Annals of Pure and Applied Logic 169 (2018) 851–871

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

Definable E_0 classes at arbitrary projective levels ☆

Vladimir Kanovei [a,b,*,1], Vassily Lyubetsky [a,c,2]

[a] *Institute for Information Transmission Problems (Kharkevich Institute) of Russian Academy of Sciences (IITP), Bolshoy Karetny per. 19, build. 1, Moscow 127051, Russia*
[b] *Russian University of Transport (MIIT), Novosuschevsky, 22, Moscow 127994, Russia*
[c] *Faculty of Mechanics and Mathematics, Moscow State University, Vorobyovy gory, 1, Main Building of MSU, 119991 Moscow, Russia*

ARTICLE INFO

Article history:
Received 9 May 2017
Accepted 18 April 2018
Available online 24 April 2018

MSC:
03E15
03E35

Keywords:
Definable Vitali classes
Jensen's forcing
Silver's forcing
Π^1_1 sets
Σ^1_1 sets

ABSTRACT

Using a modification of the invariant Jensen forcing of [11], we define a model of **ZFC**, in which, for a given $n \geq 3$, there exists a lightface Π^1_n-set of reals, which is a E_0-equivalence class, hence a countable set, and which does not contain any OD element, while every non-empty countable Σ^1_n-set of reals is constructible, hence contains only OD reals.

1. Introduction

Problems related to definability of mathematical objects, were among focal points of the famous discussion on mathematical foundations in the beginning of XIX C. In particular, Baire, Borel, Hadamard, and Lebesgue, participants of the exchange of letters published in [5], in spite of essential disagreement between them on questions related to mathematical foundations, generally agreed that the proof of existence of an element in a given set, and a direct definition (or effective construction) of such an element — are different

☆ This document is a collaborative effort.

* Corresponding author at: Institute for Information Transmission Problems (Kharkevich Institute) of Russian Academy of Sciences (IITP), Bolshoy Karetny per. 19, build. 1, Moscow 127051, Russia.
E-mail addresses: kanovei@googlemail.com (V. Kanovei), lyubetsk@iitp.ru (V. Lyubetsky).

1 Partial support of RFBR grant 17-01-00705 acknowledged.
2 Partial support of the Russian Science Foundation grant 14-50-00150 acknowledged.

https://doi.org/10.1016/j.apal.2018.04.006
0168-0072/

mathematical results, of which the second does not follow from the first. In particular, Lebesgue in his contribution to [5] pointed out the difficulties in the problem of effective choice, that is, choice of definable element in a definable non-empty set.[3]

Studies in modern set theory demonstrated that effective choice is not always possible. In particular, it is true in many well-known models (including the very first Cohen models), that the set $X = \mathbb{R} \smallsetminus \mathbf{L}$ of all Gödel-nonconstructible reals is not empty, but contains no definable elements.

One may note that if the set X is non-empty, then it has to be rather large, that is, surely of cardinality $\mathfrak{c}$, if measurable then of full measure, *etc.* Is there such an example among *small*, *e.g.* countable sets? This problem was discussed at *Mathoverflow*[4] and *Foundations of mathematics* (FOM).[5]

The problem was solved in [10] (to appear in [16]). Namely, let $\mathbf{L}[\langle a_n \rangle_{n<\omega}]$ be a $\boldsymbol{J}^\omega$-generic extension of $\mathbf{L}$, where $\boldsymbol{J}^\omega$ is the countable power (with finite support) of *Jensen's minimal forcing* $\boldsymbol{J}$ [9].[6] The key property of $\boldsymbol{J}$ is that it adds a nonconstructible generic Δ^1_3 real $a \in 2^\omega$, in fact $\{a\}$ is a Π^1_2 singleton. Accordingly $\boldsymbol{J}^\omega$ adds a sequence $\langle a_n \rangle_{n<\omega}$ of $\boldsymbol{J}$-generic reals real $a_n \in 2^\omega$ to $\mathbf{L}$, and as shown in [10,16] there is no other $\boldsymbol{J}$-generic reals in $\mathbf{L}[\langle a_n \rangle_{n<\omega}]$ except for the reals a_n. Furthermore, since "being a $\boldsymbol{J}$-generic real" is a Π^1_2 relation, it is true in $\mathbf{L}[\langle a_n \rangle_{n<\omega}]$ that $A = \{a_n : n < \omega\}$ is a countable (infinite) lightface Π^1_2 set of reals without OD (ordinal-definable) elements.

Remark 1.1. Π^1_2 is best possible definability type for such a set since every Σ^1_2 set of reals is definitely constructible and hence consists of OD elements. □

Using an uncountable product of forcing notions similar to $\boldsymbol{J}^\omega$, we defined in [12] a model in which there is a "planar" Π^1_2 set with countable vertical cross-sections, which cannot be uniformized by any real-ordinal definable (ROD) set.

For a more detailed analysis of the problem, note that the elements a_n of the set A, adjoined by the forcing $\boldsymbol{J}^\omega$, are connected to each other only by the common property of their $\boldsymbol{J}$-genericity. Does there exist a similar countable set with a more definite mathematical structure?

This question was answered in [11] by a model in which there is an equivalence class of the equivalence relation E_0[7] (a E_0-*class*, for brevity), which is a (countable) lightface Π^1_2 set in 2^ω, and does not contain OD elements. This model makes use of a forcing notion $\boldsymbol{J}^{\mathrm{inv}}$, similar to Jensen's forcing $\boldsymbol{J}$, but different from $\boldsymbol{J}$. In particular, it consists of Silver trees (rather than perfect trees of general form, as $\boldsymbol{J}$ does) and is invariant under finite transformations. Thus it can be called *an invariant Jensen forcing*. Due to the invariance, $\boldsymbol{J}^{\mathrm{inv}}$ adjoins a E_0-equivalence class of $\boldsymbol{J}^{\mathrm{inv}}$-generic reals rather than a single real, and this class turns out to be a Π^1_2 set without OD elements. And again Π^1_2 is the lowest possible type in which such a set can be found, by Remark 1.1.

A forcing similar to $\boldsymbol{J}^{\mathrm{inv}}$ was also used in [3] to define a model with a Π^1_2 Groszek–Laver pair of E_0-classes.

Our main theorem extends this research line.

Theorem 1.2. *Let $\mathfrak{n} \geq 3$. There is a generic extension $\mathbf{L}[a]$ of $\mathbf{L}$, by a real $a \in 2^\omega$, such that the following is true in $\mathbf{L}[a]$:*

(i) *$a \notin$ OD and a is minimal over $\mathbf{L}$ — hence each OD real belongs to $\mathbf{L}$;*
(ii) *the E_0-class $[a]_{\mathsf{E}_0}$ is a (countable) lightface $\Pi^1_{\mathfrak{n}}$ set — which by (i) does not contain OD elements;*

[3] Ainsi je vois déjà une difficulté dans ceci "dans un M' déterminé je puis choisir un m' déterminé", in the French original. Thus I already see a difficulty with the assertion that "in a determinate M' I can choose a determinate m'", in the translation.

[4] A question about ordinal definable real numbers. *Mathoverflow*, March 09, 2010. http://mathoverflow.net/questions/17608.

[5] Ali Enayat. Ordinal definable numbers. FOM Jul 23, 2010. http://cs.nyu.edu/pipermail/fom/2010-July/014944.html.

[6] See also 28A in [8] on this forcing. We acknowledge that the idea to use the countable power $\boldsymbol{J}^\omega$ of Jensen's forcing $\boldsymbol{J}$ to define such a model belongs to Ali Enayat [2].

[7] Recall that E_0 is defined on the Cantor space 2^ω so that $x \,\mathsf{E}_0\, y$ iff the set $\{n : x(n) \neq y(n)\}$ is finite. E_0-equivalence classes are countable sets in 2^ω, of course.

(iii) *every countable* $\Sigma^1_{\mathfrak{n}}$ *set belongs to* $\mathbf{L}$*, hence consists of* OD *elements.*

Thus the theorem asserts the existence of a model of **ZFC**, in which, for a given index $\mathfrak{n} \geq 3$, there is a lightface $\Pi^1_{\mathfrak{n}}$ E_0-class (therefore, a countable OD set) in 2^ω containing only non-OD elements, and in the same time every countable set of the dual definability type $\Sigma^1_{\mathfrak{n}}$ belongs to $\mathbf{L}$. The above mentioned result of [11] corresponds to $\mathfrak{n} = 2$ in this theorem by Remark 1.1.

2. Connections to the Vitali equivalence relation

The relation E_0 is tightly connected with the Vitali equivalence relation $\mathsf{VIT} = \mathbb{R}/\mathbb{Q}$ on the true real line $\mathbb{R}$.[8] In particular, there is a lightface Δ^1_1 (in fact of a very low class Δ^0_n) injection $\vartheta : \mathbb{R} \to 2^\omega$ which *reduces* VIT to E_0 in the sense that the equivalence $x\mathsf{VIT}y \iff \vartheta(x)\mathsf{E}_0\vartheta(y)$ holds for all $x, y \in \mathbb{R}$. (See Mycielski and Osofsky [19] for an explicit construction of ϑ, along with an inverse reduction of E_0 to VIT.) It follows that Theorem 1.2 is true with (ii) for $[a]_{\mathsf{VIT}}$ as well, since if $C \subseteq 2^\omega$ is a $\Pi^1_{\mathfrak{n}}$ E_0-class not containing OD elements, then the preimage $C' = \{x \in \mathbb{R} : \vartheta(x) \in C\}$ is a $\Pi^1_{\mathfrak{n}}$ Vitali class not containing OD elements.

The interest in Vitali classes in this context is inspired by the observation that they can be viewed as the most elementary countable sets in $\mathbb{R}$ which do not allow immediate effective choice of an element. Indeed if a set $X \subseteq \mathbb{R}$ contains at least one isolated, or even one-sided isolated point, then one of such points can be chosen effectively. However any non-empty set $X \subseteq \mathbb{R}$ without one-sided isolated points is just an everywhere dense set (not counting closed segments of the complementary set). Yet the Vitali classes, that is, shifts of the rationals $\mathbb{Q}$, are exactly the most simple and typical countable dense sets in $\mathbb{R}$.

Historically, the Vitali relation and its equivalence classes have deep roots in descriptive set theory. For instance Sierpinski [20, c. 147] and Luzin [18, Section 64] observed that the quotient set $\mathbb{R}/\mathbb{Q}$ of all Vitali classes has the property that (*) it cannot be mapped into $\mathbb{R}$ by an injective Borel map. On the other hand, as established in [15], in models of **ZF** (without the axiom of choice) the Hartogs number of the set $\mathbb{R}/\mathbb{Q}$ (the least cardinal which cannot be injectively mapped into $\mathbb{R}/\mathbb{Q}$) can be greater than the continuum. The relations E_0 and VIT play a key role in modern studies of Borel equivalence relations, being the least ones, in the sense of the Borel reducibility [6], among those satisfying (*).

3. Structure of the proof

The proof of Theorem 1.2 is organized as follows. Basic notions, related to Silver trees in the set $2^{<\omega}$ of all finite dyadic strings, are introduced in sections 4–7. Every set $\mathbf{P}$ of Silver trees T, closed under restriction to a given string $s \in T$, and under the natural action $s \cdot T$ by $s \in 2^{<\omega}$, is considered as *a forcing by Silver trees*, a *ST-forcing*, in brief. Every ST-forcing adjoins a $\mathbf{P}$-generic real $a \in 2^\omega$.

Arguing in the constructible universe $\mathbf{L}$, we define a forcing notion to prove Theorem 1.2 in Section 11 in the form $\mathbb{P} = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha$. The summands $\mathbb{P}_\alpha$ are countable ST-forcings defined by induction. Any $\mathbb{P}$-generic extension of $\mathbf{L}$ is a model for Theorem 1.2. The inductive construction of $\mathbb{P}_\alpha$ involves two key ideas.

The first idea, essentially due to Jensen [9], is to make every level $\mathbb{P}_\alpha$ of the construction *generic* in some sense over the union of lower levels $\mathbb{P}_\xi$, $\xi < \alpha$. This is based on a construction developed in sections 8–10, which includes the technique of fusion of Silver trees. A special aspect of this construction, elaborated in [11], guarantees that $\mathbb{P}$ is invariant under the group of transformations ($2^{<\omega}$ with the componentwise addition mod 2), which induces the equivalence relation E_0. This invariance implies that $\mathbb{P}$ (unlike Jensen's original forcing) adjoins a E_0-class of generic reals rather than a single such real as in [9]. And overall, the successive genericity of the levels $\mathbb{P}_\alpha$ implies that the three sets are equal in any $\mathbb{P}$-generic extension of $\mathbf{L}$:

[8] VIT is defined on $\mathbb{R}$ so that $x\mathsf{VIT}y$ iff the set $x - y$ is rational.

1) the E_0-class $[a]_{\mathsf{E}_0}$ of the principal generic real $a \in 2^\omega$, 2) the intersection $\bigcap_{\alpha<\omega_1} \bigcup_{T\in\mathbb{P}_\alpha} [T]$, and 3) the set of all $\mathbb{P}$-generic reals over $\mathbf{L}$. This equality is established in sections 12–14, and it leads to $[a]_{\mathsf{E}_0}$ being $\Pi^1_{\mathfrak{n}}$.

The second idea goes back to papers [7,14]. In $\mathbf{L}$, let $\overline{\mathbf{STF}}$ be the set of all countable sequences $\overline{\mathbf{P}} = \langle \mathbf{P}_\xi \rangle_{\xi<\alpha}$ ($\alpha < \omega_1$) compatible with the first genericity idea at each step $\xi < \alpha$. Then a whole sequence $\langle \mathbf{P}_\alpha \rangle_{\alpha<\omega_1}$ can be interpreted as a maximal chain in $\overline{\mathbf{STF}}$. It happens that if this chain is *generic*, in some sense precisely defined in Section 11 ((ii) of Theorem 11.4), with respect to all $\mathbf{\Sigma}^1_{\mathfrak{n}-1}$ subsets of $\overline{\mathbf{STF}}$, then the ensuing forcing notion $\mathbb{P} = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha$ inherits some basic forcing properties of the whole Silver forcing, up to a certain level of projective hierarchy. This includes, in particular, the invariance of the forcing relation with respect to some natural transformations of Silver trees, leading eventually to the proof of (iii) of Theorem 1.2 in sections 15–17.

4. Silver trees

Let $2^{<\omega}$ be the set of all *dyadic strings* (finite sequences) of numbers $0, 1$ — including *the empty string* Λ. If $t \in 2^{<\omega}$ and $i = 0, 1$, then $t^\frown i$ is the extension of t by i as the rightmost term. If $s, t \in 2^{<\omega}$, then $s \subseteq t$ means that the string t extends s, while $s \subset t$ means a proper extension. The length of a string s is denoted by $\mathtt{lh}(s)$, and we let $2^n = \{s \in 2^{<\omega} : \mathtt{lh}(s) = n\}$ (strings of length n).

The Cantor space 2^ω is the set of all functions $f : \omega \to 2 = \{0, 1\}$. Any string $s \in 2^{<\omega}$ *acts* on 2^ω so that $(s \cdot x)(k) = x(k) + s(k) \pmod 2$ for $k < \mathtt{lh}(s)$, and $(s \cdot x)(k) = x(k)$ otherwise. If $X \subseteq 2^\omega$ and $s \in 2^{<\omega}$, then $s \cdot X = \{s \cdot x : x \in X\}$.

Similarly, if $s \in 2^m$, $t \in 2^n$, $m \le n$, then define a string $s \cdot t \in 2^n$ by $(s \cdot t)(k) = t(k) + s(k) \pmod 2$ whenever $k < m$, and $(s \cdot t)(k) = t(k)$ whenever $m \le k < n$. If $m > n$, then let $s \cdot t = (s \restriction n) \cdot t$. In both cases, $\mathtt{lh}(s \cdot t) = \mathtt{lh}(t)$. If $T \subseteq 2^{<\omega}$, then let $s \cdot T = \{s \cdot t : t \in T\}$.

Definition 4.1. A set $T \subseteq 2^{<\omega}$ is a *Silver tree*, in symbol $T \in \mathbf{ST}$, whenever there exists an infinite sequence of strings $u_k = u_k(T) \in 2^{<\omega}$ such that T consists of all strings of the form $s = u_0{}^\frown i_0{}^\frown u_1{}^\frown i_1{}^\frown u_2{}^\frown i_2{}^\frown \dots {}^\frown u_n{}^\frown i_n$, and their substrings (including Λ), where $n < \omega$ and $i_k = 0, 1$. In this case we let $\mathtt{stem}(T) = u_0$ (*the stem* of T), and define a closed set $[T] \subseteq 2^\omega$ of all *branches* $a = u_0{}^\frown i_0{}^\frown u_1{}^\frown i_1{}^\frown u_2{}^\frown i_2{}^\frown \dots \in 2^\omega$ of T, where $i_k = 0, 1$, $\forall\, k$. Let

$$\mathtt{spl}_n(T) = \mathtt{lh}(u_0) + 1 + \mathtt{lh}(u_1) + 1 + \dots + \mathtt{lh}(u_{n-1}) + 1 + \mathtt{lh}(u_n)\,,$$

in particular, $\mathtt{spl}_0(T) = \mathtt{lh}(u_0)$, so that $\mathtt{spl}(T) = \{\mathtt{spl}_n(T) : n < \omega\} \subseteq \omega$ is the set of all *splitting levels* of $T \in \mathbf{ST}$; $\mathtt{spl}(T)$ is infinite. If $u \in T \in \mathbf{ST}$, then define the *restricted tree* $T\restriction_u = \{t \in T : u \subseteq t \vee t \subseteq u\}$; $T\restriction_u \in \mathbf{ST}$. $\square$

Example 4.2. If $s \in 2^{<\omega}$, then the tree $B[s] = \{t \in 2^{<\omega} : s \subseteq t \vee t \subset s\}$ belongs to $\mathbf{ST}$, $\mathtt{stem}(B[s]) = u_0(B[s]) = s$, $\mathtt{spl}_n(B[s]) = \mathtt{lh}(s) + n$, and if $k \ge 1$, then $u_k(B[s]) = \Lambda$. In particular $B[\Lambda] = 2^{<\omega}$. $\square$

Lemma 4.3. *Let* $T \in \mathbf{ST}$. *Then*

(i) *if a set* $\varnothing \ne X \subseteq [T]$ *is open in* $[T]$, *then there is* $s \in T$ *with* $[T\restriction_s] \subseteq X$;
(ii) *if* $h \in \mathtt{spl}(T)$ *and* $u, v \in 2^h \cap T$, *then* $T\restriction_v = (u \cdot v) \cdot T\restriction_u$ *and* $(u \cdot v) \cdot T = T$.

Proof. (ii) By definition, $h = \mathtt{lh}(u_0) + 1 + \mathtt{lh}(u_1) + 1 + \dots + \mathtt{lh}(u_{n-1}) + 1 + \mathtt{lh}(u_n)$,

$$u = u_0{}^\frown i_0{}^\frown u_1{}^\frown \dots {}^\frown u_{n-1}{}^\frown i_{n-1}{}^\frown u_n, \quad v = u_0{}^\frown j_0{}^\frown u_1{}^\frown \dots {}^\frown u_{n-1}{}^\frown j_{n-1}{}^\frown u_n$$

for some n and numbers $i_0, i_1, \dots, i_{n-1}, j_0, j_1, \dots, j_{n-1} = 0, 1$, where $u_k = u_k(T)$. Now if $y \in T{\restriction}_v$, then $y = v^\frown a$, where $a = j_n{}^\frown u_{n+1}{}^\frown j_{n+1}{}^\frown u_{n+2}{}^\frown \dots$ and $j_n, j_{n+1}, \dots = 0, 1$. Then $x = u^\frown a$ belongs to $T{\restriction}_u$, and $y = (u \cdot v) \cdot x$. $\square$

5. Splitting Silver trees

The simple splitting of a tree $T \in \mathbf{ST}$ consists of the subtrees

$$T(\to i) = T{\restriction}_{\mathtt{stem}(T)^\frown i} = \{x \in [T] : x(h) = i\}\,, \quad \text{where } h = \mathtt{lh}(\mathtt{stem}(T)),\ i = 0, 1.$$

Then we have $T(\to i) \in \mathbf{ST}$, and $u_0(T(\to i)) = u_0(T)^\frown\langle i\rangle^\frown u_1(T)$, and then $u_k(T(\to i)) = u_{k+1}(T)$ whenever $k \geq 1$, and $\mathtt{spl}(T(\to i)) = \mathtt{spl}(T) \smallsetminus \{\mathtt{spl}_0(T)\}$.

The splitting can be iterated. Namely if $s \in 2^n$, then define

$$T(\to s) = T(\to s(0))(\to s(1))(\to s(2)) \dots (\to s(n-1))\,.$$

Separately let $T(\to \Lambda) = T$, for the empty string Λ.

Lemma 5.1. *Let $T \in \mathbf{ST}$, $n < \omega$, and $h = \mathtt{spl}_n(T)$. Then*

(i) *if $s \in 2^n$, then $T(\to s) \in \mathbf{ST}$, $\mathtt{lh}(\mathtt{stem}(T(\to s))) = h$, and there is a string $u[s] \in 2^h \cap T$ such that $T(\to s) = T{\restriction}_{u[s]}$;*
(ii) *conversely if $u \in 2^h \cap T$, then there is a string $s[u] \in 2^n$ such that $T{\restriction}_u = T(\to s[u])$;*
(iii) *therefore, $\{T{\restriction}_u : u \in T\} = \{T(\to s) : s \in 2^{<\omega}\}$.*

Proof. (i) $u[s] = u_0(T)^\frown\langle s(0)\rangle^\frown \dots {}^\frown u_{n-1}(T)^\frown\langle s(n-1)\rangle^\frown u_n(T)$.

(ii) Define $s = s[u] \in 2^n$ by $s(k) = u(\mathtt{spl}_k(T))$ for all $k < n$.

(iii) Let $u \in T$. Then $\mathtt{spl}_{n-1}(T) < \mathtt{lh}(u) \leq \mathtt{spl}_n(T)$ for some n. Now, by Definition 4.1, there exists a (unique) string $v \in 2^h \cap T$, where $h = \mathtt{spl}_n(T)$, such that $T{\restriction}_u = T{\restriction}_v$. It remains to refer to (ii). $\square$

If $T, S \in \mathbf{ST}$ and $n \in \omega$, then define $S \subseteq_n T$ (the tree S *n-refines* T), whenever $S \subseteq T$ and $\mathtt{spl}_k(T) = \mathtt{spl}_k(S)$ for all $k < n$. In particular $S \subseteq_0 T$ is equivalent to just $S \subseteq T$. By definition, if $S \subseteq_{n+1} T$, then $S \subseteq_n T$ (and $S \subseteq T$).

Lemma 5.2. *If $T \in \mathbf{ST}$, $n < \omega$, $s_0 \in 2^n$, and $U \in \mathbf{ST}$, $U \subseteq T(\to s_0)$, then there is a unique tree $T' \in \mathbf{ST}$ such that $T' \subseteq_n T$ and $T'(\to s_0) = U$.*

Proof. Let $h = \mathtt{spl}_n(T)$. Pick a string $u_0 = u[s_0] \in 2^h$ by Lemma 5.1(i). Following Lemma 4.3(ii), define T' so that $T' \cap 2^h = T \cap 2^h$, and if $u \in T \cap 2^h$, then $T'{\restriction}_u = (u \cdot u_0) \cdot U$. In particular $T'{\restriction}_{u_0} = U$. $\square$

Lemma 5.3. *Let $\dots \subseteq_5 T_4 \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0$ be an infinite decreasing sequence of trees in $\mathbf{ST}$. Then*

(i) *$T = \bigcap_n T_n \in \mathbf{ST}$;*
(ii) *if $n < \omega$ and $s \in 2^{n+1}$, then $T(\to s) = T \cap T_n(\to s) = \bigcap_{m \geq n} T_m(\to s)$.*

Proof. Note that $\mathtt{spl}(T) = \{\mathtt{spl}_n(T_n) : n < \omega\}$; both claims easily follow. $\square$

Corollary 5.4. *Let $T \in \mathbf{ST}$. If a set $X \subseteq [T]$ has the Baire property inside $[T]$, but is not meager inside $[T]$, then there is a tree $S \in \mathbf{ST}$ such that $[S] \subseteq X$. If $f : [T] \to 2^\omega$ is a continuous map, then there exists a tree $S \in \mathbf{ST}$ such that $S \subseteq T$ and $f{\restriction}[S]$ is a bijection or a constant.*

Proof. We can assume that X is co-meager inside $[T]$; otherwise just replace T by $T{\restriction}_u$, where $u \in T$ and X is co-meager on $[T{\restriction}_u]$. In other words, let $X = \bigcup_n G_n$, where each $G_n \subseteq [T]$ is an open dense set. Using Lemma 5.2, we define, by induction, an infinite decreasing sequence $\ldots \subseteq_5 T_4 \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0$ of trees in $\mathbf{ST}$, such that $T_0 \subseteq T$ and $T_n \subseteq G_n$for all n.[9] Then $S = \bigcap_n T_n \in \mathbf{ST}$ by Lemma 5.3, and $[S] \subseteq \bigcup_n G_n = X$ by construction.

The second claim is known from the folklore theory of the Silver forcing. As the earliest reference, it can be extracted from some very general results in [4, §5] on the minimality of Silver's reals (especially Corollary 5.5). For a more direct argument see Theorem 8.5 in [17].

A pedestrian proof can be as follows. If $a \in 2^\omega$ and $n < \omega$, then let $\{n\} \cdot a = b \in 2^\omega$ be defined so that $b(k) = a(k)$ for all $k \neq n$, while $b(n) = 1 - a(n)$. Let $S \in \mathbf{ST}$ and $n(S) = \mathtt{lh}(\mathtt{stem}(S))$. Note that $a \in [S] \Longrightarrow \{n(S)\} \cdot a \in [S]$. Say that S is *f-symmetric* iff $f(a) = f(\{n(S)\} \cdot a)$ holds for all $a \in [S]$.

Case 1: if $S \in \mathbf{ST}, S \subseteq T$, then there is a f-symmetric subtree $R \in \mathbf{ST}, R \subseteq S$. In this case, to get a tree $S \in \mathbf{ST}, S \subseteq T$, such that $f{\restriction}[S]$ is a constant, we define, by induction, an infinite decreasing sequence $\ldots \subseteq_5 T_4 \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0 \subseteq T$ of trees in $\mathbf{ST}$, such that for each m and $s \in 2^m$, $T_m(\to s)$ is f-symmetric. Then by Lemma 5.3 the limit tree $S = \bigcap_m T_m$ belongs to $\mathbf{ST}$ and $S(\to s)$ is f-symmetric for all $s \in 2^{<\omega}$. It easily follows that if $n < \omega$, $a \in [S]$, and $\{n\} \cdot a \in [S]$ as well then $f(a) = f(\{n\} \cdot a)$. Then f is E_0-invariant on $[S]$. Prove that f is a constant on $[S]$. Indeed if $a \neq b$ belong to $[S]$ and $f(a) \neq f(b)$ then there are strings $u, v \in S$ of equal length, such that for some k and $i \neq j$ we have $f(x)(k) = i$ and $f(y)(k) = j$ for all $x \in [S{\restriction}_u]$ and $y \in [S{\restriction}_v]$. But the sets $[S{\restriction}_u]$, $[S{\restriction}_v]$ contain E_0-equivalent points, a contradiction.

Thus it remains to define a sequence $\ldots \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0 \subseteq T$ with the above defined properties. We let $T_0 \subseteq T$ be any f-symmetric subtree in $\mathbf{ST}$. Let $r = \mathtt{stem}(T_0)$ and $n_0 = \mathtt{lh}(r)$. Thus $\{n_0\} \cdot a \in [T_0]$ and $f(a) = f(\{n_0\} \cdot a)$ holds for all $a \in [T_0]$, by the f-symmetry.

Now let $U \in \mathbf{ST}, U \subseteq T_0(\to 0) = T_0{\restriction}_{r^\frown 0}$ be any f-symmetric tree. Then $V = \{n_0\} \cdot U \subseteq T_0(\to 1) = T_0{\restriction}_{r^\frown 1}$ is f-symmetric as well by the f-symmetry of T_0. On the other hand, the tree $T_1 = U \cup V$ belongs to $\mathbf{ST}$ and $T_1(\to 0) = U \subseteq T_0(\to 0)$ and $T_1(\to 1) = V \subseteq T_0(\to 1)$, hence $T_1 \subseteq_1 T_0$. Let $n_1 = \mathtt{lh}(\mathtt{stem}(U))$.

The next step: let $U_{00} \in \mathbf{ST}, U_{00} \subseteq T_1(\to 00)$ be any f-symmetric tree. Then so are the trees $U_{01} = \{n_1\} \cdot U_{00}$, $U_{10} = \{n_0\} \cdot U_{00}$, $U_{11} = \{n_1\} \cdot \{n_0\} \cdot U_{00}$. We define $T_2 = U_{00} \cup U_{01} \cup U_{10} \cup U_{11}$; then $T_2 \in \mathbf{ST}$, $T_2 \subseteq_2 T_1$.

And so on.

Case 2 = not case 1, that is, there is a tree $T' \in \mathbf{ST}$, $T' \subseteq T$, which has no f-symmetric subtrees $R \in \mathbf{ST}, R \subseteq T'$. In this case, to get a tree $S \in \mathbf{ST}, S \subseteq T$, such that $f{\restriction}[S]$ is a bijection, we define, by induction, an infinite decreasing sequence $\ldots \subseteq_5 T_4 \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0 \subseteq T'$ of trees in $\mathbf{ST}$, such that $f"[T_m(\to s^\frown 0)] \cap f"[T_m(\to s^\frown 1)] = \varnothing$ holds for each m and $s \in 2^m$. Then by Lemma 5.3 the tree $S = \bigcap_m T_m$ belongs to $\mathbf{ST}$ and we have $f"[S(\to s^\frown 0)] \cap f"[S(\to s^\frown 1)] = \varnothing$ for all $s \in 2^{<\omega}$. It follows that f is a bijection on $[S]$.

Thus it remains to define a sequence $\ldots \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0 \subseteq T'$ with the above defined properties. To begin with, the tree T' is not f-symmetric, hence there exists a point $a \in [T']$ such that $f(a) \neq f(b)$, where $b = \{n'\} \cdot a \in [T']$, and $n' = n(T') = \mathtt{lh}(r)$, $r = \mathtt{stem}(T')$. Let say $a(n') = 0$ and $b(n') = 1$ (and $a(n) = b(n)$ whenever $n \neq n'$). Indeed, as f is continuous, there are strings $u, v \in S$ of equal length $\ell = \mathtt{lh}(u) = \mathtt{lh}(v) > n'$, such that $u \subset a$, $v \subset b$ (then $u(n') = 0$, $v(n') = 1$, and $u(n) = v(n)$ for all $n < \ell, n \neq n'$), and $f"[T'{\restriction}_u] \cap f"[T'{\restriction}_v] = \varnothing$. Then the tree $T_0 = T'{\restriction}_u \cup T'{\restriction}_v$ satisfies $T_0(\to 0) = T'{\restriction}_u$ and $T_0(\to 1) = T'{\restriction}_v$, hence $f"[T_0(\to 0)] \cap f"[T_0(\to 1)] = \varnothing$ by construction.

[9] If some T_n is defined, then let $T' = T_n$ and pick $s_0 \in 2^{n+1}$. As G_{n+1} is open dense, there is a tree $U \in \mathbf{ST}$, $U \subseteq T(\to s_0) \cap G_{n+1}$. By Lemma 5.2 there is a tree $T'' \in \mathbf{ST}$ such that $T'' \subseteq_{n+1} T'$ and $T''(\to s_0) = U \subseteq G_{n+1}$. Pick the next string $s_1 \in 2^{n+1}$ and get a tree $T''' \in \mathbf{ST}$ with $T''' \subseteq_{n+1} T''$ and $T'''(\to s_1) \subseteq G_{n+1}$. And so on. In the end, we get a tree $T_{n+1} \in \mathbf{ST}$ such that $T_{n+1} \subseteq_{n+1} T_n$ and $T_{n+1}(\to s) \subseteq G_{n+1}$ for all $s \in 2^{n+1}$, hence $T_{n+1} \subseteq G_{n+1}$.

To define T_1, we similarly split the tree $T_0(\to 0)$ and apply Lemma 5.2 to get a tree $S_1 \in \mathbf{ST}$ such that $S_1 \subseteq_2 T_0$ and $f"[S_1(\to \langle 0,0\rangle)] \cap f"[S_1(\to \langle 0,1\rangle)] = \varnothing$. The other pair of splitting sets,

$$S_1(\to \langle 1,0\rangle) = S_1(\to \langle 1\rangle)(\to 0) \quad \text{and} \quad S_1(\to \langle 1,1\rangle) = S_1(\to \langle 1\rangle)(\to 1)$$

do not necessarily satisfy $f"[S_1(\to \langle 1,0\rangle)] \cap f"[S_1(\to \langle 1,1\rangle)] = \varnothing$. However, as $S_1(\to \langle 1\rangle)$ is not f-symmetric, we can apply the same shrinking procedure (followed by Lemma 5.2) to obtain a tree $T_1 \in \mathbf{ST}$ such that $T_1 \subseteq_2 S_1$ and $f"[T_1(\to \langle 1,0\rangle)] \cap f"[T_1(\to \langle 1,1\rangle)] = \varnothing$.

Then a tree $T_2 \in \mathbf{ST}, T_2 \subseteq_3 T_1$, is defined, such that $f"[T_2(\to s^\frown 0)] \cap f"[T_2(\to s^\frown 1)] = \varnothing$ for any string $s = \langle i_0, i_1\rangle \in 2^{<\omega}$ of length 2. And so on. $\square$

6. ST-forcings

Definition 6.1. Any set $\mathbf{P} \subseteq \mathbf{ST}$ satisfying

(A) if $u \in T \in \mathbf{P}$, then $T\restriction_u \in \mathbf{P}$, and
(B) if $T \in \mathbf{P}$ and $\sigma \in 2^{<\omega}$, then $\sigma \cdot T \in \mathbf{P}$,

is called *a forcing by Silver trees*, *ST-forcing* in brief. $\square$

Remark 6.2. Any ST-forcing $\mathbf{P}$ can be considered as a forcing notion ordered so that if $T \subseteq T'$, then T is a stronger condition. The forcing $\mathbf{P}$ adjoins a real $x \in 2^\omega$. More exactly if a set $G \subseteq \mathbf{P}$ is $\mathbf{P}$-generic over a given model or set universe M (and $\mathbf{P} \in M$ is assumed), then the intersection $\bigcap_{T\in G}[T]$ contains a unique real $a = \mathbf{a}[G] \in 2^\omega$, and this real satisfies $M[G] = M[\mathbf{a}[G]]$ and $G = \{T \in \mathbf{P} : \mathbf{a}[G] \in [T]\}$. Reals $\mathbf{a}[G]$ of this kind are called $\mathbf{P}$-*generic*. $\square$

Example 6.3. The following sets are ST-forcings: the set $\mathbf{P}_{\mathrm{coh}} = \{T[s] : s \in 2^{<\omega}\}$ of all trees in 4.2 — *the Cohen forcing*, and the set $\mathbf{ST}$ of all Silver trees — the *Silver forcing* itself. $\square$

Lemma 6.4. *If $\varnothing \neq Q \subseteq \mathbf{ST}$, then the following set is a ST-forcing:*

$$\mathbf{P} = \{\sigma \cdot (T\restriction_u) : u \in T \in Q \wedge \sigma \in 2^{<\omega}\} = \{\sigma \cdot (T(\to s)) : T \in Q \wedge s, \sigma \in 2^{<\omega}\}.$$

Proof. To prove 6.1(A), let $T \in Q$ and $v \in S = \sigma \cdot (T\restriction_u)$. Then $w = \sigma \cdot v \in T\restriction_u$ and $v = \sigma \cdot w$. It follows that $S\restriction_v = \sigma \cdot (T\restriction_u\restriction_w)$, where $T\restriction_u\restriction_w = T\restriction_u$ or $= T\restriction_w$, whenever accordingly $w \subseteq u$ or $u \subset w$. The second equality of the lemma follows from Lemma 5.1(iii). $\square$

Definition 6.5 *(Collages).* If $\mathbf{P} \subseteq \mathbf{ST}$, $T \in \mathbf{ST}$, $n < \omega$, and all split trees $T(\to s)$, $s \in 2^n$, belong to $\mathbf{P}$, then T is an n-*collage over* $\mathbf{P}$. The set of all n-collages over $\mathbf{P}$ is $\mathbf{Colg}_n(\mathbf{P})$. Then $\mathbf{P} = \mathbf{Colg}_0(\mathbf{P}) \subseteq \mathbf{Colg}_n(\mathbf{P}) \subseteq \mathbf{Colg}_{n+1}(\mathbf{P})$. $\square$

Lemma 6.6. *Let $\mathbf{P} \subseteq \mathbf{ST}$ be a ST-forcing, and $n < \omega$. Then:*

(i) *if $T \in \mathbf{P}$ and $s \in 2^{<\omega}$, then $T(\to s) \in \mathbf{P}$;*
(ii) *if $T \in \mathbf{ST}$ and $s_0 \in 2^n$, then $T(\to s_0) \in \mathbf{P}$ is equivalent to $T \in \mathbf{Colg}_n(\mathbf{P})$;*
(iii) *if $U \in \mathbf{Colg}_n(\mathbf{P})$, $s_0 \in 2^n$, $S \in \mathbf{P}$, and $S \subseteq U(\to s_0)$, then there is a tree $V \in \mathbf{Colg}_n(\mathbf{P})$ such that $V \subseteq_n U$ and $V(\to s_0) = S$;*
(iv) *if $U \in \mathbf{Colg}_n(\mathbf{P})$ and $D \subseteq \mathbf{P}$ is open dense in $\mathbf{P}$, then there is a tree $V \in \mathbf{Colg}_n(\mathbf{P})$ such that $V \subseteq_n U$ and $V(\to s) \in D$ for all $s \in 2^n$.*

A set $D \subseteq \mathbf{P}$ is *dense* in $\mathbf{P}$ if for any $S \in \mathbf{P}$ there is a tree $T \in D$, $T \subseteq S$, and *open dense*, if in addition $S \in D$ holds whenever $S \in \mathbf{P}$, $T \in D$, $S \subseteq T$.

Proof. To prove (i) make use of 6.1(A) and Lemma 5.1(i).

(ii) If $T \in \mathbf{Colg}_n(\mathbf{P})$, then by definition $T(\to s_0) \in \mathbf{P}$. To prove the converse let $h = \mathtt{spl}_n(T)$ and let a string $u[s] \in 2^h \cap T$ satisfy $T(\to s) = T{\restriction}_{u[s]}$ for all $s \in 2^n$ by Lemma 5.1(i). If $T(\to s_0) \in \mathbf{P}$ and $s \in 2^n$, then $T(\to s) = T{\restriction}_{u[s]} = (u[s] \cdot u[s_0]) \cdot T{\restriction}_{u[s_0]} = (u[s] \cdot u[s_0]) \cdot T(\to s_0)$ by Lemma 4.3(ii). Therefore $T(\to s) \in \mathbf{P}$ by 6.1(B). And finally $T \in \mathbf{Colg}_n(\mathbf{P})$.

(iii) By Lemma 5.2, there is a tree $V \in \mathbf{ST}$, satisfying $V \subseteq_n U$ and $V(\to s_0) = S$. However $V \in \mathbf{Colg}_n(\mathbf{P})$ by (ii).

To prove (iv) apply (iii) 2^n times (for all $s \in 2^n$). □

7. Continuous maps

Let $\mathbf{P}$ be a fixed ST-forcing in this section.

Regularity. We study the behavior of continuous maps on sets of the form $[T]$, $T \in \mathbf{P}$. The next definition highlights the case when a given continuous $f : 2^\omega \to 2^\omega$ is forced to be not equal to a map of the form $x \mapsto \sigma \cdot x$, $\sigma \in 2^{<\omega}$.

Definition 7.1. Let $T \in \mathbf{P}$. A continuous $f : 2^\omega \to 2^\omega$ is *regular on* T *inside* $\mathbf{P}$, if there is no tree $T' \in \mathbf{P}$ and string $\sigma \in 2^{<\omega}$ such that $T' \subseteq T$ and $\sigma \cdot f(x) = x$ (equivalently, $f(x) = \sigma \cdot x$) for all $x \in [T']$. □

Lemma 7.2. *Let $S, T \in \mathbf{P}$, $f : 2^\omega \to 2^\omega$ be continuous, and $\sigma \in 2^{<\omega}$. Then:*

(i) *there are trees $S', T' \in \mathbf{P}$ such that $S' \subseteq S$, $T' \subseteq T$, $[T'] \cap (\sigma \cdot f\,``[S']) = \varnothing$;*

(ii) *if $\tau \in 2^{<\omega}$, $T = \tau \cdot S$, and f is regular on S inside $\mathbf{P}$, then there exist trees $S', T' \in \mathbf{P}$ such that $S' \subseteq S$, $T' \subseteq T$, $T' = \tau \cdot S'$, and $[T'] \cap (\sigma \cdot f\,``[S']) = \varnothing$.*

Proof. (i) Let $x_0 \in [S]$ and let $y_0 \in [T]$ be different from $\sigma \cdot f(x_0)$. As f is continuous, there is $m \ge \mathtt{lh}(\sigma)$, such that $\sigma \cdot (f(x){\restriction}m) \ne y_0{\restriction}m$ whenever $x \in [S]$ and $x{\restriction}m = x_0{\restriction}m$. By 6.1(A), there are trees $S', T' \in \mathbf{P}$ such that $S' \subseteq S$, $T' \subseteq T$, $x{\restriction}m = x_0{\restriction}m$ for all $x \in [S']$, and $y{\restriction}m = y_0{\restriction}m$ for all $y \in [T']$.

(ii) Assume that $\mathtt{lh}(\sigma) = \mathtt{lh}(\tau)$ (otherwise the shorter string can be extended by zeros). The set $X = \{x \in [S] : \sigma \cdot f(x) \ne \tau \cdot x\}$ is open in $[S]$ and non-empty, by the regularity. Let $x_0 \in X$. There is a number $m \ge \mathtt{lh}(\sigma) = \mathtt{lh}(\tau)$ satisfying $\tau \cdot (x_0{\restriction}m) \ne \sigma \cdot (f(x_0){\restriction}m)$. By 6.1(A) there is a tree $S' \in \mathbf{P}$ satisfying $x_0 \in [S']$ and $x{\restriction}m = x_0{\restriction}m$, $f(x){\restriction}m = f(x_0){\restriction}m$ for all $x \in [S']$. Put $T' = \tau \cdot S'$. □

Coding continuous maps. If $f : 2^\omega \to 2^\omega$ is continuous, $k < \omega$ and $i = 0, 1$, then the set $\mathbf{D}^i_k(f) = \{x \in 2^\omega : f(x)(k) = i\}$, and $\mathbf{D}^0_k(f) \cap \mathbf{D}^1_k(f) = \varnothing$, $\mathbf{D}^0_k(f) \cup \mathbf{D}^1_k(f) = 2^\omega$. Let $\mathbf{C}^i_k(f)$ be the set of all $\subseteq$-minimal strings $s \in 2^{<\omega}$ such that $[B[s]] \subseteq \mathbf{D}^i_k(f)$. Then $\mathbf{C}^0_k(f), \mathbf{C}^1_k(f) \subseteq 2^{<\omega}$ are finite irreducible and incompatible antichains[10] in $2^{<\omega}$, and $\mathbf{D}^i_k(f) = \bigcup_{s \in \mathbf{C}^i_k(f)} [B[s]]$, while the union $\mathbf{C}^0_k(f) \cup \mathbf{C}^1_k(f)$ is a maximal antichain in $2^{<\omega}$. Let $\mathbf{code}(f) = \langle \mathbf{C}^i_k(f) \rangle_{k<\omega, i=0,1}$ — the *code* of f. Thus each $\mathbf{D}^i_k(f)$ is a clopen set, determined by a finite union of basic clopen sets $[B[s]]$, and $\mathbf{C}^i_k(f)$ is coding this finite union.

The other way around, if $\mathbf{c} = \langle \mathbf{C}^i_k \rangle_{k<\omega, i=0,1}$ is a family of finite irreducible antichains $\mathbf{C}^i_k \subseteq 2^{<\omega}$, and if $k < \omega$, then $\mathbf{C}^0_k, \mathbf{C}^1_k$ are incompatible and $\mathbf{C}^0_k \cup \mathbf{C}^1_k$ is a maximal antichain in $2^{<\omega}$, then $\mathbf{c}$ is called *a code of continuous function*. In this case, the related continuous function $f : 2^\omega \to 2^\omega$ is denoted by $f_{\mathbf{c}}$, that is, $f_{\mathbf{c}}(x)(k) = i$ whenever $x \in \bigcup_{s \in \mathbf{C}^i_k} [B[s]]$.

[10] An antichain $A \subseteq 2^{<\omega}$ is *irreducible*, if it does not contain a pair of the form $s^\frown 0, s^\frown 1$. Two antichains $A, A' \subseteq 2^{<\omega}$ are *incompatible*, if any strings $s \in A$ and $s' \in A'$ are $\subseteq$-incomparable.

Let **CCF** denote the set of all codes of continuous functions.

8. Generic extensions of ST-forcings

The forcing notion to prove Theorem 1.2 will be defined in the form of an ω_1-union of its countable parts — *levels*. The next definition presents requirements which will govern the interactions between the levels.

Definition 8.1. Let $\mathfrak{M}$ be any set and $\mathbf{P}$ be a ST-forcing. We say that another ST-forcing $\mathbf{Q}$ is an $\mathfrak{M}$-*extension* of $\mathbf{P}$, in symbol $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$, if the following holds:

(A) $\mathbf{Q}$ is dense in $\mathbf{Q} \cup \mathbf{P}$;
(B) if a set $D \in \mathfrak{M}$, $D \subseteq \mathbf{P}$ is pre-dense in $\mathbf{P}$ and $U \in \mathbf{Q}$, then $U \subseteq^{\mathtt{fin}} \bigcup D$, that is, there is a finite set $D' \subseteq D$ such that $U \subseteq \bigcup D'$;
(C) if $T_0 \in \mathbf{P}$, $f : 2^\omega \to 2^\omega$ is a continuous map with a code in $\mathfrak{M}$, regular on T_0 inside $\mathbf{P}$, and $U, V \in \mathbf{Q}$, $U \subseteq T_0$, then $[V] \cap (f``[U]) = \varnothing$. $\square$

Condition (C) claims essentially that if the regularity holds, then T_0 forces, in the sense of $\mathbf{Q} \cup \mathbf{P}$, that $f(\mathbf{a}[G])$ does not belong to $\bigcup_{V \in \mathbf{Q}} [V]$.

If $\mathfrak{M} = \varnothing$, then we write $\mathbf{P} \sqsubset \mathbf{Q}$ instead of $\mathbf{P} \sqsubset_{\varnothing} \mathbf{Q}$; in this Case (B) and (C) are trivial. Generally, we'll consider, in the role of $\mathfrak{M}$, transitive models of the theory $\mathbf{ZFC}'$ which includes all $\mathbf{ZFC}$ axioms except for the Power Set axiom, but an axiom is adjoined, which claims the existence of $\mathscr{P}(\omega)$. (Then the existence of ω_1 and sets like 2^ω and $\mathbf{ST}$ easily follows.)

Lemma 8.2. *Let $\mathfrak{M} \models \mathbf{ZFC}'$ be a transitive model, $\mathbf{P} \in \mathfrak{M}$ and $\mathbf{Q}$ be ST-forcings satisfying $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$, and $U \in \mathbf{Q}$. Then*

(i) *if $T \in \mathbf{P}$, then $[U] \cap [T]$ is clopen in $[U]$;*
(ii) *if $D \in \mathfrak{M}$, $D \subseteq \mathbf{P}$ is pre-dense in $\mathbf{P}$, then D remains pre-dense in $\mathbf{Q} \cup \mathbf{P}$;*
(iii) *if $T, T' \in \mathbf{P}$ are incompatible in $\mathbf{P}$, then T, T' are incompatible in $\mathbf{Q} \cup \mathbf{P}$, too, and moreover*
(iv) *if $R \subseteq U$ is a Silver tree, then the set $\mathbf{R} = \mathbf{Q} \cup \{\sigma \cdot (R(\to t)) : t, \sigma \in 2^{<\omega}\}$ is a ST-forcing, and still $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{R}$.*

Proof. (i) The set $D(T) = \{S \in \mathbf{P} : S \subseteq T \vee [S] \cap [T] = \varnothing\}$ belongs to $\mathfrak{M}$ and is open dense in $\mathbf{P}$ by 6.1(A). Then by 8.1(B) there is a finite set $D' \subseteq D(T)$ such that $U \subseteq \bigcup D'$. But then we have $[U] \smallsetminus [T] = \bigcup_{S \in D''}([S] \cap [U])$, where $D'' = \{S \in D' : [S] \cap [T] = \varnothing\}$. Thus $[U] \smallsetminus [T]$ is a closed set.

(ii) Let $U \in \mathbf{Q}$. Then by 8.1(B) there is a finite set $D' \subseteq D$ such that $U \subseteq \bigcup D'$. By (i) there is a tree $T \in D'$ such that $[T] \cap [U]$ has a non-empty open interior in $[U]$. By Lemma 4.3(i), there is $s \in U$ such that $[U'] \subseteq [T] \cap [U]$, where $U' = U{\restriction}_s$, thus $U' \subseteq U \cap T$. Finally $U' \in \mathbf{Q}$, as $\mathbf{Q}$ is a ST-forcing.

(iii) By the incompatibility, if $S \in \mathbf{P}$, then $S \not\subseteq T$ or $S \not\subseteq T'$, and hence there is a tree $S' \in \mathbf{P}$, $S' \subseteq S$, satisfying $[S'] \cap [T] \cap [T'] = \varnothing$. Therefore the set $D = \{S \in \mathbf{P} : [S] \cap [T] \cap [T'] = \varnothing\}$ is dense in $\mathbf{P}$ and belongs to $\mathfrak{M}$. Now if $U \in \mathbf{Q}$, then $U \subseteq^{\mathtt{fin}} \bigcup D$ by 8.1(B), thus immediately $[U] \cap [T] \cap [T'] = \varnothing$.

(iv) $\mathbf{R}$ is a ST-forcing by Lemma 6.4, so we have to prove $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{R}$. We skip a routine verification of 8.1(A),(B) and focus on (C). Let T_0, f be as in (C).

Consider trees $R' = \sigma \cdot (R(\to t)) \in \mathbb{R}$, $R' \subseteq T_0$, and $V \in \mathbf{Q}$; we have to prove that $[V] \cap (f``[R']) = \varnothing$. The problem is that while $R' \subseteq \sigma \cdot R \subseteq U' = \sigma \cdot U$, it may not be true that $U' \subseteq T_0$. But, now we claim that *there is a finite set of trees* $\{U_1, \dots, U_n\} \subseteq \mathbf{Q}$ *such that* $R' \subseteq U_1 \cup \ldots \cup U_n \subseteq T_0$. If this is established, then $[V] \cap (f``[U_i]) = \varnothing$ for all i since $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$, and we are done.

To prove the claim consider the dense set $D(T_0) \in \mathfrak{M}$ as in the proof of (i) above. Then $U' \subseteq^{\mathtt{fin}} \bigcup D$, hence there is a finite set $D' \subseteq D(T_0)$ satisfying $U' \subseteq \bigcup D'$. As by construction $R' \subseteq U' \cap T_0$, we conclude that $R' \subseteq \bigcup D''$, where $D'' = \{S \in D' : S \subseteq T_0\}$. Now if $S \in D''$, then $[S] \cap [U']$ is clopen in $[U']$, therefore $[S] \cap [U'] = \bigcup_{v \in W(S)} [U' {\restriction}_v]$, where $W(S) \subseteq U'$ is a suitable finite set. We put $\{U_1, \dots, U_n\} = \bigcup_{S \in D''} \{U' {\restriction}_v : v \in W(S)\}$. □

Theorem 8.3. *Let $\mathfrak{M} \models \mathbf{ZFC}'$ be a countable transitive model and $\mathbf{P} \in \mathfrak{M}$ be a ST-forcing. Then there exists a countable ST-forcing $\mathbf{Q}$ satisfying $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$.*

The proof of this theorem is presented below. Section 9 contains the definition of the ST-forcing $\mathbf{Q}$, and the proof of its properties follows in Section 10.

9. Construction of extending ST-forcing

The next definition formalizes a construction of countably many Silver trees by means of Lemma 5.3.

Definition 9.1. A *system* is any indexed set $\varphi = \langle \langle \nu_m^\varphi, \tau_m^\varphi \rangle \rangle_{m \in |\varphi|}$, where $|\varphi| \subseteq \omega$ is finite, and if $m \in |\varphi|$, then $\nu_m^\varphi \in \omega$, $\tau_m^\varphi = \langle T_m^\varphi(0), T_m^\varphi(1), \dots, T_m^\varphi(\nu_m^\varphi) \rangle$, each $T_m^\varphi(n)$ is a Silver tree, and $T_m^\varphi(n+1) \subseteq_{n+1} T_m^\varphi(n)$ whenever $n < \nu_m^\varphi$.

In this case, if $n \le \nu_m^\varphi$ and $s \in 2^n$, then let $T_m^\varphi(s) = T_m^\varphi(n)(\to s)$.

If $\mathbf{P}$ is a ST-forcing, then $\mathbf{Sys}(\mathbf{P})$ is the set of all system φ such that $T_m^\varphi(n) \in \mathbf{Colg}_n(\mathbf{P})$ for all $m \in |\varphi|$ and $n \le \nu_m^\varphi$. Then every tree $T_m^\varphi(s)$ belongs to $\mathbf{P}$. □

A system φ *extends* a system ψ, in symbol $\psi \preccurlyeq \varphi$, if $|\psi| \subseteq |\varphi|$, and for any index $m \in |\psi|$, first, $\nu_m^\varphi \ge \nu_m^\psi$, and second, τ_m^φ extends τ_m^ψ in the sense that simply $T_m^\varphi(n) = T_m^\psi(n)$ for all $n \le \nu_m^\psi$.

Lemma 9.2. *Let $\mathbf{P}$ be a ST-forcing and $\varphi \in \mathbf{Sys}(\mathbf{P})$.*

If $m \in |\varphi|$ and $n = \nu_m^\varphi$, then the extension φ' of the systemφ by $\nu_m^{\varphi'} = n+1$ and $T_m^{\varphi'}(n+1) = T_m^\varphi(n)$ belongs to $\mathbf{Sys}(\mathbf{P})$ and $\varphi \preccurlyeq \varphi'$.

If $m \notin |\varphi|$ and $T \in \mathbf{P}$, then the extension φ' of the systemφ by $|\varphi'| = |\varphi| \cup \{m\}$, $\nu_m^{\varphi'} = 0$ and $T_m^{\varphi'}(0) = T$, belongs to $\mathbf{Sys}(\mathbf{P})$ and $\varphi \preccurlyeq \varphi'$. □

Now, according to the formulation of Theorem 8.3, we assume that $\mathfrak{M} \models \mathbf{ZFC}'$ is a countable transitive model and $\mathbf{P} \in \mathfrak{M}$ is a (countable) ST-forcing.

Definition 9.3. (i) We fix a $\preccurlyeq$-increasing sequence $\mathbf{\Phi} = \langle \varphi(j) \rangle_{j<\omega}$ of systems $\varphi(j) \in \mathbf{Sys}(\mathbf{P})$, *generic over* $\mathfrak{M}$ in the sense that it intersects every set $D \in \mathfrak{M}, D \subseteq \mathbf{Sys}(\mathbf{P})$ dense in $\mathbf{Sys}(\mathbf{P})$. The density here means that for any system $\psi \in \mathbf{Sys}(\mathbf{P})$ there is a system $\varphi \in D$ such that $\psi \subseteq \varphi$. The goal of this definition is to define another ST-forcing $\mathbf{Q}$ from $\mathbf{\Phi}$, see item (iv) below.

(ii) If $m, n < \omega$, then the set $D_{mn} = \{\varphi \in \mathbf{Sys}(\mathbf{P}) : \nu_m^\varphi \ge n\}$ is dense by Lemma 9.2 and belongs to $\mathfrak{M}$, hence it intersects $\mathbf{\Phi}$. Therefore if $m < \omega$, then there exists an infinite sequence

$$\dots \subseteq_5 \mathbf{T}_m^{\mathbf{\Phi}}(4) \subseteq_4 \mathbf{T}_m^{\mathbf{\Phi}}(3) \subseteq_3 \mathbf{T}_m^{\mathbf{\Phi}}(2) \subseteq_2 \mathbf{T}_m^{\mathbf{\Phi}}(1) \subseteq_1 \mathbf{T}_m^{\mathbf{\Phi}}(0)$$

of trees $\mathbf{T}_m^{\mathbf{\Phi}}(n) \in \mathbf{Colg}_n(\mathbf{P})$ such that for each j, if $m \in |\varphi(j)|$ and $n \le \nu_m^{\varphi(j)}$, then $T_m^{\varphi(j)}(n) = \mathbf{T}_m^{\mathbf{\Phi}}(n)$. If $n < \omega$ and $s \in 2^n$, then let $\mathbf{T}_m^{\mathbf{\Phi}}(s) = \mathbf{T}_m^{\mathbf{\Phi}}(n)(\to s)$; then $\mathbf{T}_m^{\mathbf{\Phi}}(s) \in \mathbf{P}$ because $\mathbf{T}_m^{\mathbf{\Phi}}(n) \in \mathbf{Colg}_n(\mathbf{P})$.

(iii) In this case, by Lemma 5.3, every set

$$\mathbf{U}_m^{\mathbf{\Phi}} = \bigcap_n \mathbf{T}_m^{\mathbf{\Phi}}(n) = \bigcap_n \bigcup_{s \in 2^n} \mathbf{T}_m^{\mathbf{\Phi}}(s)$$

belongs to $\mathbf{ST}$, the same is true for all subtrees $\mathbf{U}^{\mathbf{\Phi}}_m(\to s)$, and we have

$$\mathbf{U}^{\mathbf{\Phi}}_m(\to s) = \mathbf{U}^{\mathbf{\Phi}}_m \cap \mathbf{T}^{\mathbf{\Phi}}_m(s) = \bigcap_{n \geq \mathtt{lh}(s)} \mathbf{T}^{\mathbf{\Phi}}_m(n)(\to s)\,,$$

by Lemma 5.3, and $\mathbf{U}^{\mathbf{\Phi}}_m = \mathbf{U}^{\mathbf{\Phi}}_m(\to \Lambda)$ holds by definition. In addition if strings $s, t \in 2^{<\omega}$ satisfy $t \subset s$, then $\mathbf{T}^{\mathbf{\Phi}}_m(s) \subseteq \mathbf{T}^{\mathbf{\Phi}}_m(t)$[11] and $\mathbf{U}^{\mathbf{\Phi}}_m(\to s) \subseteq \mathbf{U}^{\mathbf{\Phi}}_m(\to t)$, but if s, t are $\subseteq$-incomparable, then $[\mathbf{U}^{\mathbf{\Phi}}_m(\to s)] \cap [\mathbf{U}^{\mathbf{\Phi}}_m(\to t)] = [\mathbf{T}^{\mathbf{\Phi}}_m(s)] \cap [\mathbf{T}^{\mathbf{\Phi}}_m(t)] = \varnothing$.

(iv) Finally the set $\mathbf{Q} = \{\sigma \cdot \mathbf{U}^{\mathbf{\Phi}}_m(\to s) : m < \omega \wedge \sigma, s \in 2^{<\omega}\}$ is a countable ST-forcing by Lemma 6.4. $\square$

10. Validation of the extension property

Here we prove that $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$ in the context of Definition 9.3. We check all requirements of Definition 8.1.

8.1(A). If $T \in \mathbf{P}$, then the set $\Delta(T)$ of all systems $\varphi \in \mathbf{Sys}(\mathbf{P}) \cap \mathfrak{M}$ such that $T^{\varphi}_m(0) = T$ for some m, belongs to $\mathfrak{M}$ and is dense in $\mathbf{Sys}(\mathbf{P})$ by Lemma 9.2. Therefore $\varphi(J) \in \Delta(T)$ for some J, by the choice of $\mathbf{\Phi}$. Then $\mathbf{T}^{\mathbf{\Phi}}_m(0) = T$ for some $m < \omega$. But $\mathbf{U}^{\mathbf{\Phi}}_m(\to \Lambda) = \mathbf{U}^{\mathbf{\Phi}}_m \subseteq \mathbf{T}^{\mathbf{\Phi}}_m(0)$ and $\mathbf{U}^{\mathbf{\Phi}}_m \in \mathbf{Q}$.

8.1(B). Let $U = \sigma \cdot \mathbf{U}^{\mathbf{\Phi}}_m(\to s) \in \mathbf{Q}$, where $m < \omega$ and $s, \sigma \in 2^{<\omega}$. We have to check $U \subseteq^{\mathtt{fin}} \bigcup D$. We can assume, as above, that $\sigma = \Lambda$, that is, $U = \mathbf{U}^{\mathbf{\Phi}}_m(\to s)$. (Otherwise substitute $\sigma \cdot D = \{\sigma \cdot p : p \in D\}$ for D. If D is pre-dense, then so is $\sigma \cdot D$.) We can also assume that $s = \Lambda$, that is, $U = \mathbf{U}^{\mathbf{\Phi}}_m$, as $\mathbf{U}^{\mathbf{\Phi}}_m(\to s) \subseteq \mathbf{U}^{\mathbf{\Phi}}_m$. Thus let $U = \mathbf{U}^{\mathbf{\Phi}}_m$.

The set $\Delta \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{Sys}(\mathbf{P})$ such that $m \in |\varphi|$ and for every string $t \in 2^n$, where $n = \nu^{\varphi}_m$, there is a tree $S_t \in D$ with $T^{\varphi}_m(t) \subseteq S_t$, is dense in $\mathbf{Sys}(\mathbf{P})$ by Lemma 6.6(iv) due to the pre-density of D itself. Therefore there is an index j such that $\varphi(j) \in \Delta$. Let this be witnessed by trees $S_t \in D, t \in 2^n$, where $n = \nu^{\varphi(j)}_m$, so that $T^{\varphi(j)}_m(t) \subseteq S_t$, $\forall t$, and hence $T^{\varphi(j)}_m(n) \subseteq^{\mathtt{fin}} D$. Then $U = \mathbf{U}^{\mathbf{\Phi}}_m \subseteq T^{\mathbf{\Phi}}_m(n) = T^{\varphi(j)}_m(n) \subseteq^{\mathtt{fin}} \bigcup D$.

8.1(C). Let $T_0 \in \mathbf{P}$, f, and $U, V \in \mathbf{Q}$ be as in 8.1(C). Then $U = \tau \cdot \mathbf{U}^{\mathbf{\Phi}}_K(\to s_0)$, where $s_0, \tau \in 2^{<\omega}$ and $K < \omega$. We can wlog assume that $\tau = \Lambda$, that is, $U = \mathbf{U}^{\mathbf{\Phi}}_K(\to s_0)$, since the general case is reducible to this case by the substitution of $\tau \cdot T_0$ for T_0 and the function $f'(x) = f(\tau \cdot x)$ for f. Thus let $U = \mathbf{U}^{\mathbf{\Phi}}_K(\to s_0)$.

Similarly, generally speaking $V = \rho \cdot \mathbf{U}^{\mathbf{\Phi}}_L(\to t_0)$, where $t_0, \rho \in 2^{<\omega}$ and $L < \omega$. But this is reducible to the case $\rho = \Lambda$ by the substitution of $f'(x) = \rho \cdot f(x)$ for f. Thus let $V = \mathbf{U}^{\mathbf{\Phi}}_L(\to t_0)$. Finally, since $\mathbf{U}^{\mathbf{\Phi}}_L(\to t_0) \subseteq \mathbf{U}^{\mathbf{\Phi}}_L$, we can assume that even $V = \mathbf{U}^{\mathbf{\Phi}}_L$. Now consider the set $\Delta \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{Sys}(\mathbf{P})$ such that there is a number $m < \omega$ satisfying the following:

(I) $K, L \in |\varphi|$, $\nu^{\varphi}_K = \nu^{\varphi}_L = m$, and $\mathtt{lh}(s_0) \leq m$;
(II) if $s \in 2^m$, then $T^{\varphi}_K(s) \subseteq T_0$ or $[T^{\varphi}_K(s)] \cap [T_0] = \varnothing$;
(III) if $s \in 2^m$ and $T^{\varphi}_K(s) \subseteq T_0$, then $[T^{\varphi}_L(m)] \cap (f\text{"}[T^{\varphi}_K(s)]) = \varnothing$.

Lemma 10.1. *The set Δ is dense in* $\mathbf{Sys}(\mathbf{P})$.

Proof. Let $\psi \in \mathbf{Sys}(\mathbf{P})$; we have to define a system $\varphi \in \Delta$ satisfying $\psi \preccurlyeq \varphi$. By Lemma 9.2, we can assume that $K, L \in |\psi|$ and $\nu^{\psi}_K = \nu^{\psi}_L = m - 1$ for some $m \geq \mathtt{lh}(s_0)$. Now we define a first preliminary version of φ, by $\nu^{\varphi}_K = \nu^{\psi}_L = m$ and $T^{\varphi}_K(m) = T^{\psi}_K(m-1)$, $T^{\varphi}_L(m) = T^{\psi}_L(m-1)$, and keeping the other elements of φ equal to those of the system ψ, so that $\psi \preccurlyeq \varphi$. The trees $S = T^{\varphi}_K(m)$ and $T = T^{\varphi}_L(m)$ belong to $\mathbf{Colg}_m(\mathbf{P})$.

The set $D(T_0)$ of all trees $W \in \mathbf{P}$, such that $W \subseteq T_0$ or $[W] \cap [T_0] = \varnothing$, is open dense in $\mathbf{P}$ by 6.1(A). Therefore by Lemma 6.6(iv) there is a tree $S' \in \mathbf{Colg}_m(\mathbf{P})$ satisfying $S' \subseteq_m S$ and if $s \in 2^m$, then

[11] To prove $\mathbf{T}^{\mathbf{\Phi}}_m(s) \subseteq \mathbf{T}^{\mathbf{\Phi}}_m(t)$, note that by construction $\mathbf{T}^{\mathbf{\Phi}}_m(s) = \mathbf{T}^{\mathbf{\Phi}}_m(n)(\to s)$ and $\mathbf{T}^{\mathbf{\Phi}}_m(t) = \mathbf{T}^{\mathbf{\Phi}}_m(k)(\to t)$, where $n = \mathtt{lh}(s) > k = \mathtt{lh}(t)$ However $\mathbf{T}^{\mathbf{\Phi}}_m(n) \subseteq_k \mathbf{T}^{\mathbf{\Phi}}_m(k)$ by construction, part (ii). Therefore $\mathbf{T}^{\mathbf{\Phi}}_m(k)(\to s) \subseteq \mathbf{T}^{\mathbf{\Phi}}_m(k)(\to t) \subseteq \mathbf{T}^{\mathbf{\Phi}}_m(n)(\to t)$, as required.

$S'(\to s) \in D(T_0)$. We modify φ by putting $T^\varphi_K(m) = S'$ instead of S. It is clear that the new system still satisfies $\psi \preccurlyeq \varphi$, and in addition (II) holds by construction.

Further modification of φ towards (III) depends on whether $K = L$.

Case 1: $K \neq L$. If $s, t \in 2^m$, then by Lemma 7.2(i) there exist trees $C, D \in \mathbf{P}$ such that $C \subseteq S'(\to s)$, $D \subseteq T'(\to t)$, and $[D] \cap (f"[C]) = \varnothing$. Applying Lemma 6.6(iii), we get trees $S'', T'' \in \mathbf{Colg}_m(\mathbf{P})$ satisfying $S'' \subseteq_m S'$, $T'' \subseteq_m T'$, $C = S''(\to s)$, $D = T''(\to t)$, so that $[T''(\to t)] \cap (f"[S''(\to s)]) = \varnothing$. Iterate this construction by exhaustion of all pairs $s, t \in 2^m$. We get trees $S^*, T^* \in \mathbf{Colg}_m(\mathbf{P})$ such that $S^* \subseteq_m S'$, $T^* \subseteq_m T'$, and $[T^*(\to t)] \cap (f"[S^*(\to s)]) = \varnothing$ for all $s, t \in 2^m$, that is $[T^*] \cap (f"[S^*]) = \varnothing$. Modify φ by $T^\varphi_K(m) = S^*$ instead of S' and $T^\varphi_L(m) = T^*$ instead of T'. Now (III) holds for the modified φ.

Case 2: $K = L$, and then $S' = T' = T^\varphi_K(m) = T^\varphi_L(m)$. It suffices to show that if $s, t \in 2^m$ (strings of length m) and $S'(\to s) \subseteq T_0$, there is a tree $S'' \in \mathbf{Colg}_m(P)$ such that $S'' \subseteq_m S'$ and $[S''(\to t)] \cap (f"[S''(\to s)]) = \varnothing$. Then the iteration by exhaustion of all those pairs of strings s, t yields a tree $S^* \in \mathbf{Colg}_m(\mathbf{P})$ such that $S^* \subseteq_m S'$ and $[S^*(\to t)] \cap (f"[S^*(\to s)]) = \varnothing$ for all $s, t \in 2^m$ with $S'(\to s) \subseteq T_0$, that is, $[S^*] \cap (f"[S^*(\to s)]) = \varnothing$ whenever $s \in 2^m$ satisfies $S'(\to s) \subseteq T_0$. To achieve (III), it remains to modify the system φ by $T^\varphi_K(m) = S^*$ instead of S'.

Thus let us carry out the construction of S'' for a pair of strings $s, t \in 2^m$ with $S(\to s) \subseteq T_0$. It follows that f is regular on $S'(\to s)$ inside $\mathbf{P}$. By Lemma 5.1(i), we have $S'(\to s) = S'\restriction_u$ and $S'(\to t) = S'\restriction_v$, where $u, v \in S'$ are strings of length $h = \mathtt{lh}(u) = \mathtt{lh}(v)$, and $S'(\to s) = \tau \cdot (S'(\to t))$, where $\tau = u \cdot v$, by Lemma 4.3(ii). Lemma 7.2(ii) yields a pair of trees $U, V \in \mathbf{P}$ satisfying $U \subseteq S'(\to s)$, $V \subseteq S'(\to t)$, $V = \tau \cdot U$, and $[V] \cap (f"[U]) = \varnothing$. Now, twice reducing the tree S' by means of Lemma 6.6(iii), we get a tree $S'' \in \mathbf{Colg}_m(P)$ such that $S'' \subseteq_m S'$ and $S''(\to s) = U$, $S''(\to t) = V$, so that $[S''(\to t)] \cap (f"[S''(\to s)]) = \varnothing$, as required. $\square$ (Lemma)

Now return to the verification of 8.1(C). By the lemma, at least one system $\varphi(j)$ belongs to Δ, that is, conditions (I), (II), (III) are satisfied for $\varphi = \varphi(j)$. The tree $V = \mathbf{U}^{\mathbf{\Phi}}_L$ satisfies $V \subseteq T = T^{\varphi(j)}_L(m)$. Moreover, the tree $U = \mathbf{U}^{\mathbf{\Phi}}_K(\to s_0)$ satisfies $U \subseteq S = \bigcup_{s \in \Sigma} T^{\varphi(j)}_K(s)$, where $\Sigma = \{s \in 2^m : S(\to s) \subseteq T_0\}$, by (I), (II). However $[T] \cap (f"[S]) = \varnothing$ by (III), thus $[V] \cap (f"[U]) = \varnothing$, as required. $\square$ (Theorem 8.3)

11. The blocking sequence of ST-forcings

We argue in the constructible universe L in this section.

The forcing to prove Theorem 1.2 will be defined as the union of a ω_1-sequence of countable ST-forcings, increasing in the sense of a relation $\sqsubset$ (Definition 8.1). We here introduce the notational system to be used in this construction.

Let **STF** be the set of all *countable* ST-forcings. If $\overline{\mathbf{P}} = \langle \mathbf{P}_\xi \rangle_{\xi < \lambda}$ is a transfinite sequence of countable ST-forcings, of length $\mathtt{dom}\overline{\mathbf{P}} = \lambda < \omega_1$, let $\bigcup \overline{\mathbf{P}} = \bigcup_{\xi < \lambda} \mathbf{P}_\xi$, and let $\mathfrak{M}(\overline{\mathbf{P}})$ be the least transitive model of $\mathbf{ZFC}^-$ of the form $\mathbf{L}_\vartheta$, containing $\overline{\mathbf{P}}$, in which λ and $\bigcup \overline{\mathbf{P}}$ are countable.

Definition 11.1. If $\alpha \leq \omega_1$, let $\overline{\mathbf{STF}}_\alpha$ be the set of all α-sequences $\overline{\mathbf{P}} = \langle \mathbf{P}_\xi \rangle_{\xi < \alpha}$ of forcings $\mathbf{P}_\xi \in \mathbf{STF}$, satisfying the following:

(*) if $\gamma < \alpha = \mathtt{dom}\overline{\mathbf{P}}$, $\bigcup (\overline{\mathbf{P}} \restriction \gamma) \sqsubset_{\mathfrak{M}(\overline{\mathbf{P}} \restriction \gamma)} \mathbf{P}_\gamma$.

Let $\overline{\mathbf{STF}} = \bigcup_{\alpha < \omega_1} \overline{\mathbf{STF}}_\alpha$. $\square$

The set $\overline{\mathbf{STF}} \cup \overline{\mathbf{STF}}_{\omega_1}$ is ordered by the extension relations $\subset$ and $\subseteq$.

Lemma 11.2. *Assume that $\kappa < \lambda < \omega_1$, and $\overline{\mathbf{P}} = \langle \mathbf{P}_\xi \rangle_{\xi < \kappa} \in \overline{\mathbf{STF}}$. Then:*

(i) *the union* $\mathbf{P} = \bigcup \overline{\mathbf{P}}$ *is a countable ST-forcing;*
(ii) *there is a sequence* $\overline{\mathbf{Q}} \in \overline{\mathbf{STF}}$ *such that* $\mathtt{dom}(\overline{\mathbf{Q}}) = \lambda$ *and* $\overline{\mathbf{P}} \subset \overline{\mathbf{Q}}$.

Proof. To prove (ii) apply Theorem 8.3 by induction on λ. $\square$

Definition 11.3 *(Key definition).* A sequence $\overline{\mathbf{P}} \in \overline{\mathbf{STF}}$ *blocks* a set $W \subseteq \overline{\mathbf{STF}}$ if either $\overline{\mathbf{P}} \in W$ (the positive block case) or there is no sequence $\overline{\mathbf{Q}} \in W$ satisfying $\overline{\mathbf{P}} \subseteq \overline{\mathbf{Q}}$ (the negative block case). $\square$

Approaching the next *blocking sequence theorem*, we recall that HC is the set of all *hereditarily countable* sets, so that $\mathrm{HC} = \mathbf{L}_{\omega_1}$ in $\mathbf{L}$. See [1, Part B, Chap. 5, Section 4] on definability classes Σ^X_n, Π^X_n, Δ^X_n for any set X, and especially on Σ^{HC}_n, Π^{HC}_n, Δ^{HC}_n for $X = \mathrm{HC}$ in [13, Sections 8, 9] or elsewhere.

In particular, Σ^{HC}_n consists of all sets $U \subseteq \mathrm{HC}$, definable in HC by a parameter-free Σ_n-formula, Π^{HC}_n is defined similarly, and $\Delta^{\mathrm{HC}}_n = \Sigma^{\mathrm{HC}}_n \cap \Pi^{\mathrm{HC}}_n$. In addition, we define $\Sigma^{\mathrm{HC}}_n(\mathrm{HC})$ to contain all sets $U \subseteq \mathrm{HC}$, definable in HC by a Σ_n-formula with arbitrary parameters in HC, $\Pi^{\mathrm{HC}}_n(\mathrm{HC})$ is defined similarly, and $\Delta^{\mathrm{HC}}_n(\mathrm{HC}) = \Sigma^{\mathrm{HC}}_n(\mathrm{HC}) \cap \Pi^{\mathrm{HC}}_n(\mathrm{HC})$.

Theorem 11.4 *(Blocking sequence theorem, in* $\mathbf{L}$*). If* $\mathfrak{n} \geq 3$*, then there is a sequence* $\overline{\mathbb{P}} = \langle \mathbb{P}_\xi \rangle_{\xi<\omega_1} \in \overline{\mathbf{STF}}_{\omega_1}$ *satisfying the following two conditions:*

(i) $\overline{\mathbb{P}}$*, as the set of pairs* $\langle \xi, \mathbb{P}_\xi \rangle$*, belongs to the definability class* $\Delta^{\mathrm{HC}}_{\mathfrak{n}-1}$;
(ii) (genericity of $\overline{\mathbf{P}}$ w.r.t. $\Sigma^{\mathrm{HC}}_{\mathfrak{n}-2}(\mathrm{HC})$ sets) *if* $W \subseteq \overline{\mathbf{STF}}$ *is a* $\Sigma^{\mathrm{HC}}_{\mathfrak{n}-2}(\mathrm{HC})$ *set* (that is parameters from HC are admitted)*, then there is an ordinal* $\gamma < \omega_1$ *such that the restricted sequence* $\overline{\mathbb{P}} \restriction \gamma = \langle \mathbb{P}_\xi \rangle_{\xi<\gamma} \in \overline{\mathbf{STF}}$ *blocks* W*.*

Proof. Let $\leqslant_{\mathbf{L}}$ be the canonical Δ_1 wellordering of $\mathbf{L}$; thus its restriction to $\mathrm{HC} = \mathbf{L}_{\omega_1}$ is Δ^{HC}_1. As $\mathfrak{n} \geq 3$, there exists a universal $\Sigma^{\mathrm{HC}}_{\mathfrak{n}-2}$ set $\mathfrak{U} \subseteq \omega_1 \times \mathrm{HC}$. That is, $\mathfrak{U}$ is $\Sigma^{\mathrm{HC}}_{\mathfrak{n}-2}$ (parameter-free $\Sigma_{\mathfrak{n}-2}$ definable in HC), and for every set $X \subseteq \mathrm{HC}$ of class $\Sigma^{\mathrm{HC}}_{\mathfrak{n}-2}(\mathrm{HC})$ (sets definable in HC by a $\Sigma_{\mathfrak{n}-2}$ formula with arbitrary parameters in HC) there is an ordinal $\alpha < \omega_1$ such that $X = \mathfrak{U}_\alpha$, where $\mathfrak{U}_\alpha = \{x : \langle \alpha, x \rangle \in \mathfrak{U}\}$. The choice of ω_1 as the domain of parameters in the universality property is validated by the assumption $\mathbf{V} = \mathbf{L}$, which implies the existence of a Δ^{HC}_1 surjection $\omega_1 \xrightarrow{\text{onto}} \mathrm{HC}$.

Coming back to Definition 11.3, note that for any sequence $\overline{\mathbf{P}} \in \overline{\mathbf{STF}}$ and any set $W \subseteq \overline{\mathbf{STF}}$ there is a sequence $\overline{\mathbf{Q}} \in \overline{\mathbf{STF}}$ which satisfies $\overline{\mathbf{P}} \subset \overline{\mathbf{Q}}$ and blocks W. This allows us to define $\overline{\mathbf{Q}}_\alpha \in \overline{\mathbf{STF}}$ by induction on $\alpha < \omega_1$ so that $\overline{\mathbf{Q}}_0 = \varnothing$, $\overline{\mathbf{Q}}_\lambda = \bigcup_{\alpha<\lambda} \overline{\mathbf{Q}}_\alpha$, and each $\overline{\mathbf{Q}}_{\alpha+1}$ is equal to the $\leqslant_{\mathbf{L}}$-least sequence $\overline{\mathbf{Q}} \in \overline{\mathbf{STF}}$ which satisfies $\overline{\mathbf{Q}}_\alpha \subset \overline{\mathbf{Q}}$ and blocks $\mathfrak{U}_\alpha$. Then $\overline{\mathbb{P}} = \bigcup_{\alpha<\omega_1} \overline{\mathbf{Q}}_\alpha \in \overline{\mathbf{STF}}_{\omega_1}$.

Condition (ii) holds by construction, while (i) follows by a routine verification, based on the fact that $\overline{\mathbf{STF}} \in \Delta^{\mathrm{HC}}_1$. $\square$

Definition 11.5 *(in* $\mathbf{L}$*).* We fix a natural number $\mathfrak{n} \geq 3$, for which Theorem 1.2 is to be established. We also fix a sequence $\overline{\mathbb{P}} = \langle \mathbb{P}_\xi \rangle_{\xi<\omega_1} \in \overline{\mathbf{STF}}_{\omega_1}$, given by Theorem 11.4 for this $\mathfrak{n}$.

If $\alpha < \omega_1$, then let $\mathfrak{M}_\alpha = \mathfrak{M}(\overline{\mathbb{P}} \restriction \alpha)$ and $\mathbb{P}_{<\alpha} = \bigcup_{\xi<\alpha} \mathbb{P}_\xi$.

Let $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{P}_\xi$. $\square$

12. CCC and some other forcing properties

The ST-forcing $\mathbb{P}$ defined by 11.5 will be the forcing notion for the proof of Theorem 1.2. Here we establish some forcing properties of $\mathbb{P}$, including CCC.

We continue to argue in the conditions and notation of Definition 11.5.

Lemma 12.1. $\mathbb{P}$ *and all sets* $\mathbb{P}_\xi$, $\mathbb{P}_{<\alpha}$ *are ST-forcings. In addition:*

(i) *if* $\alpha < \omega_1$, *then* $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbb{P}_\gamma$;
(ii) *if* $\alpha < \omega_1$ *and the set* $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}_{<\alpha}$ *is pre-dense in* $\mathbb{P}_{<\alpha}$, *then it is pre-dense in* $\mathbb{P}$, *too;*
(iii) *every set* $\mathbb{P}_\alpha$ *is pre-dense in* $\mathbb{P}$;
(iv) *if* $Q \subseteq \mathbf{ST}$ *belongs to* $\Sigma^{\mathrm{HC}}_{\mathfrak{n}-2}(\mathrm{HC})$ *and* $Q^- = \{T \in \mathbf{ST} : \neg \exists S \in Q\,(S \subseteq T)\}$, *then* $\mathbb{P} \cap (Q \cup Q^-)$ *is dense in* $\mathbb{P}$;
(v) *if* $\mathbf{c} \in \mathbf{CCF}$ *is a code of continuous function and*

$$\mathbf{CB_c} = \{T \in \mathbf{ST} : f_{\mathbf{c}}{\restriction}[T] \text{ is a constant or a bijection}\},$$

then the set $\mathbb{P} \cap \mathbf{CB_c}$ *is dense in* $\mathbb{P}$;
(vi) *let* $D = \{T \in \mathbf{ST} : \mathtt{spl}(T)$ *is co-infinite*$\}$ (see Definition 4.1 on $\mathtt{spl}(T)$), *then the set* $\mathbb{P} \cap D$ *is dense in* $\mathbb{P}$.

Proof. (i) holds by (*) of Definition 11.1.

(ii) We use induction on $\gamma, \alpha \le \gamma < \omega_1$, to check that if D is pre-dense in $\mathbb{P}_{<\gamma}$, then it remains pre-dense in $\mathbb{P}_{<\gamma} \cup \mathbb{P}_\gamma = \mathbb{P}_{<\gamma+1}$ by (i) and 8.1(B). Limit steps, including the final step to $\mathbb{P}$ $(\gamma = \omega_1)$ are routine.

(iii) $\mathbb{P}_\alpha$ is dense in $\mathbb{P}_{<\alpha+1} = \mathbb{P}_{<\alpha} \cup \mathbb{P}_\alpha$ by 8.1(A). It remains to refer to (ii).

(iv) Let $T_0 \in \mathbb{P}$, that is, $T_0 \in \mathbb{P}_{<\alpha_0}$, $\alpha_0 < \omega_1$. The set W of all sequences $\overline{\mathbf{P}} \in \overline{\mathbf{STF}}$, such that $\overline{\mathbb{P}}{\restriction}\alpha_0 \subseteq \overline{\mathbf{P}}$ and $\exists T \in Q \cap (\bigcup \overline{\mathbf{P}})\,(T \subseteq T_0)$, belongs to $\Sigma^{\mathrm{HC}}_{\mathfrak{n}-2}(\mathrm{HC})$ along with Q. Therefore there is an ordinal $\alpha < \omega_1$ such that $\overline{\mathbb{P}}{\restriction}\alpha$ blocks W. We have two cases.

Case 1: $\overline{\mathbb{P}}{\restriction}\alpha \in W$. Then the related tree $T \subseteq T_0$ belongs to $Q \cap \mathbb{P}$.

Case 2: there is no sequence in W which extends $\overline{\mathbb{P}}{\restriction}\alpha$. Let $\gamma = \mathtt{max}\{\alpha, \alpha_0\}$. Then $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbb{P}_\gamma$ by (i). As $\alpha_0 \le \gamma$, there is a tree $T \in \mathbb{P}_\gamma$, $T \subseteq T_0$. We claim that $T \in Q^-$, which completes the proof in Case 2.

Suppose to the contrary that $T \notin Q^-$, thus there is a tree $S \in Q$, $S \subseteq T$. The set $\mathbf{R} = \mathbb{P}_\gamma \cup \{\sigma \cdot (S(\to t)) : t, \sigma \in 2^{<\omega}\}$ is a countable ST-forcing and $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbf{R}$ by Lemma 8.2(iv). It follows that the sequence $\overline{\mathbf{R}}$ defined by $\mathtt{dom}\overline{\mathbf{R}} = \gamma + 1$, $\overline{\mathbf{R}}{\restriction}\gamma = \overline{\mathbb{P}}{\restriction}\gamma$, and $\overline{\mathbf{R}}(\gamma) = \mathbf{R}$, belongs to $\overline{\mathbf{STF}}$, and also $\overline{\mathbf{R}} \in W$ because $S \in Q \cap \mathbf{R}$. Yet $\overline{\mathbb{P}}{\restriction}\alpha \sqsubset_{\mathfrak{M}_\gamma} \overline{\mathbf{R}}$ by construction, which contradicts to the Case 2 hypothesis.

(v) A routine verification gives $\mathbf{CB_c} \in \Sigma^{\mathrm{HC}}_1(\{\mathbf{c}\})$. The set $\mathbf{CB_c}$ is dense in $\mathbf{ST}$ by Corollary 5.4, thus $(\mathbf{CB_c})^-$ is empty. Now the result follows from (iv).

(vi) Similarly to (v), $D \in \Sigma^{\mathrm{HC}}_1$ and D is dense in $\mathbf{ST}$. □

Corollary 12.2. *If* $\alpha < \omega_1$ *and trees* $T, T' \in \mathbb{P}_{<\alpha}$ *are incompatible in* $\mathbb{P}_{<\alpha}$, *then* T, T' *are incompatible in* $\mathbb{P}$, *too.*

Proof. Prove by induction on γ that if $\alpha < \gamma \le \omega_1$, then T, T' are incompatible in $\mathbb{P}_{<\gamma}$, using Lemma 12.1(i), and Lemma 8.2(iii) on limit steps. □

To prove CCC we'll need the following lemma.

Lemma 12.3 *(in* $\mathbf{L}$*). If* $X \subseteq \mathrm{HC} = \mathbf{L}_{\omega_1}$, *then the set* $\mathscr{O}_X$ *of all ordinals* $\alpha < \omega_1$ *such that the model* $\langle \mathbf{L}_\alpha; X \cap \mathbf{L}_\alpha \rangle$ *is an elementary submodel of* $\langle \mathbf{L}_{\omega_1}; X \rangle$ *and* $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$, *is unbounded in* ω_1. *More generally, if* $X_n \subseteq \mathrm{HC}$ *for all* n, *then the set* $\mathscr{O}$ *of all ordinals* $\alpha < \omega_1$ *such that* $\langle \mathbf{L}_\alpha; \langle X_n \cap \mathbf{L}_\alpha \rangle_{n<\omega} \rangle$ *is an elementary submodel of* $\langle \mathbf{L}_{\omega_1}; \langle X_n \rangle_{n<\omega} \rangle$ *and* $\langle X_n \cap \mathbf{L}_\alpha \rangle_{n<\omega} \in \mathfrak{M}_\alpha$, *is unbounded in* ω_1.

Proof. Let $\alpha_0 < \omega_1$. There is a countable elementary submodel M of $\langle \mathbf{L}_{\omega_2}; \in \rangle$ which contains α_0, ω_1, X and is such that the set $M \cap \mathbf{L}_{\omega_1}$ is transitive. Consider the Mostowski collapse $\phi : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$. Let $\alpha = \phi(\omega_1)$. Then $\alpha_0 < \alpha < \lambda < \omega_1$ and $\phi(X) = X \cap \mathbf{L}_\alpha$ by the choice of M. We conclude that $\langle \mathbf{L}_\alpha; X \cap \mathbf{L}_\alpha \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; X \rangle$. And α is uncountable in $\mathbf{L}_\lambda$, hence $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\alpha$. It follows that $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$, as $X \cap \mathbf{L}_\alpha \in \mathbf{L}_\lambda$ by construction.

The more general claim is proved similarly. □

Corollary 12.4 *(in $\mathbf{L}$). $\mathbb{P}$ is a CCC forcing. Therefore $\mathbb{P}$-generic extensions preserve cardinals.*

Proof. Consider any maximal antichain $A \subseteq \mathbb{P}$. By Lemma 12.3 there is an ordinal α such that $\langle \mathbf{L}_\alpha; \mathbb{P}', A' \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; \mathbb{P}, A \rangle$, where $\mathbb{P}' = \mathbb{P} \cap \mathbf{L}_\alpha$ and $A' = A \cap \mathbb{P}_{<\alpha}$, and in addition $\mathbb{P}', A' \in \mathfrak{M}_\alpha$. By the elementarity, we have $\mathbb{P}' = \mathbb{P}_{<\alpha}$ and $A' = A \cap \mathbb{P}_{<\alpha} \in \mathfrak{M}_\alpha$, and A' is a maximal antichain, hence a pre-dense set, in $\mathbb{P}_{<\alpha}$. But then A' is a pre-dense set, hence, a maximal antichain, in the whole set $\mathbb{P}$ by Lemma 12.1(ii). Thus $A = A'$ is countable. □

13. Generic model

This section presents some properties of $\mathbb{P}$-generic extensions $\mathbf{L}[G]$ of $\mathbf{L}$ obtained by adjoining a $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ to $\mathbf{L}$. Recall that the forcing notion $\mathbb{P} \in \mathbf{L}$ was introduced by Definition 11.5, along with some related notation.

The next lemma involves the coding system for continuous maps introduced in Section 7. If $G \subseteq \mathbb{P}$ is generic over $\mathbf{L}$ and $\mathbf{c} \in \mathbf{CCF} \cap \mathbf{L}$, then define $\mathbf{c}[G] = f_{\mathbf{c}}(\mathbf{a}[G]) \in 2^\omega \cap \mathbf{L}[G]$; for the definition of $\mathbf{a}[G]$ see Remark 6.2.

Lemma 13.1 *(Continuous reading of names). If a set $G \subseteq \mathbb{P}$ is generic over $\mathbf{L}$ and $x \in 2^\omega \cap \mathbf{L}[G]$, then there exists a code $\mathbf{c} \in \mathbf{CCF} \cap \mathbf{L}$ such that $x = \mathbf{c}[G]$.*

Proof. One of basic forcing lemmas (Lemma 2.5 in [1, Chap. 4]) claims that there is a $\mathbb{P}$-name $t \in \mathbf{L}$ for x, satisfying $x = t[G]$ (the G-valuation of t), and it can be assumed that $\mathbb{P}$ forces that t is valuated as a real in 2^ω. Then the sets $F_{ni} = \{T \in \mathbb{P} : T \models t(n) = i\}$ ($n < \omega$ and $i = 0, 1$) satisfy the following:

(1) the indexed set $\langle F_{ni} \rangle_{n<\omega \wedge i=0,1}$ belongs to $\mathbf{L}$;
(2) if $n < \omega$, $S \in F_{n0}$, $T \in F_{n1}$, then S, T are incompatible in $\mathbb{P}$;
(3) if $n < \omega$, then the set $F_n = F_{n0} \cup F_{n1}$ is open dense in $\mathbb{P}$.

We argue in $\mathbf{L}$. Pick a maximal antichain $A_n \subseteq F_n$ in each F_n. Then all sets A_n are maximal antichains in $\mathbb{P}$ by (3), and all A_n are countable by Corollary 12.4. Therefore there is an ordinal $\alpha < \omega_1^{\mathbf{L}}$ such that the set $\bigcup_n A_n \subseteq \mathbb{P}_{<\alpha}$ and the sequence $\langle A_n \rangle_{n<\omega}$ belong to $\mathfrak{M}_\alpha$. Note that $G \cap \mathbb{P}_\alpha \neq \varnothing$ by Lemma 12.1(iii); let $U \in G \cap \mathbb{P}_\alpha$. As $\mathbb{P}_{<\alpha} \sqsubset_{\mathfrak{M}_\alpha} \mathbb{P}_\alpha$ by Lemma 12.1(i), we have $U \subseteq^{\mathtt{fin}} \bigcup A_n$ for all n, hence there is a finite set $A'_n \subseteq A_n$ such that $U \subseteq \bigcup A'_n$.

Let $A'_{ni} = A'_n \cap F_{ni}$ and $X_{ni} = [U] \cap \bigcup_{T \in A'_{ni}} [T]$, $i = 0, 1$. We claim that $X_{n0} \cap X_{n1} = \varnothing$. Indeed otherwise there exist trees $T_i \in F_{ni}$ such that $Z = [U] \cap [T_0] \cap [T_1]$ is non-empty. By Lemma 8.2(i), Z is clopen in $[U]$. Therefore there is a tree $U' \in \mathbb{P}_\alpha$ such that $[U'] \subseteq Z$, hence, T_0 and T_1 are compatible in $\mathbb{P}$, which contradicts (2) by construction. Thus indeed $X_{n0} \cap X_{n1} = \varnothing$.

As clearly $X_{n0} \cup X_{n1} = [U]$, the sets X_{ni} are relatively clopen in $[U]$. Therefore the map $g : [U] \to 2^\omega$, defined so that $g(x)(n) = i$ iff $x \in X_{ni}$, is continuous. By the Tietze extension theorem, the map g can be extended to a continuous $f : 2^\omega \to 2^\omega$, that is $f(x)(n) = i$ whenever $x \in X_{ni}$ — for all $n < \omega$ and $i = 0, 1$. We have $f = f_{\mathbf{c}}$, where $\mathbf{c} \in \mathbf{CCF} \cap \mathbf{L}$.

We argue in $\mathbf{L}[G]$. Prove that $x = \mathbf{c}[G] = f_{\mathbf{c}}(\mathbf{a}[G])$. Suppose that *e.g.* $x(n) = 0$. As $U \in G$, we have $\mathbf{a}[G] \in [U]$. Therefore $\mathbf{a}[G] \in [T]$ for some $T \in A'_n$, and then $T \in G$ as well. If $T \in A'_{n1}$, then by definition $T \models t(n) = 1$, and hence $x(n) = 1$ because $x = t[G]$ is the G-valuation of t. The contradiction obtained shows that $T \in A'_{n1}$ is impossible. It follows that $T \in A'_{n0}$. Then any $a \in [T]$ satisfies $g(a)(n) = 0$, hence, $f(a)(n) = 0$, thus we have $f_{\mathbf{c}}(\mathbf{a}[G])(n) = 0$, as required. □

Lemma 13.2. *If $G \subseteq \mathbb{P}$ is generic over* $\mathbf{L}$*, then* $\mathbf{a}[G]$ *is not* OD *in* $\mathbf{L}[G]$.

Proof. Assume to the contrary that $\vartheta(x)$ is a formula with ordinal parameters, and a tree $T \in G$ $\mathbb{P}$-forces that $\mathbf{a}[G]$ is the only real $x \in 2^\omega$ satisfying $\vartheta(x)$. Let $s = \mathtt{stem}(T)$ and $n = \mathtt{lh}(s)$. Then T contains both $s^\frown 0$ and $s^\frown 1$. Then either $s^\frown 0 \subset \mathbf{a}[G]$ or $s^\frown 1 \subset \mathbf{a}[G]$. Let say $s^\frown 0 \subset \mathbf{a}[G]$.

Let $\sigma = 0^{n\frown} 1$, so that the strings $s^\frown 0$, $s^\frown 1$, σ belong to 2^{n+1}, $s^\frown 1 = \sigma \cdot s^\frown 0$, and $\sigma \cdot T = T$ by Lemma 4.3(ii). As $\mathbb{P}$ is invariant under the action of σ, the set $G' = \sigma \cdot G = \{\sigma \cdot S : S \in G\}$ is $\mathbb{P}$-generic over $\mathbf{L}$, and $T = \sigma \cdot T \in G'$. We conclude that it is true in $\mathbf{L}[G'] = \mathbf{L}[G]$ that $\mathbf{a}[G'] = \sigma \cdot \mathbf{a}[G]$ is still the unique real in 2^ω satisfying $\vartheta(\mathbf{a}[G'])$. But $\mathbf{a}[G'] \neq \mathbf{a}[G]$! □

14. Definability of the set of generic reals

We continue to argue in the context of Definition 11.5. The goal of this section is to study the definability of the set of all $\mathbb{P}$-generic reals $x \in 2^\omega$ in $\mathbb{P}$-generic extensions of $\mathbf{L}$.

Lemma 14.1. *In a transitive model of* $\mathbf{ZF}$ *extending* $\mathbf{L}$*, it is true that a real $x \in 2^\omega$ is $\mathbb{P}$-generic over* $\mathbf{L}$ *iff x belongs to the set* $\mathit{GEN}_{\mathbb{P}} = \bigcap_{\alpha<\omega_1^{\mathbf{L}}} \bigcup_{T\in\mathbb{P}_\alpha} [T]$.

Proof. All sets $\mathbb{P}_\alpha$ are pre-dense in $\mathbb{P}$ by Lemma 12.1(iii). Therefore all $\mathbb{P}$-generic reals belong to $\mathrm{GEN}_{\mathbb{P}}$. On the other hand, any maximal antichain $A \in \mathbf{L}, A \subseteq \mathbb{P}$ is countable in $\mathbf{L}$ by Corollary 12.4, and hence $A \subseteq \mathbb{P}_{<\alpha}$ and $A \in \mathfrak{M}_\alpha$ for some index $\alpha < \omega_1^{\mathbf{L}}$. But then every tree $T \in \mathbb{P}_\alpha$ satisfies $T \subseteq^{\mathtt{fin}} \bigcup A$ by Lemma 8.2(ii). We conclude that $\bigcup_{T\in\mathbb{P}_\alpha}[T] \subseteq \bigcup_{S\in A}[S]$. □

According to the next lemma, $\mathbb{P}$-generic extensions do not contain $\mathbb{P}$-generic reals, except the real $\mathbf{a}[G]$ itself and reals connected to $\mathbf{a}[G]$ in terms of the equivalence relation E_0 (see Footnote 7). We observe that the E_0-class

$$[x]_{\mathsf{E}_0} = \{y \in 2^\omega : x\mathsf{E}_0 y\} = \{y \in 2^\omega : \exists\, s \in 2^{<\omega}(y = s \cdot x)\}$$

of any real $x \in 2^\omega$ is a countable set.

Lemma 14.2. *Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic set over* $\mathbf{L}$*. Then it is true in* $\mathbf{L}[G]$ *that* $\mathit{GEN}_{\mathbb{P}} = [\mathbf{a}[G]]_{\mathsf{E}_0}$.

Proof. The real $\mathbf{a}[G]$ is $\mathbb{P}$-generic, hence $\mathbf{a}[G] \in \mathrm{GEN}_{\mathbb{P}}$ by Lemma 14.1. Yet every set $\mathbb{P}_\alpha$ is a ST-forcing, that is by definition it is closed under the action $s \cdot T$ of any string $s \in 2^{<\omega}$. This implies $[\mathbf{a}[G]]_{\mathsf{E}_0} \subseteq \mathrm{GEN}_{\mathbb{P}}$.

To prove in the other direction, assume to the contrary that $x \in \mathbf{L}[G] \cap 2^\omega$, $x \in \mathrm{GEN}_{\mathbb{P}} \smallsetminus [\mathbf{a}[G]]_{\mathsf{E}_0}$. By Lemma 13.1, there is a code $\mathbf{c} \in \mathbf{L} \cap \mathbf{CCF}$ such that $x = \mathbf{c}[G]$. By the contrary assumption there is a tree $T_0 \in G$ which forces

$$\mathbf{c}[\underline{G}] \in \mathrm{GEN}_{\mathbb{P}} \smallsetminus [\mathbf{a}[\underline{G}]]_{\mathsf{E}_0},$$

where $\underline{G}$ is the name of G in the $\mathbb{P}$-forcing language.

Fix an ordinal $\alpha < \omega_1^{\mathbf{L}}$ such that $\mathbf{c} \in \mathfrak{M}_\alpha$. We claim that (in $\mathbf{L}$) *the function $f = f_{\mathbf{c}}$ is regular on T_0 inside $\mathbb{P}_{<\alpha}$*.

Indeed otherwise there exist $\sigma \in 2^{<\omega}$ and $T \in \mathbb{P}_{<\alpha}$ such that $T \subseteq T_0$ and $\sigma \cdot f(x) = x$ for all $x \in T$. Then T forces $\sigma \cdot \mathbf{c}[\underline{G}] = \mathbf{a}[\underline{G}]$, that is, forces $\mathbf{c}[\underline{G}] \in [\mathbf{a}[\underline{G}]]_{\mathsf{E}_0}$, contrary to the choice of T_0. The regularity is established.

Recall that $\mathbb{P}_{<\alpha} \sqsubset_{\mathfrak{M}_\alpha} \mathbb{P}_\alpha$ by Lemma 12.1(i). Therefore by 8.1(A), there is a tree $U \in \mathbb{P}_\alpha$, $U \subseteq T_0$. And by 8.1(C) if $V \in \mathbb{P}_\alpha$, then $[V] \cap (f``[U]) = \varnothing$. By Shoenfield absoluteness, the equality $[V] \cap (f``[U]) = \varnothing$ holds in any generic extension. It follows that if $V \in \mathbb{P}_\alpha$, then U forces $\mathbf{c}[\underline{G}] \notin [V]$, hence forces $\mathbf{c}[\underline{G}] \notin \mathrm{GEN}_{\mathbb{P}}$, which contradicts to the choice of T_0. $\square$

Corollary 14.3. *Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic set over $\mathbf{L}$. Then the E_0-class $[\mathbf{a}[G]]_{\mathsf{E}_0}$ is a $\Pi^1_{\mathfrak{n}}$ set in $\mathbf{L}[G]$.*

Proof. A routine verification of $\mathrm{GEN}_{\mathbb{P}} \in \Pi^1_{\mathfrak{n}}$ in $\mathbf{L}[G]$ using the Property 11.4(i) of the sequence $\overline{\mathbf{P}}$ is left to the reader. $\square$

15. Auxiliary forcing relation

Here we introduce a key tool for the proof of Claim (i) of Theorem 1.2. This is a forcing-like relation $\mathtt{forc}$. It is not explicitly connected with the forcing notion $\mathbb{P}$ (but rather connected with the full Silver forcing $\mathbf{ST}$), however it will be compatible with $\mathbb{P}$ for formulas of certain quantifier complexity (Lemma 17.1). The crucial advantage of $\mathtt{forc}$ will be its invariance under two certain groups of transformations (Lemma 16.1), a property that cannot be expected for $\mathbb{P}$. This will be the key argument in the proof of Theorem 17.2 below.

We argue in L.

We consider a *language* $\mathscr{L}$ containing variables $i, j, k, \dots$ of type 0 with the domain ω and variables $x, y, z, \dots$ of type 1 with the domain 2^ω. *Terms* are variables of type 0 and expressions like $x(k)$. Atomic formulas are those of the form $R(t_1, \dots, t_n)$, where $R \subseteq \omega^n$ is any n-ary relation on ω in $\mathbf{L}$. A formula is *arithmetic* if it does not contain quantifiers with variables of type 1. Formulas of the form

$$\exists\, x_1\, \forall\, x_2\, \exists\, x_3\, \dots\, \exists\,(\forall\,)\, x_n\, \Psi \quad \text{and} \quad \forall\, x_1\, \exists\, x_2\, \forall\, x_3\, \dots\, \forall\,(\exists\,)\, x_n\, \Psi\,,$$

where Ψ is arithmetic, are of types $\mathscr{L}\Sigma^1_n$, resp., $\mathscr{L}\Pi^1_n$.

In addition we allow codes $\mathbf{c} \in \mathbf{CCF}$ to substitute free variables of type 1. The semantics is as follows. Let $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_k)$ be an $\mathscr{L}$-formula, with all codes in $\mathbf{CCF}$ explicitly indicated, and let $x \in 2^\omega$. By $\varphi[x]$ we denote the formula $\varphi(f_{\mathbf{c}_1}(x), \dots, f_{\mathbf{c}_k}(x))$, where all $f_{\mathbf{c}_i}(x)$ are reals in 2^ω, of course.

Arithmetic formulas and those in $\mathscr{L}\Sigma^1_n \cup \mathscr{L}\Pi^1_n$, $n \ge 1$, will be called *normal*. If φ is a formula in $\mathscr{L}\Sigma^1_n$ or $\mathscr{L}\Pi^1_n$, then φ^- is the result of canonical transformation of $\neg\,\varphi$ to $\mathscr{L}\Pi^1_n$, resp., $\mathscr{L}\Sigma^1_n$ form. For arithmetic formulas, let $\varphi^- := \neg\,\varphi$.

Definition 15.1 *(in* **L***).* We define a relation $T\,\mathtt{forc}\,\varphi$ between trees $T \in \mathbf{ST}$ and closed normal $\mathscr{L}$-formulas by induction on the complexity of the formulas:

(I) if φ is a closed $\mathscr{L}$-formula, arithmetic or in $\mathscr{L}\Sigma^1_1 \cup \mathscr{L}\Pi^1_1$, then $T\,\mathtt{forc}\,\varphi$ iff $\varphi[x]$ holds for all $x \in [T]$[12];

(II) if $\varphi := \exists\, x\, \psi(x)$ is a closed $\mathscr{L}\Sigma^1_{n+1}$ formula, $n \ge 1$ (ψ being of type $\mathscr{L}\Pi^1_n$), then $T\,\mathtt{forc}\,\varphi$ iff there is a code $\mathbf{c} \in \mathbf{CCF}$ such that $T\,\mathtt{forc}\,\psi(\mathbf{c})$;

(III) if φ is a closed $\mathscr{L}\Pi^1_n$ formula, $n \ge 2$, then $T\,\mathtt{forc}\,\varphi$ iff there is no tree $S \in \mathbf{ST}$ such that $S \subseteq T$ and $S\,\mathtt{forc}\,\psi^-$. $\square$

[12] One may consider it somewhat confusing that the base of induction contains both arithmetic formulas and those of classes $\mathscr{L}\Sigma^1_1$ and $\mathscr{L}\Pi^1_1$. The key issue here is the complexity of the forcing relation for $\mathscr{L}\Pi^1_1$-formulas in Lemma 15.2. The given definition maintains it to be $\mathbf{\Pi}^1_1$, hence, $\Delta^{\mathrm{HC}}_1(\mathrm{HC})$, then we estimate it to be $\Sigma^{\mathrm{HC}}_1(\mathrm{HC})$ for $\mathscr{L}\Sigma^1_2$-formulas by (II), $\Pi^{\mathrm{HC}}_1(\mathrm{HC})$ for $\mathscr{L}\Pi^1_2$-formulas by (III), and so on. If alternatively we keep (I) only for arithmetic formulas, and handle $\mathscr{L}\Sigma^1_1$ by (II) and $\mathscr{L}\Pi^1_1$ by (III), then we get class $\Delta^{\mathrm{HC}}_1(\mathrm{HC})$ for arithmetic formulas by (I), $\Sigma^{\mathrm{HC}}_1(\mathrm{HC})$ for $\mathscr{L}\Sigma^1_1$-formulas by (II), and $\Pi^{\mathrm{HC}}_1(\mathrm{HC})$ for $\mathscr{L}\Pi^1_1$-formulas by (III), then again $\Sigma^{\mathrm{HC}}_2(\mathrm{HC})$ for $\mathscr{L}\Sigma^1_2$-formulas and $\Pi^{\mathrm{HC}}_2(\mathrm{HC})$ for $\mathscr{L}\Pi^1_2$-formulas by (II), (III) — which is essentially worse than we have now by Lemma 15.2.

If $\varphi(x_1, \dots, x_n)$ is a normal $\mathscr{L}$-formula then let

$$\mathbf{Forc}(\varphi) = \{\langle \mathbf{c}_1, \dots, \mathbf{c}_n, T\rangle \in \mathbf{CCF}^n \times \mathbf{ST} : T\,\mathtt{forc}\,\varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)\}$$

and $\mathbf{Des}(\varphi) = \mathbf{Forc}(\varphi) \cup \mathbf{Forc}(\varphi^-)$. In particular, $\mathbf{Forc}(\varphi) = \{T \in \mathbf{ST} : T\,\mathtt{forc}\,\varphi\}$ for closed formulas φ.

Lemma 15.2 *(in* **L***). If φ is a formula in $\mathscr{L}\Pi^1_1$, then the set* $\mathbf{Forc}(\varphi)$ *belongs to $\Delta^{\mathrm{HC}}_1(\mathrm{HC})$. If $m \ge 2$ and φ is a formula in $\mathscr{L}\Sigma^1_m$, resp., $\mathscr{L}\Pi^1_m$, then the set* $\mathbf{Forc}(\varphi)$ *belongs to $\Sigma^{\mathrm{HC}}_{m-1}(\mathrm{HC})$, resp., $\Pi^{\mathrm{HC}}_{m-1}(\mathrm{HC})$.*

Proof. For $\mathscr{L}\Pi^1_1$ formulas, Definition 15.1(I) implies $\mathbf{Forc}(\varphi) \in \mathbf{\Pi}^1_1$.

It follows that $\mathbf{Forc}(\varphi)$ belongs to $\Delta^{\mathrm{HC}}_1(\mathrm{HC})$. Indeed, to get a Σ^{HC}_1 definition from a Π^1_1 definition, recall that the latter can be represented in the form of the well-foundedness of a certain tree in $\omega^{<\omega}$, which is equivalent to the existence of a map from the tree onto a countable ordinal, which (map) witness the well-foundedness. This yields a Σ^1_{HC} definition required.

As soon as the result for $\mathscr{L}\Pi^1_1$-formulas is established, the inductive proof for higher levels goes on straightforwardly by Definition 15.1(II), (III), the inductive hypothesis for any step $m \mapsto m+1$ $(m \ge 1)$ being that $\mathbf{Forc}(\varphi)$ belongs to $\Delta^{\mathrm{HC}}_m(\mathrm{HC})$ $(m = 1)$ or $\Pi^{\mathrm{HC}}_{m-1}(\mathrm{HC})$ $(m \ge 2)$, for any formula $\varphi \in \mathscr{L}\Pi^1_m$. □

Recall that a number $\mathbb{n} \ge 3$ is fixed by Definition 11.5.

Lemma 15.3 *(in* **L***). Let φ be a closed normal $\mathscr{L}$-formula. Then the set* $\mathbf{Des}(\varphi)$ *is dense in* $\mathbf{ST}$*. If φ is of type $\mathscr{L}\Sigma^1_m$, $m < \mathbb{n}$, then* $\mathbf{Des}(\varphi) \cap \mathbb{P}$ *is dense in* $\mathbb{P}$*.*

Proof. It suffices to establish the density of $\mathbf{Des}(\varphi)$ for formulas φ as in (I). If φ is such, then the set $X(\varphi) = \{x \in [T] : \varphi[x]\}$ belongs to $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$, that is, it has the Baire property inside $[T]$. Therefore at least one of the two complimentary sets $X(\varphi)$, $X(\varphi^-)$ is not meager in $[T]$. It remains to apply Corollary 5.4.

The second claim follows from the first one by Lemmas 15.2 and 12.1(iv). □

16. Invariance

It happens that the relation $\mathtt{forc}$ is invariant under two rather natural groups of transformations of $\mathbf{ST}$. Here we prove the invariance. **We still argue in L.**

First group. Let $h \subseteq \omega$. If $x \in 2^\omega$, then a real $h \cdot x \in 2^\omega$ is defined by $(h \cdot x)(j) = 1 - x(j)$ for $j \in h$, but $(h \cdot x)(j) = x(j)$ for $j \notin h$. If $X \subseteq 2^\omega$, then let $h \cdot X = \{h \cdot x : x \in X\}$. Accordingly, if $s \in 2^{<\omega}$ and $n = \mathtt{lh}(s)$, then a string $h \cdot s \in 2^{<\omega}$ is defined by $\mathtt{dom}(h \cdot s) = n = \mathtt{dom}s$ and if $j < n$, then $(h \cdot x)(j) = 1 - x(j)$ for $j \in h$, but $(h \cdot x)(j) = x(j)$ for $j \notin h$. If $T \subseteq 2^{<\omega}$, then let $h \cdot T = \{h \cdot s : s \in T\}$. Then obviously $T \in \mathbf{ST}$ iff $h \cdot T \in \mathbf{ST}$.

If $f : 2^\omega \to 2^\omega$, then a function $h \cdot f : 2^\omega \to 2^\omega$ is defined by $(h \cdot f)(x) = f(h \cdot x)$, equivalently, $(h \cdot f)(h \cdot x) = f(x)$. If f is continuous, then $f = f_{\mathbf{c}}$, where $\mathbf{c} \in \mathbf{CCF}$, and there is a canonical definition of a code $h \cdot \mathbf{c} \in \mathbf{CCF}$ such that $h \cdot (f_{\mathbf{c}}) = f_{h \cdot \mathbf{c}}$.

Finally if $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_k)$ is a $\mathscr{L}$-formula, then let $h\varphi$ be the formula $\varphi(h \cdot \mathbf{c}_1, \dots, h \cdot \mathbf{c}_k)$. Then $(h\varphi)[h \cdot x]$ coincides with $\varphi[x]$.

Second group. Let IB be the set of all idempotent bijections $b : \omega \xrightarrow{\text{onto}} \omega$, that is, we require that $b(j) = b^{-1}(j), \forall j$. If $x \in 2^\omega$, then define $b \cdot x \in 2^\omega$ by $(b \cdot x)(j) = x(b(j)), \forall j$. Let $b \cdot X = \{b \cdot x : x \in X\}$, for $X \subseteq 2^\omega$. If $T \in \mathbf{ST}$, then put $b \cdot T = \{x \restriction m : x \in (b \cdot [T]) \wedge m < \omega\}$. Clearly $T \in \mathbf{ST}$ iff $b \cdot T \in \mathbf{ST}$.

If $f : 2^\omega \to 2^\omega$, then a function $b \cdot f : 2^\omega \to 2^\omega$ is defined similarly to the above by $(b \cdot f)(x) = f(b \cdot x)$, equivalently, $(b \cdot f)(b \cdot x) = f(x)$. If f is continuous, then $f = f_{\mathbf{c}}$, where $\mathbf{c} \in \mathbf{CCF}$, and still there is a canonical definition of a code $b \cdot \mathbf{c} \in \mathbf{CCF}$ such that $b \cdot (f_{\mathbf{c}}) = f_{b \cdot \mathbf{c}}$.

And finally if $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_k)$ is a $\mathscr{L}$-formula, then let $b\varphi$ be the formula $\varphi(b \cdot \mathbf{c}_1, \dots, b \cdot \mathbf{c}_k)$. Then $(b\varphi)[b \cdot x]$ coincides with $\varphi[x]$.

Lemma 16.1 *(in* $\mathbf{L}$*). Let* $T \in \mathbf{ST}$ *and* φ *be a closed normal* $\mathscr{L}$*-formula. Then*

(i) *if* $h \subseteq \omega$*, then* $T\,\mathtt{forc}\,\varphi$ *iff* $(h \cdot T)\mathtt{forc}\,h\varphi$*;*
(ii) *if* $b \in \mathrm{IB}$*, then* $T\,\mathtt{forc}\,\varphi$ *iff* $(b \cdot T)\mathtt{forc}\,b\varphi$*.*

Proof. (i) If φ is of type 15.1(I), then it suffices to note that, first, $[h \cdot T] = \{h \cdot x : x \in [T]\}$, and second, if $x \in [T]$, then $\varphi[x]$ coincides with $(h\varphi)[h \cdot x]$. A routine induction based on Definition 15.1(II), (III) is left to the reader. $\square$

Corollary 16.2. *Let* φ *be a closed normal* $\mathscr{L}$*-formula, such that: if a code* $\mathbf{c}$ *occurs in* φ*, then* $f_{\mathbf{c}}$ *is a constant. Assume that* $S, T \in \mathbf{ST}$ *and the splitting sets* $\mathtt{spl}(S)$*,* $\mathtt{spl}(T)$ (see Section 4) *are both co-infinite. Then* $S\,\mathtt{forc}\,\varphi$ *iff* $T\,\mathtt{forc}\,\varphi$*.*

Proof. Assume that $S\,\mathtt{forc}\,\varphi$. As both $\mathtt{spl}(S)$ and $\mathtt{spl}(T)$ are co-infinite (and they are infinite anyway), there is a bijection $b \in \mathrm{IB}$ such that $b"(\mathtt{spl}(S)) = \mathtt{spl}(T)$. Then the tree $S' = b \cdot S$ satisfies $\mathtt{spl}(S') = \mathtt{spl}(T)$, and still $S'\mathtt{forc}\,\varphi$ by Lemma 16.1(ii), as all occurring codes define constant functions, and hence $b\varphi$ and φ essentially coincide. Now, as $\mathtt{spl}(S') = \mathtt{spl}(T)$, there is a set $h \subseteq \omega \smallsetminus \mathtt{spl}(S') = \omega \smallsetminus \mathtt{spl}(T)$ such that $T = h \cdot S'$. Then $T\,\mathtt{forc}\,\varphi$ by Lemma 16.1(i) and the hypothesis, that all codes involved are codes of constant functions, just as above. $\square$

17. The final argument

Recall that $\mathtt{n} \geq 3$ is fixed by Definition 11.5.

The last part of the proof of Theorem 1.2 will be Lemma 17.2. Note the key ingredient of the proof: we surprisingly approximate the forcing $\mathbb{P}$, definitely non-invariant under the transformations considered in Section 16, by the invariant relation $\mathtt{forc}$, using the next lemma.

Lemma 17.1. *Assume that* $1 \leq n < \mathtt{n}$*,* $\varphi \in \mathbf{L}$ *is a closed formula in* $\mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$*, and a set* $G \subseteq \mathbb{P}$ *is generic over* $\mathbf{L}$*. Then the sentence* $\varphi[\mathbf{a}[G]]$ *is true in* $\mathbf{L}[G]$ *if and only if* $\exists T \in G\,(T\,\mathtt{forc}\,\varphi)$*.*

Proof. Base of induction: φ is arithmetic or belongs to $\mathscr{L}\Sigma^1_1 \cup \mathscr{L}\Pi^1_1$, as in 15.1(I). If $T \in G$ and $T\,\mathtt{forc}\,\varphi$, then $\varphi[\mathbf{a}[G]]$ holds by the Shoenfield absoluteness theorem, as $\mathbf{a}[G] \in [T]$. The inverse holds by Lemma 15.3.

Step $\mathscr{L}\Pi^1_n \Longrightarrow \mathscr{L}\Sigma^1_{n+1}$**.** Let φ be $\exists\,x\,\psi(x)$ where ψ is of type $\mathscr{L}\Pi^1_n$. Assume that $T \in G$ and $T\,\mathtt{forc}\,\varphi$. Then by Definition 15.1(II) there is a code $\mathbf{c} \in \mathbf{CCF} \cap \mathbf{L}$ such that $T\,\mathtt{forc}\,\psi(\mathbf{c})$. By the inductive hypothesis, the formula $\psi(\mathbf{c})[\mathbf{a}[G]]$, that is, $\psi[G](f_{\mathbf{c}}(\mathbf{a}[G]))$, is true in $\mathbf{L}[G]$. But then $\varphi[\mathbf{a}[G]]$ is obviously true as well.

Conversely assume that $\varphi[\mathbf{a}[G]]$ is true. Then there is a real $y \in \mathbf{L}[G] \cap 2^\omega$ such that $\psi[\mathbf{a}[G]](y)$ is true. By Lemma 13.1, $y = f_{\mathbf{c}}(\mathbf{a}[G])$ for a code $\mathbf{c} \in \mathbf{CCF} \cap \mathbf{L}$. But then $\psi(\mathbf{c})[\mathbf{a}[G]]$ is true in $\mathbf{L}[G]$. By the inductive hypothesis, there is a tree $T \in G$ satisfying $T\,\mathtt{forc}\,\psi(\mathbf{c})$. Then $T\,\mathtt{forc}\,\varphi$ as well.

Step $\mathscr{L}\Sigma^1_n \Longrightarrow \mathscr{L}\Pi^1_n$**.** Let φ be a $\mathscr{L}\Pi^1_n$ formula, $n \geq 2$. By Lemma 15.3, there is a tree $T \in G$ such that either $T\,\mathtt{forc}\,\varphi$ or $T\,\mathtt{forc}\,\varphi^-$. If $T\,\mathtt{forc}\,\varphi^-$, then $\varphi^-[\mathbf{a}[G]]$ is true by the inductive hypothesis, hence $\varphi[\mathbf{a}[G]]$ is false. Now assume that $T\,\mathtt{forc}\,\varphi$. We have to prove that $\varphi[\mathbf{a}[G]]$ is true. Suppose otherwise. Then $\varphi^-[\mathbf{a}[G]]$ is true. By the inductive hypothesis, there is a tree $S \in G$ such that $S\,\mathtt{forc}\,\varphi^-$. But the trees S, T belong to the same generic set G, hence they are compatible, which leads to a contradiction with the assumption $T\,\mathtt{forc}\,\varphi$, according to Definition 15.1(III). $\square$

Lemma 17.2. *If a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$, then it is true in $\mathbf{L}[G]$, that every countable $\Sigma^1_{\mathfrak{n}}$ set $Y \subseteq 2^\omega$ satisfies $Y \in \mathbf{L}$.*

Proof. We work in the context of Definition 11.5. The first part of the proof is to show that $Y \subseteq \mathbf{L}$. Suppose to the contrary that it holds in $\mathbf{L}[G]$ that $Y \subseteq 2^\omega$ is a countable $\Sigma^1_{\mathfrak{n}}$ set, but $Y \not\subseteq \mathbf{L}$. We have $Y = \{y \in 2^\omega : \varphi(y)\}$ where $\varphi(y) := \exists\, z\, \psi(y, z)$ is a $\Sigma^1_{\mathfrak{n}}$ formula, that is, $\mathscr{L}\Sigma^1_{\mathfrak{n}}$ formula without codes in **CCF**. There is a tree $T_0 \in G$ which $\mathbb{P}$-forces that "$\{y \in 2^\omega : \varphi(y)\}$ is a countable set and $\exists\, y(\varphi(y) \wedge y \notin \mathbf{L})$". Our goal is to derive a contradiction.

By Lemma 13.1, there exist codes $\mathbf{c}, \mathbf{d} \in \mathbf{CCF} \cap \mathbf{L}$ such that the real $y_0 = \mathbf{c}[G] = f_{\mathbf{c}}(\mathbf{a}[G])$ belongs to Y, so that $y_0 \notin \mathbf{L}$ and $\varphi(\mathbf{c})[\mathbf{a}[G]]$ holds, that is, $\exists\, z\, \psi(\mathbf{c}, z)[\mathbf{a}[G]]$, and finally $\mathbf{d}$ witnesses the existence quantifier, so that the sentence $\psi(\mathbf{c}, \mathbf{d})[\mathbf{a}[G]]$ holds in $\mathbf{L}[G]$. By Lemma 17.1 as ψ is a $\mathscr{L}\Pi^1_{\mathfrak{n}-1}$ formula, there is a tree $T_1 \in G$ satisfying $T_1 \,\mathtt{forc}\, \psi(\mathbf{c}, \mathbf{d})$.

We can wlog assume that $T_1 \subseteq T_0$ and that, in $\mathbf{L}$, the map $f_{\mathbf{c}}$ is either a constant or a bijection on $[T_1]$, by Lemma 12.1(v).

Case 1: $f_{\mathbf{c}} \restriction [T_1]$ is a constant, that is, there exists a real $y_1 \in 2^\omega \cap \mathbf{L}$ such that $f_{\mathbf{c}}(x) = y_1$ for all $x \in [T_1]$. But then $y_1 = f_{\mathbf{c}}(\mathbf{a}[G]) = y_0$, however $y_0 \notin \mathbf{L}$ while $y_1 \in \mathbf{L}$, which is a contradiction.

Case 2: $f_{\mathbf{c}} \restriction [T_1]$ is a bijection. As T_1 is a Silver tree, the set $H = \mathtt{spl}(T_1) \subseteq \omega$ of all its splitting levels is infinite (Definition 4.1). Let $h \in \mathbf{L}$, $h \subseteq H$. Then $h \cdot T_1 = T_1$, and we have $T_1 \mathtt{forc}\, \psi(h \cdot \mathbf{c}, h \cdot \mathbf{d})$ By Lemma 16.1. Therefore the formula $\psi(h \cdot \mathbf{c}, h \cdot \mathbf{d})[\mathbf{a}[G]]$ is true in $\mathbf{L}[G]$ by Lemma 17.1. But this formula coincides with $\psi(f_{h \cdot \mathbf{c}}(\mathbf{a}[G]), f_{h \cdot \mathbf{d}}(\mathbf{a}[G]))$, hence we have $\varphi(f_{h \cdot \mathbf{c}}(\mathbf{a}[G]))$ in $\mathbf{L}[G]$. This implies $f_{h \cdot \mathbf{c}}(\mathbf{a}[G]) \in Y$, or equivalently, $f_{\mathbf{c}}(h \cdot \mathbf{a}[G]) \in Y$.

However, if sets $h, h' \in \mathbf{L}$, $h \cup h' \subseteq H$, satisfy $h \neq h'$, then $h \cdot \mathbf{a}[G] \neq h' \cdot \mathbf{a}[G]$, and hence $f_{\mathbf{c}}(h \cdot \mathbf{a}[G]) \neq f_{\mathbf{c}}(h' \cdot \mathbf{a}[G])$, as $f_{\mathbf{c}}$ is a bijection on $[T_1]$ (and $\mathbf{a}[G] \in [T_1]$). Thus picking different sets $h \in \mathbf{L} \cap \mathscr{P}(H)$ we get uncountably many different elements of the set Y in $\mathbf{L}[G]$, which contradicts to the choice of Y.

The proof of $Y \subseteq \mathbf{L}$ is accomplished.

To prove $Y \in \mathbf{L}$, a stronger statement, it suffices now to show that if $y_0 \in 2^\omega \cap \mathbf{L}$, then $y_0 \in Y$ iff $\exists\, T \in \mathbf{ST}\, (T \mathtt{forc}\, \varphi(\mathbf{c}_0))$, where $\mathbf{c}_0 \in \mathbf{CCF} \cap \mathbf{L}$ is the code of the constant function $f_{\mathbf{c}_0}(x) = y_0$, $\forall\, x \in 2^\omega$.

If $y_0 \in Y$, then the formula $\varphi(y_0)$, equal to $\varphi(\mathbf{c}_0)[x]$ for any x, is true in $\mathbf{L}[G]$ by the choice of φ. It follows by Lemma 17.1 that there is a tree $T \in G$ satisfying $T \mathtt{forc}\, \varphi(\mathbf{c}_0)$, as required.

Now suppose that $T \in \mathbf{ST}$ (not necessarily $\in \mathbb{P}$!) and $T \mathtt{forc}\, \varphi(\mathbf{c}_0)$. As the set $D = \{T \in \mathbf{ST} : \mathtt{spl}(T)$ is co-infinite$\}$ (see Lemma 12.1(vi)) is open dense in $\mathbf{ST}$, we can assume that $\mathtt{spl}(T)$ is co-infinite. On the other hand, it follows from Lemma 12.1(vi) that there is a tree $S \in G \cap D$, so that $\mathtt{spl}(S)$ is co-infinite as well. Now we have $S \mathtt{forc}\, \varphi(\mathbf{c}_0)$ by Corollary 16.2, and then $\varphi(\mathbf{c}_0)[\mathbf{a}[G]]$ is true in $\mathbf{L}[G]$ by Lemma 17.1, that is, $\varphi(y_0)$ holds in $\mathbf{L}[G]$, and $y_0 \in Y$, as required. □

Proof (Theorem 1.2, the main theorem) We assert that any $\mathbb{P}$-generic extension $\mathbf{L}[G] = \mathbf{L}[\mathbf{a}[G]]$ satisfies conditions (i), (ii), (iii) of the theorem. That $\mathbf{a}[G] \notin \mathrm{OD}$ in (i) follows by Lemma 13.2. The minimality follows from Lemma 12.1(v) by Lemma 13.1 (continuous reading of names). We further have (ii) by Corollary 14.3, and we have (iii) by Lemma 17.2. □

Acknowledgements

The authors are thankful to the anonymous reviewer for a number of helpful and important remarks and suggestions allowed to substantially improve the text.

References

[1] J. Barwise (Ed.), Handbook of Mathematical Logic, Stud. Logic Found. Math., vol. 90, North-Holland, Amsterdam, 1977.

[2] Ali Enayat, On the Leibniz–Mycielski axiom in set theory, Fund. Math. 181 (3) (2004) 215–231.
[3] M. Golshani, V. Kanovei, V. Lyubetsky, A Groszek–Laver pair of undistinguishable E_0 classes, MLQ Math. Log. Q. 63 (1–2) (2017) 19–31.
[4] S. Grigorieff, Combinatorics on ideals and forcing, Ann. Math. Logic 3 (4) (1971) 363–394.
[5] J. Hadamard, R. Baire, H. Lebesgue, E. Borel, Cinq lettres sur la théorie des ensembles, Bull. Soc. Math. France 33 (1905) 261–273.
[6] L.A. Harrington, A.S. Kechris, A. Louveau, A Glimm–Effros dichotomy for Borel equivalence relations, J. Amer. Math. Soc. 3 (4) (1990) 903–928.
[7] Leo Harrington. The constructible reals can be anything. Preprint dated May 1974 with several addenda dated up to October 1975:
(A) Models where Separation principles fail, May 74;
(B) Separation without Reduction, April 75;
(C) The constructible reals can be (almost) anything, Part II, May 75.
[8] Thomas Jech, Set Theory, The third millennium revised and expanded edition, Springer-Verlag, Berlin, Heidelberg, New York, 2003.
[9] Ronald Jensen, Definable sets of minimal degree, in: Yehoshua Bar-Hillel (Ed.), Math. Logic Found. Set Theory, Proc. Int. Colloqu., Jerusalem 1968, North-Holland, Amsterdam, London, 1970, pp. 122–128.
[10] V. Kanovei, V. Lyubetsky, A countable definable set of reals containing no definable elements. ArXiv e-prints, August 2014.
[11] V. Kanovei, V. Lyubetsky, A definable E_0-class containing no definable elements, Arch. Math. Logic 54 (5) (2015) 711–723.
[12] V. Kanovei, V. Lyubetsky, Counterexamples to countable-section Π^1_2 uniformization and Π^1_3 separation, Ann. Pure Appl. Logic 167 (4) (2016) 262–283.
[13] V.G. Kanovei, Projective hierarchy of Luzin: modern state of the theory, in: J. Barwise (Ed.), Handbook of Mathematical Logic. Part 2. Set Theory (Spravochnaya Kniga po Matematicheskoj Logike. Chast' 2. Teoriya Mnozhestv), Nauka, Moscow, 1982, pp. 273–364. Transl. from the English.
[14] V.G. Kanovei, On the nonemptiness of classes in axiomatic set theory, Math. USSR, Izv. 12 (1978) 507–535.
[15] V.G. Kanovei, Cardinality of the set of Vitali equivalence classes, Math. Notes 49 (4) (1991) 370–374.
[16] V.G. Kanovei, V.A. Lyubetsky, A definable countable set not containing definable elements, Math. Notes 102 (3–4) (2017) 338–349.
[17] Vladimir Kanovei, Martin Sabok, Jindřich Zapletal, Canonical Ramsey Theory on Polish Spaces, Cambridge University Press, Cambridge, 2013.
[18] Nicolas Lusin, Sur les ensembles analytiques, Fund. Math. 10 (1927) 1–95.
[19] Jan Mycielski, Barbara Osofsky, Avoiding the axiom of choice, Amer. Math. Monthly 82 (3) (1975) 308–309.
[20] Wacław Sierpiński, L'axiome de M. Zermelo et son rôle dans la théorie des ensembles et l'analyse, Bull. Acad. Sci. Cracovie (1918) 97–152.