambda[u]$ such that $x' \approx_T x$ and $\mathbf{h}(x') \restriction \lambda = u$.) Moreover, the *Becker sets*

$$B_\delta(i) = \{x : \delta < \mathtt{dom}\,\mathfrak{h}(x) \wedge \mathfrak{h}(x)(\delta) = i\}, \quad \text{where } \delta < \omega_1 \text{ and } i = 0, 1$$

are $\mathbf{\Pi}^1_1$. (Indeed, $x \in B_\delta(i)$ iff $\mathfrak{h}(x) \notin \bigcup_{\lambda \le \delta} R_\lambda$, which amounts to $\mathbf{\Pi}^1_1$ by the above, and $\mathbf{h}(x)(\delta) = i$, which is Borel.) It would be nice to have the sets $B_\delta(i)$ Borel: we refer to question 1 in [7, Section 7]. Finally, the map $\mathfrak{h} = \bigcup_{\lambda < \omega_1} F_\lambda$ becomes an intersection of a $\mathbf{\Sigma}^1_1$ set and a $\mathbf{\Pi}^1_1$ set if we code ordinals by reals as usual.

As for the more interesting reduction $\mathfrak{h}'$, a careful analysis of the construction in Section 5.5 shows that the related sets R'_λ and maps F'_λ remain resp. Borel and analytic, $\mathfrak{h}'$ itself remains an intersection of $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_1$, but the Becker sets B'_δ are not, in general, co-analytic any more. [17] Note that if all B'_δ are co-analytic then, as the range of $\mathfrak{h}'$ is an antichain, all $\approx_T$-classes are co-analytic, too, hence Borel, which, generally, is not the case.

6. Linearization in the Solovay model

The basic universe, in this section, will be a Solovay model: in other words, we shall assume that $\Omega = \omega_1$ is an inaccessible cardinal in L, the constructible universe, while the whole universe V is a generic extension of L via the Levy–Solovay forcing (which collapses to ω all L-cardinals less than Ω). Relevant properties of the Solovay model are summarized in [8].

Let $\preccurlyeq$ be the PQO which we deal with in Theorem 6. It is known that in the Solovay model every ROD set of reals is ω_1-Souslin (hence ω_1-co-Souslin as well), moreover, for any real parameter p, every $\mathrm{OD}(p)$ set of reals is ω_1-Souslin via a tree which belongs to $\mathrm{L}[p]$. It follows that there exist trees $S, T \subseteq (\omega \times \omega \times \Omega)^{<\omega}$ such that $\preccurlyeq = \preccurlyeq_S = \preccurlyeq_T$, where $\preccurlyeq_T = \mathfrak{p}[T]$ while $\preccurlyeq_S = \complement\mathfrak{p}[S]$, and T, S belong to $\mathrm{L}[p]$ for one and the same $p \in \mathcal{N}$. Let us fix T and S.

We want to apply Theorem 5 (for $\lambda = \Omega$). But this requires:

[17] We thank the referee for pointing out a miscalculation here in the original version.

Lemma 30. $\complement\mathfrak{p}[S]$ *is a* PQO *in Cohen generic extensions of the universe.*

Proof. It is known that the universe is a Solovay-like extension of $\mathrm{L}[p]$. Therefore, the property "$\complement\mathfrak{p}[S]$ is a PQO" is Solovay-forced over $\mathrm{L}[p]$ by any forcing condition. (Recall that the forcing which generates Solovay's model is homogeneous.) However, Cohen generic extensions of the Solovay model are Solovay's models themselves. This easily implies the result. □

Thus Theorem 5 is applicable, so that at least one of conditions (I^λ), (II) in Section 1.3 is satisfied. Now, to complete the proof of Theorem 6, it is sufficient to verify that, in the Solovay model, first, $(\mathrm{I}^{\mathrm{s}})$ and (II) are incompatible, and second, (I^λ) of Section 1.3 implies $(\mathrm{I}^{\mathrm{s}})$ of Theorem 6.

6.1. Incompatibility

Suppose on the contrary that both $(\mathrm{I}^{\mathrm{s}})$ and (II) hold in the Solovay model. Then the composition of the maps involved is a ROD h.o.p map $f:\langle 2^\omega;\preccurlyeq\rangle\to\langle A;\leqslant_{\mathrm{lex}}\rangle$ satisfying $a \not\mathsf{E}_0\, b \Rightarrow f(a)\neq_{\mathrm{lex}} f(b)$. Assume that f is $\mathrm{OD}(p)$, where $p\in\mathcal{N}$.

Then the f-image $\varphi(a)=\{f(b): b\,\mathsf{E}_0\,a\}$ of every E_0-class $[a]_{\mathsf{E}_0}$ is a subset of the antichain A, ordered similarly to a subset of $\mathbb{Z}$ (the integers) and $\varphi(a)\cap\varphi(b)=\emptyset$ whenever $a\not\mathsf{E}_0\, b$. As there is an OD assignment of an element $a(X)\in X$ to any set $X\subseteq 2^{<\omega_1}$, $\leqslant_{\mathrm{lex}}$-ordered similarly to a subset of $\mathbb{Z}$ (see [11]), we have an $\mathrm{OD}(p)$ map $\vartheta:2^\omega\to A$ such that $\vartheta(a)\in\varphi(a)$ for all a, and $\vartheta(a)=\vartheta(b)$ whenever $a\,\mathsf{E}_0\,b$. For any $x\in\Theta=\{\vartheta(a): a\in 2^\omega\}$, the pre-image $\varphi^{-1}(x)\subseteq 2^\omega$ is a countable $\mathrm{OD}(p,x)$ set, essentially a E_0-class, hence, by the known properties of the Solovay model, $\varphi^{-1}(x)\subseteq\mathrm{L}[p,x]$. Using the Gödel well ordering of $\mathrm{L}[p,x]$, we obtain an $\mathrm{OD}(p)$ map $\psi:\Theta\to 2^\omega$, such that $\psi(x)\in\varphi^{-1}(x)$ for all x. The full image of ψ is clearly an $\mathrm{OD}(p)$ set $C\subseteq 2^\omega$, having exactly one point in common with every E_0-class. Thus C is a non-measurable ROD set, which is impossible in the Solovay model.

6.2. Reduction to short sequences

We continue to argue in the Solovay model.

Say that a set X is $\mathrm{OD}(p)$-*continual* iff it is the full image of $\mathcal{N}$ via an $\mathrm{OD}(p)$ function. The following theorem clearly suffices to derive $(\mathrm{I}^{\mathrm{s}})$ of Theorem 6 from (I^λ) of Section 1.3 for $\lambda=\Omega$.

Theorem 31. *Assume that* $p\in\mathcal{N}$ *and* $\vartheta<\Omega^+$. *Then any* $\mathrm{OD}(p)$-*continual set* $X\subseteq 2^\vartheta$ *is* $<_{\mathrm{lex}}$-*order-isomorphic to an antichain* $A(X)\subseteq 2^{<\Omega}$ *via an* $\mathrm{OD}(p)$ *isomorphism* $i(X)$, *so that the map, sending* X *to* $A(X)$ *and* $i(X)$, *is* $\mathrm{OD}(p)$.

Proof. We argue by induction on ϑ. The only essential part is the induction step for cofinality Ω. Thus let $\vartheta=\bigcup_{\alpha<\Omega}\vartheta_\alpha$, for an increasing $\mathrm{OD}(p)$ sequence of ordinals

ϑ_α. Let $I_\alpha = [\vartheta_\alpha, \vartheta_{\alpha+1})$ Then, by the induction hypothesis, for any $\alpha < \Omega$ the set $X_\alpha = \{S \upharpoonright I_\alpha : S \in X\} \subseteq 2^{I_\alpha}$ is $<_{\text{lex}}$-order-isomorphic to an antichain $A_\alpha \subseteq 2^{<\Omega}$ via an OD(p) isomorphism i_α, and the map, which sends α to A_α and i_α, is OD(p). It follows that the map which sends each $S \in X$ to the concatenation of all sequences $i_\alpha(x \upharpoonright I_\alpha)$, is an OD($p$) $<_{\text{lex}}$-order-isomorphism X onto an antichain in 2^Ω. Therefore, it suffices to prove the theorem for $\vartheta = \Omega$. Thus consider an OD(p)-continual set $X \subseteq 2^\Omega$. The construction we apply is just another version of the method used in Section 5.5 in order to obtain $\mathfrak{h}'$ from $\mathfrak{h}$.

First of all, note that each $S \in X$ is ROD. Lemma 7 in [8] shows that, in this case, we have $S \in \mathrm{L}[S \upharpoonright \eta]$ for an ordinal $\eta < \Omega$. Let $\eta(S)$ be the least such an ordinal, and $f(S) = S \upharpoonright \eta(S)$, so that $f(S)$ is a countable initial segment of S and $S \in \mathrm{L}[f(S)]$. Note that f is still OD(p).

Consider the set $R = \mathtt{ran}\, f \subseteq 2^{<\Omega}$. We can assume that every sequence $r \in R$ has a limit ordinal as its length. Then $R = \bigcup_{\gamma < \Omega} R_\gamma$, where $R_\gamma = R \cap 2^{\omega\gamma}$. (As usual, $\omega\gamma$ is the γth limit ordinal.) For $r \in R_\gamma$, let $\gamma = \gamma_r$.

Lemma 32. *For any $r \in R = \mathtt{ran}\, f$, the set $X_r = \{S \in X\colon f(S) = r\}$ belongs to* L[r] *and is of cardinality $\leqslant \Omega$ in* L[r].

Proof. Let $X = \{F(a) : a \in \mathcal{N}\}$, where F is an OD(p) function. By definition, $X_r \subseteq \mathrm{L}[r]$. It follows that, for each $S \in X_r$, the set $D_S = F^{-1}(S) \subseteq \mathcal{N}$ is an OD(r) set of reals. As r itself is ROD, it follows from the known properties of the Solovay model (where we argue) that there exist only Ω-many non-empty sets of the form D_S, $S \in X_r$, hence only Ω-many different elements in X_r. $\square$

Fix an enumeration $X_r = \{S_r(\alpha)\colon \gamma_r \leqslant \alpha < \Omega\}$ for all $r \in R$. We can assume that the map $\alpha, r \mapsto S_r(\alpha)$ is OD(p). For all $r \in R$ and $\gamma_r \leqslant \alpha < \Omega$, we define a shorter sequence, $s_r(\alpha) \in 3^{\omega\alpha+1}$, as follows:

(i) $s_r(\alpha)(\xi + 1) = S_r(\alpha)(\xi)$ for any $\xi < \omega\alpha$.

(ii) $s_r(\alpha)(\omega\alpha) = 1$.

(iii) Let $\delta < \alpha$. If $S_r(\alpha) \upharpoonright \omega\delta = S_q(\delta) \upharpoonright \omega\delta$ for some $q \in R$ (equal to or different from r) then $s_r(\alpha)(\omega\delta) = 0$ whenever $S_r(\alpha) <_{\text{lex}} S_q(\delta)$, and $s_r(\alpha)(\omega\delta) = 2$ whenever $S_q(\delta) <_{\text{lex}} S_r(\alpha)$.

(iv) Otherwise (i.e., if there is no such q), $s_r(\alpha)(\omega\delta) = 1$.

To demonstrate that (iii) is consistent, we show that $S_{r'}(\delta) \upharpoonright \omega\delta = S_{r''}(\delta) \upharpoonright \omega\delta$ implies $r' = r''$. Indeed, as by definition $r' \subset S_{r'}(\delta)$ and $r'' \subset S_{r''}(\delta)$, r' and r'' must be $\subseteq$-comparable: let, say, $r' \subseteq r''$. Now, by definition, $S_{r''}(\delta) \in \mathrm{L}[r'']$, therefore $\in \mathrm{L}[S_{r'}(\delta)]$ because $r'' \subseteq S_{r''}(\delta) \upharpoonright \omega\delta = S_{r'}(\delta) \upharpoonright \omega\delta$, finally $\in \mathrm{L}[r']$, which shows that $r' = r''$ as $S_{r''}(\delta) \in X_{r''}$.

We are going to prove that the map $(*)$ $S_r(\alpha) \mapsto s_r(\alpha)$ is a $<_{\text{lex}}$-order isomorphism, so that $S_q(\beta) <_{\text{lex}} S_r(\alpha)$ implies $s_q(\beta) <_{\text{lex}} s_r(\alpha)$.

We first observe that $s_q(\beta)$ and $s_r(\alpha)$ are $\subseteq$-incomparable. Indeed assume that $\beta<\alpha$. If $S_r(\alpha)\restriction\omega\beta\neq S_q(\beta)\restriction\omega\beta$ then clearly $s_q(\beta)\not\subseteq s_r(\alpha)$ by (i). If $S_r(\alpha)\restriction\omega\beta=S_q(\beta)\restriction\omega\beta$ then $s_r(\alpha)(\omega\beta)=0$ or 2 by (iii) while $s_q(\beta)(\omega\beta)=1$ by (ii). Thus all $s_r(\alpha)$ are mutually $\subseteq$-incomparable, so that it suffices to show that conversely $s_q(\beta)<_{\rm lex} s_r(\alpha)$ implies $S_q(\beta)<_{\rm lex} S_r(\alpha)$. Let v be the least ordinal such that $s_q(\beta)(v)<s_r(\alpha)(v)$; then $s_r(\alpha)\restriction v=s_q(\beta)\restriction v$ and $v\leqslant\mathtt{min}\{\omega\alpha,\omega\beta\}$.

The case when $v=\xi+1$ is clear: then by definition $S_r(\alpha)\restriction\xi=S_q(\beta)\restriction\xi$ while $S_q(\beta)(\xi)<S_r(\alpha)(\xi)$, so let us suppose that $v=\omega\delta$, where $\delta\leqslant\mathtt{min}\{\alpha,\beta\}$. Then obviously $S_r(\alpha)\restriction\omega\delta=S_q(\beta)\restriction\omega\delta$. Assume that one of the ordinals α,β is equal to δ, say, $\beta=\delta$. Then $s_q(\beta)(\omega\delta)=1$ while $s_r(\alpha)(\omega\delta)$ is computed by (iii). Now, as $s_q(\beta)(\omega\delta)<s_r(\alpha)(\omega\delta)$, we conclude that $s_r(\alpha)(\omega\delta)=2$, hence $S_q(\beta)<_{\rm lex} S_r(\alpha)$, as required. Assume now that $\delta<\mathtt{min}\{\alpha,\beta\}$. Then easily α and β appear in one and the same class (iii) or (iv) with respect to the δ. However this cannot be (iv) because $s_q(\beta)(\omega\delta)\neq s_r(\alpha)(\omega\delta)$. Hence we are in (iii), so that, for some (unique) $w\in R$, $0=S_q(\beta)<_{\rm lex} S_w(\delta)<_{\rm lex} S_r(\alpha)=2$, as required.

This ends the proof of the theorem, except for the fact that the sequences $s_r(\alpha)$ belong to $3^{<\Omega}$, but improvement to $2^{<\Omega}$ is easy. □ (Theorem 31)

□ (Theorem 6)

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