Annals of Pure and Applied Logic 175 (2024) 103426

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

journal homepage: www.elsevier.com/locate/apal

Full Length Article

A good lightface Δ^1_n well-ordering of the reals does not imply the existence of boldface $\mathbf{\Delta}^1_{n-1}$ well-orderings [☆]

Vladimir Kanovei [*], Vassily Lyubetsky

Kharkevich Institute for Information Transmission Problems, Russian Academy of Sciences, Bolshoy Karetny per. 19, build. 1, Moscow, 127051, Russian Federation

A R T I C L E I N F O

Article history:
Received 13 October 2022
Received in revised form 8 February 2024
Accepted 21 February 2024
Available online 28 February 2024

MSC:
03E15
03E35
28A05
03E40

Keywords:
Projective classes
Well-orderings
Jensen's forcing

A B S T R A C T

We make use of a finite support product of the Jensen-type forcing notions to define a model of the set theory **ZFC** in which, for a given $n \geq 3$, there exists a good lightface Δ^1_n well-ordering of the reals but there are no any (not necessarily good) well-orderings in the boldface class $\mathbf{\Delta}^1_{n-1}$.

Contents

☆ The research was supported by the Russian Science Foundation grant No. 24-44-00099.
* Corresponding author.
E-mail addresses: kanovei@iitp.ru (V. Kanovei), lyubetsk@iitp.ru (V. Lyubetsky).

https://doi.org/10.1016/j.apal.2024.103426
0168-0072/

1. Introduction

The problem of well-orderability of the continuum $\mathbb{R}$ has been known in set theory since the time of Cantor and Hilbert. Zermelo's axiom of choice **AC** directly postulates the existence of a well-ordering of $\mathbb{R}$ (and of any other set of course), but this is far from an effective construction of a concrete, "nameable" well-ordering of $\mathbb{R}$. We refer to the famous *Sinq Lettres* [18] in matters of the discussion on these issues in early set theory.

Somewhat later, using the methods of descriptive set theory, which had just emerged, it became possible to prove that *no well-ordering $\preccurlyeq$ of $\mathbb{R}$ could belong to the first-level projective classes $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$* — and then to $\mathbf{\Delta}^1_1$ since $x \preccurlyeq y$ iff $x = y$ or $y \not\preccurlyeq x$. This is an easy consequence of Luzin's theorem [34] that sets in $\mathbf{\Sigma}^1_1 \cup \mathbf{\Pi}^1_1$ are Lebesgue measurable, see e.g. Sierpinski [37].

To shorten terminology, if Γ is a class of subsets of Polish spaces (as, e.g., any projective class $\mathbf{\Delta}^1_n = \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$, or *effective* projective class $\varDelta^1_n = \varSigma^1_n \cap \varPi^1_n$) then, following [3], let $\Gamma(\mathrm{WO})$ be the statement: *there is a well-ordering, of the set $\mathbb{R}$ of all reals, which as a set of pairs belongs to* Γ. Then the result above is summarized as $\neg\, \mathbf{\Delta}^1_1(\mathrm{WO})$.

The next key result was obtained by Gödel [16]: it is true in Gödel's constructible universe $\mathbf{L}$ that there exists a $\varDelta^1_2$ well-ordering $\leqslant_{\mathbf{L}}$ of the reals. In other words, the statement $\varDelta^1_2(\mathrm{WO})$ follows from *the axiom of constructibility* $\mathbf{V} = \mathbf{L}$, and hence $\varDelta^1_2(\mathrm{WO})$ is consistent with the axioms of the Zermelo–Fraenkel set theory **ZFC** (containing the axiom of choice **AC**) because the axiom of constructibility $\mathbf{V} = \mathbf{L}$ itself is consistent by [16].

Addison [2] singled out an important additional property of the Gödel well-ordering $\leqslant_{\mathbf{L}}$. Namely, a $\varDelta^1_n$-*good* well-ordering is any $\varDelta^1_n$ well-ordering $\preccurlyeq$ such that for any binary $\varDelta^1_n$ relation $P(y,x)$ on the reals, the relations

$$Q(z,x) := \exists\, y \preccurlyeq x\, P(z,y) \quad \text{and} \quad R(z,x) := \forall\, y \preccurlyeq x\, P(z,y)$$

belong to $\varDelta^1_n$ as well, so that the class $\varDelta^1_n$ is closed under $\preccurlyeq$-bounded quantification. (See Moschovakis [36, Section 5A].) In these terms, the Gödel – Addison result then says that $\leqslant_{\mathbf{L}}$ is a $\varDelta^1_2$-good well-ordering of the reals in $\mathbf{L}$, and hence the existence of such a well-ordering follows from $\mathbf{V} = \mathbf{L}$ and is consistent with **ZFC**. The property of $\varDelta^1_2$-goodness of $\leqslant_{\mathbf{L}}$ is behind many key results on projective sets in Gödel's universe $\mathbf{L}$, see [36, Section 5A].

In the opposite direction, it was established in the early years of modern set theory (see, e.g., Levy [33] and Solovay [38]) that the non-existence statement $\neg\mathbf{\Sigma}^1_\infty(\mathrm{WO})$ is also consistent with **ZFC**, where $\mathbf{\Sigma}^1_\infty = \bigcup_n \mathbf{\Sigma}^1_n$ is the class of all projective sets, and moreover a much stronger statement $\neg\,\mathbf{ROD}(\mathrm{WO})$, saying that there is no **ROD** well-ordering of the reals, is consistent as well. Here **ROD** is the class of all *real-ordinal definable* sets, i.e., those defined by any set-theoretic formula with arbitrary reals and ordinals as parameters; $\mathbf{\Sigma}^1_\infty$ is a rather small part of **ROD**.

Recent studies on projective well-orderings explore such topics as

- connections with forcing axioms [5,6];
- connections with large cardinals [4,13];
- connections with cardinal characteristics of the continuum [8,9];
- relations to the structure and properties of projective sets [3,10,30,32];

and others. The following theorem contributes to these studies. We investigate the interrelations between the hypotheses $\Delta^1_n(\mathrm{WO})$, $\mathbf{\Delta}^1_n(\mathrm{WO})$ for different values of n. Note that $\Delta^1_{n-1}(\mathrm{WO})$ implies $\Delta^1_n(\mathrm{WO})$, and $\mathbf{\Delta}^1_{n-1}(\mathrm{WO})$ implies $\mathbf{\Delta}^1_n(\mathrm{WO})$. The theorem shows that these implications are irreversible.

Theorem 1.1. *Let $\mathtt{n} \ge 3$. There is a generic extension of $\mathbf{L}$, the constructible universe, in which it is true that*

(i) *there exists a lightface $\Delta^1_{\mathtt{n}}$-good well-ordering of the reals, of length ω_1;*
(ii) $\mathbf{\Delta}^1_{\mathtt{n}-1}(\mathrm{WO})$ *fails: there are no boldface $\mathbf{\Delta}^1_{\mathtt{n}-1}$ well-ordering of the reals.*

Thus the lightface $\Delta^1_{\mathtt{n}}(\mathrm{WO})$ does not imply even the boldface $\mathbf{\Delta}^1_{\mathtt{n}-1}(\mathrm{WO})$.

This theorem is the main result of this paper. It improves our earlier result in [32], where it is established that there is a generic extension of $\mathbf{L}$ with a lightface $\Delta^1_{\mathtt{n}}$-good well-ordering of the reals, but no lightface $\Delta^1_{\mathtt{n}-1}$-good well-orderings of the reals. Thus Theorem 1.1 strengthens this earlier result by removing the goodness in part (ii) and extending the lightface class $\Delta^1_{\mathtt{n}-1}$ to the boldface class $\mathbf{\Delta}^1_{\mathtt{n}-1}$ also in part (ii).

2. Outline of the proof

Given $\mathtt{n} \ge 3$, our plan is to make use of a generic extension of $\mathbf{L}$, originally defined in [31] in order to get a model where the Separation principle fails for both classes $\mathbf{\Sigma}^1_{\mathtt{n}}$ and $\mathbf{\Pi}^1_{\mathtt{n}}$, and then applied in [32] to prove the aforementioned weaker version of Theorem 1.1. This extension utilizes a sequence of forcing notions $\mathbb{P}_\xi$, $\xi < \omega_1$ (or $\Pi(\xi)$, as in Section 14 below), defined in $\mathbf{L}$ so that the finite-support product $\mathbb{P} = \prod_\xi \mathbb{P}_\xi$ satisfies CCC and adjoins a sequence of generic reals $x_\xi \in 2^\omega$, such that the binary relation "$x \in 2^\omega$ is a real $\mathbb{P}_\xi$-generic over $\mathbf{L}$" (with arguments x, ξ) is $\Pi^1_{\mathtt{n}-1}$ in $\mathbf{L}[G] = \mathbf{L}[\langle x_\xi\rangle_{\xi<\omega_1}]$. This will suffice to define a well-ordering satisfying Theorem 1.1(i).

Claim (ii) of Theorem 1.1 involves another crucial property: the $\mathbb{P}$-forcing relation of $\Sigma^1_{\mathtt{n}-1}$ formulas can be suitably approximated by an auxiliary forcing relation `forc` invariant w.r.t. the permutations of indices $\xi < \omega_1$. The $\mathbb{P}$-forcing relation itself is not permutation-invariant since all forcing notions $\mathbb{P}_\xi$ are pairwise different.

Each factor forcing $\mathbb{P}_\xi$ consists of *perfect trees* in $2^{<\omega}$ and is a clone of Jensen's minimal forcing defined in [22], see also [20, Section 28A] on this forcing. The idea of finite-support products of Jensen's forcing, which we owe to Enayat [7], has been exploited recently to obtain generic models with counterexamples to the separation theorem for both $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$ [25], counterexamples to the axiom of choice [14], and a model in which every non-empty $\varSigma^1_\infty$ set of reals contains a $\varSigma^1_\infty$ real but there are no $\varSigma^1_\infty$ well-orderings of the reals [30], to name a few examples.

Sections 3 and 4: perfect trees in $2^{<\omega}$, arboreal forcing notions, multitrees (finite tuples of trees), and multiforcings (countable products of arboreal forcing notions) are considered.

Section 5: we explore the refinement relation and properties of refinements of arboreal forcing notions and multiforcings. Then in Section 6 we introduce an important property of *sealing* of dense sets by Jensen's refinements, i.e., a dense set in the original multiforcing remains pre-dense in the extended one. Then we consider some other types of refinements, related to properties of real names, in Section 7. Some applications of refinements with various properties to the reals in according generic extensions are established in Section 8.

Jensen's construction of generic refinements is introduced in Section 9. We prove in Section 10 that it indeed gives a refinement of a given multiforcing, which satisfies those extra properties considered above.

Transfinite sequences of small multiforcings, increasing in the sense of the refinement relation, are considered in Section 12. We follow our earlier paper [31] in Sections 14, 15, 16 to introduce the key product forcing notion $\mathbb{P}$ for Theorem 1.1 by means of a specially constructed in $\mathbf{L}$ refinement-increasing ω_1-sequence of small multiforcings. Theorem 16.1 shows that $\mathbb{P}$ provides (i) of Theorem 1.1. See Remarks 9.2 and 9.3 on some deviations from the technical construction given in [31].

After a short introduction into claim (ii) of Theorem 1.1 in Section 17, we define and explore an auxiliary forcing relation $\mathtt{forc}$ in Sections 18–22. It is proved that $\mathtt{forc}$ is tail-invariant (Theorem 20.1), permutation-invariant (Theorem 21.1), and approximates the truth in $\mathbb{P}$-generic extensions for $\varSigma^1_{\mathbb{n}-1}$-formulas and below (Theorem 22.3). We also prove that the relation $\mathtt{forc}$ restricted to any class $\varSigma^1_m$ or $\varPi^1_m$, $m \ge 2$, is $\varSigma^1_m$, resp., $\varPi^1_m$ itself (Lemma 19.1).

The final part of the paper (Sections 23–27) contains a lengthy proof of Theorem 17.1 that leads to claim (ii) of Theorem 1.1 in $\mathbb{P}$-generic models. The conclusive argument in Section 27 will show that the contrary assumption, of the existence of a $\mathbf{\Sigma}^1_{n-1}$ well-ordering, say $<$, of the reals in the extension, leads to the existence of a non-empty set of reals which does not have a $<$-least element, a contradiction. This is similar to some other theorems of this kind. (See, e.g., Theorem 25.39 in [20].) Yet here the flow of arguments involves a lot of different details and is way more complex.

This paper is a sequel of [31] in many details, in particular the model we consider is more or less the same model as defined in [31] for different purposes. However some important adjustments will be made in basic constructions, see e.g. Remark 9.2. This forces us to present the whole construction anew in all necessary detail.

3. Arboreal forcing notions

Let $2^{<\omega}$ be the set of all tuples (finite sequences) of numbers $0, 1$. If $t \in 2^{<\omega}$ and $i = 0, 1$ then $t^\frown i$ denotes the extension of t by i as the rightmost term. If $s, t \in 2^{<\omega}$ then $s \subseteq t$ means that t extends s, while $s \subset t$ means proper extension.[1] By $\mathtt{lh}(t)$ we denote the length of a tuple t, and we put $2^n = \{t \in 2^{<\omega} : \mathtt{lh}(t) = n\}$ (tuples of length n).

PT is the set of all *perfect* trees $\varnothing \ne T \subseteq 2^{<\omega}$. Thus a tree $\varnothing \ne T \subseteq 2^{<\omega}$ belongs to **PT** iff it has no endpoints and no isolated branches. In this case

$$[T] = \{a \in 2^\omega : \forall n\, (a \restriction n \in T)\} \subseteq 2^\omega$$

is a perfect set. If $s \in T \in \mathbf{PT}$ then put

$$T \restriction_s = \{t \in T : s \subseteq t \vee t \subseteq s\}; \text{ then } T \restriction_s \in \mathbf{PT}.$$

[1] In this paper, $\subset$ means a **proper** or **strict** inclusion to the expense of equality "=" in all cases, i.e., the same as $\subsetneqq$. The improper inclusion is $\subseteq$.

Definition 3.1. Trees S, T are called *incompatible*, in symbol $S \perp T$, if $[S] \cap [T] = \varnothing$, and *compatible* ($S \not\perp T$) otherwise. Note that $S \perp T$ is equivalent to $S \cap T$ being finite.

We call an *antichain* any set $A \subseteq \mathbf{PT}$ of pairwise incompatible trees. □

Let an *arboreal forcing* be any set $P \subseteq \mathbf{PT}$ such that if $u \in T \in P$ then $T \restriction_u \in P$. Let $\mathbf{AF}$ be the set of all arboreal forcings P. Any $P \in \mathbf{AF}$ is:

- *regular*, if, for any $S, T \in P$, the intersection $[S] \cap [T]$ is clopen in $[S]$ or in $[T]$ (or in both $[S]$ and $[T]$ simultaneously);
- *special*, if there is a *finite or countable* antichain $A \subseteq P$ such that $P = \{T \restriction_s : s \in T \in A\}$ — A is unique and P is countable in this case.

Note that every special arboreal forcing is regular.

Lemma 3.2. *Let P be a regular arboreal forcing. Then any $S, T \in P$ are P-compatible* (that is, there is a tree $R \in P$ with $R \subseteq S \cap T$) *iff just $S \not\perp T$.*

Proof. By the regularity, let $X = [S] \cap [T]$ be clopen in say $[T]$. Then there is a tuple $s \in T$ such that $[T \restriction s] \subseteq X$. But $T \restriction s \in P$ as $P \in \mathbf{AF}$. □

Splitting. Consider pairs of the form $\langle n, T \rangle$, where $n < \omega$ and $T \in \mathbf{PT}$. Following [1], the set $\omega \times \mathbf{PT}$ of such pairs is ordered by a relation $\preccurlyeq$ so that $\langle n, T \rangle \preccurlyeq \langle m, S \rangle$ (reads: $\langle n, T \rangle$ *extends* $\langle m, S \rangle$) iff $m \leq n$, $T \subseteq S$, and $T \cap 2^m = S \cap 2^m$. The role of the number m in a pair $\langle m, S \rangle$ is to preserve the value $S \cap 2^m$ under $\preccurlyeq$-extensions. We underline that this definition does not contain any explicit splitting condition. This is why one needs the genericity requirement in Lemma 3.3 to get actual splitting.

The implication $m > n \Longrightarrow \langle m, T \rangle \preccurlyeq \langle n, T \rangle$ (the same T!) always holds, but $S \subseteq T \Longrightarrow \langle n, S \rangle \preccurlyeq \langle n, T \rangle$ is not necessarily true: we also need $T \cap 2^n = S \cap 2^n$.

Lemma 3.3 *(Fusion lemma, see [1]). Let $\ldots \preccurlyeq \langle n_2, T_2 \rangle \preccurlyeq \langle n_1, T_1 \rangle \preccurlyeq \langle n_0, T_0 \rangle$ be a decreasing sequence in $\omega \times \mathbf{PT}$, with $n_0 \leq n_1 \leq n_2 \leq \ldots \to \infty$, minimally generic in the sense that it meets every set of the form*

$$D_t = \{\langle n, T \rangle \in \omega \times \mathbf{PT} : t \notin T \vee \exists s \in T\,(t \subseteq s \wedge s^\frown 0, s^\frown 1 \in T)\}\,,\ t \in 2^{<\omega}.$$

Then $T = \bigcap_n T_n \in \mathbf{PT}$, and if $i < \omega$ then we have $\langle n_i, T \rangle \preccurlyeq \langle n_i, T_i \rangle$. □

Finite unions. To carry out splitting constructions, as in Lemma 3.3, over a forcing $P \in \mathbf{AF}$, we make use of a bigger forcing notion $\bigcup^{\mathtt{fin}} P \in \mathbf{AF}$, that consists of all finite unions of trees in P. Then P is dense in $\bigcup^{\mathtt{fin}} P$, so the forcing properties of both sets coincide. Yet $\bigcup^{\mathtt{fin}} P$ is more flexible w.r.t. tree constructions.

Lemma 3.4. *Let $P \in \mathbf{AF}$, $n < \omega$, $T \in \bigcup^{\mathtt{fin}} P$, $s_0 \in 2^n \cap T$, and $U \subseteq T \restriction_{s_0}$, $U \in P$. There is a tree $R \in \bigcup^{\mathtt{fin}} P$ such that $\langle n, R \rangle \preccurlyeq \langle n, T \rangle$ and $R \restriction_{s_0} = U$.*

Proof. We let R consist of all tuples $r \in T$ such that either (1) $\mathtt{lh}(r) \leq n$, or (2) $\mathtt{lh}(r) > n$ and $r \restriction n \neq s_0$, or (3) $\mathtt{lh}(r) > n$ and $r \restriction n = s_0$ and $r \in U$. □

4. Multiforcings

Let a **multiforcing** be any map $\boldsymbol{\pi} : |\boldsymbol{\pi}| \to \mathbf{AF}$, where $|\boldsymbol{\pi}| = \mathtt{dom}\,\boldsymbol{\pi} \subseteq \omega_1$. Let $\mathbf{MF}$ be the collection of all multiforcings. Every $\boldsymbol{\pi} \in \mathbf{MF}$ can be presented as an indexed set $\boldsymbol{\pi} = \langle P_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$, where $P_\xi \in \mathbf{AF}$ for all $\xi \in |\boldsymbol{\pi}|$, so that each set $P_\xi = P^{\boldsymbol{\pi}}_\xi = \boldsymbol{\pi}(\xi)$, $\xi \in |\boldsymbol{\pi}|$, is an arboreal forcing. Such a $\boldsymbol{\pi}$ is:

- *small*, if both $|\boldsymbol{\pi}|$ and each forcing $\boldsymbol{\pi}(\xi) = P^{\boldsymbol{\pi}}_\xi$, $\xi \in |\boldsymbol{\pi}|$, are countable;
- *special*, if each $\boldsymbol{\pi}(\xi) = P^{\boldsymbol{\pi}}_\xi$ is special in the sense of Section 3;
- *regular*, if each $\boldsymbol{\pi}(\xi) = P^{\boldsymbol{\pi}}_\xi$ is regular in the sense of Section 3.

A **multitree** is a function $\boldsymbol{p} : |\boldsymbol{p}| \to \mathbf{PT}$ with a *finite* support $|\boldsymbol{p}| = \mathtt{dom}\, p \subseteq \omega_1$. Let $\mathbf{MT}$ be the set of all multitrees. We represent multitrees $\boldsymbol{p} \in \mathbf{MT}$ as indexed sets $\boldsymbol{p} = \langle T^{\boldsymbol{p}}_\xi \rangle_{\xi \in |\boldsymbol{p}|}$, where $T^{\boldsymbol{p}}_\xi = \boldsymbol{p}(\xi) \in \mathbf{PT}$ for all $\xi \in |\boldsymbol{p}|$. Put

$$[\boldsymbol{p}] = \prod_{\xi \in |\boldsymbol{p}|} [T^{\boldsymbol{p}}_\xi] = \{x \in (2^\omega)^{|\boldsymbol{p}|} : \forall\, i \in |\boldsymbol{p}|\, (x(i) \in [T^{\boldsymbol{p}}_\xi])\},$$

this is a perfect product in $(2^\omega)^{|\boldsymbol{p}|}$ provided $\boldsymbol{p} \in \mathbf{MT}$.

We order $\mathbf{MT}$ componentwise: $\boldsymbol{q} \leqslant \boldsymbol{p}$ ($\boldsymbol{q}$ is stronger than $\boldsymbol{p}$) iff $|\boldsymbol{p}| \subseteq |\boldsymbol{q}|$ and $T^{\boldsymbol{q}}_\xi \subseteq T^{\boldsymbol{p}}_\xi$ for all $\xi \in |\boldsymbol{p}|$. In particular, if just $\boldsymbol{p} \subseteq \boldsymbol{q}$ then $\boldsymbol{q} \leqslant \boldsymbol{p}$.

Assume that $\boldsymbol{\pi} = \langle P_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ is a multiforcing. Let a $\boldsymbol{\pi}$*-multitree* be any multitree $\boldsymbol{p} \in \mathbf{MT}$ such that $|\boldsymbol{p}| \subseteq |\boldsymbol{\pi}|$, and if $\xi \in |\boldsymbol{p}|$ then the tree $\boldsymbol{p}(\xi) = T^{\boldsymbol{p}}_\xi$ belongs to P_ξ. The set $\mathbf{MT}(\boldsymbol{\pi})$ of all $\boldsymbol{\pi}$-multitrees can be identified with the *finite support* product $\prod_{\xi \in |\boldsymbol{\pi}|} P_\xi$ of the arboreal forcings P_ξ involved.

Definition 4.1. Multitrees $\boldsymbol{p}, \boldsymbol{q}$ are *incompatible*, in symbol $\boldsymbol{p} \perp \boldsymbol{q}$, if there is an index $\xi \in |\boldsymbol{p}| \cap |\boldsymbol{q}|$ such that $[T^{\boldsymbol{p}}_\xi] \cap [T^{\boldsymbol{q}}_\xi] = \varnothing$, and *compatible* otherwise. Any set $A \subseteq \mathbf{MT}$ of pairwise incompatible multitrees is an *antichain.* □

Corollary 4.2 *(of Lemma 3.2). Let $\boldsymbol{\pi}$ be a regular multiforcing. Then any multitrees $\boldsymbol{p}, \boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ are $\mathbf{MT}(\boldsymbol{\pi})$-compatible* (i.e., there is $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$ with $\boldsymbol{r} \leqslant \boldsymbol{p}$ and $\boldsymbol{r} \leqslant \boldsymbol{q}$) *iff $\boldsymbol{p}, \boldsymbol{q}$ are compatible in the sense of Definition 4.1. Thus a set $A \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is a $\mathbf{MT}(\boldsymbol{\pi})$-antichain* (that is, a set of pairwise $\mathbf{MT}(\boldsymbol{\pi})$-incompatible trees) *iff A is an antichain as in Definition 4.1.* □

If $\boldsymbol{\pi}, \boldsymbol{\varrho}$ are multiforcings then a multiforcing $\boldsymbol{\sigma} = \boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho}$ (the **componentwise union**) is defined so that $|\boldsymbol{\sigma}| = |\boldsymbol{\pi}| \cup |\boldsymbol{\varrho}|$ and

$$\boldsymbol{\sigma}(\xi) = \begin{cases} \boldsymbol{\pi}(\xi) \text{ in case } \xi \in |\boldsymbol{\pi}| \smallsetminus |\boldsymbol{\varrho}|\,, \\ \boldsymbol{\varrho}(\xi) \text{ in case } \xi \in |\boldsymbol{\varrho}| \smallsetminus |\boldsymbol{\pi}|\,, \\ \boldsymbol{\pi}(\xi) \cup \boldsymbol{\varrho}(\xi) \text{ in case } \xi \in |\boldsymbol{\varrho}| \cap |\boldsymbol{\pi}|\,. \end{cases}$$

Lemma 4.3. *If $\boldsymbol{\pi}, \boldsymbol{\varrho}$ coincide on the common domain $d = |\boldsymbol{\pi}| \cap |\boldsymbol{\varrho}|$, i.e., $\boldsymbol{\pi} \restriction d = \boldsymbol{\varrho} \restriction d$, then $\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho} = \boldsymbol{\pi} \cup \boldsymbol{\varrho}$, the ordinary union.* (This includes the case of disjoint domains $|\boldsymbol{\pi}| \cap |\boldsymbol{\varrho}| = \varnothing$, of course.) □

Given any sequence $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha < \lambda}$ of multiforcings, we similarly define *the componentwise union* $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}} \vec{\boldsymbol{\pi}} = \bigcup^{\mathtt{cw}}_{\alpha < \lambda} \boldsymbol{\pi}_\alpha \in \mathbf{MF}$ so that $|\boldsymbol{\pi}| = \bigcup_{\alpha < \lambda} |\boldsymbol{\pi}_\alpha|$ and $\boldsymbol{\pi}(\xi) = \bigcup_{\alpha < \lambda,\, \xi \in |\boldsymbol{\pi}_\alpha|} \boldsymbol{\pi}_\alpha(\xi)$ for $\xi \in |\boldsymbol{\pi}|$.

Remark 4.4. Any arboreal forcing $P \in \mathbf{AF}$ is considered as a forcing notion (if $T \subseteq T'$ then T is a stronger condition); such a forcing adds a real in 2^ω.

Accordingly any forcing notion of the form $\mathbf{MT}(\boldsymbol{\pi})$, where $\boldsymbol{\pi} = \langle P_\xi \rangle_{\xi \in |\boldsymbol{\pi}|} \in \mathbf{MF}$, adjoins a generic sequence $\langle x_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$, each $x_\xi = x_\xi[G] \in 2^\omega$ being a P_ξ-generic real. Reals of the form $x_\xi[G]$ are *principal generic reals* in $\mathbf{V}[G]$. □

5. Refinements

The following definition reminds the notions related to density in the forcing context.

Definition 5.1. If $P \subseteq R \subseteq \mathbf{PT}$ then the set P is

— *dense* in R iff $\forall T \in R \, \exists S \in P \, (S \subseteq T)$,
— *open dense* in R iff in addition $\forall T \in R \, \forall S \in P \, (T \subseteq S \implies T \in P)$,
— *pre-dense* in R iff the set $P' = \{T \in R : \exists S \in P (T \subseteq S)\}$ is dense.

In the case of multitrees, if $\boldsymbol{P} \subseteq \boldsymbol{R} \subseteq \mathbf{MT}$ then similarly the set $\boldsymbol{P}$ is

— *dense* in $\boldsymbol{R}$ iff $\forall \boldsymbol{q} \in R \, \exists \boldsymbol{p} \in \boldsymbol{P} \, (\boldsymbol{p} \leqslant \boldsymbol{q})$,
— *open dense* in $\boldsymbol{R}$ iff in addition $\forall \boldsymbol{q} \in \boldsymbol{R} \, \forall \boldsymbol{p} \in \boldsymbol{P} \, (\boldsymbol{q} \leq \boldsymbol{p} \implies \boldsymbol{q} \in \boldsymbol{P})$,
— *pre-dense* in $\boldsymbol{R}$ iff the set $\boldsymbol{P}' = \{\boldsymbol{q} \in \boldsymbol{R} : \exists \boldsymbol{p} \in \boldsymbol{P}(\boldsymbol{q} \leq \boldsymbol{p})\}$ is dense. □

Now let $P, Q \in \mathbf{AF}$ be arboreal forcings. We say that Q is a **refinement** of P (in symbol $P \sqsubset Q$) if

(1) the set Q is dense in $P \cup Q$: if $T \in P$ then $\exists Q \in Q \, (Q \subseteq T)$;
(2) if $T \in Q$ then $T \subseteq^{\mathtt{fin}} \bigcup P$, that is, there is a finite set $D \subseteq P$ such that $T \subseteq \bigcup D$, or equivalently $[T] \subseteq \bigcup_{S \in D} [S]$;
(3) if $T \in Q$ and $S \in P$ then $[S] \cap [T]$ is clopen in $[T]$ and $S \not\subseteq T$.

Lemma 5.2.

(i) *If $P \sqsubset Q$ and $S \in P$, $T \in Q$, then $[S] \cap [T]$ is meager in $[S]$, therefore $P \cap Q = \varnothing$ and Q is open dense in $P \cup Q$;*
(ii) *if $P \sqsubset Q \sqsubset R$ then $P \sqsubset R$, thus $\sqsubset$ is a strict partial order on $\mathbf{AF}$;*
(iii) *if $\langle P_\alpha \rangle_{\alpha<\lambda}$ is a $\sqsubset$-increasing sequence in $\mathbf{AF}$ and $0 < \mu < \lambda$ then $P = \bigcup_{\alpha<\mu} P_\alpha \sqsubset Q = \bigcup_{\mu \leq \alpha < \lambda} P_\alpha$;*
(iv) *if $\langle P_\alpha \rangle_{\alpha<\lambda}$ is a $\sqsubset$-increasing sequence in $\mathbf{AF}$ and each P_α is special then $P = \bigcup_{\alpha<\lambda} P_\alpha \in \mathbf{AF}$, P is regular, and all P_γ are pre-dense in P.*

Proof. (i) Otherwise there is a string $u \in S$ such that $[S{\restriction}_u] \subseteq [T] \cap [S]$. But $S{\restriction}_u \in P$, which contradicts to (3) above.

To prove (ii) it suffices to verify (3). Let $S \in P$ and $T \in R$. By (2) for $Q \sqsubset R$, there is a finite $D \subseteq Q$ such that $T \subseteq \bigcup D$. If $U \in D$ then $[S] \cap [U]$ is clopen in $[U]$, so $[S] \cap [\bigcup D]$ is clopen in $[\bigcup D]$. We conclude that $[S] \cap [T]$ is clopen in $[T]$. Moreover, if $U \in D$ then $[U] \cap [T]$ is meager in $[U]$ while $[S] \cap [U]$ is meager in $[S]$, by (i). Thus $[S] \cap [T]$ is meager in $[S]$.

To prove (iii) in part (3), let $S \in P$ and $T \in Q$, so that $S \in P_\alpha$ and $T \in P_\gamma$, $\alpha < \mu \leq \gamma$. But then $P_\alpha \sqsubset P_\gamma$.

(iv) To check the regularity, let $S \in P_\alpha$, $T \in P_\beta$, $\alpha \leq \beta$. If $\alpha = \beta$ then, as P_α is special, the trees S, T either satisfy $S \perp T$ or are $\subseteq$-comparable. If $\alpha < \beta$ then $[S] \cap [T]$ is clopen in $[T]$ by (3) above.

To check the pre-density of P_γ, let $S \in P_\alpha$, $\alpha \neq \gamma$. If $\alpha < \gamma$ then by (1) above there is a tree $T \in P_\gamma$, $T \subseteq S$. Now let $\gamma < \alpha$. Then $S \subseteq^{\mathtt{fin}} \bigcup P_\gamma$ by (2), hence there is a tree $T \in P_\gamma$ such that $[S] \cap [T] \neq \varnothing$. However $[S] \cap [T]$ is clopen in $[S]$ by (3) above. Therefore $S{\restriction}_u \subseteq T$ for a string $u \in S$. Finally $S{\restriction}_u \in P_\alpha$ since $P_\alpha \in \mathbf{AF}$. □

In the case of multiforcings, a multiforcing $\boldsymbol{\varrho}$ is a **refinement** of $\boldsymbol{\pi}$ (in symbol $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$) if $|\boldsymbol{\pi}| \subseteq |\boldsymbol{\varrho}|$ and $\boldsymbol{\pi}(\xi) \sqsubset \boldsymbol{\varrho}(\xi)$ in $\mathbf{AF}$ for all $\xi \in |\boldsymbol{\pi}|$.

Corollary 5.3 *(of Lemma 5.2(ii)).* *$\sqsubset$ is a strict partial order on* $\mathbf{MF}$. □

Lemma 5.4. *If $\boldsymbol{\pi}' \subseteq \boldsymbol{\pi} \sqsubset \boldsymbol{\varrho} \subseteq \boldsymbol{\varrho}'$ are multiforcings then $\boldsymbol{\pi}' \sqsubset \boldsymbol{\varrho}'$.*

Proof. By definition, $|\boldsymbol{\pi}'| \subseteq |\boldsymbol{\pi}| \subseteq |\boldsymbol{\varrho}| \subseteq |\boldsymbol{\varrho}'|$ holds, and if $\xi \in |\boldsymbol{\pi}'|$ then $\boldsymbol{\pi}'(\xi) = \boldsymbol{\pi}(\xi) \sqsubset \boldsymbol{\varrho}(\xi) = \boldsymbol{\varrho}(\xi')$. □

Lemma 5.5. *Let* $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ *be a* $\sqsubset$*-increasing sequence of special multiforcings, and* $0 < \mu < \lambda$*. Then the componentwise union* $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}}_{\alpha<\lambda} \boldsymbol{\pi}_\alpha$ *is a regular multiforcing, each* $\mathbf{MT}(\boldsymbol{\pi}_\alpha)$ *is pre-dense in* $\mathbf{MT}(\boldsymbol{\pi})$*, and we have* $\bigcup^{\mathtt{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha = \boldsymbol{\pi}_{<\mu} \sqsubset \boldsymbol{\pi}_{\ge\mu} = \bigcup^{\mathtt{cw}}_{\mu\le\alpha<\lambda} \boldsymbol{\pi}_\alpha$.

Proof. If $\xi \in \bigcup_{\alpha<\lambda} |\boldsymbol{\pi}_\alpha|$ then $\boldsymbol{\pi}(\xi) = \bigcup_{\alpha<\lambda, \xi\in|\boldsymbol{\pi}_\alpha|} \boldsymbol{\pi}_\alpha(\xi)$ is a regular arboreal forcing by Lemma 5.2(iv). The pre-density claim also follows from Lemma 5.2(iv). To prove the last claim make use of Lemma 5.2(iii). □

6. Sealing dense sets

A key property of Jensen's refinement construction is that it allows to *seal* dense sets, i.e., keep them to be still pre-dense after a refinement is adjoined. In the easier case of arboreal forcings this is based on the following definition.

Definition 6.1. Let $P, Q \in \mathbf{AF}$ and $D \subseteq P$. Say that Q *seals* D *over* P, symbolically $P \sqsubset_D Q$, if $P \sqsubset Q$ holds and every tree $S \in Q$ satisfies $S \subseteq^{\mathtt{fin}} \bigcup D$, that is, $S \subseteq \bigcup D'$ for a finite subset $D' \subseteq D$. □

As we'll see now, a sealed set is pre-dense after the refinement. The additional importance of sealing refinements lies in the fact that, once established, it is preserved under further simple refinements, as in (ii) of the following lemma:

Lemma 6.2.

(i) *If* $P \sqsubset_D Q$*,* $D \subseteq P$*, then* D *is pre-dense in* $P \cup Q$*;*
(ii) *if* $\langle P_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence in* $\mathbf{AF}$*,* $0 < \mu < \lambda$*, and* $P = \bigcup_{\alpha<\mu} P_\alpha \sqsubset_D P_\mu$*, then* $P \sqsubset_D Q = \bigcup_{\mu\le\alpha<\lambda} P_\alpha$.

Proof. (i) Let $T_0 \in P \cup Q$. By (1) in Section 5, there is a tree $T \in Q$, $T \subseteq T_0$. Then $T \subseteq^{\mathtt{fin}} \bigcup D$, in particular, there is a tree $S \in D$ with $X = [S] \cap [T] \ne \varnothing$. However X is clopen in $[S]$ by (3) in Section 5. Therefore there is a tree $T' \in Q$ with $[T'] \subseteq X$, thus $T' \subseteq S \in D$ and $T' \subseteq T \subseteq T_0$. We conclude that T_0 is compatible with $S \in D$ in $P \cup Q$.

(ii) By Lemma 5.2(ii), $P \sqsubset Q$. Now let $T' \in Q$. So for some $\mu \le \alpha < \lambda$, $T' \in P_\alpha$, and since $P_\mu \sqsubset P_\alpha$, we have $T' \subseteq^{\mathtt{fin}} \bigcup P_\mu$. Thus for some finite $D' \subseteq P_\mu$, $T' \subseteq \bigcup D$. But, as $P \sqsubset_D P_\mu$, for each $T \in D'$ the relation$T \subseteq^{\mathtt{fin}} \bigcup D$ holds, hence $T' \subseteq^{\mathtt{fin}} \bigcup D$.[2] □

In the case of **multiforcings**, we modify the above definitions as follows. First of all, if $\boldsymbol{u}$ is a multitree and $\boldsymbol{D}$ a collection of multitrees, then $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}$ will mean that there is a finite set $\boldsymbol{D}' \subseteq \boldsymbol{D}$ satisfying 1) $|\boldsymbol{v}| = |\boldsymbol{u}|$ for all $\boldsymbol{v} \in \boldsymbol{D}'$, and 2) $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v}\in\boldsymbol{D}'} [\boldsymbol{v}]$. (See Section 4 on $[\boldsymbol{u}]$.)

Definition 6.3. Let $\boldsymbol{\pi}, \boldsymbol{\varrho}$ be multiforcings. Say that $\boldsymbol{\varrho}$ *seals a set* $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$ over $\boldsymbol{\pi}$, symbolically $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho}$, if $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$ and the following holds:

(∗) if $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varrho})$, $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$, then there is $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, still $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, and $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$, where $\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}} = \{\boldsymbol{u}' \in \mathbf{MT}(\boldsymbol{\pi}) : |\boldsymbol{u}'| = |\boldsymbol{u}| \text{ and } \boldsymbol{u}' \cup \boldsymbol{q} \in \boldsymbol{D}\}$. □

Lemma 6.4. *Let* $\boldsymbol{\pi}, \boldsymbol{\varrho}, \boldsymbol{\sigma}$ *be multiforcings and* $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$*. Then:*

[2] This proof of (ii), correcting our original one, was suggested by the anonymous referee.

(i) *if* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho}$ *then* $\boldsymbol{D}$ *is pre-dense in* $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$*;*

(ii) *if* $\boldsymbol{\pi}$ *is regular,* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}_i} \boldsymbol{\varrho}$ *for* $i = 1, \dots, n$*, all sets* $\boldsymbol{D}_i \subseteq \mathbf{MT}(\boldsymbol{\pi})$ *are open dense in* $\mathbf{MT}(\boldsymbol{\pi})$*, and* $\boldsymbol{D} = \bigcap_i \boldsymbol{D}_i$*, then* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho}$*;*

(iii) *if* $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence of special multiforcings,* $0 < \mu < \lambda$*,* $\boldsymbol{\pi}_{<\mu} = \bigcup^{\mathtt{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha$*,* $\boldsymbol{D}$ *is open dense in* $\mathbf{MT}(\boldsymbol{\pi}_{<\mu})$*, and* $\boldsymbol{\pi}_{<\mu} \sqsubset_{\boldsymbol{D}} \boldsymbol{\pi}_\mu$*, then* $\boldsymbol{\pi}_{<\mu} \sqsubset_{\boldsymbol{D}} \boldsymbol{\pi}_{\geq\mu} = \bigcup^{\mathtt{cw}}_{\mu\leq\alpha<\lambda} \boldsymbol{\pi}_\alpha$*.*

Proof. (i) Let $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$. Due to the product character of the multiforcing $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$, we can assume that $|\boldsymbol{r}| \subseteq |\boldsymbol{\pi}|$. Let

$$X = \{\xi \in |\boldsymbol{r}| : T^{\boldsymbol{r}}_\xi \in \mathbf{MT}(\boldsymbol{\varrho})\}, \quad Y = \{\xi \in |\boldsymbol{r}| : T^{\boldsymbol{r}}_\xi \in \mathbf{MT}(\boldsymbol{\pi})\}.$$

Then $\boldsymbol{r} = \boldsymbol{u} \cup \boldsymbol{p}$, where $\boldsymbol{u} = \boldsymbol{r} \restriction X \in \mathbf{MT}(\boldsymbol{\varrho})$, $\boldsymbol{p} = \boldsymbol{r} \restriction Y \in \mathbf{MT}(\boldsymbol{\pi})$. As $\boldsymbol{\varrho}$ seals $\boldsymbol{D}$, there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, and $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. Easily there is a multitree $\boldsymbol{u}' \in \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ compatible with $\boldsymbol{u}$ in $\mathbf{MT}(\boldsymbol{\varrho})$; let $\boldsymbol{w} \in \mathbf{MT}(\boldsymbol{\varrho})$, $\boldsymbol{w} \leqslant \boldsymbol{u}$, $\boldsymbol{w} \leqslant \boldsymbol{u}'$, $|\boldsymbol{w}| = |\boldsymbol{u}'| = |\boldsymbol{u}|$. Then the multitree $\boldsymbol{r}' = \boldsymbol{w} \cup \boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$ satisfies $\boldsymbol{r}' \leqslant \boldsymbol{r}$ and $\boldsymbol{r}' \leqslant \boldsymbol{u}' \cup \boldsymbol{q} \in \boldsymbol{D}$.

(ii) Let $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varrho})$, $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$. Iterating $(*)$ for $\boldsymbol{D}_i$, $i = 1, \dots, n$, we find a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, the equality $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$ holds, and $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee (\boldsymbol{D}_i)^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ for all i, where

$$(\boldsymbol{D}_i)^{|\boldsymbol{u}|}_{\boldsymbol{q}} = \{\boldsymbol{u}' \in \mathbf{MT}(\boldsymbol{\pi}) : |\boldsymbol{u}'| = |\boldsymbol{u}| \text{ and } \boldsymbol{u}' \cup \boldsymbol{q} \in \boldsymbol{D}_i\}.$$

Thus there exist finite sets $U_i \subseteq (\boldsymbol{D}_i)^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ such that $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v}\in U_i} [\boldsymbol{v}]$ for all i. Using the regularity assumption, we get a finite set $W \subseteq \mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{w}| = |\boldsymbol{u}|$ for all $\boldsymbol{w} \in W$, the equality $\bigcap_i \bigcup_{\boldsymbol{v}\in U_i} [\boldsymbol{v}] = \bigcup_{\boldsymbol{w}\in W} [\boldsymbol{w}]$ holds, and if $i = 1, \dots, n$ and $\boldsymbol{w} \in W$ then $[\boldsymbol{w}] \subseteq [\boldsymbol{v}]$ for some $\boldsymbol{v} \in U_i$ — hence $\boldsymbol{w} \cup \boldsymbol{q} \in \boldsymbol{D}_i$. We conclude that if $\boldsymbol{w} \in W$ then $\boldsymbol{w} \cup \boldsymbol{q} \in \boldsymbol{D}$, hence $\boldsymbol{w} \in \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. Thus $W \subseteq \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. However $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{w}\in W} [\boldsymbol{w}]$ by the choice of W. Thus $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$.

(iii) Both $\boldsymbol{\pi}_{<\mu}$ and $\boldsymbol{\pi}_{\geq\mu}$ are regular multiforcings by Lemma 5.5. To check that $\boldsymbol{\pi}_{\geq\mu}$ seals $\boldsymbol{D}$ over $\boldsymbol{\pi}_{<\mu}$, let $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\pi}_{\geq\mu})$, $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}_{<\mu}|$, $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}_{<\mu})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$. There is a finite set $U \subseteq \mathbf{MT}(\boldsymbol{\pi}_\mu)$, such that $|\boldsymbol{v}| = |\boldsymbol{u}|$ for all $\boldsymbol{v} \in U$, and $[\boldsymbol{u}] \subseteq \bigcup_{\boldsymbol{v}\in U} [\boldsymbol{v}]$. As $\boldsymbol{\pi}_{<\mu} \sqsubset_{\boldsymbol{D}} \boldsymbol{\pi}_\mu$, by iterated application of Definition 6.3$(*)$ we get a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi}_{<\mu})$ such that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, and if $\boldsymbol{v} \in U$ then $\boldsymbol{v} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$, where

$$\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}} = \{\boldsymbol{v}' \in \mathbf{MT}(\boldsymbol{\pi}_{<\mu}) : |\boldsymbol{v}'| = |\boldsymbol{v}| = |\boldsymbol{u}| \wedge \boldsymbol{v}' \cup \boldsymbol{q} \in \boldsymbol{D}\}.$$

And finally $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee U$ by construction, hence $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ as well. □

7. Sealing real names

In this section we present another extension of the refinement technique, discovered in [31], related to the structure of **real names**, i.e., names of reals in 2^ω in the context of forcing notions of the form $\mathbf{MT}(\boldsymbol{\pi})$.

Let a *real name* be any set $\mathbf{c} \subseteq \mathbf{MT} \times (\omega \times 2)$ such that the sets $K^{\mathbf{c}}_{ni} = \{\boldsymbol{p} \in \mathbf{MT} : \langle \boldsymbol{p}, n, i \rangle \in \mathbf{c}\}$ satisfy the following: if $n < \omega$ and $\boldsymbol{p} \in K^{\mathbf{c}}_{n0}$, $\boldsymbol{q} \in K^{\mathbf{c}}_{n1}$, then the multitrees $\boldsymbol{p}, \boldsymbol{q}$ are *incompatible* in the sense of Definition 4.1.

Let $K^{\mathbf{c}}_n = K^{\mathbf{c}}_{n0} \cup K^{\mathbf{c}}_{n1}$; then $K^{\mathbf{c}}_n \subseteq \mathbf{MT}$.

A real name $\mathbf{c}$ is *small* if each $K^{\mathbf{c}}_n$ is at most countable — then the set $|\mathbf{c}| = \bigcup_n \bigcup_{\boldsymbol{p}\in K^{\mathbf{c}}_n} |\boldsymbol{p}|$, and $\mathbf{c}$ itself, are countable, too.

Now let $\boldsymbol{\pi}$ be a multiforcing. A real name $\mathbf{c}$ is $\boldsymbol{\pi}$*-complete* if the set

$$K^{\mathbf{c}}_n \!\uparrow\! \boldsymbol{\pi} = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \exists\, \boldsymbol{q} \in K^{\mathbf{c}}_n\, (\boldsymbol{p} \leq \boldsymbol{q})\}$$

is dense in $\mathbf{MT}(\boldsymbol{\pi})$ for each n. (We do not require here that $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times (\omega \times 2)$, or equivalently, $K_n^{\mathbf{c}} \subseteq \mathbf{MT}(\boldsymbol{\pi})$, $\forall n$.) In this case, if a set (a filter) $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is $\mathbf{MT}(\boldsymbol{\pi})$-generic over the family of all sets $K_n^{\mathbf{c}}\!\uparrow\!\boldsymbol{\pi}$, $n < \omega$, then we define a real $\mathbf{c}[G] \in 2^\omega$ so that $\mathbf{c}[G](n) = i$ iff $G \cap (K_n^{\mathbf{c}}\!\uparrow\!\boldsymbol{\pi}) \neq \varnothing$.

Example 7.1. Assume that $\xi < \omega_1$. Define a real name $\dot{\boldsymbol{x}}_\xi \in \mathbf{L}$ such that each set $K_{ni}^{\dot{\boldsymbol{x}}_\xi}$ consists of a single multitree $\boldsymbol{P}_{ni}^\xi$, where $|\boldsymbol{P}_{ni}^\xi| = \{\xi\}$ (the domain), $\boldsymbol{P}_{ni}^\xi(\xi) = T_{ni}$, and $T_{ni} = \{s \in 2^{<\omega} : \mathtt{lh}(s) \leq n \vee s(n) = i\}$.

We leave it as a routine **exercise** to prove that, in $\mathbf{L}$, $\dot{\boldsymbol{x}}_\xi$ is a small real name, $\boldsymbol{\pi}$-complete for any multiforcing $\boldsymbol{\pi}$, and if a set $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is $\mathbf{MT}(\boldsymbol{\pi})$-generic over $\mathbf{L}$, then the real $\dot{\boldsymbol{x}}_\xi[G]$ is identic to $x_\xi[G]$ defined as in Remark 4.4. Thus $\dot{\boldsymbol{x}}_\xi$ is a *canonical name* for the generic real $x_\xi[G]$. □

Direct forcing. Assume that $\mathbf{c}$ is a real name. Say that a multitree $\boldsymbol{p}$:

- *directly forces* $\mathbf{c}(n) = i$, where $n < \omega$ and $i = 0, 1$, iff there is a multitree $\boldsymbol{q} \in K_{ni}^{\mathbf{c}}$ such that $\boldsymbol{p} \leqslant \boldsymbol{q}$;
- *directly forces* $s \subset \mathbf{c}$, where $s \in 2^{<\omega}$, iff for all $n < \mathtt{lh}(s)$, $\boldsymbol{p}$ directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;
- *directly forces* $\mathbf{c} \notin [T]$, where $T \in \mathbf{PT}$, iff there is a string $s \in 2^{<\omega} \smallsetminus T$ such that $\boldsymbol{p}$ directly forces $s \subset \mathbf{c}$.

The definition of direct forcing is not explicitly associated with any concrete forcing notion, but in fact it is compatible with any multiforcing.

Lemma 7.2. *Assume that $\boldsymbol{\pi}$ is a multiforcing, $\mathbf{c}$ is a $\boldsymbol{\pi}$-complete real name, and $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$. If $n < \omega$ then there exists $i = 0, 1$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, which directly forces $\mathbf{c}(n) = i$. If $T \in \mathbf{PT}$ then there exists $s \in T$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, which directly forces $\mathbf{c} \notin [T\!\restriction_s]$.*

Proof. To prove the first claim use the density of sets $K_n^{\mathbf{c}}\!\uparrow\!\boldsymbol{\pi}$ by the definition of completeness. To prove the second claim, pick n such that $T \cap 2^n$ contains at least two strings. By the first claim, there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, and a string $t \in T \cap 2^n$ such that $\boldsymbol{q}$ directly forces $t \subset \mathbf{c}$. Now take any $s \in T \cap 2^n$, $s \neq t$. □

Sealing names. The next definition extends Definition 6.3 to real names.

Definition 7.3. Assume that $\boldsymbol{\pi}, \boldsymbol{\wp}$ are multiforcings, $\mathbf{c}$ is a real name, and $\boldsymbol{\pi} \sqsubset \boldsymbol{\wp}$. Say that $\boldsymbol{\wp}$ *seals* $\mathbf{c}$ *over* $\boldsymbol{\pi}$, symbolically $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\wp}$, if $\boldsymbol{\wp}$ seals, over $\boldsymbol{\pi}$, each set $K_n^{\mathbf{c}}\!\uparrow\!\boldsymbol{\pi} = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \exists\, \boldsymbol{q} \in K_n^{\mathbf{c}}\, (\boldsymbol{p} \leq \boldsymbol{q})\}$. □

Non-principal names. The following definition presents conditions which will work towards a given real name $\mathbf{c}$ being NOT a name of a real of the form $x_\xi[G]$ in the context of Remark 4.4.

Definition 7.4. Let $\boldsymbol{\pi}$ be a multiforcing, $\xi \in |\boldsymbol{\pi}|$. A real name $\mathbf{c}$ is *non-principal over* $\boldsymbol{\pi}$ *at* ξ, if the following set is open dense in $\mathbf{MT}(\boldsymbol{\pi})$:

$$\boldsymbol{D}_\xi^{\boldsymbol{\pi}}(\mathbf{c}) = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \xi \in |\boldsymbol{p}| \wedge \boldsymbol{p} \text{ directly forces } \mathbf{c} \notin [T_\xi^{\boldsymbol{p}}]\}\,.$$

Let $\boldsymbol{\wp}$ be another multiforcing, and $\boldsymbol{\pi} \sqsubset \boldsymbol{\wp}$. Say that $\boldsymbol{\wp}$ *avoids a real name* $\mathbf{c}$ *over* $\boldsymbol{\pi}$ *at* ξ, in symbol $\boldsymbol{\pi} \sqsubset_\xi^{\mathbf{c}} \boldsymbol{\wp}$, if for each $Q \in \boldsymbol{\wp}(\xi)$, $\boldsymbol{\wp}$ seals the set

$$\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi}) = \{\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi}) : \xi \in |\boldsymbol{r}| \wedge \boldsymbol{r} \text{ directly forces } \mathbf{c} \notin [Q]\}\,,$$

over $\boldsymbol{\pi}$ in the sense of Definition 6.3 — that is formally $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},Q,\boldsymbol{\pi})} \boldsymbol{\wp}$. □

8. Consequences for generic extensions

We first prove a lemma on adequately representation of reals in $\mathbf{MT}(\boldsymbol{\pi})$-generic extensions by real names. Then Theorem 8.2 will show corollaries for non-principal names.

Lemma 8.1. *Suppose that $\boldsymbol{\pi}$ is a regular multiforcing. If $\mathbf{MT}(\boldsymbol{\pi})$ is a CCC forcing notion, $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is $\mathbf{MT}(\boldsymbol{\pi})$-generic over the ground* **ZFC** *universe* $\mathbf{V}$*, and $x \in \mathbf{V}[G] \cap 2^\omega$, then there is a small $\boldsymbol{\pi}$-complete real name Here we follow the 'Forcing over the Universe' approach as in Kunen, Set Theory, IV.5.2. $\mathbf{c} \in \mathbf{V}$, $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times \omega \times 2$, such that $x = \mathbf{c}[G]$.*

Proof. It is an instance of a general forcing theorem that there is a (not necessarily small) $\boldsymbol{\pi}$-complete real name $\mathbf{d} \in \mathbf{V}$, $\mathbf{d} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times \omega \times 2$, such that $x = \mathbf{d}[G]$. To get a small name, extend each set $K_n^{\mathbf{d}} \subseteq \mathbf{MT}(\boldsymbol{\pi})$ to an open dense set

$$K_n^{\mathbf{d}} \uparrow \boldsymbol{\pi} = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \exists\, \boldsymbol{q} \in K_n^{\mathbf{d}}\, (\boldsymbol{p} \le \boldsymbol{q})\}\,,$$

choose maximal (countable by CCC) antichains $A_n \subseteq K_n^{\mathbf{d}} \uparrow \boldsymbol{\pi}$ in those sets, and then let $A_{ni} = \{\boldsymbol{p} \in A_n : \exists\, \boldsymbol{q} \in K_{ni}^{\mathbf{d}}\, (\boldsymbol{p} \le \boldsymbol{q})\}$ and $\mathbf{c} = \{\langle \boldsymbol{p}, n, i\rangle : \boldsymbol{p} \in A_{ni}\}$. $\square$

Theorem 8.2. *Let $\boldsymbol{\pi}$ be a regular multiforcing and $\xi \in |\boldsymbol{\pi}|$. Then*

(i) *if $\mathbf{MT}(\boldsymbol{\pi})$ is CCC, a set $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is $\mathbf{MT}(\boldsymbol{\pi})$-generic over the ground set universe $\mathbf{V}$, and $x \in \mathbf{V}[G] \cap 2^\omega$, $x \ne x_\xi[G]$, then, in $\mathbf{V}$, there is a small $\boldsymbol{\pi}$-complete real name $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times (\omega \times 2)$, non-principal over $\boldsymbol{\pi}$ at ξ and such that $x = \mathbf{c}[G]$;*
(ii) *if $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times (\omega \times 2)$ is a $\boldsymbol{\pi}$-complete real name, $\boldsymbol{\varrho}$ is a multiforcing, $\boldsymbol{\pi} \sqsubset_\xi^{\mathbf{c}} \boldsymbol{\varrho}$, and a set $G \subseteq \mathbf{MT}(\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varrho})$ is $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varrho})$-generic over $\mathbf{V}$, then $\mathbf{c}[G] \notin \bigcup_{Q \in \boldsymbol{\varrho}(\xi)} [Q]$.*

Proof. (i) Let $x \ne x_\xi[G]$. By Lemma 8.1, there is a $\boldsymbol{\pi}$-complete real name $\mathbf{c}$ such that $x = \mathbf{c}[G]$ and $\mathbf{MT}(\boldsymbol{\pi})$ forces that $\mathbf{c} \ne x_\xi[\underline{G}]$. It remains to show that $\mathbf{c}$ is a non-principal name over $\boldsymbol{\pi}$ at ξ, that is, the set

$$\boldsymbol{D}_\xi^{\boldsymbol{\pi}}(\mathbf{c}) = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \xi \in |\boldsymbol{p}| \wedge \boldsymbol{p} \text{ directly forces } \mathbf{c} \notin [T_\xi^{\boldsymbol{p}}]\}\,.$$

is open dense in $\mathbf{MT}(\boldsymbol{\pi})$. The openness is clear, let us prove the density. Consider any $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$. Then $\boldsymbol{q}$ $\mathbf{MT}(\boldsymbol{\pi})$-forces $\mathbf{c} \ne x_\xi[\underline{G}]$ by the choice of $\mathbf{c}$, hence we can assume that, for some n, $\mathbf{c}(n) \ne x_\xi[\underline{G}](n)$ is $\mathbf{MT}(\boldsymbol{\pi})$-forced by $\boldsymbol{q}$. Then by Lemma 7.2 there is a multitree $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{p} \leqslant \boldsymbol{q}$, and $s \in \omega^{n+1}$, such that $\boldsymbol{p}$ directly forces $s \subseteq \mathbf{c}$. Now it suffices to show that $s \notin T_\xi^{\boldsymbol{p}}$. Suppose otherwise: $s \in T_\xi^{\boldsymbol{p}}$. Then the tree $T = T_\xi^{\boldsymbol{p}} \restriction_s$ still belongs to $\mathbf{MT}(\boldsymbol{\pi})$. Therefore the multitree $\boldsymbol{r}$ defined by $T_\xi^{\boldsymbol{r}} = T$ and $T_{\xi'}^{\boldsymbol{r}} = T_{\xi'}^{\boldsymbol{p}}$ for each $\xi' \ne \xi$, belongs to $\mathbf{MT}(\boldsymbol{\pi})$ and satisfies $\boldsymbol{r} \leqslant \boldsymbol{p} \leqslant \boldsymbol{q}$. However $\boldsymbol{r}$ directly forces both $\mathbf{c}(n)$ and $x_\xi[\underline{G}](n)$ to be equal to one and the same value $\ell = s(n)$, which contradicts to the choice of n.

(ii) Suppose towards the contrary that $Q \in \boldsymbol{\varrho}(\xi)$ and $\mathbf{c}[G] \in [Q]$. By definition, $\boldsymbol{\varrho}$ seals, over $\boldsymbol{\pi}$, the set

$$\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi}) = \{\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi}) : \xi \in |\boldsymbol{r}| \wedge \boldsymbol{r} \text{ directly forces } \mathbf{c} \notin [Q]\}\,.$$

Therefore $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ is pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varrho})$ by Lemma 6.4(i), and hence $G \cap \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi}) \ne \varnothing$, i.e., there is a multitree $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$ which directly forces $\mathbf{c} \notin [Q]$. Then $\mathbf{c}[G] \notin [Q]$, which is a contradiction. $\square$

9. Refinement construction by Jensen

The splitting/fusion construction was originally invented as a method to obtain perfect sets in Polish spaces. Jensen modified it in [22] in order to get refinements of arboreal forcing notions. The definition below in this Section, taken from [31], introduces essentially a product version of Jensen's refinements. As we deal with *finite* support products, the standard technique in the theory of countable-support Sacks products, as e.g. in [17] or [24], is not fully applicable. The notion of a *system* in the next definition will be the key instrument. That finite-support products of Jensen-style forcing notions are CCC, preserve cardinals (unlike finite-support Sacks products), and admit a suitable version of splitting/fusion technique, was demonstrated in [26].

9A. Systems. Suppose that $\boldsymbol{\pi} = \langle P_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ is a multiforcing. Let a $\boldsymbol{\pi}$-*system* be any map φ defined on a finite set $|\varphi| \subseteq |\boldsymbol{\pi}| \times \omega$ such that if $\langle \xi, k \rangle \in |\varphi|$ then $\varphi(\xi,k) = \langle n^\varphi_{\xi k}, T^\varphi_{\xi k} \rangle \in \omega \times \bigcup^{\mathtt{fin}} P_\xi$, where $\bigcup^{\mathtt{fin}} P_\xi$ consists of all finite unions of trees in P_ξ, as above. A system φ is *antichain-like* if for any two different pairs of indices $\langle \xi, k \rangle \neq \langle \eta, m \rangle$ in $|\varphi|$ we have $T^\varphi_{\xi k} \perp T^\varphi_{\eta m}$.

We order the set $\mathbf{Sys}(\boldsymbol{\pi})$ of all $\boldsymbol{\pi}$-systems componentwise: $\varphi \preccurlyeq \psi$ (φ extends ψ) iff $|\psi| \subseteq |\varphi|$ and $\langle n^\varphi_{\xi k}, T^\varphi_{\xi k} \rangle \preccurlyeq \langle n^\psi_{\xi k}, T^\psi_{\xi k} \rangle$ for all $\langle \xi, k \rangle \in |\psi|$ in $\omega \times \mathbf{PT}$ (Section 3).

A set $\Delta \subseteq \mathbf{Sys}(\boldsymbol{\pi})$ is *dense* if for any $\psi \in \mathbf{Sys}(\boldsymbol{\pi})$ there is $\varphi \in \Delta$ with $\varphi \preccurlyeq \psi$.

Lemma 9.1. *If $\boldsymbol{\pi}$ is a multiforcing and $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ then there is an antichain-like system $\psi \in \mathbf{Sys}(\boldsymbol{\pi})$ such that $|\psi| = |\varphi|$ and $\psi \preccurlyeq \varphi$.*

Proof. If $S, T \in \mathbf{PT}$ then there are $\subset$-incomparable tuples $u \in S$, $v \in T$. The claim on this line is true in case S=T as well, so no change is necessary Then the trees $S' = S \restriction_u$, $T' = T \restriction_v$ satisfy $S' \perp T'$. Moreover if $S \in P$ and $T \in Q$, where P, Q are arboreal forcings, then still $S' \in P$ and $T' \in Q$. We get an antichain-like system $\psi \preccurlyeq \varphi$ by consecutive shrinking trees in φ. □

Remark 9.2. This definition of systems somewhat differs from the one considered in [31, Definition 7.1]. The difference is that the version of [31] requires that all numbers $n^\varphi_{\xi k}$ are equal to one and the same number $n = n_\varphi$. Call such a system *uniform* and let $\mathbf{Sys}_{\mathrm{unif}}(\boldsymbol{\pi}) \subseteq \mathbf{Sys}(\boldsymbol{\pi})$ be the set of all uniform systems. We may note that $\mathbf{Sys}_{\mathrm{unif}}(\boldsymbol{\pi})$ is dense in $\mathbf{Sys}(\boldsymbol{\pi})$: if $\psi \in \mathbf{Sys}(\boldsymbol{\pi})$ then there is a system $\varphi \in \mathbf{Sys}_{\mathrm{unif}}(\boldsymbol{\pi})$, $\varphi \preccurlyeq \psi$. (This is because clearly $\langle n, T \rangle \preccurlyeq \langle m, T \rangle$ for any tree T in case $n > m$.) Therefore each $\mathfrak{M}$-generic sequence Φ of systems in $\mathbf{Sys}(\boldsymbol{\pi})$ contains an $\mathfrak{M}$-generic infinite subsequence Φ' in $\mathbf{Sys}_{\mathrm{unif}}(\boldsymbol{\pi})$. The version developed here will allow us to use product-generic arguments in the proof of Theorem 26.1 below, ineffective in the $\mathbf{Sys}_{\mathrm{unif}}$ version. □

9B. A power-free subtheory. Let $\mathbf{ZFC}^-_1$ be the theory $\mathbf{ZFC}$ with the powerset axiom removed, the collection scheme in place of the replacement scheme, the version of the axiom of choice which states that every set can be well-ordered, plus the axiom "$\mathscr{P}(\omega)$ exists".[3] Using the "$\mathscr{P}(\omega)$ existence" principle (which accounts for the index 1), $\mathbf{ZFC}^-_1$ proves that *continual* objects, such as ω^ω, ω_1, $\mathbf{PT}$, $\mathbf{MT}$, $\mathbf{sMF}$ (small multiforcings) do exist as sets.

9C. Generic sequences of systems. Let $\mathfrak{M}$ be a countable transitive model (CTM) of $\mathbf{ZFC}^-_1$. Suppose that $\boldsymbol{\pi} = \langle P_\xi \rangle_{\xi \in |\boldsymbol{\pi}|} \in \mathfrak{M}$ is a multiforcing.

(I) As $\mathfrak{M}$ is countable, there is a $\preccurlyeq$-decreasing sequence $\Phi = \langle \varphi_j \rangle_{j<\omega}$ of systems $\varphi_j \in \mathbf{Sys}(\boldsymbol{\pi})$, $\mathfrak{M}$-*generic* in the sense that it intersects every set $\Delta \in \mathfrak{M}$, $\Delta \subseteq \mathbf{Sys}(\boldsymbol{\pi})$, dense in $\mathbf{Sys}(\boldsymbol{\pi})$. (See Subsection 9A on the density.)

Fix any such an $\mathfrak{M}$-generic $\preccurlyeq$-decreasing sequence Φ of systems in $\mathbf{Sys}(\boldsymbol{\pi})$.

[3] It is known that without the powerset axiom, the collection and replacement schemes are not equivalent and neither are the various forms of the axiom of choice equivalent over $\mathbf{ZF}$. See more in [15] on power-free subtheories of $\mathbf{ZFC}$.

(II) By definition, we have $\varphi_j = \langle\langle n_\xi^{\varphi_j} k, T_\xi^{\varphi_j} k\rangle\rangle_{\langle\xi,k\rangle\in|\varphi_j|}$ for all j, where $|\varphi_j| \subseteq |\boldsymbol{\pi}| \times \omega$ is finite, and each tree $T_{\xi k}^{\varphi_j}$ belongs to $\bigcup^{\mathtt{fin}} P_\xi$. We have $n_{\xi k}^{\varphi_j} \to \infty$ with $j \to \infty$ monotonously for each ξ, k.

(III) Let $\xi \in |\boldsymbol{\pi}|$, $k < \omega$. By the genericity assumption, there is a number $j(\xi, k)$ such that if $j \geq j(\xi, k)$ then $\langle\xi, k\rangle \in |\varphi_j|$, hence the pair $\varphi_j(\xi, k) = \langle n_{\xi k}^{\varphi_j}, T_{\xi k}^{\varphi_j}\rangle$ is defined, $T_{\xi k}^{\varphi_j} \in \bigcup^{\mathtt{fin}} P_\xi$, and we have

$$\ldots \preccurlyeq \langle n_{\xi k}^{\varphi_{j(\xi,k)+2}}, T_{\xi k}^{\varphi_{j(\xi,k)+2}}\rangle \preccurlyeq \langle n_{\xi k}^{\varphi_{j(\xi,k)+1}}, T_{\xi k}^{\varphi_{j(\xi,k)+1}}\rangle \preccurlyeq \langle n_{\xi k}^{\varphi_{j(\xi,k)}}, T_{\xi k}^{\varphi_{j(\xi,k)}}\rangle,$$

with $n_{\xi k}^{\varphi_{j(\xi,k)}} \leq n_{\xi k}^{\varphi_{j(\xi,k)+1}} \leq n_{\xi k}^{\varphi_{j(\xi,k)+2}} \leq \ldots \to \infty$, by (II) above.

(IV) Then it follows by Lemma 3.3 that each intersection $\boldsymbol{Q}_{\xi k}^{\Phi} = \bigcap_{j \geq j(\xi,k)} T_{\xi k}^{\varphi_j}$ is a tree in **PT** (not necessarily in P_ξ), and

$$\langle n_{\xi k}^{\varphi_{j(\xi,k)}}, \boldsymbol{Q}_{\xi k}^{\Phi}\rangle \preccurlyeq \langle n_{\xi k}^{\varphi_{j(\xi,k)}}, T_{\xi k}^{\varphi_j}\rangle$$

holds for all $j \geq j(\xi, k)$. We put $\mathbb{Q}_\xi^\Phi = \{\boldsymbol{Q}_{\xi k}^\Phi \restriction_s : k < \omega \wedge s \in \boldsymbol{Q}_{\xi k}^\Phi\}$.

(V) We put $\boldsymbol{\varrho} = \lim[\Phi] := \langle \mathbb{Q}_\xi^\Phi\rangle_{\xi\in|\boldsymbol{\pi}|}$; then $\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho} = \langle P_\xi \cup \mathbb{Q}_\xi^\Phi\rangle_{\xi\in|\boldsymbol{\pi}|}$. If $\boldsymbol{\varrho} = \lim[\Phi]$ is obtained from an $\mathfrak{M}$-generic sequence Φ as in (I)–(V) above, then $\boldsymbol{\varrho}$ is called an **$\mathfrak{M}$-generic refinement** of $\boldsymbol{\pi}$.

Remark 9.3. A somewhat stronger notion of genericity was considered in [31]. Let HC be the set of all *hereditarily countable* sets; $X \in \mathrm{HC}$ iff the transitive closure $\mathrm{TC}(X)$ is at most countable. Let $\mathfrak{M}^+$ be the set of all sets $X \subseteq \mathrm{HC}$ definable in HC by an $\in$-formula with sets $x \in \mathfrak{M}$ as parameters; clearly $\mathfrak{M} \subsetneqq \mathfrak{M}^+$. The notion of an $\mathfrak{M}^+$-*generic sequence* of systems in $\mathbf{Sys}(\boldsymbol{\pi})$ is introduced similarly to (I), with the condition of non-empty intersection with every dense set $\Delta \in \mathfrak{M}^+$, as well as the ensuing definition of an $\mathfrak{M}^+$-*generic refinement* similar to (V).

Yet it will be demonstrated below that the $\mathfrak{M}$-genericity suffices to infer all crucial consequences of the stronger $\mathfrak{M}^+$-genericity obtained in [31]. □

Lemma 9.4. *If $\mathfrak{M} \models \mathbf{ZFC}_1^-$ is a CTM and $\boldsymbol{\pi} \in \mathfrak{M}$ is a multiforcing, then there exists an $\mathfrak{M}$-generic sequence Φ of systems in $\mathbf{Sys}(\boldsymbol{\pi})$ as in* (I)*, and hence there is a $\mathfrak{M}$-generic refinement $\boldsymbol{\varrho}$ of $\boldsymbol{\pi}$ as in* (V)*.*

Proof. Use the countability of $\mathfrak{M}$. □

10. Jensen's construction indeed yields a refinement

It is not immediately clear that the construction of $\mathfrak{M}$-generic refinements indeed results in refinements in the sense of Section 5. This is a subject of the next theorem, proved in [31] under the stronger assumption of $\mathfrak{M}^+$-genericity.

Theorem 10.1. *Assume that $\mathfrak{M} \models \mathbf{ZFC}_1^-$ is a CTM, $\boldsymbol{\pi} = \langle P_\xi\rangle_{\xi\in|\boldsymbol{\pi}|} \in \mathfrak{M}$ is a regular multiforcing, $\Phi = \langle\varphi_j\rangle_{j<\omega}$ is an $\mathfrak{M}$-generic $\preccurlyeq$-decreasing sequence of systems in $\mathbf{Sys}(\boldsymbol{\pi})$, as in* (I) *of Section 9, and $\boldsymbol{\varrho} = \lim[\Phi] = \langle\mathbb{Q}_\xi\rangle_{\xi\in|\boldsymbol{\pi}|}$ is an $\mathfrak{M}$-generic refinement of $\boldsymbol{\pi}$, as in* (V) *of Section 9. Then in the notation of items* (I)–(V) *of Section 9 we have:*

(i) *$\boldsymbol{\varrho}$ is a small special multiforcing, $|\boldsymbol{\varrho}| = |\boldsymbol{\pi}|$, and $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$;*
(ii) *if pairs $\langle\xi, k\rangle \neq \langle\eta, \ell\rangle$ belong to $|\boldsymbol{\pi}| \times \omega$ then $[\boldsymbol{Q}_\xi^\Phi k] \cap [\boldsymbol{Q}_{\eta\ell}^\Phi] = \varnothing$;*
(iii) *if $\xi \in |\boldsymbol{\pi}|$, $S \in \mathbb{Q}_\xi$ and $T \in P_\xi$ then $[S] \cap [T]$ is clopen in $[S]$ and $T \not\subseteq S$, in particular, $\mathbb{Q}_\xi \cap P_\xi = \varnothing$;*
(iv) *if $\xi \in |\boldsymbol{\pi}|$ then the set $\mathbb{Q}_\xi$ is open dense in $\mathbb{Q}_\xi \cup P_\xi$;*
(v) *if $\xi \in |\boldsymbol{\pi}|$ and $D \in \mathfrak{M}$, $D \subseteq P_\xi$ is pre-dense in P_ξ then $P_\xi \sqsubset_D \mathbb{Q}_\xi$;*
(vi) *if in addition $\boldsymbol{\pi} = \bigcup_{\alpha<\lambda}^{\mathtt{cw}} \boldsymbol{\pi}_\alpha$, where the ordinal $\lambda < \omega_1$ is limit and $\langle\boldsymbol{\pi}_\alpha\rangle_{\alpha<\lambda} \in \mathfrak{M}$ is a $\sqsubset$-increasing sequence of small special multiforcings, then $\boldsymbol{\pi}_\alpha \sqsubset \boldsymbol{\varrho}$ for all$\alpha < \lambda$, and $|\boldsymbol{\varrho}| = \bigcup_{\alpha<\lambda} |\boldsymbol{\pi}_\alpha|$;*

(vii) *if* $\boldsymbol{D} \in \mathfrak{M}$*,* $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$*, and* $\boldsymbol{D}$ *is open dense in* $\mathbf{MT}(\boldsymbol{\pi})$*, then* $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho}$*, and hence* $\boldsymbol{D}$ *is pre-dense in* $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho})$ *by Lemma 6.4;*

(viii) *if* $\mathbf{c} \in \mathfrak{M}$ *is a* $\boldsymbol{\pi}$*-complete real name then* $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho}$*.*

(ix) *if* $\eta \in |\boldsymbol{\pi}|$ *and* $\mathbf{c} \in \mathfrak{M}$ *is a* $\boldsymbol{\pi}$*-complete real name non-principal over* $\boldsymbol{\pi}$ *at* η*, then* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\eta} \boldsymbol{\varrho}$*.*

Proof. We argue in the notation of items (I)–(V) of Section 9.

(ii) By Lemma 9.1, the set D of all antichain-like systems $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$, such that $|\varphi|$ contains both $\langle \xi, k\rangle$ and $\langle \eta, \ell\rangle$, is dense in $\mathbf{Sys}(\boldsymbol{\pi})$, and obviously $D \in \mathfrak{M}$. Thus $\varphi_j \in D$ for some $j < \omega$. Then $T^{\varphi_j}_{\xi k} \perp T^{\varphi_j}_{\eta\ell}$ since φ_j is antichain-like. But $\boldsymbol{Q}^{\Phi}_{\xi} k \subseteq T^{\varphi_j}_{\xi k}$, $\boldsymbol{Q}^{\Phi}_{\eta\ell} \subseteq T^{\varphi_j}_{\eta\ell}$ by construction.

(iii) Let $k < \omega$ and $S = \boldsymbol{Q}^{\Phi}_{\xi} k$. (We don't need to consider all trees $S = \boldsymbol{Q}^{\Phi}_{\xi k} {\restriction}_s$, $s \in \boldsymbol{Q}^{\Phi}_{\xi k}$, since any $[\boldsymbol{Q}^{\Phi}_{\xi k} {\restriction}_s]$ is clopen in $[\boldsymbol{Q}^{\Phi}_{\xi k} {\restriction}_s]$.) To prove the clopenness claim, note that the set $D(T)$ of all systems $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ such that

$$\langle \xi, k\rangle \in |\varphi|, \text{ and if } s \in U_\varphi = 2^{n^{\varphi}_{\xi k}} \cap T^{\varphi}_{\xi k} \text{ then either } T^{\varphi}_{\xi k}{\restriction}_s \subseteq T \text{ or } T^{\varphi}_{\xi k}{\restriction}_s \perp T,$$

is dense in $\mathbf{Sys}(\boldsymbol{\pi})$, and obviously $D(T) \in \mathfrak{M}$. Thus $\varphi_j \in D(T)$ for some $j < \omega$. Then the disjoint sets

$$U^+ = \{s \in U_{\varphi_j} : T^{\varphi}_{\xi} k{\restriction}_s \subseteq T\} \quad \text{and} \quad U^- = \{s \in U_{\varphi_j} : T^{\varphi}_{\xi} k{\restriction}_s \perp T\}$$

satisfy $U^+ \cup U^- = U_{\varphi_j}$, and obviously the set $[T^{\varphi}_{\xi k}] \cap [T] = \bigcup_{s \in U^+} [T^{\varphi}_{\xi k}{\restriction}_s]$ is clopen in $[T^{\varphi}_{\xi k}]$. However $\boldsymbol{Q}^{\Phi}_{\xi} k \subseteq T^{\varphi_j}_{\xi k}$ by construction.

To prove $T \not\subseteq S$, note that the set $D'(T) \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$, such that $\langle \xi, k\rangle \in |\varphi|$ and $T \not\subseteq T^{\varphi}_{\xi k}$, is dense. Then argue as above.

(iv) The openness easily follows from (iii). To prove the density, let $T \in P_\xi$. The set $\Delta(T)$ of all systems $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$, such that $\langle \xi, k\rangle \in |\varphi|$ and $T^{\varphi}_{\xi k} = T$ for some $k < \omega$, belongs to $\mathfrak{M}$ and is dense in $\mathbf{Sys}(\boldsymbol{\pi})$.

(i) By construction, the sets $\boldsymbol{\varrho}(\xi) = \mathbb{Q}^{\Phi}_{\xi}$ are special arboreal forcings, and hence $\boldsymbol{\varrho}$ is a small special multiforcing, and $|\boldsymbol{\varrho}| = |\boldsymbol{\pi}|$. To establish $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$, let $\xi \in |\boldsymbol{\pi}|$. The relation $P_\xi \sqsubset \mathbb{Q}_\xi$ follows from (iv) and (iii).

(v) Assume that $\xi \in |\boldsymbol{\pi}|$, $k < \omega$, $D \in \mathfrak{M}$ is pre-dense in P_ξ. Then the set $D' = \{T \in \mathbb{P}_\xi : \exists S \in D(T \subseteq S)\}$ is open dense in P_ξ, and hence the set $\Delta \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$, such that $\langle \xi, k\rangle \in |\varphi|$ and $T^{\varphi}_{\xi k}{\restriction}_s \in D'$ for all $s \in 2^{n^{\varphi}_{\xi k}} \cap T^{\varphi}_{\xi k}$, is dense in $\mathbf{Sys}(\boldsymbol{\pi})$. Thus $\varphi_j \in \Delta$ for some j, and this implies $\boldsymbol{Q}^{\Phi}_{\xi} k \subseteq T^{\varphi_j}_{\xi k} \subseteq^{\mathtt{fin}} \bigcup D$.

(vi) We have to prove that $\boldsymbol{\pi}_\alpha(\xi) \sqsubset \boldsymbol{\varrho}(\xi)$ whenever $\xi \in |\boldsymbol{\pi}_\alpha|$. By (iv) and (iii) already established, it suffices to show that $\boldsymbol{Q}^{\Phi}_{\xi} k \subseteq^{\mathtt{fin}} \bigcup \boldsymbol{\pi}_\alpha(\xi)$ for any k. Note that the set $\boldsymbol{\pi}_\alpha(\xi) \in \mathfrak{M}$ is pre-dense in $\boldsymbol{\pi}(\xi) = P_\xi$ by Lemma 5.2. We conclude that the set $D = \{T \in P_\xi : \exists S \in \boldsymbol{\pi}_\alpha(\xi)(T \subseteq S)\} \in \mathfrak{M}$ is open dense in P_ξ. This implies that if $S \in \bigcup^{\mathtt{fin}} P_\xi$ and $m < \omega$ then there is a tree $T \in \bigcup^{\mathtt{fin}} P_\xi$ satisfying $\langle m, T\rangle \preccurlyeq \langle m, S\rangle$ and such that if $u \in T \cap 2^m$ then $T{\restriction}_u \in D$, and hence overall $T \subseteq^{\mathtt{fin}} \bigcup D$. It follows that the set $\Delta \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$, such that $\langle \xi, k\rangle \in |\varphi|$ and $T^{\varphi}_{\xi k} \subseteq^{\mathtt{fin}} \bigcup D$, is dense in $\mathbf{Sys}(\boldsymbol{\pi})$. Therefore $\boldsymbol{Q}^{\Phi}_{\xi} k \subseteq^{\mathtt{fin}} \bigcup D$ by construction, and then obviously $\boldsymbol{Q}^{\Phi}_{\xi} k \subseteq^{\mathtt{fin}} \bigcup \boldsymbol{\pi}_\alpha(\xi)$, as required.

(vii) Suppose that $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varrho})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$, as in $(*)$ of Definition 6.3; the extra condition $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$ holds automatically as $|\boldsymbol{\varrho}| = |\boldsymbol{\pi}|$. We have to find a multitree $\boldsymbol{q}$ which witnesses 6.3 $(*)$ for $\boldsymbol{u}$.

Each term $T^{\boldsymbol{u}}_{\xi}$ of $\boldsymbol{u}$ ($\xi \in |\boldsymbol{u}|$) is equal to some $\boldsymbol{Q}^{\Phi}_{\xi,k_\xi}{\restriction}_{t_\xi}$, where $k_\xi < \omega$ and $t_\xi \in \boldsymbol{Q}^{\Phi}_{\xi,k_\xi}$. We w.l.o.g. assume that just $t_\xi = \Lambda$, so $T^{\boldsymbol{u}}_{\xi} = \boldsymbol{Q}^{\Phi}_{\xi,k_\xi}$, $\forall\,\xi$.

Let $\mathbf{Sys}_{\boldsymbol{u}}(\boldsymbol{\pi})$ be the set of all systems $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ such that $\langle \xi, k_\xi\rangle \in |\varphi|$ for all $\xi \in |\boldsymbol{u}|$, and $T^{\varphi}_{\xi k}{\restriction}_t \in P_\xi = \boldsymbol{\pi}(\xi)$ (not just $\in \bigcup^{\mathtt{fin}} P_\xi$!) for all $\langle \xi, k\rangle \in |\varphi|$ and all $t \in T^{\varphi}_{\xi k}$ with $\mathtt{lh}(t) = n^{\varphi}_{\xi k}$. If $\varphi \in \mathbf{Sys}_{\boldsymbol{u}}(\boldsymbol{\pi})$ then let $\mathbf{S}^{\varphi}_{\boldsymbol{u}}$ consist of all multituples $\mathbf{s} = \langle s_\xi\rangle_{\xi \in |\boldsymbol{u}|}$ such that $s_\xi \in T^{\varphi}_{\xi,k_\xi}$ and $\mathtt{lh}(s_\xi) = n^{\varphi}_{\xi,k_\xi}$ for all $\xi \in |\boldsymbol{u}|$. If $\mathbf{s} = \langle s_\xi\rangle_{\xi \in |\boldsymbol{u}|} \in \mathbf{S}^{\varphi}_{\boldsymbol{u}}$ then define a multitree $\boldsymbol{v}^{\varphi}_{\mathbf{s}} \in \mathbf{MT}(\boldsymbol{\pi})$ by $|\boldsymbol{v}^{\varphi}_{\mathbf{s}}| = |\boldsymbol{u}|$ and $T^{\boldsymbol{v}^{\varphi}_{\mathbf{s}}}_{\xi} = T^{\varphi}_{\xi,k_\xi}{\restriction}_{s_\xi}$ for all $\xi \in |\boldsymbol{u}|$.

Now assume that $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$ and $|\boldsymbol{r}| \cap |\boldsymbol{u}| = \varnothing$. Consider the set $\Delta_{\boldsymbol{r}} \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{Sys}_{\boldsymbol{u}}(\boldsymbol{\pi})$, such that there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying $\boldsymbol{q} \leqslant \boldsymbol{r}$, still $|\boldsymbol{u}| \cap |\boldsymbol{q}| = \varnothing$, and

(†) if $\mathbf{s} \in \mathbf{S}^{\varphi}_{\boldsymbol{u}}$ then $\boldsymbol{v}^{\varphi}_{\mathbf{s}} \cup \boldsymbol{q} \in \boldsymbol{D}$.

Lemma 10.2. *Under the assumptions of the theorem, if $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{r}| \cap |\boldsymbol{u}| = \varnothing$, then the set $\Delta_{\boldsymbol{r}}$ belongs to $\mathfrak{M}$ and is dense in $\mathbf{Sys}(\boldsymbol{\pi})$.*

Proof *(Lemma).* $\Delta_{\boldsymbol{r}} \in \mathfrak{M}$ follows from $\boldsymbol{\pi} \in \mathfrak{M} \models \mathbf{ZFC}^-_1$.

To prove the density, let $\psi \in \mathbf{Sys}(\boldsymbol{\pi})$. We'll find a system $\varphi \in \Delta_{\boldsymbol{r}}$, $\varphi \preccurlyeq \psi$. We w.l.o.g. assume that $\psi \in \mathbf{Sys}_{\boldsymbol{u}}(\boldsymbol{\pi})$. (If not then adjoin each $\langle \xi, k_\xi \rangle \notin |\varphi|$ to $|\psi|$ and define $T^{\psi}_{\xi,k_\xi} \in P_\xi$ arbitrarily. If $\langle \xi, k \rangle \in |\psi|$ and $t \in T^{\psi}_{\xi k}$, $\mathtt{lh}(t) = n^{\psi}_{\xi k}$, but $T^{\psi}_{\xi k} \restriction_t \in \bigcup^{\mathtt{fin}} P_\xi \smallsetminus P_\xi$, then shrink $T^{\psi}_{\xi k} \restriction_t$ to a tree in P_ξ.)

Let $\mathbf{s} = \langle s_\xi \rangle_{\xi \in |\boldsymbol{u}|} \in \mathbf{S}^{\psi}_{\boldsymbol{u}}$. Consider the multitree $\boldsymbol{v}^{\psi}_{\mathbf{s}} \in \mathbf{MT}(\boldsymbol{\pi})$. As $\boldsymbol{D}$ is dense, there are multitrees $\boldsymbol{r}', \boldsymbol{v} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{v}| = |\boldsymbol{u}|$, $\boldsymbol{v} \leqslant \boldsymbol{v}^{\psi}_{\mathbf{s}}$, $|\boldsymbol{r}'| \cap |\boldsymbol{u}| = \varnothing$, $\boldsymbol{r}' \leqslant \boldsymbol{r}$, and $\boldsymbol{v} \cup \boldsymbol{r}' \in \boldsymbol{D}$. Define a system $\psi' \in \mathbf{Sys}(\boldsymbol{\pi})$ with $|\psi'| = |\psi|$ extending ψ, by putting $n^{\psi'}_{\xi k} = n^{\psi}_{\xi k}$ for all $\langle \xi, k \rangle \in |\psi|$, and shrinking each tree $T^{\psi}_{\xi,k_\xi} \restriction_{s_\xi}$ to $T^{\boldsymbol{v}}_{\xi}$, so that $T^{\psi'}_{\xi,k_\xi} \restriction_{s_\xi} = T^{\boldsymbol{v}}_{\xi}$, but $T^{\psi'}_{\xi,k_\xi} \restriction_t = T^{\psi}_{\xi,k_\xi} \restriction_t$ for all $t \in T^{\psi}_{\xi,k_\xi}$ such that $\mathtt{lh}(t) = n^{\psi}_{\xi,k_\xi}$ and $t \neq s_\xi$, and finally $T^{\psi'}_{\eta k} = T^{\psi}_{\eta k}$ if $\langle \eta, k \rangle \in |\psi|$ does not have the form $\langle \xi, k_\xi \rangle$, where $\xi \in |\boldsymbol{u}|$. Then $\psi' \preccurlyeq \psi$ and $\mathbf{S}^{\psi'}_{\boldsymbol{u}} = \mathbf{S}^{\psi}_{\boldsymbol{u}}$ by construction.

This construction can be iterated, so that all strings $\mathbf{s} \in \mathbf{S}^{\psi}_{\boldsymbol{u}}$ are considered consecutively one by one. This results in a system $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$, such that $|\varphi| = |\psi|$, $\varphi \preccurlyeq \psi$, and $\mathbf{S}^{\varphi}_{\boldsymbol{u}} = \mathbf{S}^{\psi}_{\boldsymbol{u}}$, and a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ with $\boldsymbol{q} \leqslant \boldsymbol{r}$ and still $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, such that if $\mathbf{s} \in \mathbf{S}^{\psi}_{\boldsymbol{u}}$ then the multitree $\boldsymbol{v}^{\psi}_{\mathbf{s}}$ satisfies $\boldsymbol{v}^{\psi}_{\mathbf{s}} \cup \boldsymbol{q} \in \boldsymbol{D}$. Thus (†) holds and $\boldsymbol{q}$ witnesses that $\varphi \in \Delta_r$. This completes the proof of the lemma. □

Coming back to (vii) of the theorem, we have $\varphi = \varphi_j \in \Delta_{\boldsymbol{p}}$ for some j by the lemma and the genericity of the sequence of systems φ_j. This is witnessed by some $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, so that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{u}| \cap |\boldsymbol{q}| = \varnothing$, and (†) of Lemma 10.2 holds for φ. Then $[\boldsymbol{u}] \subseteq \bigcup_{\mathbf{s} \in \mathbf{S}^{\varphi}_{\boldsymbol{u}}} [\boldsymbol{v}^{\varphi}_{\mathbf{s}}]$. Yet $\boldsymbol{v}^{\varphi}_{\mathbf{s}} \in \boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$, $\forall \mathbf{s}$, by (†).

(viii) Each set $K^{\mathbf{c}}_n \!\uparrow\! \boldsymbol{\pi}$ belongs to $\mathfrak{M}$ (as so do $\mathbf{c}$ and $\boldsymbol{\pi}$) and is open dense in $\mathbf{MT}(\boldsymbol{\pi})$, so it remains to apply (vii) already established.

(ix) Let $Q \in \boldsymbol{\varphi}(\eta)$; we have to prove that $\boldsymbol{\varphi}$ seals the set $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ over $\boldsymbol{\pi}$. By construction $Q = \boldsymbol{Q}^{\Phi}_{\eta K} \restriction_s$ for some $K < \omega$ and $s \in \boldsymbol{Q}^{\Phi}_{\eta K}$; it suffices to consider only the case $Q = \boldsymbol{Q}^{\Phi}_{\eta K}$. Following the proof of Theorem 10.1(vii), we suppose that $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varphi})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$, and $T^{\boldsymbol{u}}_{\xi} = \boldsymbol{Q}^{\Phi}_{\xi,k_\xi}$, for each $\xi \in |\boldsymbol{u}|$. We have to find a multitree $\boldsymbol{q}$ which witnesses 6.3 $(*)$ for $\boldsymbol{u}$, $\boldsymbol{p}$, $\boldsymbol{D} = \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$. In the remainder of the proof, we use **the notation in the proof of (vii) of Theorem 10.1**, in particular, $\mathbf{Sys}_{\boldsymbol{u}}(\boldsymbol{\pi})$, $\mathbf{S}^{\varphi}_{\boldsymbol{u}}$, $\boldsymbol{v}^{\varphi}_{\mathbf{s}}$, k_ξ.

Note that η may or may not belong to the set $|\boldsymbol{u}|$, and even if $\eta \in |\boldsymbol{u}|$, so k_η is defined (with $T^{\boldsymbol{u}}_{\eta} = \boldsymbol{Q}^{\Phi}_{\eta,k_\eta}$), then K may or may not be equal to k_η.

Assume that $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{r}| \cap |\boldsymbol{u}| = \varnothing$. Consider the set $\Delta_{\boldsymbol{r}} \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{Sys}_{\boldsymbol{u}}(\boldsymbol{\pi})$, such that $\langle \eta, K \rangle \in |\varphi|$, and there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying $\boldsymbol{q} \leqslant \boldsymbol{r}$, still $|\boldsymbol{u}| \cap |\boldsymbol{q}| = \varnothing$, and

(†′) if $\mathbf{s} \in \mathbf{S}^{\varphi}_{\boldsymbol{u}}$ and $t \in T^{\varphi}_{\eta K} \cap 2^{n^{\varphi}_{\eta K}}$ then $\boldsymbol{v}^{\varphi}_{\mathbf{s}} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\varphi}_{\eta K} \restriction_t]$.

Condition (†′) is somewhat similar to (†) of Lemma 10.2, and will play the same role. The direct forcing of $\mathbf{c} \notin [Q]$ cannot be used in (†′) as Q is not necessarily an element of $\mathfrak{M}$, but $\mathbf{c} \notin [T^{\varphi}_{\eta K}]$ will be an effective replacement.

Lemma 10.3. *If $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{r}| \cap |\boldsymbol{u}| = \varnothing$, then $\Delta_{\boldsymbol{r}}$ is dense in $\mathbf{Sys}(\boldsymbol{\pi})$.*

Proof. Following the proof of Lemma 10.2, let $\psi \in \mathbf{Sys}(\boldsymbol{\pi})$. We wlog assume that $\psi \in \mathbf{Sys}_{\boldsymbol{u}}(\boldsymbol{\pi})$ (see Lemma 10.2), so $\langle \xi, k_\xi \rangle \in |\psi|$ for all $\xi \in |\boldsymbol{u}|$ and $T^{\psi}_{\xi k} \restriction_t \in P_\xi$ for all $\langle \xi, k \rangle \in |\psi|$ and $t \in 2^{n^{\psi}_{\xi k}} \cap T^{\psi}_{\xi k}$, and $\langle \eta, K \rangle \in |\psi|$ as well.

We have to define a system $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ such that $\varphi \preccurlyeq \psi$ and $\varphi \in \Delta_r$. As in the proof of Lemma 10.2, it suffices to fulfill (†′) for **one particular pair** of $\mathbf{s} = \langle s_\xi \rangle_{\xi \in |\boldsymbol{u}|} \in \mathbf{S}^{\psi}_{\boldsymbol{u}}$ and $t \in T^{\psi}_{\eta K} \cap 2^{n^{\psi}_{\eta K}}$; the final goal is then achieved by simple iteration through all such pairs. We have two cases.

Case 1: $\eta \in |\boldsymbol{u}|$, $K = k_\eta$, $t = s_\eta$. Consider the multitree $\boldsymbol{v}^{\psi}_{\mathbf{s}} \in \mathbf{MT}(\boldsymbol{\pi})$. The set $\boldsymbol{D}^{\boldsymbol{\pi}}_{\eta}(\mathbf{c})$, as in Definition 7.4, is dense by the non-principality of $\mathbf{c}$. It follows that there are multitrees $\boldsymbol{q}, \boldsymbol{v} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{v}| = |\boldsymbol{u}|$, $\boldsymbol{v} \leqslant \boldsymbol{v}^{\psi}_{\mathbf{s}}$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, $\boldsymbol{q} \leqslant \boldsymbol{r}$, and $\boldsymbol{v} \cup \boldsymbol{q} \in \boldsymbol{D}^{\boldsymbol{\pi}}_{\eta}(\mathbf{c})$. Therefore $\boldsymbol{v} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\boldsymbol{q}}_{\eta}]$. Define a system $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ with $|\varphi| = |\psi|$, from ψ by:

(a) shrinking each tree $T^{\psi}_{\xi, k_\xi} \restriction_{s_\xi}$ $(\xi \in |\boldsymbol{u}|)$ to $T^{\boldsymbol{v}}_{\xi}$, so that $T^{\varphi}_{\xi, k_\xi} \restriction_{s_\xi} = T^{\boldsymbol{v}}_{\xi}$,
(b) in particular, shrinking $T^{\psi}_{\eta K} \restriction_t$ to $T^{\boldsymbol{v}}_{\eta}$, so that $T^{\varphi}_{\eta K} \restriction_t = T^{\boldsymbol{v}}_{\eta}$,

and no other changes. We have $\langle n, \varphi \rangle \preccurlyeq \langle n, \psi \rangle$, $\boldsymbol{v}^{\varphi}_{\mathbf{s}} = \boldsymbol{v}$, and $T^{\varphi}_{\eta K} \restriction_t = T^{\boldsymbol{v}}_{\eta}$ by construction. Thus $\boldsymbol{v}^{\varphi}_{\mathbf{s}} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\varphi}_{\eta K} \restriction_t]$, and (†′) holds.

Case 2: not Case 1. By Lemma 7.2, there exist multitrees $\boldsymbol{q}, \boldsymbol{v} \in \mathbf{MT}(\boldsymbol{\pi})$ and a tree $T \in P_\eta$ such that $T \subseteq T^{\psi}_{\eta K} \restriction_t$, $|\boldsymbol{v}| = |\boldsymbol{u}|$, $\boldsymbol{v} \leqslant \boldsymbol{v}^{\psi}_{\mathbf{s}}$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, $\boldsymbol{q} \leqslant \boldsymbol{r}$, and $\boldsymbol{v} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T]$. Define a system $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ with $|\varphi| = |\psi|$, that extends ψ, by (a) above and

(c) shrinking $T^{\psi}_{\eta K} \restriction_t$ to T, so that $T^{\varphi}_{\eta K} \restriction_t = T$,

and no other changes. Note that (a) and (c) do not contradict each other since $\langle \eta, T, t \rangle \neq \langle \xi, k_\xi, s_\xi \rangle$ for all $\xi \in \boldsymbol{u}$ by the Case 2 hypothesis. We have $\varphi \preccurlyeq \psi$, $\boldsymbol{v}^{\varphi}_{\mathbf{s}} = \boldsymbol{v}$, and $T^{\varphi}_{\eta K} \restriction_t = T^{\boldsymbol{v}}_{\eta}$ by construction. In particular, $\boldsymbol{v}^{\varphi}_{\mathbf{s}} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\varphi}_{\eta K} \restriction_t]$, thus (†′) holds. This completes the proof of the lemma. □

Come back to (ix) of the theorem. As $\Delta_{\boldsymbol{p}} \in \mathfrak{M}$, we have $\varphi = \varphi_j \in \Delta_{\boldsymbol{p}}$ for some j by the lemma. Let this be witnessed by a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, so that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{u}| \cap |\boldsymbol{q}| = \varnothing$, and (†′) holds for $\varphi = \varphi_j$. In particular, as $T^{\varphi}_{\eta K} = \bigcup_{t \in T^{\varphi}_{\eta K} \cap 2^n} T^{\varphi}_{\eta K} \restriction_t$, where $n = n^{\varphi}_{\xi k}$, the multitree $\boldsymbol{v}^{\varphi}_{\mathbf{s}} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\varphi_j}_{\eta K}]$ whenever $\mathbf{s} \in \mathbf{S}^{\varphi}_{\boldsymbol{u}}$, hence directly forces $\mathbf{c} \notin [Q]$ as well, because $Q = \boldsymbol{Q}^{\Phi}_{\eta K} \subseteq T^{\varphi}_{\eta K}$ by construction. Thus if $\mathbf{s} \in \mathbf{S}^{\varphi}_{\boldsymbol{u}}$ then $\boldsymbol{v}^{\varphi}_{\mathbf{s}} \cup \boldsymbol{q} \in \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$, and hence $\boldsymbol{v}^{\varphi}_{\mathbf{s}} \in \boldsymbol{D}(\mathbf{c}, Q, \varphi)^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. On the other hand, $[\boldsymbol{u}] \subseteq \bigcup_{\mathbf{s} \in \mathbf{S}^{\varphi}_{\boldsymbol{u}}} [\boldsymbol{v}^{\varphi}_{\mathbf{s}}]$, so that $\boldsymbol{u} \subseteq^{\mathtt{fin}} \bigvee \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})^{|\boldsymbol{u}|}_{\boldsymbol{q}}$, as required. □

11. Combining refinement types

Here we summarize the properties of generic refinements considered above. The next definition combines the refinement types $\sqsubset_D$, $\sqsubset_{\boldsymbol{D}}$, $\sqsubset_{\mathbf{c}}$, $\sqsubset^{\mathbf{c}}_{\xi}$.

Definition 11.1. Suppose that $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$ are multiforcings and $\mathfrak{M}$ is any set. Let $\boldsymbol{\pi} \sqsubset^{*}_{\mathfrak{M}} \boldsymbol{\varrho}$ mean that $\boldsymbol{\varrho}$ is an $\mathfrak{M}$-generic refinement of $\boldsymbol{\pi}$.

Let $\boldsymbol{\pi} \sqsubset_{\mathfrak{M}} \boldsymbol{\varrho}$ mean that the four following requirements hold:

(1) if $\xi \in |\boldsymbol{\pi}|$, $D \in \mathfrak{M}$, $D \subseteq \boldsymbol{\pi}(\xi)$, D is dense in $\boldsymbol{\pi}(\xi)$, then $\boldsymbol{\pi}(\xi) \sqsubset_D \boldsymbol{\varrho}(\xi)$;
(2) if $\boldsymbol{D} \in \mathfrak{M}$, $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{D}$ is open dense in $\mathbf{MT}(\boldsymbol{\pi})$, then $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho}$;
(3) if $\mathbf{c} \in \mathfrak{M}$ is a $\boldsymbol{\pi}$-complete real name then $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho}$;
(4) if $\xi \in |\boldsymbol{\pi}|$ and $\mathbf{c} \in \mathfrak{M}$ is a $\boldsymbol{\pi}$-complete real name, *non-principal over* $\boldsymbol{\pi}$ *at* ξ, then $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\varrho}$, that is, $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})} \boldsymbol{\varrho}$ for all $Q \in \boldsymbol{\varrho}(\xi)$. □

Corollary 11.2. *If $\mathfrak{M} \models \mathbf{ZFC}_1^-$ is a CTM, $\boldsymbol{\pi} \in \mathfrak{M}$ is a small multiforcing. Then $\boldsymbol{\pi} \sqsubset\!\!\sqsubset^*_{\mathfrak{M}} \boldsymbol{\varphi}$ implies $\boldsymbol{\pi} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \boldsymbol{\varphi}$.*

Proof. We have (1), (2), (3), (4) of Definition 11.1 by resp. (v), (vii), (viii), (ix) of Theorem 10.1. □

Corollary 11.3. *Assume that $\mathfrak{M} \models \mathbf{ZFC}_1^-$ is a CTM, $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda}$ is a $\sqsubset$-increasing sequence in* **MF**, $0 < \mu < \lambda$, $\boldsymbol{\pi}_{<\mu} = \bigcup^{\mathtt{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha$, *and* $\boldsymbol{\pi}_{<\mu} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \boldsymbol{\pi}_\mu$, *then* $\boldsymbol{\pi}_{<\mu} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \boldsymbol{\pi}_{\geq\mu} = \bigcup^{\mathtt{cw}}_{\mu\leq\alpha<\lambda} \boldsymbol{\pi}_\alpha$.

Proof. We have (1), (2) of Definition 11.1 for the relation $\boldsymbol{\pi}_{<\mu} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \boldsymbol{\pi}_{\geq\mu}$ by resp. Lemma 6.2(ii) and Lemma 6.4(iii). This also implies (3) of Definition 11.1 since this is a particular case of (2), see the proof of Theorem 10.1(viii). It remains to establish (4) of Definition 11.1. Thus assume that $\xi \in |\boldsymbol{\pi}_{<\mu}|$ and $\mathbf{c} \in \mathfrak{M}$ is a $\boldsymbol{\pi}_{<\mu}$-complete real name, *non-principal over $\boldsymbol{\pi}_{<\mu}$ at ξ*; we have to prove that $\boldsymbol{\pi}_{<\mu} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\pi}_{\geq\mu}$, that is, $\boldsymbol{\pi}_{<\mu} \sqsubset_{\boldsymbol{D}(\mathbf{c},S,\boldsymbol{\pi}_{<\mu})} \boldsymbol{\pi}_{\geq\mu}$ for any tree $S \in \boldsymbol{\pi}_{\geq\mu}(\xi)$.

Then there is a finite set $\{Q_1, \dots, Q_m\} \subseteq \boldsymbol{\pi}_\mu(\xi)$ such that $S \subseteq Q_1 \cup \dots \cup Q_m$. We have $\boldsymbol{\pi}_{<\mu} \sqsubset_{\boldsymbol{D}(\mathbf{c},Q_i,\boldsymbol{\pi}_{<\mu})} \boldsymbol{\pi}_\mu$ for all i since $\boldsymbol{\pi}_{<\mu} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \boldsymbol{\pi}_\mu$. It follows that $\boldsymbol{\pi}_{<\mu} \sqsubset_D \boldsymbol{\pi}_\mu$ by Lemma 6.4(ii), where $D = \bigcap_i \boldsymbol{D}(\mathbf{c}, Q_i, \boldsymbol{\pi}_{<\mu})$, and further $\boldsymbol{\pi}_{<\mu} \sqsubset_D \boldsymbol{\pi}_{\geq\mu}$ by Lemma 6.4(iii). However $D \subseteq \boldsymbol{D}(\mathbf{c}, S, \boldsymbol{\pi}_{<\mu})$ because $S \subseteq \bigcup_i Q_i$. We conclude that $\boldsymbol{\pi}_{<\mu} \sqsubset_{\boldsymbol{D}(\mathbf{c},S,\boldsymbol{\pi}_{<\mu})} \boldsymbol{\pi}_{\geq\mu}$, as required. □

12. Increasing sequences of multiforcings

Recall that **MF** is the set of all multiforcings. Let

$$\mathbf{spMF} = \{\boldsymbol{\pi} \in \mathbf{MF} : \boldsymbol{\pi} \text{ is a } \textit{special}\text{, hence small multiforcing}\}.$$

Thus a multiforcing $\boldsymbol{\pi} \in \mathbf{MF}$ belongs to **spMF** if $|\boldsymbol{\pi}| \subseteq \omega_1$ is (at most) countable and if $\xi \in |\boldsymbol{\pi}|$ then $\boldsymbol{\pi}(\xi)$ is a special forcing in **AF** (see Sections 3, 4).

- If $\kappa \leq \omega_1$ then let $\overrightarrow{\mathbf{MF}}_\kappa$ be the set of all $\sqsubset$-increasing sequences $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\kappa}$ of multiforcings $\boldsymbol{\pi}_\alpha \in \mathbf{spMF}$, of *length* $\mathtt{dom}(\vec{\boldsymbol{\pi}}) = \kappa$, *domain-continuous* so that if $\lambda < \kappa$ is a limit ordinal then $|\boldsymbol{\pi}_\lambda| = \bigcup_{\alpha<\lambda} |\boldsymbol{\pi}_\alpha|$.
- Let $\overrightarrow{\mathbf{MF}} = \bigcup_{\kappa<\omega_1} \overrightarrow{\mathbf{MF}}_\kappa$ ($\sqsubset$-increasing sequences of countable length).

The set $\overrightarrow{\mathbf{MF}} \cup \overrightarrow{\mathbf{MF}}_{\omega_1}$ is ordered by the relations $\subseteq$, $\subset$ of the extension of sequences.

Lemma 12.1. *Assume that $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}_{\kappa+1}$, $\kappa < \omega_1$, $\boldsymbol{\pi} = \vec{\boldsymbol{\pi}}(\kappa)$* (the last term), *and $\xi < \omega_1$. Then there is a sequence $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}}_{\kappa+1}$ such that $\vec{\boldsymbol{\varphi}} \restriction \kappa = \vec{\boldsymbol{\pi}} \restriction \kappa$, $\boldsymbol{\pi} \subseteq \vec{\boldsymbol{\varphi}}(\kappa)$, and $|\vec{\boldsymbol{\varphi}}(\kappa)| = |\boldsymbol{\pi}| \cup \{\xi\}$.*

Proof. If $\xi \in |\boldsymbol{\pi}|$ then put $\vec{\boldsymbol{\varphi}} = \vec{\boldsymbol{\pi}}$. If $\xi \notin |\boldsymbol{\pi}|$ then define $\boldsymbol{\varphi} = \vec{\boldsymbol{\varphi}}(\kappa)$ by $|\boldsymbol{\varphi}| = |\boldsymbol{\pi}| \cup \{\xi\}$, $\boldsymbol{\varphi}(\eta) = \boldsymbol{\pi}(\eta)$ for $\eta \neq \xi$, and $\boldsymbol{\varphi}(\xi) = \mathbb{P}_{\text{coh}}$, where $\mathbb{P}_{\text{coh}}$ (Cohen's forcing) consists of all trees $[s] = \{t \in 2^{<\omega} : s \subset t \vee t \subseteq s\}$, $s \in 2^{<\omega}$. □

Definition 12.2. Assume that $\mathfrak{M} \models \mathbf{ZFC}_1^-$ is a CTM, sequences $\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\varphi}}$ belong to $\overrightarrow{\mathbf{MF}}$, and $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}} \vec{\boldsymbol{\pi}} = \bigcup^{\mathtt{cw}}_{\alpha<\kappa} \vec{\boldsymbol{\pi}}(\alpha)$. We define:

$\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$, if $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varphi}}$ and $\boldsymbol{\pi} \sqsubset\!\!\sqsubset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}(\kappa)$, where $\kappa = \mathtt{dom}(\vec{\boldsymbol{\pi}}) < \mathtt{dom}(\vec{\boldsymbol{\varphi}}) \leq \omega_1$, (note that $\vec{\boldsymbol{\varphi}}(\kappa)$ is the first term in $\vec{\boldsymbol{\varphi}}$ missing in $\vec{\boldsymbol{\pi}}$);

$\vec{\boldsymbol{\pi}} \subset^*_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$, if $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varphi}}$ and $\boldsymbol{\pi} \sqsubset\!\!\sqsubset^*_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}(\kappa)$, where $\kappa = \mathtt{dom}(\vec{\boldsymbol{\pi}}) < \mathtt{dom}(\vec{\boldsymbol{\varphi}})$. □

We observe that $\vec{\boldsymbol{\pi}} \subset^*_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$ implies $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$ by Corollary 11.2.

Theorem 12.3. *If $\mathfrak{M} \models \mathbf{ZFC}_1^-$ is a CTM, $\kappa < \lambda \le \omega_1$, and $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}_\kappa \cap \mathfrak{M}$, then there is a sequence $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}}_\lambda$ with $\vec{\boldsymbol{\pi}} \subset^*_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$.*

Proof. We define terms $\vec{\boldsymbol{\varphi}}(\alpha)$ of the sequence $\vec{\boldsymbol{\varphi}}$ by induction.

Naturally put $\vec{\boldsymbol{\varphi}}(\alpha) = \vec{\boldsymbol{\pi}}(\alpha)$ for each $\alpha < \kappa$.

To define $\vec{\boldsymbol{\varphi}}(\kappa)$, let $\boldsymbol{\varphi}$ be an $\mathfrak{M}$-generic refinement $\boldsymbol{\varphi}$ of $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$; such a $\boldsymbol{\varphi}$ exists by Lemma 9.4. Corollary 11.2 implies $\boldsymbol{\pi} \sqsubset_{\mathfrak{M}} \boldsymbol{\varphi}$. By Theorem 10.1(vi), adding $\boldsymbol{\varphi}$ as the last term to $\vec{\boldsymbol{\pi}}$, results in a sequence $\vec{\boldsymbol{\varphi}} \restriction (\kappa+1) \in \overrightarrow{\mathbf{MF}}_{\kappa+1}$ satisfying $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varphi}} \restriction (\kappa+1)$.

Assume that $\kappa < \beta < \lambda$, all terms $\vec{\boldsymbol{\varphi}}(\alpha)$, $\alpha < \beta$, are defined, and the sequence $\vec{\boldsymbol{\varphi}} \restriction \beta = \langle \vec{\boldsymbol{\varphi}}(\alpha) \rangle_{\alpha<\beta}$ satisfies $(\vec{\boldsymbol{\varphi}} \restriction (\kappa+1)) \subset (\vec{\boldsymbol{\varphi}} \restriction \beta)$. Pick any CTM $\mathfrak{N} \models \mathbf{ZFC}_1^-$ containing $\vec{\boldsymbol{\varphi}} \restriction \beta$. By Lemma 9.4, there is an $\mathfrak{N}$-generic refinement $\boldsymbol{\varphi}$ of $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$. By Theorem 10.1(vi), adding $\boldsymbol{\varphi}$ as the last term to $\vec{\boldsymbol{\varphi}} \restriction \beta$, yields a sequence $\vec{\boldsymbol{\varphi}} \restriction (\beta+1) \in \overrightarrow{\mathbf{MF}}_{\beta+1}$ with $\vec{\boldsymbol{\varphi}} \restriction \beta \subset \vec{\boldsymbol{\varphi}} \restriction (\beta+1)$.

This construction results in a sequence $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}}_\lambda$ satisfying $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varphi}}$ and $\vec{\boldsymbol{\varphi}}(\kappa) = \boldsymbol{\varphi}$. Then we have $\vec{\boldsymbol{\pi}} \subset^*_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$ by the choice of $\boldsymbol{\varphi}$. □

13. The key sequence

The forcing notion to prove Theorem 1.1, defined (modulo minor details) in our earlier paper [31], will be introduced in the next section. It will have the form $\mathbf{MT}(\Pi)$, for a certain multiforcing $\Pi \in \mathbf{L}$ with $|\Pi| = \omega_1$ in $\mathbf{L}$. The multiforcing Π itself will be equal to the componentwise union of terms of a certain sequence $\vec{\Pi} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ which we present in Definition 13.6.

Recall that HC is the set of all *hereditarily countable* sets; $\mathrm{HC} = \mathbf{L}_{\omega_1}$ in $\mathbf{L}$. We use Σ^{HC}_n, Π^{HC}_n, Δ^{HC}_n (note slanted Σ, Π, Δ) for classes of *lightface* definability in HC (no parameters allowed), and $\mathbf{\Sigma}_n(\mathrm{HC})$, $\mathbf{\Pi}_n(\mathrm{HC})$, $\mathbf{\Delta}_n(\mathrm{HC})$ for *boldface* definability in HC (parameters in HC allowed).

Lemma 13.1 *(Lemma 25.25 in [20]). If $n \ge 1$ and $X \subseteq 2^\omega$ then*

$$X \in \Sigma^{\mathrm{HC}}_n \iff X \in \Sigma^1_{n+1}, \quad \text{and} \quad X \in \mathbf{\Sigma}_n(\mathrm{HC}) \iff X \in \mathbf{\Sigma}^1_{n+1},$$

and the same for Π, $\mathbf{\Pi}$, Δ, $\mathbf{\Delta}$. □

Definition 13.2 *(in $\mathbf{L}$).* Let $\mathbf{ZFL}^-$ be the theory $\mathbf{ZFC}_1^-$, as in Subsection 9B, with the axiom of constructibility $\mathbf{V} = \mathbf{L}$ added.

If $x \in \mathrm{HC}$ then let $\mathfrak{L}(x)$ be the least CTM of $\mathbf{ZFL}^-$ containing x and satisfying $x \in (\mathrm{HC})^{\mathfrak{L}(x)}$. It necessarily has the form $\mathfrak{L}(x) = \mathbf{L}_\mu$ for some $\mu = \mu_x < \omega_1$.

An ordinal $\xi < \kappa$ is a **crucial ordinal* of a sequence $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\kappa} \in \overrightarrow{\mathbf{MF}}_\kappa$ if $\vec{\boldsymbol{\pi}} \restriction \xi \subset^*_{\mathfrak{L}(\vec{\boldsymbol{\pi}} \restriction \xi)} \vec{\boldsymbol{\pi}}$ holds, that is, (1) $\vec{\boldsymbol{\pi}}(\xi)$ is an $\mathfrak{L}(\vec{\boldsymbol{\pi}} \restriction \xi)$-generic refinement of $\boldsymbol{\pi} = \bigcup^{\mathsf{cw}}_{\eta<\xi} \vec{\boldsymbol{\pi}}(\eta)$, and hence (2) $\boldsymbol{\pi} \sqsubset_{\mathfrak{L}(\vec{\boldsymbol{\pi}} \restriction \xi)} \vec{\boldsymbol{\pi}}(\xi)$ by Corollary 11.2. □

The superscript * is added in the notion of *crucial ordinal to distinguish it from the notion of a crucial ordinal in [31, Definition 14.3], where it was required that, somewhat weaker, $\vec{\boldsymbol{\pi}} \restriction \xi \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}} \restriction \xi)} \vec{\boldsymbol{\pi}}$.

Definition 13.3. A sequence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ *blocks* a set W if either $\vec{\boldsymbol{\pi}} \in W$ (*positive* block) or there is no $\vec{\boldsymbol{\varphi}} \in W$ extending $\vec{\boldsymbol{\pi}}$ (*negative* block). □

Theorem 13.4 *(in $\mathbf{L}$). Let $\mathtt{n} \ge 3$. There exists a sequence $\vec{\Pi} = \langle \Pi_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ satisfying $|\bigcup^{\mathsf{cw}} \vec{\Pi}| = \omega_1$ and the following requirements:*

(i) *the sequence $\vec{\Pi}$ belongs to the definability class $\Delta^{\mathrm{HC}}_{\mathtt{n}-2}$;*

(ii) *if* $\mathfrak{n} \geq 4$ *and* $W \subseteq \overrightarrow{\mathbf{MF}}$ *is a boldface* $\mathbf{\Sigma}_{\mathfrak{n}-3}(\mathrm{HC})$ *set then there is an ordinal* $\gamma < \omega_1$ *such that the sequence* $\vec{\pi} \restriction \gamma$ *blocks* W*;*

(iii) *there is a closed unbounded set* $\mathbb{C} \subseteq \omega_1$ *such that every* $\gamma \in \mathbb{C}$ *is a limit ordinal and a* **crucial ordinal for* $\vec{\pi}$ *in the sense of Definition 13.2.*

Proof. We argue under V = L. If $\mathfrak{n} \geq 4$ then let $\mathbf{un}_{\mathfrak{n}}(p, x)$ be a canonical universal $\Sigma_{\mathfrak{n}-3}$ formula, so that the family of all $\mathbf{\Sigma}_{\mathfrak{n}-3}(\mathrm{HC})$ sets $X \subseteq \mathrm{HC}$ is equal to the family of all sets $\Upsilon_{\mathfrak{n}}(p) = \{x \in \mathrm{HC} : \mathrm{HC} \models \mathbf{un}_{\mathfrak{n}}(p, x)\}$, $p \in \mathrm{HC}$.

Claim 13.5. *If* $\mathfrak{n} \geq 4$ *then the following set* $B_{\mathfrak{n}}$ *is* $\Delta^{\mathrm{HC}}_{\mathfrak{n}-2}$*:*

$$B_{\mathfrak{n}} = \{\langle \vec{\boldsymbol{\pi}}, p\rangle : \vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}} \wedge p \in \mathrm{HC} \wedge \vec{\boldsymbol{\pi}} \text{ block s } \Upsilon_{\mathfrak{n}}(p)\}.$$

Proof *(Claim).* We skip a routine check that $\overrightarrow{\mathbf{MF}}$ is Δ^{HC}_1. If $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ and $p \in \mathrm{HC}$ then for $\vec{\boldsymbol{\pi}}$ to block $\Upsilon_{\mathfrak{n}}(p)$ it is necessary and sufficient that

$$\underbrace{\vec{\boldsymbol{\pi}} \in \Upsilon_{\mathfrak{n}}(p)}_{\Sigma^{\mathrm{HC}}_{\mathfrak{n}-3}} \;\vee\; \underbrace{\neg\,\exists\, \vec{\boldsymbol{\varphi}}\,\big(\underbrace{\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}} \wedge \vec{\boldsymbol{\varphi}} \text{ extends } \vec{\boldsymbol{\pi}}}_{\Delta^{\mathrm{HC}}_1} \wedge \underbrace{\vec{\boldsymbol{\varphi}} \in \Upsilon_{\mathfrak{n}}(p)}_{\Sigma^{\mathrm{HC}}_{\mathfrak{n}-3}}\big)}_{\Pi^{\mathrm{HC}}_{\mathfrak{n}-3}},$$

so this is a disjunction of $\Sigma^{\mathrm{HC}}_{\mathfrak{n}-3}$ and $\Pi^{\mathrm{HC}}_{\mathfrak{n}-3}$, hence, $\Delta^{\mathrm{HC}}_{\mathfrak{n}-2}$. □

For $\alpha < \omega_1$, define a sequence $\vec{\boldsymbol{\pi}}[\alpha] \in \overrightarrow{\mathbf{MF}}$ by induction as follows.

We let $\vec{\boldsymbol{\pi}}[0] = \varnothing$, the empty sequence.

Step $\alpha \to \alpha + 1$**.** Suppose that $\vec{\boldsymbol{\pi}}[\alpha] \in \overrightarrow{\mathbf{MF}}$ is defined, $\kappa = \mathtt{dom}\,\vec{\boldsymbol{\pi}}[\alpha]$, $\boldsymbol{\pi}[\alpha] = \bigcup^{\mathtt{cw}} \vec{\boldsymbol{\pi}}[\alpha] = \bigcup^{\mathtt{cw}}_{\gamma<\alpha} \vec{\boldsymbol{\pi}}[\alpha](\gamma)$, $\mathfrak{M} = \mathfrak{L}(\vec{\boldsymbol{\pi}}[\alpha])$, and p_α is the α-th element of $\mathrm{HC} = \mathbf{L}_{\omega_1}$ in the sense of the Gödel well-ordering $\leqslant_{\mathbf{L}}$.

By Lemma 9.4, there exists an $\mathfrak{M}$-generic sequence Φ of systems in $\mathbf{Sys}(\boldsymbol{\pi}[\alpha])$. We let $\Phi[\alpha]$ be the $\leq_{\mathbf{L}}$-least of them. Then $\boldsymbol{\varphi} = \lim[\Phi[\alpha]]$ is an $\mathfrak{M}$-generic refinement of $\boldsymbol{\pi}[\alpha]$, and by Theorem 10.1(i),(vi), adjoining $\boldsymbol{\varphi}$ to $\boldsymbol{\pi}[\alpha]$ as the last term results in the sequence $\vec{\boldsymbol{\tau}} = \vec{\boldsymbol{\pi}}[\alpha]^\frown\boldsymbol{\varphi} \in \overrightarrow{\mathbf{MF}}_{\kappa+1}$ satisfying $\vec{\boldsymbol{\pi}}[\alpha] \subset^*_{\mathfrak{M}} \vec{\boldsymbol{\tau}}$. By Lemma 12.1, there is a sequence $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}}_{\kappa+2}$ satisfying $\vec{\boldsymbol{\tau}} \subset \vec{\boldsymbol{\varphi}}$ and $\alpha \in |\vec{\boldsymbol{\varphi}}(\kappa+1)|$. Finally if $\mathfrak{n} \geq 4$ then there is a sequence $\vec{\boldsymbol{\pi}}[\alpha+1] \in \overrightarrow{\mathbf{MF}}$ satisfying $\vec{\boldsymbol{\varphi}} \subset \vec{\boldsymbol{\pi}}[\alpha+1]$ and blocking the set $\Upsilon_{\mathfrak{n}}(p_\alpha)$, while if $\mathfrak{n} = 3$ then put $\vec{\boldsymbol{\pi}}[\alpha+1] = \vec{\boldsymbol{\varphi}}$. To conclude the step, we have:

(‡) $\vec{\boldsymbol{\pi}}[\alpha] \subset^*_{\mathfrak{M}} \vec{\boldsymbol{\pi}}[\alpha+1]$, $\kappa + 1 < \mathtt{dom}\,\vec{\boldsymbol{\pi}}[\alpha+1]$, $\alpha \in |\vec{\boldsymbol{\pi}}[\alpha+1](\kappa+1)|$, and if $\mathfrak{n} \geq 4$ then $\vec{\boldsymbol{\pi}}[\alpha+1]$ blocks $\Upsilon_{\mathfrak{n}}(p_\alpha)$.

Note that the axiom $\mathbf{V} = \mathbf{L}$ is a sine qua non of this construction since otherwise the $\leqslant_{\mathbf{L}}$-least choice of $\Phi[\alpha]$ would not be necessarily possible.

Limit step. If $\lambda < \omega_1$ is limit then we naturally define $\vec{\boldsymbol{\pi}}[\lambda] = \bigcup_{\alpha<\lambda} \vec{\boldsymbol{\pi}}[\alpha]$.

Overall, we have $\alpha < \beta \implies \vec{\boldsymbol{\pi}}[\alpha] \subset \vec{\boldsymbol{\pi}}[\beta]$ by construction. It follows that $\vec{\pi} = \bigcup_\alpha \vec{\boldsymbol{\pi}}[\alpha] \in \overrightarrow{\mathbf{MF}}_{\omega_1}$. This completes the construction.

To prove (i), note first of all that the relation

$$R(\boldsymbol{\pi}, \mathfrak{M}, \Phi) := \text{“}\Phi \text{ is an } \mathfrak{M}\text{-generic sequence in } \mathbf{Sys}(\boldsymbol{\pi})\text{”}$$

is absolute for all transitive models of $\mathbf{ZFC}^-_1$, hence R is Δ^{HC}_1. Easily the assignment $\vec{\boldsymbol{\pi}} \mapsto \mathfrak{L}(\vec{\boldsymbol{\pi}})$ is Δ^{HC}_1 as well. And “to block $\Upsilon_{\mathfrak{n}}(p)$” is a $\Delta^{\mathrm{HC}}_{\mathfrak{n}-2}$ relation by Claim 13.5. On the other hand, it is known that, under

$\mathbf{V} = \mathbf{L}$, choosing the $\leqslant_{\mathbf{L}}$-least element in every non-empty section of a Δ^{HC}_k set, $k \geq 1$, results in a set (transversal) of the same class Δ^{HC}_k. This allows to routinely complete the verification of (i).

To check $|\bigcup^{\mathtt{cw}} \vec{\sqcap}| = \omega_1$, note that $\alpha \in |\bigcup^{\mathtt{cw}} \vec{\boldsymbol{\pi}}[\alpha+1]|$ by construction.

To check (ii) ($\mathfrak{n} \geq 4$), note that any boldface $\mathbf{\Sigma}_{\mathfrak{n}-3}(\mathrm{HC})$ set $W \subseteq \overrightarrow{\mathbf{MF}}$ is equal to $\Upsilon_{\mathfrak{n}}(p_\alpha)$ for some $\alpha < \omega_1$, so $\gamma = \mathtt{dom}\,\vec{\boldsymbol{\pi}}[\alpha+1]$ is as required.

(iii) The set $\mathbb{C} = \{\mathtt{dom}\,\vec{\boldsymbol{\pi}}[\alpha] : \alpha < \omega_1\}$ is closed unbounded by the limit step of the construction. Moreover if $\gamma = \mathtt{dom}\,\vec{\boldsymbol{\pi}}[\alpha] \in \mathbb{C}$ then $\vec{\sqcap}\restriction\gamma = \vec{\boldsymbol{\pi}}[\alpha]$, and hence γ is *crucial for $\vec{\sqcap}$ by construction. This ends the proof of Theorem 13.4. □

Definition 13.6 *(in* **L***).* From now on we fix **a number** $\mathfrak{n} \geq 3$ as in Theorem 1.1. We also **fix a sequence** $\vec{\sqcap} = \langle \sqcap_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ satisfying Theorem 13.4 for this $\mathfrak{n}$. In particular, $\bigcup_\alpha |\sqcap_\alpha| = \omega_1$, and conditions (i), (ii), (iii) (with an according club $\mathbb{C}$) of Theorem 13.4 hold. We call this fixed $\vec{\sqcap} \in \mathbf{L}$ **the key sequence**. □

Corollary 13.7. *If $\mathfrak{n} \geq 4$ and $W \subseteq \overrightarrow{\mathbf{MF}}$ is a $\mathbf{\Sigma}_{\mathfrak{n}-3}(\mathrm{HC})$ set dense in $\overrightarrow{\mathbf{MF}}$ then there is an ordinal $\gamma < \omega_1$ such that $\vec{\sqcap}\restriction\gamma \in W$.*

Proof. By construction, $\vec{\sqcap}$ satisfies (ii) of Theorem 13.4, hence there is an ordinal $\gamma < \omega_1$ such that $\vec{\sqcap}\restriction\gamma$ blocks W. The negative block is impossible by the density of W, hence in fact $\vec{\sqcap}\restriction\gamma \in W$. □

Remark 13.8. Theorem 13.4 just proved is a clone of Theorem 15.3 in [31]. In fact it is a bit stronger, because of the stronger relation $\subset^*$, instead of $\subset$ as in [31], in the notion of a *crucial ordinal. Therefore all consequences of the choice of $\vec{\sqcap}$ by [31, Theorem 15.3] remain valid in our setting here. □

14. The key forcing notion

The following definition introduces some derived notions.

Definition 14.1 *(in* **L***).* Using the key sequence $\vec{\sqcap} = \langle \sqcap_\alpha \rangle_{\alpha<\omega_1}$ as in Definition 13.6, we define:

— the multiforcing $\sqcap = \bigcup^{\mathtt{cw}}_{\alpha<\omega_1} \sqcap_\alpha \in \mathbf{MF}$, and

— the forcing notion $\mathbb{P} = \mathbf{MT}(\sqcap) = \mathbf{MT}(\vec{\sqcap})$.

If $\xi < \omega_1$ then, following the equality $\bigcup_\alpha |\sqcap_\alpha| = \omega_1$ in Definition 13.6, let $\alpha(\xi) < \omega_1$ be the least ordinal α satisfying $\xi \in |\sqcap_\alpha|$. Thus a forcing notion $\sqcap_\alpha(\xi) \in \mathbf{AF}$ is defined whenever α satisfies $\alpha(\xi) \leq \alpha < \omega_1$, and $\langle \sqcap_\alpha(\xi) \rangle_{\alpha(\xi)\leq\alpha<\omega_1}$ is a $\sqsubset$-increasing sequence of special forcings in $\mathbf{AF}$, hence $\sqcap(\xi) = \bigcup_{\alpha(\xi)\leq\alpha<\omega_1} \sqcap_\alpha(\xi) \in \mathbf{AF}$. □

In the remainder, $\sqcap$ will be referred to as the key multiforcing, whereas the set $\mathbb{P} = \mathbf{MT}(\sqcap)$ will be our **key forcing notion**.

Corollary 14.2 *(in* **L***). $\sqcap$ is a regular multiforcing and $|\sqcap| = \omega_1$, thus $\mathbb{P} = \prod_{\xi<\omega_1} \sqcap(\xi)$ (with finite support).* □

Corollary 14.3 *(in* **L***). The sequence of ordinals $\langle \alpha(\xi) \rangle_{\xi<\omega_1}$ and the array of forcings $\langle \sqcap_\alpha(\xi) \rangle_{\xi<\omega_1,\, \alpha(\xi)\leq\alpha<\omega_1}$ are $\Delta^{\mathrm{HC}}_{\mathfrak{n}-2}$.*

Proof. By construction the following double equivalence holds:

$$\begin{aligned} \alpha(\xi) \leq \alpha &\iff \exists\,\boldsymbol{\pi}(\boldsymbol{\pi} = \sqcap_\alpha \wedge \xi \in \mathtt{dom}\,\boldsymbol{\pi}) \\ &\iff \forall\,\boldsymbol{\pi}(\boldsymbol{\pi} = \sqcap_\alpha \implies \xi \in \mathtt{dom}\,\boldsymbol{\pi}) \ . \end{aligned}$$

However $\boldsymbol{\pi} = \Pi_\alpha$ is a $\Delta^{\mathrm{HC}}_{\mathfrak{n}-2}$ relation by Theorem 13.4(i). It follows that so is the sequence $\langle \alpha(\xi) \rangle_{\xi<\omega_1}$. The second claim is similar. □

Corollary 14.4 *(in* $\mathbf{L}$*, of Lemma 5.2(iv)). If* $\xi < \omega_1$ *and* $\alpha(\xi) \le \alpha < \omega_1$ *then the set* $\Pi_\alpha(\xi)$ *is pre-dense in* $\Pi(\xi)$ *and in* Π*.* □

In spite of Corollary 14.2, the sets $|\Pi_{<\gamma}|$ can be quite arbitrary (countable) subsets of ω_1. However we get the next corollary:

Corollary 14.5 *(in* $\mathbf{L}$*, of Corollary 14.2).* $\mathbb{C}' = \{\gamma < \omega_1 : |\Pi_{<\gamma}| = \gamma\}$ *is closed unbounded in* ω_1*.* □

To prove the CCC property, we'll need the following result.

Lemma 14.6 *(in* $\mathbf{L}$*). If* $X \subseteq \mathrm{HC} = \mathbf{L}_{\omega_1}$ *then the set* $\mathscr{O}_X$ *of all ordinals* $\gamma < \omega_1$*, such that* $\langle \mathbf{L}_\gamma ; X \cap \mathbf{L}_\gamma \rangle$ *is an elementary submodel of* $\langle \mathbf{L}_{\omega_1} ; X \rangle$ *and* $X \cap \mathbf{L}_\gamma \in \mathfrak{L}(\vec{\Pi} \restriction \gamma)$*, is stationary, hence unbounded in* ω_1*.*

Proof. Let $C \subseteq \omega_1$ be a club. Let M be a countable elementary submodel of $\mathbf{L}_{\omega_2}$ containing $C, \omega_1, X, \vec{\Pi}$, and such that $M \cap \mathbf{L}_{\omega_1}$ is transitive. Let $\phi : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ be the Mostowski collapse, and $\gamma = \phi(\omega_1)$. Then

$$\gamma < \lambda < \omega_1, \quad \phi(X) = X \cap \mathbf{L}_\gamma, \quad \phi(C) = C \cap \gamma, \quad \phi(\vec{\Pi}) = \vec{\Pi} \restriction \gamma$$

by the choice of M. It follows that $\langle \mathbf{L}_\gamma ; X \cap \mathbf{L}_\gamma, C \cap \gamma, \vec{\Pi} \restriction \gamma \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1} ; X, C, \vec{\Pi} \rangle$, so $\gamma \in \mathscr{O}_X$. Moreover, γ is uncountable in $\mathbf{L}_\lambda$, hence $\mathbf{L}_\lambda \subseteq \mathfrak{L}(\vec{\Pi} \restriction \gamma)$. (See Definition 13.2 on models $\mathfrak{L}(\vec{\boldsymbol{\pi}}) \models \mathbf{ZFL}^-$.) We conclude that $X \cap \mathbf{L}_\gamma \in \mathfrak{L}(\vec{\Pi} \restriction \gamma)$ since $X \cap \mathbf{L}_\gamma \in \mathbf{L}_\lambda$ by construction. On the other hand, $C \cap \gamma$ is unbounded in γ by the elementarity, therefore $\gamma \in C$, as required. □

Corollary 14.7 *(in* $\mathbf{L}$*). The forcing* $\mathbb{P}$ *satisfies CCC. Therefore* $\mathbb{P}$*-generic extensions of* $\mathbf{L}$ *preserve cardinals.*

Proof. Suppose that $A \subseteq \mathbb{P} = \mathbf{MT}(\vec{\Pi})$ is a maximal antichain. By 13.6 and Theorem 13.4(iii), there is a closed unbounded set $\mathbb{C} \subseteq \omega_1$ such that every $\gamma \in \mathbb{C}$ is a *crucial ordinal for $\vec{\Pi}$. By Lemma 14.6, there is an ordinal $\gamma \in \mathbb{C}$ such that $A' = A \cap \mathbb{P}_{<\gamma}$ is a maximal antichain in $\mathbb{P}_{<\gamma} = \mathbf{MT}(\vec{\Pi} \restriction \gamma)$ and $A' \in \mathfrak{L}(\vec{\Pi} \restriction \gamma)$. It follows that the set $D(A') = \{\boldsymbol{p} \in \mathbb{P}_{<\gamma} : \exists\, \boldsymbol{q} \in A\, (p \le q)\} \in \mathfrak{L}(\vec{\Pi} \restriction \gamma)$ is open dense in $\mathbb{P}_{<\gamma}$.

Yet γ is a *crucial ordinal for $\vec{\Pi}$, therefore by Lemma 6.4(iii) both the set $D(A')$, and hence A' itself as well, remain pre-dense in the whole set $\mathbb{P} = \mathbf{MT}(\vec{\Pi})$. We conclude that $A = A'$ is countable. □

Corollary 14.8 *(in* $\mathbf{L}$*). If a set* $D \subseteq \mathbb{P}$ *is pre-dense in* $\mathbb{P}$ *then there is an ordinal* $\gamma < \omega_1$ *such that* $D \cap \mathbb{P}_{<\gamma}$ *is already pre-dense in* $\mathbb{P}$*.*

Proof. We can assume that D is dense. Let $A \subseteq D$ be a maximal antichain in D; then A is a maximal antichain in $\mathbb{P}$ because of the density of D. Then $A \subseteq \mathbb{P}_{<\gamma}$ for some $\gamma < \omega_1$ by Corollary 14.7. But A is pre-dense in $\mathbb{P}$. □

15. The key model

Our final goal will be to prove Theorem 1.1 by means of $\mathbb{P}$-generic extensions of $\mathbf{L}$. These extensions we'll call **key models**.

From now on, we'll typically argue in $\mathbf{L}$ and in $\omega_1^{\mathbf{L}}$-preserving generic extensions $\mathbf{L}$, in particular, in $\mathbb{P}$-generic extensions (see Theorem 14.7). Thus it will always be the case that $\omega_1^{\mathbf{L}} = \omega_1$. This allows us to think that $|\Pi| = \omega_1$ (rather than $\omega_1^{\mathbf{L}}$).

Definition 15.1. Let a set $G \subseteq \mathbb{P}$ be generic over the constructible universe $\mathbf{L}$. If $\xi < \omega_1$, then following Remark 4.4 at the end of Section 4,

— we define $G(\xi) = \{T^{\boldsymbol{p}}_\xi : \boldsymbol{p} \in G \wedge \xi \in |\boldsymbol{p}|\} \subseteq \sqcap(\xi)$;

— we let $x_\xi = x_\xi[G] \in 2^\omega$ be the only real in $\bigcap_{T\in G(\xi)}[T]$;

— we let $\boldsymbol{X}[G] = \langle x_\xi[G]\rangle_{\xi<\omega_1} = \{\langle \xi, x_\xi[G]\rangle : \xi < \omega_1\}$.

Thus $\mathbb{P}$ adjoins an array $\boldsymbol{X}[G]$ of reals to $\mathbf{L}$, where each $x_\xi = x_\xi[G] \in 2^\omega \cap \mathbf{L}[G]$ is a $\sqcap(\xi)$-generic real over $\mathbf{L}$, and $\mathbf{L}[G] = \mathbf{L}[\boldsymbol{X}[G]]$. □

Theorem 15.2. *Let $G \subseteq \mathbb{P}$ be $\mathbb{P}$-generic over $\mathbf{L}$. Then, in $\mathbf{L}[G]$, $\boldsymbol{X}[G]$ is a set of definability class $\Pi^{\mathrm{HC}}_{\mathtt{n}-2}$, hence, also of class $\Pi^1_{\mathtt{n}-1}$ by Lemma 13.1.*

Proof. Assume that $\xi < \omega_1$, and $x \in \mathbf{L}[G] \cap 2^\omega$. The following are equivalent:

(1) $x = x_\xi[G]$;

(2) x is $\sqcap(\xi)$-generic over $\mathbf{L}$;

(3) $x \in \bigcap_{\alpha(\xi)\le\alpha<\omega_1} \bigcup_{T\in\sqcap_\alpha(\xi)}[T]$.

Indeed (1) $\Longrightarrow$ (2) is routine (see Remark 4.4). To check (2) $\Longrightarrow$ (3) recall that each set $\sqcap_\alpha(\xi)$ is pre-dense in $\sqcap(\xi)$ by Lemma 5.2(iv).

It remains to establish (3) $\Longrightarrow$ (1). Suppose that (1) fails, that is, $x \ne x_\xi[G]$. By Theorem 8.2(i) there is a small (recall that $\mathbb{P} = \mathbf{MT}(\sqcap)$ is CCC by Corollary 14.7) $\sqcap$-complete real name $\mathbf{c} \in \mathbf{L}$, such that $\mathbf{c} \subseteq \mathbb{P} \times \omega \times 2$, $x = \mathbf{c}[G]$, and $\mathbf{c}$ is non-principal over $\sqcap$ at ξ, meaning that the set

$$\boldsymbol{D}^{\sqcap}_\xi(\mathbf{c}) = \{\boldsymbol{p} \in \mathbb{P} = \mathbf{MT}(\sqcap) : \xi \in |\boldsymbol{p}| \wedge \boldsymbol{p} \text{ directly forces } \mathbf{c} \notin [T^{\boldsymbol{p}}_\xi]\}$$

is open dense in $\mathbb{P} = \mathbf{MT}(\sqcap)$. By the smallness of $\mathbf{c}$ and Corollary 14.8, there is an ordinal $\gamma < \omega_1$ such that $\mathbf{c}$ is a $\sqcap_{<\gamma}$-complete real name and $\boldsymbol{D}^{\sqcap}_\xi(\mathbf{c}) \cap \mathbb{P}_{<\gamma}$ is pre-dense in $\mathbb{P}$, therefore, open dense in $\mathbb{P}_{<\gamma}$ — and then $\mathbf{c}$ is non-principal over $\sqcap_{<\gamma}$ at ξ. We can further assume that $\mathbf{c} \in \mathfrak{L}(\vec{\sqcap}\restriction\gamma)$. (If not then take a bigger γ.) Finally, we can assume that γ belongs to the set $\mathbb{C}$ of Theorem 13.4(iii) because $\mathbb{C}$ is a club. Then γ is *crucial for $\vec{\boldsymbol{\pi}}$, that is, $\sqcap_{<\gamma} \sqsubset^*_{\mathfrak{L}(\vec{\sqcap}\restriction\gamma)} \sqcap_\gamma$. It follows that $\sqcap_{<\gamma} \sqsubset^*_{\mathfrak{L}(\vec{\sqcap}\restriction\gamma)} \sqcap_{\ge\gamma}$ by Corollary 11.3. Then $\sqcap_{<\gamma} \sqsubset^{\mathbf{c}}_\xi \sqcap_{\ge\gamma}$ holds as well by Corollary 11.2, since $\mathbf{c} \in \mathfrak{L}(\vec{\sqcap}\restriction\gamma)$ and because of the non-principality of $\mathbf{c}$. Now Theorem 8.2(ii) with $\boldsymbol{\pi} = \sqcap_{<\gamma}$ and $\boldsymbol{\varrho} = \sqcap_{\ge\gamma}$ (note that $\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\varrho} = \sqcap$) implies $x = \mathbf{c}[G] \notin \bigcup_{Q\in\sqcap_{\ge\gamma}(\xi)}[Q]$, in particular, $x \notin \bigcup_{Q\in\sqcap_\gamma(\xi)}[Q]$. In other words, (3) fails as well.

Thus the equivalence (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (3) is established.

We conclude that the following holds in $\mathbf{L}[G]$: $\langle \xi, x\rangle \in \boldsymbol{X}[G]$ iff

$$\forall\,\alpha < \omega_1\, \exists\, T \in \sqcap_\alpha(\xi)\big(\alpha(\xi) \le \alpha \Longrightarrow x \in [T]\big),$$

which can be re-written as

$$\forall\,\alpha < \omega_1\, \forall\,\mu < \omega_1\, \forall\, Y\, \exists\, T \in Y\big(\mu = \alpha(\xi) \wedge Y = \sqcap_\alpha(\xi) \wedge \mu \le \alpha \Longrightarrow x \in [T]\big).$$

Here the equality $\mu = \alpha(\xi)$ is $\Delta^{\mathrm{HC}}_{\mathtt{n}-2}$ by Corollary 14.3, and so is the equality $Y = \sqcap_\alpha(\xi)$ by Corollary 14.3. It follows that the whole relation is $\Pi^{\mathrm{HC}}_{\mathtt{n}-2}$, since the quantifier $\exists\, T \in Y$ is bounded. □

16. $\boldsymbol{\Delta}^1_{\mathtt{n}}$-good well-orderings in the key model

The next theorem proves that the key model satisfies (i) of Theorem 1.1. The reals in Theorem 1.1 are treated here as points of the *Cantor space* 2^ω.

Theorem 16.1. *If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$ then it holds in $\mathbf{L}[G]$ that there is a $\Delta^1_{\mathfrak{n}}$-good well-ordering of 2^ω of length ω_1.*

Proof. We argue in $\mathbf{L}[G]$. Let $\boldsymbol{X} = \boldsymbol{X}[G]$. If $\gamma < \omega_1$ then let $\boldsymbol{X}\restriction\gamma = \langle x_\xi[G]\rangle_{\xi<\gamma}$. The map $\gamma \mapsto \boldsymbol{X}\restriction\gamma$ is $\Pi^{\rm HC}_{\mathfrak{n}-2}$ in $\mathbf{L}[G]$ by Proposition 15.2 since

$$Y = \boldsymbol{X}\restriction\gamma \iff Y \text{ is a function on } \gamma \wedge \forall\,\xi < \gamma\,(\langle\xi, Y(\xi)\rangle \in \boldsymbol{X}).$$

Now if $x \in 2^\omega$ (in $\mathbf{L}[G]$) then $x \in \mathbf{L}[\boldsymbol{X}\restriction\gamma]$ for some $\gamma < \omega_1$ by Proposition 15.2, hence we let $\gamma(x)$ be the least $\gamma < \omega_1$ such that $x \in \mathbf{L}[\boldsymbol{X}\restriction\gamma]$, and $\nu(x) < \omega_1$ be the index of x in the canonical $\Delta^{\rm HC}_1(\{\boldsymbol{X}\restriction\gamma\})$ well-ordering $\le_{\boldsymbol{X}\restriction\gamma}$ of 2^ω in $\mathbf{L}[\boldsymbol{X}\restriction\gamma]$ (by Gödel). We claim that the maps $x \mapsto \gamma(x)$ and $x \mapsto \nu(x)$ are $\Delta^{\rm HC}_{\mathfrak{n}-1}$. Indeed,

$$\begin{aligned}\gamma = \gamma(x) &\iff \exists\,Y\,\big(Y = \boldsymbol{X}\restriction\gamma \wedge x \in \mathbf{L}[Y] \wedge \forall\,\gamma' < \gamma\,(x \notin \mathbf{L}[Y\restriction\gamma'])\big)\\ &\iff \forall\,Y\,\big(Y = \boldsymbol{X}\restriction\gamma \implies x \in \mathbf{L}[Y] \wedge \forall\,\gamma' < \gamma\,(x \notin \mathbf{L}[Y\restriction\gamma']\big).\end{aligned}$$

This easily yields the result for the map $x \mapsto \gamma(x)$. The result for the other map follows by a similar rather routine estimation.

Now let $\preccurlyeq$ be the well-ordering of the set $2^\omega \cap \mathbf{L}[G]$ according to the lexicographical well-ordering of the triples $\langle \mathtt{max}\{\gamma(x), \nu(x)\}, \gamma(x), \nu(x)\rangle$. It easily follows from the results for maps $x \mapsto \gamma(x)$ and $x \mapsto \nu(x)$ that $\preccurlyeq$ is $\Delta^{\rm HC}_{\mathfrak{n}-1}$, hence $\Delta^1_{\mathfrak{n}}$ by Lemma 13.1.

Finally to check the $\Delta^1_{\mathfrak{n}}$-goodness, it remains to prove that, given a $\Delta^1_{\mathfrak{n}}$ set $P \subseteq 2^\omega \times 2^\omega$, the set $Q = \{\langle z, x\rangle : \forall\,y \preccurlyeq x\,\neg\,P(z,y)\}$ has to be $\Delta^1_{\mathfrak{n}}$ too. The class $\Pi^1_{\mathfrak{n}}$ is obvious as $\preccurlyeq$ is already shown to be $\Delta^1_{\mathfrak{n}}$. Thus we have to verify the class $\Sigma^1_{\mathfrak{n}}$, or equivalently, class $\Sigma^{\rm HC}_{\mathfrak{n}-1}$, for Q. But this is true as $Q(z,x)$ is equivalent to

> for all $\gamma', \nu' \le \mathtt{max}\{\gamma(x), \nu(x)\}$, if the triple $\langle \mathtt{max}\{\gamma', \nu'\}, \gamma', \nu'\rangle$ non-strictly precedes $\langle \mathtt{max}\{\gamma(x), \nu(x)\}, \gamma(x), \nu(x)\rangle$ lexicographically, then there is a real $y \in 2^\omega$ such that $\gamma' = \gamma(y)$, $\nu' = \nu(y)$, and $\neg\,P(z,y)$.

However the quoted formula is essentially $\Sigma^{\rm HC}_{\mathfrak{n}-1}$ since the bounded quantifiers $\forall\,\gamma', \nu' \le \mathtt{max}\{\gamma(x), \nu(x)\}$ do not destroy Σ-classes over HC. □

17. Non-existence of $\boldsymbol{\Sigma}^1_{\mathfrak{n}-1}$ well-orderings in the key model

We begin here a lengthy proof of the claim that the key model also satisfies (ii) of Theorem 1.1. This amounts to the following theorem.

Theorem 17.1. *If $\mathfrak{n} \ge 4$ and a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$ then it holds in $\mathbf{L}[G]$ that there is no $\boldsymbol{\Sigma}^1_{\mathfrak{n}-1}$ well-orderings of the reals, and even more, there is no $\boldsymbol{\Sigma}^1_{\mathfrak{n}-1}$ binary relation which well-orders the set $\{x_\xi[\underline{G}] : \xi < \omega_1^{\mathbf{L}}\}$.*

The theorem leaves aside the case $\mathfrak{n} = 3$ in (ii) of Theorem 1.1 which thereby needs a separate consideration to justify the assumption $n \ge 4$.

Case $\mathfrak{n} = 3$**.** We claim that (ii) of Theorem 1.1 **holds** in the key model $\mathbf{L}[G]$, where G is $\mathbb{P}$-generic over $\mathbf{L}$. Suppose to the contrary that (ii) of Theorem 1.1 **fails**, so that (as $\mathfrak{n} = 3$) there is a $\boldsymbol{\Delta}^1_2$ well-ordering of the reals (even not necessarily good) in $\mathbf{L}[G]$. Then by Theorem 25.39 in [20] there is a real $x \in 2^\omega \cap \mathbf{L}[G]$ such that $2^\omega \subseteq \mathbf{L}[x]$ in $\mathbf{L}[G]$. But this is definitely not the case for the key model $\mathbf{L}[G]$ we consider.

Indeed, arguing in $\mathbf{L}[G]$, suppose to the contrary that a real $x \in 2^\omega \cap \mathbf{L}[G] = \mathbf{L}[\langle x_\xi[G]\rangle_{\xi<\omega_1}]$ satisfies $2^\omega \cap \mathbf{L}[G] \subseteq \mathbf{L}[x]$. It follows by Corollary 14.7 that there is an ordinal $\lambda < \omega_1 = \omega_1^{\mathbf{L}}$ such that $x \in \mathbf{L}[\langle x_\xi[G]\rangle_{\xi<\lambda}]$. However the real $y = x_\lambda[G]$ does not belong to $\mathbf{L}[\langle x_\xi[G]\rangle_{\xi<\lambda}]$ by the product forcing theory. Therefore $y \notin \mathbf{L}[x]$, contrary to the choice of x. $\square$(Case $\mathbb{n} = 3$)

The proof of Theorem 17.1 involves several technical definitions and results from [31], introduced in sections 18 to 22. Beginning the proof, our plan will be to infer a contradiction from the following contrary assumption.

Assumption 17.2. Assume to the contrary that a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$, and it holds in $\mathbf{L}[G]$ that there is a $\mathbf{\Sigma}^1_{\mathbb{n}-1}$ binary relation which strictly well-orders the set $\boldsymbol{A}[G] = \{x_\xi[G] : \xi < \omega_1^{\mathbf{L}}\}$ — so that there is a $\Sigma^1_{\mathbb{n}-1}$ parameterfree formula $\Phi(\cdot,\cdot,\cdot)$, and a parameter $u \in 2^\omega \cap \mathbf{L}[G]$ such that the relation $<^u_\Phi$ iff $\Phi(x, y, u)$, strictly well-orders $\boldsymbol{A}[G]$. $\square$

Under this assumption, Theorem 8.1 implies that, in $\mathbf{L}$, there exist:

- a small Π-complete real name $\mathbb{c} \in \mathbf{L}$, $\mathbb{c} \subseteq \mathbb{P} \times \omega \times 2$, such that $u = \mathbb{c}[G]$,
- a condition $\mathbb{p} \in G$ which $\mathbb{P}$-forces, over $\mathbf{L}$, that:
  "the relation $<^{\mathbb{c}[\underline{G}]}_\Phi$, defined by $x <^{\mathbb{c}[\underline{G}]}_\Phi y$ iff $\Phi(x, y, \mathbb{c}[\underline{G}])$, well-orders the set $\{x_\xi[\underline{G}] : \xi < \omega_1^{\mathbf{L}}\}$ in $\mathbf{L}[\underline{G}]$".
- limit ordinals $\mathbb{a}, \mathbb{b} < \omega_1^{\mathbf{L}}$ such that $\mathbb{p} \in \mathbf{MT}(\vec{\mathbb{n}}(\mathbb{a}))$, $\mathbb{c} \subseteq \mathbf{MT}(\vec{\mathbb{n}} \restriction \mathbb{a}) \times \omega \times 2$, and $|\vec{\mathbb{n}}(\mathbb{a})| \subseteq \mathbb{b}$, so that $|\mathbb{c}| \cup |\mathbb{p}| \subseteq \mathbb{b}$.

We fix $\mathbb{c}, \mathbb{p}, \mathbb{a}, \mathbb{b}$ henceforth. We'll work towards a contradiction. The contradiction will be obtained in the form of a non-empty subset X' of the set $\boldsymbol{A}[G]$ as above, containing no $<^{\mathbb{c}[G]}_\Phi$-least real, see Section 27.

18. An auxiliary forcing relation

Here we introduce an auxiliary forcing relation, not explicitly connected with any particular forcing notion, in particular, with the key forcing $\mathbb{P}$.

We argue in L. Consider the language of 2nd order arithmetic, with variables $k, l, m, n, \ldots$ of type 0 over ω and variables $a, b, x, y, \ldots$ of type 1 over 2^ω, whose atomic formulas are those of the form $x(k) = n$. Let $\mathscr{L}$ be the extension of this language, which allows to substitute variables of type 0 with natural numbers and variables of type 1 with **small real names** (see Section 7) $\mathbf{c} \in \mathbf{L}$. We consider the natural classes $\mathscr{L}\Sigma^1_n$, $\mathscr{L}\Pi^1_n$ $(n \geq 1)$ of $\mathscr{L}$-formulas. Let $\mathscr{L}(\Sigma\Pi)^1_1$ be the closure of $\mathscr{L}\Sigma^1_1 \cup \mathscr{L}\Pi^1_1$ under $\neg, \wedge, \vee$ and quantifiers over ω.

A relation $\boldsymbol{p}\, \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\, \varphi$ between multitrees $\boldsymbol{p}$, sequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$, and closed $\mathscr{L}$-formulas φ in $\mathscr{L}(\Sigma\Pi)^1_1$ or $\mathscr{L}\Sigma^1_n \cup \mathscr{L}\Pi^1_n$, $n \geq 2$, was defined in [31, §22] by induction on the complexity of φ as follows. Inductive steps 2° and 3° demonstrate similarities with various conventional forcing notions.

1°. Let $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol{p} \in \mathbf{MT}$ (not necessarily $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$), and φ is a closed $\mathscr{L}(\Sigma\Pi)^1_1$ formula. We define $\boldsymbol{p}\, \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\, \varphi$ iff there is a CTM $\mathfrak{M} \models \mathbf{ZFL}^-$ (recall Definition 13.2 on $\mathbf{ZFL}^-$), an ordinal $\vartheta < \mathtt{dom}\, \vec{\boldsymbol{\pi}}$, and a multitree $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$, such that
   (1) $\boldsymbol{p} \leqslant \boldsymbol{p}_0$ (meaning: $\boldsymbol{p}$ is stronger),
   (2) $\mathfrak{M}$ contains $\vec{\boldsymbol{\pi}} \restriction \vartheta$ (then contains $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$ and $\boldsymbol{p}_0$ as well),
   (3) every name $\mathbf{c}$ in φ belongs to $\mathfrak{M}$ and is $(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-complete,
   (4) $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\mathfrak{M}} \vec{\boldsymbol{\pi}}$ in the sense of Section 12, and
   (5) $\boldsymbol{p}_0$ $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$ in the usual sense.

2°. If $\varphi(x)$ is a $\mathscr{L}\Pi^1_n$ formula, $n \geq 1$, then $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\exists x\,\varphi(x)$ iff there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi(\mathbf{c})$.

3°. If φ is a closed $\mathscr{L}\Pi^1_n$ formula, $n \geq 2$, then $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ iff there is no sequence $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{MF}}$ and multitree $\boldsymbol{p}' \in \mathbf{MT}(\vec{\boldsymbol{\tau}})$ such that $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\tau}}$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and $\boldsymbol{p}'\,\mathtt{forc}_{\vec{\boldsymbol{\tau}}}\,\varphi^-$, where φ^- is the result of the canonical transformation of $\neg\varphi$ to a $\mathscr{L}\Sigma^1_n$ form.

It is not immediately clear that the definition is consistent in part 1°, i.e., it is impossible that both $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ and $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\neg\varphi$ hold via two different triples of $\vartheta, \boldsymbol{p}_0, \mathfrak{M}$. This will be subject of Corollary 18.5.

Remark 18.1. The condition "$\boldsymbol{p}_0$ $\mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$" in 1° does not depend on the choice of a CTM $\mathfrak{M}$ containing $\vec{\boldsymbol{\pi}}\restriction\vartheta$ and φ, since if φ is $\mathscr{L}(\Sigma\Pi)^1_1$ then all transitive models agree on the formula $\varphi[G]$ by the Mostowski absoluteness theorem [20, Theorem 25.4]. □

The following lemma discovers the monotone character of $\mathtt{forc}$.

Lemma 18.2 *(in* $\mathbf{L}$*). If sequences* $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\varphi}}$ *belong to* $\overrightarrow{\mathbf{MF}}$*,* $\boldsymbol{q} \leqslant \boldsymbol{p}$ *are multitrees,* φ *is a closed formula in* $\mathscr{L}(\Sigma\Pi)^1_1$ *or in* $\mathscr{L}\Sigma^1_n$*,* $\mathscr{L}\Pi^1_n$ $(n \geq 2)$*, and* $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$*, then* $\boldsymbol{q}\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}}\,\varphi$*.*

Proof. Let φ be a closed $\mathscr{L}(\Sigma\Pi)^1_1$ formula, and $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ is witnessed by $\mathfrak{M}$, ϑ, $\boldsymbol{p}_0$ containing the relevant information as in 1°. Then the same $\mathfrak{M}$, ϑ, $\boldsymbol{p}_0$ witness $\boldsymbol{q}\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}}\,\varphi$.

The induction step $\exists$, as in 2°, is pretty elementary.

If φ is a $\mathscr{L}\Pi^1_n$-formula, $n \geq 2$, and $\boldsymbol{q}\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}}\,\varphi$ fails, then by 3° there is a sequence $\vec{\boldsymbol{\varphi}}' \in \overrightarrow{\mathbf{MF}}$ and multitree $\boldsymbol{q}' \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ such that $\vec{\boldsymbol{\varphi}} \subseteq \vec{\boldsymbol{\varphi}}'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$, and $\boldsymbol{q}'\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}'}\,\varphi^-$. Then $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\varphi}}'$ and $\boldsymbol{q}' \leqslant \boldsymbol{p}$, so $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ fails by 3°. □

The next lemma presents a useful connection with the usual forcing.

Lemma 18.3 *(in* $\mathbf{L}$*). Assume that* $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$*,* $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$*,* φ *is a formula in* $\mathscr{L}(\Sigma\Pi)^1_1$*,* $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}} \cup \overrightarrow{\mathbf{MF}}_{\omega_1}$*,* $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$*, and* $\mathfrak{N} \models \mathbf{ZFL}^-$ *is a transitive model containing* $\vec{\boldsymbol{\varphi}}, \varphi$*. Then* $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$*-forces* $\varphi[\underline{G}]$ *over* $\mathfrak{N}$ *in the usual sense.*

Proof. By definition there is an ordinal $\vartheta < \mathtt{dom}\,\vec{\boldsymbol{\pi}}$, a multitree $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$, and a CTM $\mathfrak{M} \models \mathbf{ZFL}^-$ satisfying (1)–(5) above. Let's first consider the case $\mathfrak{M} \subseteq \mathfrak{N}$. Suppose that $G \subseteq \mathbf{MT}(\vec{\boldsymbol{\varphi}})$ is a set $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$-generic over $\mathfrak{N}$, and $\boldsymbol{p} \in G$ — then $\boldsymbol{p}_0 \in G$, too. We have to prove that $\varphi[G]$ is true in $\mathfrak{N}[G]$.

We claim that the set $G' = G \cap \mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$ is $\mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$-generic over $\mathfrak{M}$. Indeed, let a set $\boldsymbol{D} \in \mathfrak{M}$, $\boldsymbol{D} \subseteq \mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$, be open dense in $\mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction\vartheta)$. Then, as $\vec{\boldsymbol{\pi}}\restriction\vartheta \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$, it follows by Lemma 6.4 and Theorem 10.1(vii) that $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$. Moreover $\boldsymbol{D} \in \mathfrak{M} \subseteq \mathfrak{N}$. We conclude $G \cap \boldsymbol{D} \neq \varnothing$ by the choice of G. It follows that $G' \cap \boldsymbol{D} \neq \varnothing$.

Now if $\mathbf{c}$ is a name in φ then $\mathbf{c} \in \mathfrak{M}$ and $\mathbf{c}$ is $\vec{\boldsymbol{\pi}}\restriction\vartheta$-complete, hence $\mathbf{c}[G'] \in 2^\omega$ is defined. Therefore $\mathbf{c}[G] = \mathbf{c}[G']$, because $G' \subseteq G$. Thus $\varphi[G]$ coincides with $\varphi[G']$. Note also that $\boldsymbol{p}_0 \in G'$. We conclude that $\varphi[G']$ holds in $\mathfrak{M}[G']$ as $\boldsymbol{p}_0$ forces $\varphi[\underline{G}]$ over $\mathfrak{M}$. The same formula $\varphi[G]$ holds in $\mathfrak{N}[G]$ by the Mostowski absoluteness theorem [20, Thm 25.4], as required.

Now suppose that $\mathfrak{M} \not\subseteq \mathfrak{N}$. Then $\mathfrak{N} \subseteq \mathfrak{M}$. By the part already established, $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$ in the usual sense. Assume towards the contrary that $\boldsymbol{p}$ does not $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$-force $\varphi[\underline{G}]$ over $\mathfrak{N}$. Then there is a condition $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\varphi}})$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, which $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$-forces $\neg\varphi[\underline{G}]$ over $\mathfrak{N}$. Consider an arbitrary set $G \subseteq \mathbf{MT}(\vec{\boldsymbol{\varphi}})$, $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$-generic over $\mathfrak{M}$ – then over $\mathfrak{N}$ as well since $\mathfrak{N} \subseteq \mathfrak{M}$, and containing $\boldsymbol{q}$, hence, $\boldsymbol{p}$ too. Then the same formula $\varphi[G]$ is true in $\mathfrak{M}[G]$ (as $\boldsymbol{p} \in G$) but false in $\mathfrak{N}[G]$ (as $\boldsymbol{q} \in G$). But this contradicts the Mostowski absoluteness theorem. □

Lemma 18.4 *(in* $\mathbf{L}$*). Assume that* $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{MF}}$*,* $\boldsymbol p \in \mathbf{MT}(\vec{\boldsymbol\pi})$*,* φ *is a formula in* $\mathscr{L}(\Sigma\Pi)^1_1$*,* $\mathfrak{M} \models \mathbf{ZFL}^-$ *is a transitive model containing* $\vec{\boldsymbol\pi}$*, each name* $\mathbf{c}$ *in* φ *belongs to* $\mathfrak{M}$ *and is* $\vec{\boldsymbol\pi}$*-complete, and* $\boldsymbol p$ $\mathbf{MT}(\vec{\boldsymbol\pi})$*-forces* $\varphi[\underline{G}]$ *over* $\mathfrak{M}$*. Then there exists a sequence* $\vec{\boldsymbol\varphi} \in \overrightarrow{\mathbf{MF}}$ *such that* $\vec{\boldsymbol\pi} \subset_{\mathfrak{M}} \vec{\boldsymbol\varphi}$ *and* $\boldsymbol p \,\mathtt{forc}_{\vec{\boldsymbol\varphi}}\, \varphi$*.*

Proof. Theorem 12.3 yields a sequence $\vec{\boldsymbol\varphi} \in \overrightarrow{\mathbf{MF}}$ such that $\vec{\boldsymbol\pi} \subset_{\mathfrak{M}} \vec{\boldsymbol\varphi}$. Then the triple of $\mathfrak{M}$, $\vartheta = \mathtt{dom}(\vec{\boldsymbol\varphi})$, $\boldsymbol p_0 = \boldsymbol p$ witnesses that $\boldsymbol p \,\mathtt{forc}_{\vec{\boldsymbol\varphi}}\, \varphi$ by 1°. □

Corollary 18.5 *(in* $\mathbf{L}$*). Let* $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{MF}}$*,* $\boldsymbol p \in \mathbf{MT}(\vec{\boldsymbol\pi})$*,* φ *be a closed formula in* $\mathscr{L}(\Sigma\Pi)^1_1$ *or* $\mathscr{L}\Sigma^1_n$*,* $n \ge 2$*. Then* $\boldsymbol p \,\mathtt{forc}_{\vec{\boldsymbol\pi}}\, \varphi$ *and* $\boldsymbol p \,\mathtt{forc}_{\vec{\boldsymbol\pi}}\, \varphi^-$ *cannot hold together.*

Proof. Let $\varphi \in \mathscr{L}(\Sigma\Pi)^1_1$. If both $\boldsymbol p \,\mathtt{forc}_{\vec{\boldsymbol\pi}}\, \varphi$ and $\boldsymbol p \,\mathtt{forc}_{\vec{\boldsymbol\pi}}\, \varphi^-$ then, by Lemma 18.3, $\boldsymbol p$ $\mathbf{MT}(\vec{\boldsymbol\pi})$-forces both $\varphi[\underline{G}]$ and $\varphi^-[\underline{G}]$ over a large enough CTM $\mathfrak{M}$, a contradiction. If $\varphi \in \mathscr{L}\Sigma^1_n$ then the result follows by 3°. □

19. Definability of the auxiliary forcing

If K is one of the classes $\mathscr{L}(\Sigma\Pi)^1_1$, $\mathscr{L}\Sigma^1_n$, $\mathscr{L}\Pi^1_n$ ($n \ge 2$), then let $\mathbf{FORC}[K]$ consist of all triples $\langle \vec{\boldsymbol\pi}, \boldsymbol p, \varphi\rangle$ such that $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol p \in \mathbf{MT}$, φ is a formula in K, and $\boldsymbol p \,\mathtt{forc}_{\vec{\boldsymbol\pi}}\, \varphi$. Then $\mathbf{FORC}[K]$ is a subset of HC.

Lemma 19.1 *(in* $\mathbf{L}$*).* $\mathbf{FORC}[\mathscr{L}(\Sigma\Pi)^1_1] \in \Delta^{\mathrm{HC}}_1$*, whereas if* $n \ge 2$ *then* $\mathbf{FORC}[\mathscr{L}\Sigma^1_n]$ *belongs to* $\Sigma^{\mathrm{HC}}_{n-1}$ *and* $\mathbf{FORC}[\mathscr{L}\Pi^1_n]$ *belongs to* Π^{HC}_{n-1}*.* □

Proof. Relations like $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{MF}}$, "being a formula in $\mathscr{L}(\Sigma\Pi)^1_1$", $\boldsymbol p \in \mathbf{MT}(\vec{\boldsymbol\rho})$, forcing over a CTM, etc. are definable in HC by bounded formulas, hence Δ^{HC}_1. Moreover, the model $\mathfrak{M}$ can be tied by both $\exists$ and $\forall$ in 1°, see Remark 18.1. This wraps up the Δ^{HC}_1 estimation for $\mathscr{L}(\Sigma\Pi)^1_1$.

The inductive step by 2° is quite simple.

Now the step by 3°. Assume that $n \ge 2$, and it is already established that $\mathbf{FORC}[\mathscr{L}\Sigma^1_n] \in \Sigma^{\mathrm{HC}}_{n-1}$. Then $\langle \vec{\boldsymbol\pi}, \boldsymbol p, \varphi\rangle \in \mathbf{FORC}[\mathscr{L}\Pi^1_n]$ iff $\vec{\boldsymbol\pi} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol p \in \mathbf{MT}$, φ is a closed $\mathscr{L}\Pi^1_n$ formula, and, by 3°, there exist no triple $\langle \vec{\boldsymbol\tau}, \boldsymbol p', \psi\rangle \in \mathbf{FORC}[\mathscr{L}\Sigma^1_n]$ such that $\vec{\boldsymbol\tau} \in \overrightarrow{\mathbf{MF}}$, $\vec{\boldsymbol\pi} \subseteq \vec{\boldsymbol\tau}$, $\boldsymbol p' \in \mathbf{MT}(\vec{\boldsymbol\tau})$, $\boldsymbol p' \leqslant \boldsymbol p$, and ψ is φ^-. We easily get the required estimation Π^{HC}_{n-1} of $\mathbf{FORC}[\mathscr{L}\Pi^1_n]$. □

20. Tail invariance

Invariance theorems are rather typical for all kinds of forcing. We present here an invariance theorem on the auxiliary forcing $\mathtt{forc}$. It deals with the tail invariance, and it is considerably stronger than a tail invariance theorem established in [31]. Another invariance theorem (Section 21 below) explores the permutational invariance.

Arguing in $\mathbf{L}$, if $\vec{\boldsymbol\pi} = \langle \boldsymbol\pi_\alpha\rangle_{\alpha<\lambda} \in \overrightarrow{\mathbf{MF}}$ and $\gamma < \lambda = \mathtt{dom}\,\vec{\boldsymbol\pi}$, then let the γ-*tail* $\vec{\boldsymbol\pi}{\restriction}_{\ge\gamma}$ be the restriction $\vec{\boldsymbol\pi}{\restriction}[\gamma,\lambda)$ to the semiinterval $[\gamma,\lambda) = \{\alpha : \gamma \le \alpha < \lambda\}$. in ω_1. Thus essentially $\vec{\boldsymbol\pi}{\restriction}_{\ge\gamma}$ is a $\sqsubset$-increasing sequence of length $\lambda - \gamma$ (= the unique ordinal β with $\gamma + \beta = \lambda$), whose domain is shifted to the right in ω_1. In this case the set $\mathbf{MT}(\vec{\boldsymbol\pi}{\restriction}_{\ge\gamma}) = \bigcup^{\mathtt{cw}}_{\gamma\le\alpha<\lambda} \vec{\boldsymbol\pi}(\alpha)$ is open dense in $\mathbf{MT}(\vec{\boldsymbol\pi})$.

Therefore it can be expected that if $\vec{\boldsymbol\varphi}$ is another sequence of length $\mu = \mathtt{dom}\,\vec{\boldsymbol\varphi}$, and $\delta < \mu$ is such that $\mu - \delta = \lambda - \gamma$ and the according restriction $\vec{\boldsymbol\varphi}{\restriction}_{\ge\delta}$ is *shift-equal* to $\vec{\boldsymbol\pi}{\restriction}_{\ge\gamma}$ in the sense that $\vec{\boldsymbol\varphi}{\restriction}_{\ge\delta}(\delta+\alpha) = \vec{\boldsymbol\pi}{\restriction}_{\ge\gamma}(\gamma+\alpha)$ for all $\alpha < \mu - \delta = \lambda - \gamma$, then the according dense sets $\mathbf{MT}(\vec{\boldsymbol\pi}{\restriction}_{\ge\gamma})$ and $\mathbf{MT}(\vec{\boldsymbol\varphi}{\restriction}_{\ge\delta})$ coincide, and we can expect that the relations $\mathtt{forc}_{\vec{\boldsymbol\pi}}$ and $\mathtt{forc}_{\vec{\boldsymbol\varphi}}$ coincide too. And indeed this turns out to be the case.

Theorem 20.1 *(in* $\mathbf{L}$*). Assume that* $\vec{\boldsymbol\pi}, \vec{\boldsymbol\varphi}$ *are sequences in* $\overrightarrow{\mathbf{MF}}$*,* $\gamma < \lambda = \mathtt{dom}\,\vec{\boldsymbol\pi}$*,* $\delta < \mu = \mathtt{dom}\,\vec{\boldsymbol\varphi}$*,* $\vec{\boldsymbol\varphi}{\restriction}_{\ge\gamma}$ *is shift-equal to* $\vec{\boldsymbol\pi}{\restriction}_{\ge\delta}$*,* $\boldsymbol p \in \mathbf{MT}$*,* $n \ge 2$*, and* φ *is a formula in* $\mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$*. Then* $\boldsymbol p \,\mathtt{forc}_{\vec{\boldsymbol\pi}}\, \varphi$ *iff* $\boldsymbol p \,\mathtt{forc}_{\vec{\boldsymbol\varphi}}\, \varphi$*.*

Proof. Part 1: the $\mathscr{L}\Pi^1_2$ case. Let $\psi(x)$ be a $\mathscr{L}\Sigma^1_1$ formula. Suppose that $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}}\,\forall x\,\psi(x)$ **fails**, so there is $\vec{\boldsymbol{\varphi}}' \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ such that $\vec{\boldsymbol{\varphi}} \subseteq \vec{\boldsymbol{\varphi}}'$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, and $\boldsymbol{q}\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}'}\,\exists x\,\psi^-(x)$. We can assume that $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}'\restriction_{\geq\delta})$. By definition there is a small real name $\mathbf{c}$ such that $\boldsymbol{q}\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}'}\,\psi^-(\mathbf{c})$.

Let $\mu' = \mathtt{dom}\,\vec{\boldsymbol{\varphi}}'$. Define a sequence $\vec{\boldsymbol{\pi}}'$ so that $\mathtt{dom}\,\vec{\boldsymbol{\pi}}' = \lambda' = \lambda + (\mu' - \mu)$, $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$, and $\vec{\boldsymbol{\pi}}'(\lambda+\alpha) = \vec{\boldsymbol{\varphi}}'(\mu+\alpha)$ for all $\alpha < \mu' - \mu$, so that $\vec{\boldsymbol{\pi}}'\restriction_{\geq\lambda}$ is shift-equal to $\vec{\boldsymbol{\varphi}}'\restriction_{\geq\mu}$. Then $\vec{\boldsymbol{\pi}}'\restriction_{\geq\gamma}$ is shift-equal to $\vec{\boldsymbol{\varphi}}'\restriction_{\geq\delta}$ either, hence $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}}'\restriction_{\geq\gamma}) \subseteq \mathbf{MT}(\vec{\boldsymbol{\pi}}')$.

Consider any CTM $\mathfrak{N} \models \mathbf{ZFL}^-$ containing ψ, $\mathbf{c}$, $\vec{\boldsymbol{\pi}}'$, $\vec{\boldsymbol{\varphi}}'$. Then $\boldsymbol{q}$ $\mathbf{MT}(\vec{\boldsymbol{\varphi}}')$-forces $\psi^-(\mathbf{c})[\underline{G}]$ over $\mathfrak{N}$ by Lemma 18.3. Yet the forcing notions $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$, $\mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ contain one and the same dense set $\mathbf{MT}(\vec{\boldsymbol{\pi}}'\restriction_{\geq\gamma}) = \mathbf{MT}(\vec{\boldsymbol{\varphi}}'\restriction_{\geq\delta})$. Thus $\boldsymbol{q}$ also $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$-forces $\psi^-(\mathbf{c})[\underline{G}]$ over $\mathfrak{N}$. By definition $\boldsymbol{q}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}'}\,\psi^-(\mathbf{c})$ and $\boldsymbol{q}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}'}\,\exists x\,\psi^-(x)$, hence $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\forall x\,\psi(x)$ **fails**, as required.

Part 2: the step $\mathscr{L}\Pi^1_n \to \mathscr{L}\Sigma^1_{n+1}$, $n \geq 2$. Let $\varphi(x)$ be a formula in $\mathscr{L}\Pi^1_n$. Assume that $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\exists x\,\varphi(x)$. By definition (see 2° in Section 18), there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi(\mathbf{c})$. Then we have $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}}\,\varphi(\mathbf{c})$ by the inductive hypothesis, thus $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}}\,\exists x\,\psi(x)$.

Part 3: the step $\mathscr{L}\Sigma^1_n \to \mathscr{L}\Pi^1_n$, $n \geq 3$. Assume that φ is a $\mathscr{L}\Pi^1_n$ formula, and $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}}\,\varphi$ **fails**. Then by 3° of Section 18, there is a sequence $\vec{\boldsymbol{\varphi}}' \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{p}' \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ such that $\vec{\boldsymbol{\varphi}} \subseteq \vec{\boldsymbol{\varphi}}'$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and $\boldsymbol{p}'\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}'}\,\varphi^-$. As $\vec{\boldsymbol{\varphi}}'$ is $\sqsubset$-increasing, there is a multitree $\boldsymbol{r} \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}'\restriction_{\geq\gamma})$, $\boldsymbol{r} \leqslant \boldsymbol{p}'$. Then $\boldsymbol{r} \leqslant \boldsymbol{p}$ and $\boldsymbol{r}\,\mathtt{forc}_{\vec{\boldsymbol{\varphi}}'}\,\varphi^-$. Define a sequence $\vec{\boldsymbol{\pi}}' \in \overrightarrow{\mathbf{MF}}$ as in Part 1 so that $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$, and $\vec{\boldsymbol{\pi}}'\restriction_{\geq\lambda}$ is shift-equal to $\vec{\boldsymbol{\varphi}}'\restriction_{\geq\mu}$. Then $\boldsymbol{r} \in \mathbf{MT}(\vec{\boldsymbol{\pi}}'\restriction_{\geq\gamma})$, $\boldsymbol{r} \leqslant \boldsymbol{p}$, and also $\boldsymbol{r}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}'}\,\varphi^-$ by the inductive hypothesis. We conclude that $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ **fails** as well. □

Let $\boldsymbol{\pi}$ be a multiforcing in **spMF** (not a sequence of multiforcings). Theorem 20.1 allows us to meaningfully define $\boldsymbol{p}\,\mathtt{forc}_{\boldsymbol{\pi}}\,\varphi$ iff $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$, where $\vec{\boldsymbol{\pi}} = \langle\boldsymbol{\pi}\rangle \in \overrightarrow{\mathbf{MF}}$ is a one-term sequence defined by $\mathtt{dom}(\vec{\boldsymbol{\pi}}) = 1$ and $\vec{\boldsymbol{\pi}}(0) = \boldsymbol{\pi}$.

Corollary 20.2 *(in* **L***). Let* $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$, $\mathtt{dom}\,\vec{\boldsymbol{\pi}} = \lambda + 1$, $\boldsymbol{\pi} = \vec{\boldsymbol{\pi}}(\lambda)$, $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $n \geq 2$, φ *is a formula in* $\mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$. *Then* $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ *iff* $\boldsymbol{p}\,\mathtt{forc}_{\boldsymbol{\pi}}\,\varphi$. □

21. Permutation invariance

Arguing in L, let PERM be the set of all permutations of indices, that is, all bijections $\boldsymbol{h} : \omega_1 \xrightarrow{\text{onto}} \omega_1$, such that $\boldsymbol{h} = \boldsymbol{h}^{-1}$ and the *non-identity domain* $\mathbf{NI}(\boldsymbol{h}) = \{\xi : \boldsymbol{h}(\xi) \neq \xi\}$ is at most countable. Elements of PERM are called *permutations.*

Let $\boldsymbol{h} \in$ PERM. We extend the action of $\boldsymbol{h}$ as follows.

- if $\boldsymbol{p}$ is a multitree then $\boldsymbol{hp}$ is a multitree, $|\boldsymbol{hp}| = \boldsymbol{h}``|\boldsymbol{p}| = \{\boldsymbol{h}(\xi) : \xi \in |\boldsymbol{p}|\}$, and $(\boldsymbol{hp})(\boldsymbol{h}(\xi)) = \boldsymbol{p}(\xi)$ whenever $\xi \in |\boldsymbol{p}|$;
- if $\boldsymbol{\pi} \in \mathbf{MT}$ is a multiforcing then $\boldsymbol{h}\cdot\boldsymbol{\pi} = \boldsymbol{\pi} \circ (\boldsymbol{h}^{-1})$ is a multiforcing, $|\boldsymbol{h}\cdot\boldsymbol{\pi}| = \boldsymbol{h}``\boldsymbol{\pi}$ and $(\boldsymbol{h}\cdot\boldsymbol{\pi})(\boldsymbol{h}(\xi)) = \boldsymbol{\pi}(\xi)$ whenever $\xi \in |\boldsymbol{\pi}|$;
- if $\mathbf{c} \subseteq \mathbf{MT} \times (\omega \times \omega)$ is a real name, then put $\boldsymbol{h}\mathbf{c} = \{\langle \boldsymbol{hp}, n, i\rangle : \langle \boldsymbol{p}, n, i\rangle \in \mathbf{c}\}$, thus easily $\boldsymbol{h}\mathbf{c}$ is a real name as well;
- if $\vec{\boldsymbol{\pi}} = \langle\boldsymbol{\pi}_\alpha\rangle_{\alpha<\kappa} \in \overrightarrow{\mathbf{MF}}$, then $\boldsymbol{h}\vec{\boldsymbol{\pi}} = \langle \boldsymbol{h}\cdot\boldsymbol{\pi}_\alpha\rangle_{\alpha<\kappa}$, still a sequence in $\overrightarrow{\mathbf{MF}}$;
- if $\varphi := \varphi(\mathbf{c}_1, \ldots, \mathbf{c}_n)$ is an $\mathscr{L}$-formula (with all names explicitly indicated), then $\boldsymbol{h}\varphi$ is $\varphi(\boldsymbol{h}\mathbf{c}_1, \ldots, \boldsymbol{h}\mathbf{c}_n)$.

Many notions and relations defined above are clearly PERM-invariant, e.g., $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$ iff $\boldsymbol{hp} \in \mathbf{MT}(\boldsymbol{h}\cdot\boldsymbol{\pi})$, $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$ iff $\boldsymbol{h}\cdot\boldsymbol{\pi} \sqsubset \boldsymbol{h}\cdot\boldsymbol{\varphi}$, *et cetera.* The invariance also takes place with respect to the relation $\mathtt{forc}$ itself.

Theorem 21.1 *(in* $\mathbf{L}$*). Assume that* $\vec{\pi} \in \overrightarrow{\mathbf{MF}}$*,* $\boldsymbol{p} \in \mathbf{MT}(\vec{\pi})$*,* $\boldsymbol{h} \in \mathrm{PERM}$*,* $n \geq 2$*, and* $\varphi \in \mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$*. Then* $\boldsymbol{p}\,\mathtt{forc}_{\vec{\pi}}\,\varphi$ *iff* $(\boldsymbol{hp})\,\mathtt{forc}_{\boldsymbol{h}\vec{\pi}}\,(\boldsymbol{h}\varphi)$*.*

Proof. Let $\vec{\varrho} = \boldsymbol{h}\vec{\pi}$, $\boldsymbol{q} = \boldsymbol{hp}$.

Part 1: the $\mathscr{L}\Pi^1_2$ case. Assume that $\varphi(x)$ is a $\mathscr{L}\Sigma^1_1$ formula, $\psi(x) := \boldsymbol{h}\varphi(x)$, and $\boldsymbol{q}\,\mathtt{forc}_{\vec{\varrho}}\,\forall x\,\psi(x)$ **fails**. Then by definition (3° and 2° in Section 18) there is a sequence $\vec{\varrho}\,' \in \overrightarrow{\mathbf{MF}}$, a multitree $\boldsymbol{q}' \in \mathbf{MT}(\vec{\varrho}\,')$, and a small real name $\mathbf{d}$, such that $\vec{\varrho} \subset \vec{\varrho}\,'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$, and $\boldsymbol{q}'\,\mathtt{forc}_{\vec{\varrho}\,'}\,\psi^-(\mathbf{d})$. The sequence $\vec{\pi}\,' = \boldsymbol{h}^{-1}\vec{\varrho}\,'$ then satisfies $\vec{\pi} \subset \vec{\varrho}$, the multitree $\boldsymbol{p}' = \boldsymbol{h}^{-1}\boldsymbol{q}'$ belongs to $\mathbf{MT}(\vec{\pi}\,')$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$, and $\mathbf{c} = \boldsymbol{h}^{-1}\mathbf{d}$ is a small real name. However we cannot claim immediately that $\boldsymbol{p}'\,\mathtt{forc}_{\vec{\pi}\,'}\,\varphi^-(\mathbf{c})$, since the existence of $\mathfrak{M}$ and ϑ as in 1° in Section 18 is not necessarily preserved by the action of $\boldsymbol{h}^{-1}$ or $\boldsymbol{h}$.

To circumvent this difficulty, we make use of two lemmas above. Let $\mathfrak{M} \models \mathbf{ZFL}^-$ be a CTM containing $\vec{\pi}\,', \vec{\varrho}\,', \boldsymbol{h}, \mathbf{c}, \mathbf{d}$ and (all names in) φ, ψ. Then $\boldsymbol{q}'$ $\mathbf{MT}(\vec{\varrho}\,')$-forces $\psi^-(\mathbf{d})[\underline{G}]$ over $\mathfrak{M}$ by Lemma 18.3. Then $\boldsymbol{p}'$ $\mathbf{MT}(\vec{\pi}\,')$-forces $\varphi^-(\mathbf{c})[\underline{G}]$ over $\mathfrak{M}$, by the standard theorems of forcing. Lemma 18.4 yields a sequence $\vec{\tau} \in \overrightarrow{\mathbf{MF}}$ with $\vec{\pi}\,' \subset \vec{\tau}$, such that $\boldsymbol{p}'\,\mathtt{forc}_{\vec{\tau}}\,\varphi^-(\mathbf{c})$, hence $\boldsymbol{p}'\,\mathtt{forc}_{\vec{\tau}}\,\exists x\,\varphi^-(x)$ by 2°. However $\vec{\pi} \subset \vec{\pi}\,' \subset \vec{\tau}$ and $\boldsymbol{p}' \leqslant \boldsymbol{p}$. Thus $\boldsymbol{p}\,\mathtt{forc}_{\vec{\pi}}\,\forall x\,\varphi(x)$ **fails** by 3°, as required.

Part 2: the step $\mathscr{L}\Pi^1_n \to \mathscr{L}\Sigma^1_{n+1}$, $2 \leq n$. Let $\varphi(x)$ be a formula in $\mathscr{L}\Pi^1_n$ and $\psi(x) := \boldsymbol{h}\varphi(x)$. Assume that $\boldsymbol{p}\,\mathtt{forc}_{\vec{\pi}}\,\exists x\,\varphi(x)$. By definition (2° in Section 18), there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\,\mathtt{forc}_{\vec{\pi}}\,\varphi(\mathbf{c})$. Then we have $\boldsymbol{q}\,\mathtt{forc}_{\vec{\varrho}}\,\psi(\mathbf{d})$ by inductive assumption, where $\mathbf{d} = \boldsymbol{h}\mathbf{c}$ is a small real name itself. Thus $\boldsymbol{q}\,\mathtt{forc}_{\vec{\varrho}}\,\exists x\,\psi(x)$.

Part 3: the step $\mathscr{L}\Sigma^1_n \to \mathscr{L}\Pi^1_n$, $n \geq 3$. Let φ be a formula in $\mathscr{L}\Pi^1_n$, and $\boldsymbol{q}\,\mathtt{forc}_{\vec{\varrho}}\,\psi$ **fails**, where $\boldsymbol{q} = \boldsymbol{hp}$, $\vec{\varrho} = \boldsymbol{h}\vec{\pi}$, and ψ is $\boldsymbol{h}\varphi$, as above. By 3°, there is a sequence $\vec{\varrho}\,' \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{q}' \in \mathbf{MT}(\vec{\varrho}\,')$ such that $\vec{\varrho} \subseteq \vec{\varrho}\,'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$, and $\boldsymbol{q}'\,\mathtt{forc}_{\vec{\varrho}\,'}\,\psi^-$. Now let $\boldsymbol{p}' = \boldsymbol{h}^{-1}\boldsymbol{q}'$ and $\vec{\pi}\,' = \boldsymbol{h}^{-1}\vec{\varrho}\,'$, so that $\boldsymbol{p}' \leqslant \boldsymbol{p}$ and $\vec{\pi} \subseteq \vec{\pi}\,'$. We have $\boldsymbol{p}'\,\mathtt{forc}_{\vec{\pi}\,'}\,\varphi^-$ by inductive assumption. We conclude that $\boldsymbol{p}\,\mathtt{forc}_{\vec{\pi}}\,\varphi$ **fails**, as required. □

22. Forcing inside the key sequence

Theorem 22.3 below shows that the forcing relation $\mathtt{forc}_{\vec{\pi}}$, considered with countable initial segments $\vec{\pi} = \vec{\mathbb{n}}\restriction\alpha$ of the key sequence $\vec{\mathbb{n}}$, coincides with the true $\mathbb{P}$-forcing relation (see Definition 14.1) up to the level $\mathscr{L}\Sigma^1_{\mathfrak{n}-1}$.

We argue in $\mathbf{L}$. Recall that the key sequence $\vec{\mathbb{n}} = \langle \mathbb{n}_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$, satisfying $|\bigcup^{\mathtt{cw}} \vec{\mathbb{n}}| = \omega_1$ and (i), (ii), (iii) of Theorem 13.4, was introduced by Definition 13.6, and $\mathbb{P} = \mathbf{MT}(\vec{\mathbb{n}})$ is our forcing notion. In addition, $\mathfrak{n} \geq 4$ by the conditions of Theorem 17.1.

Definition 22.1. We write $\boldsymbol{p}\,\mathtt{forc}_\alpha\,\varphi$ instead of $\boldsymbol{p}\,\mathtt{forc}_{\vec{\mathbb{n}}\restriction\alpha}\,\varphi$, for the sake of brevity. Let $\boldsymbol{p}\,\mathtt{forc}_\infty\,\varphi$ mean: $\boldsymbol{p}\,\mathtt{forc}_\alpha\,\varphi$ for some $\alpha < \omega_1$. □

The following technical lemma contains some useful results.

Lemma 22.2 *(in* $\mathbf{L}$*). Assume that* $\boldsymbol{p} \in \mathbb{P}$*,* $\alpha < \omega_1$*, and* $\boldsymbol{p}\,\mathtt{forc}_\alpha\,\varphi$*. Then:*

(i) *if* $\alpha \leq \beta < \omega_1$*,* $\boldsymbol{q} \in \mathbb{P}_{<\beta} = \mathbf{MT}(\vec{\mathbb{n}}\restriction\beta)$*, and* $\boldsymbol{q} \leqslant \boldsymbol{p}$*, then* $\boldsymbol{q}\,\mathtt{forc}_\beta\,\varphi$*;*
(ii) *if* $\boldsymbol{q} \in \mathbb{P}$*,* $\boldsymbol{q} \leqslant \boldsymbol{p}$*, then* $\boldsymbol{q}\,\mathtt{forc}_\beta\,\varphi$ *for some* β*;* $\alpha \leq \beta < \omega_1$*;*
(iii) *if* $\boldsymbol{q} \in \mathbb{P}$ *and* $\boldsymbol{q}\,\mathtt{forc}_\infty\,\varphi^-$ *then* $\boldsymbol{p} \perp \boldsymbol{q}$ *in the sense of Definition 4.1;*
(iv) *therefore, 1st, if* $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}$*,* $\boldsymbol{q} \leqslant \boldsymbol{p}$*, and* $\boldsymbol{p}\,\mathtt{forc}_\infty\,\varphi$ *then* $\boldsymbol{q}\,\mathtt{forc}_\infty\,\varphi$*, and 2nd,* $\boldsymbol{p}\,\mathtt{forc}_\infty\,\varphi$*,* $\boldsymbol{p}\,\mathtt{forc}_\infty\,\varphi^-$ *cannot hold together.*

Proof. To prove (i) apply Lemma 18.2. To prove (ii) pick β such that $\alpha < \beta < \omega_1$ and $\boldsymbol{q} \in \mathbf{MT}(\vec{\mathbb{n}} \restriction \beta)$, and apply (i). To prove (iii) note that $\boldsymbol{p}, \boldsymbol{q}$ are incompatible in $\mathbb{P}$, as otherwise (i) leads to contradiction, but the incompatibility in $\mathbb{P}$ implies $\perp$ by Corollary 4.2. □

Theorem 22.3. *If φ is a closed $\mathscr{L}$-formula in $\mathscr{L}(\Sigma\Pi)^1_1 \cup \mathscr{L}\Sigma^1_2 \cup \mathscr{L}\Pi^1_2 \cup \ldots \cup \mathscr{L}\Sigma^1_{\mathbb{n}-2} \cup \mathscr{L}\Pi^1_{\mathbb{n}-2} \cup \mathscr{L}\Sigma^1_{\mathbb{n}-1}$ and $\boldsymbol{p} \in \mathbb{P}$, then $\boldsymbol{p}$ $\mathbb{P}$-forces $\varphi[\underline{G}]$ over $\mathbf{L}$ in the usual sense, if and only if $\boldsymbol{p}\, \mathtt{forc}_\infty\, \varphi$.*

Proof. Let $\Vdash$ denote the usual $\mathbb{P}$-forcing relation over $\mathbf{L}$.

Part 1: φ is a formula in $\mathscr{L}(\Sigma\Pi)^1_1$. If $\boldsymbol{p}\, \mathtt{forc}_\infty\, \varphi$ then $\boldsymbol{p}\, \mathtt{forc}_{\vec{\mathbb{n}}\restriction\gamma}\, \varphi$ for some $\gamma < \omega_1$, and then $\boldsymbol{p} \Vdash \varphi[\underline{G}]$ by Lemma 18.3 with $\vec{\boldsymbol{\varphi}} = \vec{\mathbb{n}}$ and $\mathfrak{N} = \mathbf{L}$.

Suppose now that $\boldsymbol{p} \Vdash \varphi[\underline{G}]$. There is an ordinal $\gamma_0 < \omega_1$ such that $\boldsymbol{p} \in \mathbb{P}_{\gamma_0} = \mathbf{MT}(\vec{\mathbb{n}} \restriction \gamma_0)$ and φ belongs to $\mathfrak{L}(\vec{\mathbb{n}} \restriction \gamma_0)$. (Recall Definition 13.2 on models $\mathfrak{L}(x) \models \mathbf{ZFL}^-$.) The set U of all sequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ such that $\gamma_0 < \mathtt{dom}\, \vec{\boldsymbol{\pi}}$ and there is an ordinal ϑ, $\gamma_0 < \vartheta < \mathtt{dom}\, \vec{\boldsymbol{\pi}}$, such that $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)} \vec{\boldsymbol{\pi}}$, is dense in $\overrightarrow{\mathbf{MF}}$ by Theorem 12.3, and is $\boldsymbol{\Delta}_1(\mathrm{HC})$. Therefore by Corollary 13.7 there is an ordinal $\gamma < \omega_1$ such that $\vec{\boldsymbol{\pi}} = \vec{\mathbb{n}} \restriction \gamma \in U$. Let this be witnessed by an ordinal ϑ, so that $\gamma_0 < \vartheta < \gamma = \mathtt{dom}\, \vec{\boldsymbol{\pi}}$ and $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)} \vec{\boldsymbol{\pi}}$. We claim that $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{L}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$ in the usual sense — then by definition $\boldsymbol{p}\, \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\, \varphi$, as required.

To prove the claim, assume otherwise. Then there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\mathbb{n}} \restriction \vartheta)$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, which $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-forces $\neg\, \varphi[\underline{G}]$ over $\mathfrak{L}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$. Then by definition (1° in Section 18) $\boldsymbol{q}\, \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\, \neg\, \varphi$ holds, hence $\boldsymbol{q}\, \mathtt{forc}_\infty\, \neg\, \varphi$, and then $\boldsymbol{q} \Vdash \neg\, \varphi[\underline{G}]$ (see above), with a contradiction to $\boldsymbol{p} \Vdash \varphi[\underline{G}]$.

Part 2: the step $\mathscr{L}\Pi^1_m \to \mathscr{L}\Sigma^1_{m+1}$ $(1 \le m \le \mathbb{n} - 2)$. Consider a $\mathscr{L}\Pi^1_m$ formula $\varphi(x)$. Assume $\boldsymbol{p}\, \mathtt{forc}_\infty\, \exists\, x\, \varphi(x)$. By definition there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\, \mathtt{forc}_\infty\, \varphi(\mathbf{c})$. By inductive hypothesis, $\boldsymbol{p} \Vdash \varphi(c)[\underline{G}]$, that is, $\boldsymbol{p} \Vdash \exists\, x\, \varphi(x)[\underline{G}]$. Conversely, assume that $\boldsymbol{p} \Vdash \exists\, x\, \varphi(x)[\underline{G}]$. As $\mathbb{P}$ is CCC, there is a small real name $\mathbf{c}$ (in $\mathbf{L}$) such that $\boldsymbol{p} \Vdash \varphi(\mathbf{c})[\underline{G}]$. We have $\boldsymbol{p}\, \mathtt{forc}_\infty\, \varphi(\mathbf{c})$ by the inductive hypothesis, hence $\boldsymbol{p}\, \mathtt{forc}_\infty\, \exists\, x\, \varphi(x)$.

Part 3: the step $\mathscr{L}\Sigma^1_m \to \mathscr{L}\Pi^1_m$ $(2 \le m \le \mathbb{n} - 2)$. Assume that φ is a closed $\mathscr{L}\Sigma^1_m$ formula, and $\boldsymbol{p}\, \mathtt{forc}_\infty\, \varphi^-$. By Lemma 22.2(iv), there is no multitree $\boldsymbol{q} \in \mathbb{P}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, with $\boldsymbol{q}\, \mathtt{forc}_\infty\, \varphi$. This implies $\boldsymbol{p} \Vdash \varphi^-$ by the inductive hypothesis.

Conversely, let $\boldsymbol{p} \Vdash \varphi^-$. There is an ordinal $\gamma_0 < \omega_1$ such that $\boldsymbol{p} \in \mathbb{P}_{\gamma_0} = \mathbf{MT}(\vec{\mathbb{n}} \restriction \gamma_0)$ and φ belongs to $\mathfrak{L}(\vec{\mathbb{n}} \restriction \gamma_0)$. Consider the set U of all sequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ such that $\mathtt{dom}\, \vec{\boldsymbol{\pi}} > \gamma_0$ and there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ satisfying $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\boldsymbol{q}\, \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\, \varphi$. Then U belongs to $\boldsymbol{\Sigma}_{m-1}(\mathrm{HC})$ (φ, $\boldsymbol{p}_0$ as parameters) by Lemma 19.1, hence to $\boldsymbol{\Sigma}_{\mathbb{n}-3}(\mathrm{HC})$. Recall that $\mathbb{n} \ge 4$ by the conditions of Theorem 17.1. Thus by Definition 13.6 (and (ii) of Theorem 13.4) there is an ordinal $\gamma < \omega_1$ such that $\vec{\mathbb{n}} \restriction \gamma$ blocks U.

Case 1: $\vec{\mathbb{n}} \restriction \gamma \in U$. Let this be witnessed by a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, so that in particular $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\gamma > \gamma_0$. Thus $\boldsymbol{q} \in \mathbf{MT}(\vec{\mathbb{n}} \restriction \gamma)$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, and $\boldsymbol{q}\, \mathtt{forc}_{\vec{\mathbb{n}}\restriction\gamma}\, \varphi$, that is, $\boldsymbol{q} \Vdash \varphi[\underline{G}]$ by the inductive hypothesis, contrary to the choice of $\boldsymbol{p}$. Therefore Case 1 cannot happen, and we have:

Case 2: no sequence in U extends $\vec{\mathbb{n}} \restriction \gamma$. We can assume that $\gamma > \gamma_0$. (If not, replace γ by $\gamma_0 + 1$.) We claim that $\boldsymbol{p}\, \mathtt{forc}_\gamma\, \varphi^-$. Indeed otherwise by 3° of Section 18 there is a sequence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, such that $\vec{\mathbb{n}} \restriction \gamma \subseteq \vec{\boldsymbol{\pi}}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$, and $\boldsymbol{q}\, \mathtt{forc}_{\vec{\boldsymbol{\varphi}}}\, \varphi$. But then $\vec{\boldsymbol{\pi}}$ belongs to U. On the other hand, $\vec{\mathbb{n}} \restriction \gamma \subseteq \vec{\boldsymbol{\pi}}$, contrary to the Case 2 assumption. Thus indeed $\boldsymbol{p}\, \mathtt{forc}_\infty\, \varphi^-$, as required. □

23. Embedding multiforcings in the key sequence

We argue in L. The following lemma proves that any special multiforcing in **spMF** admits an embedding into a layer of the key sequence $\vec{\mathbb{n}}$, by means of an appropriate permutation, due to the generic properties of $\vec{\mathbb{n}}$.

We make use of the *semiinterval notation* $[\alpha, \beta) = \{\gamma : \alpha \le \gamma < \beta\}$.

If $\beta < \vartheta < \omega_1$ then define permutations $\boldsymbol{h}_1[\vartheta], \boldsymbol{h}_2[\vartheta] \in \text{PERM}$ so that

$$\left.\begin{array}{c} \mathbf{NI}(\boldsymbol{h}_1[\vartheta]) = [\beta, \vartheta) \cup [\vartheta + \beta, \vartheta{\cdot}2), \;\; \mathbf{NI}(\boldsymbol{h}_2[\vartheta]) = [\beta, \vartheta) \cup [\vartheta{\cdot}2 + \beta, \vartheta{\cdot}3), \\ \left.\begin{array}{rcl} \boldsymbol{h}_1[\vartheta](\xi) = \boldsymbol{h}_1[\vartheta]^{-1}(\xi) &=& \vartheta + \xi \\ \boldsymbol{h}_2[\vartheta](\xi) = \boldsymbol{h}_2[\vartheta]^{-1}(\xi) &=& \vartheta{\cdot}2 + \xi \end{array}\right\} \text{ whenever } \beta \le \xi < \vartheta, \end{array}\right\} \qquad (*)$$

where as usual, $\vartheta{\cdot}2 = \vartheta + \vartheta$ and $\vartheta{\cdot}3 = \vartheta + \vartheta + \vartheta$.

Note that the inclusion $\boldsymbol{\varrho} \subseteq \boldsymbol{\pi}$ between multiforcings $\boldsymbol{\varrho}$ and $\boldsymbol{\pi}$ means simply that $d = |\boldsymbol{\varrho}| \subseteq |\boldsymbol{\pi}|$ and $\boldsymbol{\varrho} = \boldsymbol{\pi} \restriction d$, that is, $\boldsymbol{\varrho}(\xi) = \boldsymbol{\pi}(\xi)$ for all $\xi \in d$.

The next lemma has two applications below. One of them (Lemma 24.1) utilizes the effect of both $\boldsymbol{h}_1[\vartheta]$ and $\boldsymbol{h}_2[\vartheta]$. The other one (Section 27) involves only $\boldsymbol{h}_1[\theta]$.

Lemma 23.1 *(in* $\mathbf{L}$*). Assume that* $\alpha_0 < \omega_1$*, and* $\boldsymbol{\sigma} \in \mathbf{spMF}$*,* $\boldsymbol{\sigma} \restriction \beta = \vec{\mathbb{n}}(\alpha_0) \restriction \beta$*. Then there is a refinement* $\boldsymbol{\varrho} \in \mathbf{spMF}$*,* $\boldsymbol{\sigma} \sqsubset \boldsymbol{\varrho}$*, and ordinals* $\nu > \alpha_0$ *and* θ*,* $\beta < \theta < \omega_1$*, such that* $|\boldsymbol{\varrho}| \subseteq \theta$*, still* $\boldsymbol{\varrho} \restriction \beta = \vec{\mathbb{n}}(\nu) \restriction \beta$*, and the multiforcings* $\boldsymbol{\varrho}_1 = \boldsymbol{h}_1[\theta] \cdot \boldsymbol{\varrho}$*,* $\boldsymbol{\varrho}_2 = \boldsymbol{h}_2[\theta] \cdot \boldsymbol{\varrho}$ *satisfy* $\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2 \subseteq \vec{\mathbb{n}}(\nu)$*.*

Proof. Arguing in $\mathbf{L}$, let U be the set of all sequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ such that:

(†) $\mathtt{dom}(\vec{\boldsymbol{\pi}}) = \nu + 1$, where $\nu > \alpha_0$, and there is an ordinal θ, $\beta < \theta < \omega_1$, and a multiforcing $\boldsymbol{\varrho} \in \mathbf{spMF}$ such that $\boldsymbol{\sigma} \sqsubset \boldsymbol{\varrho}$, $|\boldsymbol{\varrho}| \subseteq [0, \theta)$, $\boldsymbol{\varrho} \restriction \beta = \vec{\mathbb{n}}(\nu) \restriction \beta$, and the shifted multiforcings $\boldsymbol{\varrho}_1 = \boldsymbol{h}_1[\theta] \cdot \boldsymbol{\varrho}$, $\boldsymbol{\varrho}_2 = \boldsymbol{h}_2[\theta] \cdot \boldsymbol{\varrho}$ satisfy $\boldsymbol{\varrho}_1, \boldsymbol{\varrho}_2 \subseteq \vec{\boldsymbol{\pi}}(\nu)$.

By routine estimation, U is a $\mathbf{\Sigma}_1(\text{HC})$ set (with $\boldsymbol{\sigma}, \alpha_0, \beta$ as the only parameters of the straightforward Σ_1 definition in HC), hence definitely a $\mathbf{\Sigma}_{\mathbb{n}-3}(\text{HC})$ set. Thus by (ii) of Definition 13.6 there is an ordinal $\nu < \omega_1$ such that $\vec{\mathbb{n}} \restriction \nu$ blocks U.

We can w.l.o.g. assume that $\nu = \lambda + 1 > \alpha_0$ is a successor ordinal.

Case 1: no sequence in U extends $\vec{\mathbb{n}} \restriction \nu$. To show that this cannot happen, let $\boldsymbol{\tau} = \vec{\mathbb{n}}(\lambda)$. Let $\boldsymbol{\tau}' \in \mathbf{spMF}$ satisfy $\boldsymbol{\tau} \sqsubset \boldsymbol{\tau}'$ and $|\boldsymbol{\tau}| = |\boldsymbol{\tau}'|$. Let $\boldsymbol{\sigma}' \in \mathbf{spMF}$ satisfy $\boldsymbol{\sigma} \sqsubset \boldsymbol{\sigma}'$ and $|\boldsymbol{\sigma}| = |\boldsymbol{\sigma}'|$. We put $\boldsymbol{\varrho} = (\boldsymbol{\tau}' \restriction \beta) \cup (\boldsymbol{\sigma}' \restriction (\omega_1 \smallsetminus \beta))$. Let $\theta < \omega_1$ be the least ordinal satisfying $|\boldsymbol{\tau}| \cup |\boldsymbol{\sigma}| \subseteq [0, \theta)$ and $\beta < \theta$. We define $\boldsymbol{\varrho}_1 = \boldsymbol{h}_1[\theta] \cdot \boldsymbol{\varrho}$, $\boldsymbol{\varrho}_2 = \boldsymbol{h}_2[\theta] \cdot \boldsymbol{\varrho}$, $\boldsymbol{\pi} = \boldsymbol{\tau}' \cup \boldsymbol{\varrho}_1 \cup \boldsymbol{\varrho}_2$.

We claim that $\boldsymbol{\pi}$ is a special multiforcing. Indeed as all three summands are special multiforcings, it suffices, by Lemma 4.3, to check that they pairwise coincide on common domains. Note that by construction $|\boldsymbol{\varrho}| \subseteq \beta = [0, \beta)$, and hence by $(*)$ we have $|\boldsymbol{\varrho}| \cap |\boldsymbol{\varrho}_1| \cap |\boldsymbol{\varrho}_2| \subseteq [0, \beta)$ whereas outside the interval $[0, \beta)$ all three domains $|\boldsymbol{\varrho}|, |\boldsymbol{\varrho}_1|, |\boldsymbol{\varrho}_2|$ are pairwise disjoint. Furthermore $\boldsymbol{\varrho}_1 \restriction [0, \beta) = \boldsymbol{\varrho}_2 \restriction [0, \beta) = \boldsymbol{\varrho} \restriction [0, \beta)$ still by $(*)$, so we have got the pairwise coincidence on common domains, as required.

Moreover, as $\vec{\mathbb{n}}(\lambda) = \boldsymbol{\tau} \sqsubset \boldsymbol{\tau}' \subseteq \boldsymbol{\pi}$, we have $\boldsymbol{\tau} \sqsubset \boldsymbol{\pi}$ by Lemma 5.4.

Now let $\vec{\boldsymbol{\pi}}$ be the extension of $\vec{\mathbb{n}} \restriction \nu$ by $\vec{\boldsymbol{\pi}}(\nu) = \boldsymbol{\pi}$. It follows from the above that $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}_{\nu+1}$. We claim that $\vec{\boldsymbol{\pi}} \in U$ via the condition (†) witnessed by $\boldsymbol{\varrho}$ and θ. Indeed the only part of (†) not immediately clear is $\boldsymbol{\sigma} \sqsubset \boldsymbol{\varrho}$. But this follows from the definition of $\boldsymbol{\varrho}$ since $\boldsymbol{\sigma} \sqsubset \boldsymbol{\sigma}'$ and $\boldsymbol{\sigma} \restriction \beta = \vec{\mathbb{n}}(\alpha_0) \restriction \beta \sqsubset \boldsymbol{\tau} = \vec{\mathbb{n}}(\lambda) \sqsubset \boldsymbol{\tau}'$. Thus indeed $\vec{\boldsymbol{\pi}} \in U$.

On the other hand, $\vec{\mathbb{n}} \restriction \nu \subset \vec{\boldsymbol{\pi}}$. This contradicts the Case 1 assumption.

Case 2: $\vec{\boldsymbol{\pi}} = \vec{\mathbb{n}} \restriction \nu \in U$. Let this be witnessed by $\theta, \boldsymbol{\varrho}$ *etc.* as in (†). Then $\boldsymbol{\varrho}_1 \cup \boldsymbol{\varrho}_2 \subseteq \vec{\mathbb{n}}(\nu)$ immediately by (†). □

24. The non-existence claim, part I

The next lemma continues **the proof of Theorem 17.1**.

Lemma 24.1 *(in* $\mathbf{L}$*). Suppose that* $\alpha_0 < \omega_1$, $\boldsymbol{\sigma} \in \mathbf{spMF}$, $\boldsymbol{\sigma}\restriction\mathbb{b} = \vec{\mathbb{\pi}}(\alpha_0)\restriction\mathbb{b}$, $\boldsymbol{s} \in \mathbf{MT}(\boldsymbol{\sigma})$, $\boldsymbol{s} \leqslant \mathbb{p}$, *and* $\xi \in |\boldsymbol{\sigma}|$, $\xi \geq \mathbb{b}$. *Then there is a refinement* $\boldsymbol{\pi} \in \mathbf{spMF}$, $\boldsymbol{\sigma} \sqsubset \boldsymbol{\pi}$, *a condition* $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{p} \leqslant \boldsymbol{s}$, *and ordinals* $\mu > \alpha_0$ *and* $\eta \in |\boldsymbol{\pi}|$, $\eta \geq \mathbb{b}$, *such that still* $\boldsymbol{\pi}\restriction\mathbb{b} = \vec{\mathbb{\pi}}(\mu)\restriction\mathbb{b}$ *and* $\boldsymbol{p}\;\mathtt{forc}_{\boldsymbol{\pi}}\;\mathbb{\Phi}(\dot{\boldsymbol{x}}_\eta, \dot{\boldsymbol{x}}_\xi, \mathbb{c})$.

See Section 17 on the definition of sets $\mathbb{p}, \mathbb{c}, \mathbb{b}, \mathbb{a}$ occurring in this lemma. A club $\mathbb{C} \in \mathbf{L}$, $\mathbb{C} \subseteq \omega_1$ is fixed by (iii) of Theorem 13.4 and Definition 13.6.

Proof. We argue in $\mathbf{L}$. By Lemma 23.1, there exist:

(1) ordinals $\nu > \alpha_0$ and $\mathbb{\theta}$, $\mathbb{b} < \mathbb{\theta} < \omega_1$, and a refinement $\boldsymbol{\wp} \in \mathbf{spMF}$, $\boldsymbol{\sigma} \sqsubset \boldsymbol{\wp}$, such that $|\boldsymbol{\wp}| \subseteq \mathbb{\theta}$, still $\vec{\boldsymbol{\wp}}\restriction\mathbb{b} = \vec{\mathbb{\pi}}(\nu)\restriction\mathbb{b}$, and the derived multiforcings $\boldsymbol{\wp}_1 = \boldsymbol{h}_1[\mathbb{\theta}]\cdot\boldsymbol{\wp}$, $\boldsymbol{\wp}_2 = \boldsymbol{h}_2[\mathbb{\theta}]\cdot\boldsymbol{\wp}$ satisfy $\boldsymbol{\wp}_1, \boldsymbol{\wp}_2 \subseteq \vec{\mathbb{\pi}}(\nu)$.

Let $\xi_1 = \boldsymbol{h}_1[\mathbb{\theta}](\xi)$, $\xi_2 = \boldsymbol{h}_2[\mathbb{\theta}](\xi)$. Note that $\mathbb{b} \leq \xi < \mathbb{\theta}$ by construction, and hence we have $\mathbb{\theta} + \mathbb{b} \leq \xi_1 < \mathbb{\theta}\cdot 2$ and $\mathbb{\theta}\cdot 2 + \mathbb{b} \leq \xi_2 < \mathbb{\theta}\cdot 3$ by $(*)$ in Section 23.

Pick a condition $\boldsymbol{q} \in \boldsymbol{\wp}$ with $\boldsymbol{q} \leqslant \boldsymbol{s}$. Then

$$\boldsymbol{q}_1 = \boldsymbol{h}_1[\mathbb{\theta}]\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\wp}_1) \quad \text{and} \quad \boldsymbol{q}_2 = \boldsymbol{h}_2[\mathbb{\theta}]\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\wp}_2),$$

hence, $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \mathbb{P}$ by (1). (Recall that $\mathbb{P}$ is the key forcing notion, Section 14.) Note that $|\boldsymbol{q}_1| \cap |\boldsymbol{q}_2| \subseteq [0, \mathbb{b})$ by $(*)$ in Section 23. However both $\boldsymbol{h}_1[\mathbb{\theta}]$ and $\boldsymbol{h}_2[\mathbb{\theta}]$ are equal to the identity on the domain $\mathbb{b} = [0, \mathbb{b})$. It follows that $\boldsymbol{q}_1\restriction[0, \mathbb{b}) = \boldsymbol{q}_2\restriction[0, \mathbb{b}) = \boldsymbol{q}\restriction[0, \mathbb{b})$. We conclude that $\boldsymbol{p}' = \boldsymbol{q}_1 \cup \boldsymbol{q}_2$ is a multitree in $\mathbf{MT}(\vec{\mathbb{\pi}}(\nu)) \subseteq \mathbb{P}$ by (1).

It follows by the choice of $\mathbb{p}$, that there is a condition $\boldsymbol{p}_1 \in \mathbb{P}$, $\boldsymbol{p}_1 \leqslant \boldsymbol{p}'$, which $\mathbb{P}$-forces either $\mathbb{\Phi}(x_{\xi_1}[\underline{G}], x_{\xi_2}[\underline{G}], \mathbb{c}[\underline{G}])$ or $\mathbb{\Phi}(x_{\xi_2}[\underline{G}], x_{\xi_1}[\underline{G}], \mathbb{c}[\underline{G}])$. Let $\boldsymbol{p}_1$ $\mathbb{P}$-force say $\mathbb{\Phi}(x_{\xi_2}[\underline{G}], x_{\xi_1}[\underline{G}], \mathbb{c}[\underline{G}])$ over $\mathbf{L}$. By Theorem 22.3, we have

(2) $\boldsymbol{p}_1\;\mathtt{forc}_{\vec{\mathbb{\pi}}\restriction(\mu+1)}\;\mathbb{\Phi}(\dot{\boldsymbol{x}}_{\xi_2}, \dot{\boldsymbol{x}}_{\xi_1}, \mathbb{c})$, or equivalently, by Corollary 20.2,
$\boldsymbol{p}_1\;\mathtt{forc}_{\vec{\mathbb{\pi}}(\mu)}\;\mathbb{\Phi}(\dot{\boldsymbol{x}}_{\xi_2}, \dot{\boldsymbol{x}}_{\xi_1}, \mathbb{c})$, — for some $\mu \geq \nu$,

where, by Lemma 18.2, $\boldsymbol{p}_1$ and the ordinal μ can be chosen so that $\boldsymbol{p}_1 \in \mathbf{MT}(\vec{\mathbb{\pi}}(\mu))$ and $[0, \mathbb{\theta}\cdot 3) \subseteq |\vec{\mathbb{\pi}}(\mu)|$. Acting by $\boldsymbol{h}_1[\mathbb{\theta}] = (\boldsymbol{h}_1[\mathbb{\theta}])^{-1}$ on (2), we get

(3) $\boldsymbol{p}\;\mathtt{forc}_{\boldsymbol{\pi}}\;\mathbb{\Phi}(\dot{\boldsymbol{x}}_{\xi_2}, \dot{\boldsymbol{x}}_{\xi}, \mathbb{c})$

by Theorem 21.1, where $\boldsymbol{\pi} = \boldsymbol{h}_1[\mathbb{\theta}](\vec{\mathbb{\pi}}(\mu))$ and $\boldsymbol{p} = \boldsymbol{h}_1[\mathbb{\theta}]\boldsymbol{p}_1$, since $\boldsymbol{h}_1[\mathbb{\theta}](\xi_1) = (\boldsymbol{h}_1[\mathbb{\theta}])^{-1}(\xi_1) = \xi$, $\boldsymbol{h}_1[\mathbb{\theta}](\xi_2) = \xi_2$, and $\boldsymbol{h}_1[\mathbb{\theta}]\mathbb{c} = \mathbb{c}$ (because $|\mathbb{c}| \subseteq \mathbb{b}$ and $\boldsymbol{h}_1[\mathbb{\theta}]$ is the identity on $\mathbb{b} = [0, \mathbb{b})$).

Now we observe that $\boldsymbol{h}_1[\mathbb{\theta}]\boldsymbol{\wp} = \boldsymbol{\wp}_1 \subseteq \vec{\mathbb{\pi}}(\nu) \sqsubset \vec{\mathbb{\pi}}(\mu)$, therefore $\boldsymbol{h}_1[\mathbb{\theta}]\boldsymbol{\wp} \sqsubset \vec{\mathbb{\pi}}(\mu)$ by Lemma 5.4, and then we have $\boldsymbol{\wp} \sqsubset \boldsymbol{\pi}$ since $\boldsymbol{h}_1[\mathbb{\theta}] = \boldsymbol{h}_1[\mathbb{\theta}]^{-1}$, so that $\boldsymbol{\sigma} \sqsubset \boldsymbol{\pi}$ as well by (1). We have $\boldsymbol{p} \leqslant \boldsymbol{s}$ by similar reasons. And finally, $\boldsymbol{\pi}\restriction\mathbb{b} = \vec{\mathbb{\pi}}(\mu)\restriction\mathbb{b}$ because $\boldsymbol{h}_1[\mathbb{\theta}]$ is the identity on $[0, \mathbb{b})$.

This ends the proof of the lemma, with $\eta = \xi_2$. □

25. The non-existence claim, part II

In this section our goal will be to strengthen Lemma 24.1 so that not only one condition $\boldsymbol{p}$ but a whole dense set of conditions with the same property will be obtained. Recall that $\mathbb{a}, \mathbb{b}, \mathbb{p}, \mathbb{c}$ were introduced in Section 17 whereas a club $\mathbb{C} \in \mathbf{L}$, $\mathbb{C} \subseteq \omega_1$ is fixed by (iii) of Theorem 13.4 and Definition 13.6.

Lemma 25.1 *(in* $\mathbf{L}$*). There is a limit ordinal* $\lambda \in \mathbb{C}$, *a sequence* $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda} \in \overrightarrow{\mathbf{MF}}_\lambda$, *and an increasing continuous sequence of ordinals* ν_α, $\alpha < \lambda$, *such that:*

(i) $\vec{\pi}(\alpha) \subseteq \boldsymbol{\pi}_0$ *and* $|\boldsymbol{\pi}_0| = |\vec{\pi}(\alpha)| \cup \{\beta\}$*;*
(ii) $\boldsymbol{\pi}_\alpha \restriction \beta = \vec{\pi}(\nu_\alpha) \restriction \beta$*;*
(iii) *if* $\xi \in |\vec{\boldsymbol{\pi}}|$, $\xi \geq \beta$, *and* $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, $\boldsymbol{q} \leqslant \mathbb{p}$, *then there is a condition* $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, $\boldsymbol{p} \leqslant \boldsymbol{q}$, *and an ordinal* $\eta \in |\vec{\boldsymbol{\pi}}|$, $\eta \geq \beta$ *such that* $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \Phi(\dot{\boldsymbol{x}}_\eta, \dot{\boldsymbol{x}}_\xi, \mathbb{c})$*, or in other words, the set*

$$D_\xi[\vec{\boldsymbol{\pi}}] = \{\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}}) : \exists\, \eta \in |\vec{\boldsymbol{\pi}}| \smallsetminus \beta\, \big(\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \Phi(\dot{\boldsymbol{x}}_\eta, \dot{\boldsymbol{x}}_\xi, \mathbb{c})\big)\}$$

is dense (then open dense by Lemma 18.2) *in* $\mathbf{MT}(\vec{\boldsymbol{\pi}})$ *below* $\mathbb{p}$
(iv) $\sup_{\alpha<\lambda} \nu_\alpha = \lambda$ *and* $\vec{\boldsymbol{\pi}} \in \mathfrak{L}(\vec{\pi} \restriction \lambda)$*.* (See Definition 13.2 on $\mathfrak{L}(x)$.)

Proof. Following (i), we define $\boldsymbol{\pi}_0$ so that $|\boldsymbol{\pi}_0| = |\vec{\pi}(\alpha)| \cup \{\beta\}$, $\boldsymbol{\pi}_0(\xi) = \vec{\pi}(\alpha)(\xi)$ for all $\xi \in |\vec{\pi}(\alpha)|$, and finally $\boldsymbol{\pi}_0(\beta) = \mathbb{P}_{\text{coh}}$, where $\mathbb{P}_{\text{coh}}$ is the Cohen forcing, as in the proof of Lemma 12.1.

Successor step. Assume that $\boldsymbol{\pi}_\alpha$ and ν_α are already defined and satisfy (ii). Then for a certain specially picked (see below) pair of $\xi_\alpha \in |\boldsymbol{\pi}_\alpha| \smallsetminus \beta$ and $\boldsymbol{q}_\alpha \in \mathbf{MT}(\boldsymbol{\pi}_\alpha)$ with $\boldsymbol{q}_\alpha \leqslant \mathbb{p}$, a multiforcing $\boldsymbol{\pi}_{\alpha+1} \in \mathbf{spMF}$ and an ordinal $\nu_{\alpha+1} > \nu_\alpha$ are chosen by Lemma 24.1 so that $\boldsymbol{\pi}_\alpha \sqsubset \boldsymbol{\pi}_{\alpha+1}$, $\boldsymbol{\pi}_{\alpha+1} \restriction \beta = \vec{\pi}(\nu_{\alpha+1}) \restriction \beta$, and

$$\exists\, \boldsymbol{p}_\alpha \in \mathbf{MT}(\boldsymbol{\pi}_{\alpha+1})\, \exists\, \eta_\alpha \in |\boldsymbol{\pi}_{\alpha+1}| \smallsetminus \beta\, \big(\boldsymbol{p}_\alpha \leqslant \boldsymbol{q}_\alpha \wedge \boldsymbol{p}_\alpha\ \mathtt{forc}_{\boldsymbol{\pi}_{\alpha+1}}\ \Phi(\dot{\boldsymbol{x}}_{\eta_\alpha}, \dot{\boldsymbol{x}}_{\xi_\alpha}, \mathbb{c})\big).$$

Limit step. Assume that $\kappa < \omega_1$ is limit, and $\boldsymbol{\pi}_\alpha$ and ν_α are already defined for all $\alpha < \kappa$, so that $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\kappa} \in \overrightarrow{\mathbf{MF}}_\kappa$ and (ii) holds for all $\alpha < \kappa$. Let $\nu_\kappa = \sup_{\alpha<\kappa} \nu_\alpha$. By Theorem 12.3, there is a sequence $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}}_{\kappa+1}$ with $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\kappa} \subset \vec{\boldsymbol{\varphi}}$. Basically, $\vec{\boldsymbol{\varphi}}$ extends $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\kappa}$ by an extra term $\boldsymbol{\varphi} = \vec{\boldsymbol{\varphi}}(\kappa) \in \mathbf{spMF}$ satisfying $\boldsymbol{\pi}_\alpha \sqsubset \boldsymbol{\varphi}$ for all $\alpha < \kappa$. But $\boldsymbol{\varphi} \restriction \beta$ may not be equal to $\vec{\pi}(\nu_\kappa) \restriction \beta$. To fix this issue, we define $\boldsymbol{\pi}_\kappa$ so that $\boldsymbol{\pi}_\kappa \restriction \beta = \vec{\pi}(\nu_\kappa) \restriction \beta$, but $\boldsymbol{\pi}_\kappa(\xi) = \boldsymbol{\varphi}(\xi)$ for all $\xi \in |\boldsymbol{\varphi}|$, $\xi \geq \beta$. Then still $\boldsymbol{\pi}_\alpha \sqsubset \boldsymbol{\pi}_\kappa$ for all $\alpha < \kappa$ since the sequence $\vec{\pi}$ is $\sqsubset$-increasing.

Choice of ξ_α and $\boldsymbol{q}_\alpha$. This can be arranged so that for each ordinal α' in the process, and every $\xi \in |\boldsymbol{\pi}_{\alpha'}| \smallsetminus \beta$ and every condition $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi}_{\alpha'})$, $\boldsymbol{q} \leqslant \mathbb{p}$, there is $m < \omega$ such that $\xi_\alpha = \xi$ and $\boldsymbol{q}_\alpha \leqslant \boldsymbol{q}$, where $\alpha = \alpha' + m$.

Choice of λ. The construction above is supposed to run in principle over the whole domain $\alpha < \omega_1$. However, as the set $\mathbb{C} \subseteq \omega_1$ is a club, its subset $\mathbb{C}' = \{\lambda \in \mathbb{C} : \nu_\lambda = \lambda\}$ is a club either. Now let's find $\lambda \in \mathbb{C}'$ such that (iv) holds. Consider the $\in$-structure $\langle \mathbf{L}_{\omega_2}; \omega_1, \mathbb{C}', \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\omega_1} \rangle$, a model of $\mathbf{ZFL}^-$ (see Definition 13.2). It has a countable elementary submodel $M \subseteq \mathbf{L}_{\omega_2}$ containing ω_1, $\mathbb{C}'$, and $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\omega_1}$. The latter admits a Mostowski collapse map $h : M \xrightarrow{\text{onto}} \mathbf{L}_\vartheta \models \mathbf{ZFL}^-$, where $\vartheta < \omega_1$. Then $h(\omega_1) = \lambda < \vartheta$, $h(\mathbb{C}') = \mathbb{C}' \cap \lambda$ — and hence $\lambda \in \mathbb{C}'$ because $\mathbb{C}'$ is a club, — and $h(\langle \vec{\boldsymbol{\pi}}_\alpha \rangle_{\alpha<\omega_1}) = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda} \in \mathbf{L}_\vartheta$ by construction. On the other hand, λ is uncountable in $\mathbf{L}_\vartheta$ by the elementarity. It follows that surely $\mathbf{L}_\vartheta \subseteq \mathfrak{L}(\vec{\pi} \restriction \lambda)$, because by definition λ is countable in the model $\mathfrak{L}(\vec{\pi} \restriction \lambda) = \mathbf{L}_\mu$ for some $\mu < \omega_1$. We conclude that $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda} \in \mathfrak{L}(\vec{\pi} \restriction \lambda)$, as required.

Finalization. It remains to check (iii) with $\lambda \in \mathbb{C}$ chosen as above. Thus assume that $\xi \in |\vec{\boldsymbol{\pi}}| \smallsetminus \beta$, and $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, $\boldsymbol{q} \leqslant \mathbb{p}$. There exists an ordinal $\alpha' < \lambda$ such that $\xi \in |\boldsymbol{\pi}_{\alpha'}| \smallsetminus \beta$ and a condition $\boldsymbol{q}' \in \mathbf{MT}(\boldsymbol{\pi}_{\alpha'})$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$. By construction, there is an ordinal $\alpha = \alpha' + m$, $m < \omega$, such that $\xi_\alpha = \xi$ and $\boldsymbol{q}_\alpha \leqslant \boldsymbol{q}'$. Moreover (see the successor step) there exists an ordinal $\eta \in |\boldsymbol{\pi}_{\alpha+1}| \smallsetminus \beta$ and a condition $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}_{\alpha+1})$ such that $\boldsymbol{p} \leqslant \boldsymbol{q}_\alpha \leqslant \boldsymbol{q}' \leqslant \boldsymbol{q}$ and $\boldsymbol{p}\ \mathtt{forc}_{\boldsymbol{\pi}_{\alpha+1}}\ \Phi(\dot{\boldsymbol{x}}_\eta, \dot{\boldsymbol{x}}_\xi, \mathbb{c})$. By Theorem 20.1 this is equivalent to $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}} \restriction \alpha+2}\ \Phi(\dot{\boldsymbol{x}}_\eta, \dot{\boldsymbol{x}}_\xi, \mathbb{c})$. This implies $\boldsymbol{p}\ \mathtt{forc}_{\vec{\boldsymbol{\pi}}}\ \Phi(\dot{\boldsymbol{x}}_\eta, \dot{\boldsymbol{x}}_\xi, \mathbb{c})$ by Lemma 18.2. □

Let a *25.1-type sequence* be any $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ satisfying Lemma 25.1. Thus if $\xi \in |\vec{\boldsymbol{\pi}}| \smallsetminus \beta$ then $D_\xi[\vec{\boldsymbol{\pi}}]$ is open dense in $\mathbf{MT}(\vec{\boldsymbol{\pi}})$ below $\mathbb{p}$ in this case.

26. The non-existence claim, part III

The next theorem makes use of the results of Section 25.

Theorem 26.1 *(in* $\mathbf{L}$*). Assume that* $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ *is a 25.1-type sequence,* $\lambda = \mathtt{dom}(\vec{\boldsymbol{\pi}})$*. Then there is an extension* $\vec{\boldsymbol{\varrho}} \in \overrightarrow{\mathbf{MF}}_{\lambda+1}$*,* $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varrho}}$*, by an extra term* $\boldsymbol{\varrho} = \vec{\boldsymbol{\varrho}}(\lambda)$*, such that* $\boldsymbol{\varrho}\restriction\mathbb{\beta} = \vec{\Pi}(\lambda)\restriction\mathbb{\beta}$*, and if* $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{MF}}$ *and* $\vec{\boldsymbol{\varrho}} \subseteq \vec{\boldsymbol{\tau}}$ *then every set* $D_\xi[\vec{\boldsymbol{\pi}}]$*,* $\xi \in |\vec{\boldsymbol{\pi}}| \smallsetminus \mathbb{\beta}$*, is pre-dense in* $\mathbf{MT}(\vec{\boldsymbol{\tau}})$ *below* $\mathbb{p}$*.*

Therefore, if $\boldsymbol{\tau} \in \mathbf{spMF}$ *satisfies* $\boldsymbol{\varrho} \sqsubset \boldsymbol{\tau}$*, and* $\xi \in |\vec{\boldsymbol{\pi}}| \smallsetminus \mathbb{\beta}$*, then the set*

$$D_\xi[\boldsymbol{\tau}] = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\tau}) : \exists\, \eta \in |\vec{\boldsymbol{\pi}}| \smallsetminus \mathbb{\beta}\, \big(\boldsymbol{p} \,\mathtt{forc}_{\boldsymbol{\tau}}\, \Phi(\dot{\boldsymbol{x}}_\eta, \dot{\boldsymbol{x}}_\xi, \mathbb{c})\big)\}$$

is dense in $\mathbf{MT}(\boldsymbol{\tau})$ *below* $\mathbb{p}$*.*

Proof. We argue in $\mathbf{L}$. We know that $\vec{\boldsymbol{\pi}} \in \mathfrak{M} = \mathfrak{L}(\vec{\Pi}\restriction\lambda) \models \mathbf{ZFL}^-$. In particular, the multiforcing $\boldsymbol{\pi} = \bigcup^{\mathtt{cw}}_{\alpha<\lambda} \vec{\boldsymbol{\pi}}(\alpha)$ also belongs to $\mathfrak{M}$.

Our plan is to add a λ-th layer $\boldsymbol{\varrho}$ to $\vec{\boldsymbol{\pi}}$ so that the extended sequence $\vec{\boldsymbol{\varrho}} = \vec{\boldsymbol{\pi}}^\frown\boldsymbol{\varrho} \in \overrightarrow{\mathbf{MF}}_{\lambda+1}$ satisfies $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varrho} = \vec{\boldsymbol{\varrho}}(\lambda)$ for any set $\boldsymbol{D} \in \mathfrak{M}$, $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$, open dense in $\mathbf{MT}(\boldsymbol{\pi})$, and still satisfies $\boldsymbol{\varrho}\restriction\mathbb{\beta} = \vec{\Pi}(\lambda)\restriction\mathbb{\beta}$. However a direct application of Lemma 9.4 and then Theorem 10.1(vii) does not yield the result required since Lemma 9.4 does not provide $\boldsymbol{\varrho}\restriction\mathbb{\beta} = \vec{\Pi}(\lambda)\restriction\mathbb{\beta}$. Thus we need a more elaborate construction of $\boldsymbol{\varrho}$. Basically, as the value of $\boldsymbol{\pi}\restriction\mathbb{\beta} = \vec{\Pi}(\lambda)\restriction\mathbb{\beta}$ is predetermined, we'll have to define the complementary part $\boldsymbol{\varrho}\restriction[\mathbb{\beta}, \omega_1)$.

For that purpose, let's split the given sequence $\vec{\boldsymbol{\pi}}$ at $\mathbb{\beta}$ into sequences $\vec{\boldsymbol{\varepsilon}} = \vec{\boldsymbol{\pi}}_{<\mathbb{\beta}}$ and $\vec{\boldsymbol{\delta}} = \vec{\boldsymbol{\pi}}_{\ge\mathbb{\beta}}$, in $\overrightarrow{\mathbf{MF}}_\lambda \cap \mathfrak{M}$, such that $\mathtt{dom}(\vec{\boldsymbol{\varepsilon}}) = \mathtt{dom}(\vec{\boldsymbol{\delta}}) = \lambda$ and $\vec{\boldsymbol{\varepsilon}}(\alpha) = \vec{\boldsymbol{\pi}}(\alpha)\restriction\mathbb{\beta}$, $\vec{\boldsymbol{\delta}}(\alpha) = \vec{\boldsymbol{\pi}}(\alpha)\restriction[\mathbb{\beta}, \omega_1)$ for all $\alpha < \lambda$. By the way, $\vec{\boldsymbol{\varepsilon}}(\alpha) = \vec{\Pi}(\nu_\alpha)\restriction\mathbb{\beta}$ for all α, where $\nu_\alpha < \lambda$ and $\sup_{\alpha<\lambda} \nu_\alpha = \lambda$ as $\vec{\boldsymbol{\pi}}$ is a 25.1-type sequence. Therefore the multiforcings $\Pi_{<\lambda} = \bigcup^{\mathtt{cw}}_{\alpha<\lambda} \vec{\Pi}(\alpha)$ and $\boldsymbol{\varepsilon} = \bigcup_{\alpha<\lambda} \vec{\boldsymbol{\varepsilon}}(\alpha)$ are connected so that $\boldsymbol{\varepsilon} \subseteq \Pi_{<\lambda}\restriction\mathbb{\beta}$ and $\boldsymbol{\varepsilon}$ is dense in $\Pi_{<\lambda}\restriction\mathbb{\beta}$.

Note that as $\lambda = \mathtt{dom}(\vec{\boldsymbol{\pi}}) \in \mathbb{C}$, the ordinal λ is limit and *crucial for $\vec{\Pi}$. This means that $\vec{\Pi}(\lambda) = \lim[\Phi]$ is an $\mathfrak{M}$-generic refinement of $\Pi_{<\lambda}$ via an $\mathfrak{M}$-generic $\preccurlyeq$-decreasing sequence $\Phi = \langle\varphi_j\rangle_{j<\omega}$ of systems $\varphi_j \in \mathbf{Sys}(\Pi_{<\lambda})$, as in (I)–(V) of Section 9. Here each $\varphi_j \in \mathbf{Sys}(\Pi_{<\lambda})$ is a map defined on a *finite* set $|\varphi_j| \subseteq |\Pi_{<\lambda}| \times \omega$ such that if $\langle\xi, k\rangle \in |\varphi_j|$ then $\varphi_j(\xi, k) = \langle n^{\varphi_j}_{\xi k}, T^{\varphi_j}_{\xi k}\rangle$, where $n^{\varphi_j}_{\xi k} < \omega$, $T^{\varphi_j}_{\xi k} \in \bigcup^{\mathtt{fin}} \Pi_{<\lambda}(\xi)$, and $\bigcup^{\mathtt{fin}} \Pi_{<\lambda}(\xi)$ consists of all finite unions of trees in $\Pi_{<\lambda}(\xi)$, as in Subsection 9A.

To restrict Φ to the domain $\mathbb{\beta} = [0, \mathbb{\beta})$, we let $\Phi\restriction\mathbb{\beta} = \langle\varphi_j\restriction\mathbb{\beta}\rangle_{j<\omega}$, where $\varphi_j\restriction\mathbb{\beta}$ is just the restriction of φ_j to $\{\langle\xi, k\rangle \in |\varphi_j| : \xi < \mathbb{\beta}\}$. Thus $\Phi\restriction\mathbb{\beta}$ is a $\preccurlyeq$-decreasing sequence of systems $\varphi_j\restriction\mathbb{\beta}$ in $\mathbf{Sys}(\Pi_{<\lambda}\restriction\mathbb{\beta})$. Moreover $\Phi\restriction\mathbb{\beta}$ is an $\mathfrak{M}$-generic sequence by the product forcing theory, since Φ itself is generic.

However $\boldsymbol{\varepsilon}$ is dense in $\Pi_{<\lambda}\restriction\mathbb{\beta}$, see above, therefore $\mathbf{Sys}(\boldsymbol{\varepsilon})$ is dense in $\mathbf{Sys}(\Pi_{<\lambda}\restriction\mathbb{\beta})$. It follows that the sequence $\Phi\restriction\mathbb{\beta}$ contains an infinite cofinal subsequence $\Phi' = \langle\varphi_{j_m}\restriction\mathbb{\beta}\rangle_{m<\omega}$ of systems $\varphi_{j_m}\restriction\mathbb{\beta} \in \mathbf{Sys}(\boldsymbol{\varepsilon})$, Φ' is $\mathfrak{M}$-generic along with $\Phi\restriction\mathbb{\beta}$ by the cofinality, and we have $\lim[\Phi'] = \lim[\Phi\restriction\mathbb{\beta}] = \vec{\Pi}(\lambda)\restriction\mathbb{\beta}$ since generally $\vec{\Pi}(\lambda) = \lim[\Phi]$.

Now consider the other half $\vec{\boldsymbol{\delta}} = \vec{\boldsymbol{\pi}}_{\ge\mathbb{\beta}}$ of the given sequence $\vec{\boldsymbol{\pi}}$, and the according multiforcing $\boldsymbol{\delta} = \bigcup^{\mathtt{cw}}_{\alpha<\kappa} \vec{\boldsymbol{\delta}}(\alpha)$, where $\vec{\boldsymbol{\delta}}(\alpha) = \vec{\boldsymbol{\pi}}(\alpha)\restriction[\mathbb{\beta}, \omega_1)$. Let Ψ be an arbitrary $\mathfrak{M}[\Phi']$-generic and $\preccurlyeq$-decreasing sequence of systems $\psi_j \in \mathbf{Sys}(\boldsymbol{\delta})$, as above. Here each $\psi_j \in \mathbf{Sys}(\boldsymbol{\delta})$ is a map defined on a finite set $|\psi_j| \subseteq |\boldsymbol{\delta}| \times \omega$ such that if $\langle\xi, k\rangle \in |\psi_j|$ then $\psi_j(\xi, k) = \langle n^{\psi_j}_{\xi k}, T^{\psi_j}_{\xi k}\rangle$, where $n^{\psi_j}_{\xi k} < \omega$, $T^{\psi_j}_{\xi k} \in \bigcup^{\mathtt{fin}} \boldsymbol{\delta}(\xi)$, and the set $\bigcup^{\mathtt{fin}} \boldsymbol{\delta}(\xi)$ consists of all finite unions of trees in $\boldsymbol{\delta}(\xi)$.

Note that $|\vec{\boldsymbol{\varepsilon}}| \subseteq \mathbb{\beta}$ and $|\vec{\boldsymbol{\delta}}| \subseteq [\mathbb{\beta}, \omega_1)$ are disjoint sets by construction. It follows that if $j < \omega$ then $|\varphi_j\restriction\mathbb{\beta}| \cap |\psi_j| = \varnothing$, therefore the union $\vartheta_j = (\varphi_j\restriction\mathbb{\beta}) \cup \psi_j$ of functions with disjoint domains is a system in $\mathbf{Sys}(\boldsymbol{\pi})$. Moreover $\Theta = \langle\vartheta_j\rangle_{j<\omega}$ is a $\preccurlyeq$-decreasing sequence of systems $\vartheta_j \in \mathbf{Sys}(\boldsymbol{\pi})$, and $\mathfrak{M}$-generic by the product forcing theorem since $\Phi\restriction\mathbb{\beta}$ is $\mathfrak{M}$-generic while Ψ is $\mathfrak{M}[\Phi]$-generic.

It follows from Theorem 10.1 that the according limit multiforcing $\boldsymbol{\varrho} = \lim[\Theta]$ is a small special multiforcing satisfying $\vec{\boldsymbol{\pi}}(\alpha) \sqsubset \boldsymbol{\varrho}$ for all $\alpha < \lambda$. We conclude that the extended sequence $\vec{\boldsymbol{\varrho}} = \vec{\boldsymbol{\pi}}^\frown\boldsymbol{\varrho}$ belongs to $\overrightarrow{\mathbf{MF}}_{\kappa+1}$; note that $\boldsymbol{\varrho} = \vec{\boldsymbol{\varrho}}(\kappa)$ is the last term of $\vec{\boldsymbol{\varrho}}$. Moreover, as the $(<\mathbb{\beta})$-part of Θ is equal to $\Phi\restriction\mathbb{\beta}$, we have $\boldsymbol{\varrho}\restriction\mathbb{\beta} = \lim[\Phi\restriction\mathbb{\beta}] = \vec{\Pi}(\kappa)\restriction\mathbb{\beta}$. It follows that $\vec{\boldsymbol{\varrho}}\restriction_{<\mathbb{\beta}} \subset \vec{\Pi}\restriction_{<\mathbb{\beta}}$.

Now let $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{MF}}$ be any sequence with $\vec{\boldsymbol{\varphi}} \subseteq \vec{\boldsymbol{\tau}}$. Suppose that $\xi \in |\vec{\boldsymbol{\pi}}|$. Then the set $D_\xi[\vec{\boldsymbol{\pi}}] \in \mathfrak{M}$ is open dense in $\mathbf{MT}(\boldsymbol{\pi})$ below $\mathbb{p}$ because $\vec{\boldsymbol{\pi}}$ is an 25.1-type sequence. Therefore the bigger set $\boldsymbol{D} = D_\xi[\vec{\boldsymbol{\pi}}] \cup \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) : \boldsymbol{p} \perp \mathbb{p}\} \in \mathfrak{M}$ is simply open dense in $\mathbf{MT}(\boldsymbol{\pi})$. (Recall that $\perp$ means incompatibility.) Therefore $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi}$ by Theorem 10.1(vii). It follows by Lemma 6.4 that $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\tau}_{\geq\lambda}$, where $\boldsymbol{\tau}_{\geq\kappa} = \bigcup^{\mathtt{cw}}_{\kappa\leq\alpha<\mathtt{dom}(\vec{\boldsymbol{\tau}})} \vec{\boldsymbol{\tau}}(\alpha)$, and hence $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\tau}_{\geq\kappa})$. However $\mathbf{MT}(\vec{\boldsymbol{\tau}}) = \mathbf{MT}(\boldsymbol{\pi} \cup^{\mathtt{cw}} \boldsymbol{\tau}_{\geq\kappa})$. Thus $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\vec{\boldsymbol{\tau}})$. We conclude that $D_\xi[\vec{\boldsymbol{\pi}}]$ is pre-dense in $\mathbf{MT}(\vec{\boldsymbol{\tau}})$ below $\mathbb{p}$.

To prove the 'therefore' part of the theorem, let $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{MF}}_{\lambda+2}$ be the extension of $\vec{\boldsymbol{\varphi}}$ by the extra term $\vec{\boldsymbol{\tau}}(\lambda+1) = \boldsymbol{\tau}$. Let $\xi \in |\vec{\boldsymbol{\pi}}| \smallsetminus \mathbb{B}$. Assume that $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\tau})$, $\boldsymbol{r} \leqslant \mathbb{p}$. By the main part of the theorem, $\boldsymbol{r}$ is compatible with some $\boldsymbol{p} \in D_\xi[\vec{\boldsymbol{\pi}}]$ in $\mathbf{MT}(\vec{\boldsymbol{\tau}})$, so that there exists $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\tau}})$ with $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\boldsymbol{q} \leqslant \boldsymbol{r}$. We may assume that $\boldsymbol{q}$ belongs to $\boldsymbol{\tau}$ (the top level of $\vec{\boldsymbol{\tau}}$).

Furthermore, as $\boldsymbol{p} \in D_\xi[\vec{\boldsymbol{\pi}}]$, we have $\boldsymbol{p}\,\mathtt{forc}_{\vec{\boldsymbol{\pi}}}\,\Phi(\dot{\boldsymbol{x}}_\eta, \dot{\boldsymbol{x}}_\xi, \mathbb{c})$ for an ordinal $\eta \in |\vec{\boldsymbol{\pi}}| \smallsetminus \mathbb{B}$. Then $\boldsymbol{q}\,\mathtt{forc}_{\vec{\boldsymbol{\tau}}}\,\Phi(\dot{\boldsymbol{x}}_\eta, \dot{\boldsymbol{x}}_\xi, \mathbb{c})$ holds by Lemma 18.2 and $\boldsymbol{q}\,\mathtt{forc}_{\boldsymbol{\tau}}\,\Phi(\dot{\boldsymbol{x}}_\eta, \dot{\boldsymbol{x}}_\xi, \mathbb{c})$ by Theorem 20.1 since $\boldsymbol{\tau}$ is just the top term of $\vec{\boldsymbol{\tau}}$. Thus $\boldsymbol{q} \in D_\xi[\vec{\boldsymbol{\pi}}]$. □

Remark 26.2. In the context of Theorem 26.1, $|\boldsymbol{\varphi}| \smallsetminus \mathbb{B} \neq \varnothing$. Indeed, by (i) of Theorem 25.1, we have $\mathbb{B} \in |\boldsymbol{\pi}_0| \subseteq |\vec{\boldsymbol{\pi}}| \subseteq |\boldsymbol{\varphi}|$. □

27. The non-existence claim, part IV, finalization

Arguing in L under the contrary Assumption 17.2, we proceed as follows.

(A) In $\mathbf{L}$, pick a 25.1-type sequence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}_\lambda$, $\lambda < \omega_1$, by Lemma 25.1.
(B) Then, still in $\mathbf{L}$, pick $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}}_{\lambda+1}$ by Theorem 26.1, so that $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varphi}}$, the multiforcing $\boldsymbol{\varphi} = \vec{\boldsymbol{\varphi}}(\lambda)$ satisfies $\boldsymbol{\varphi}{\restriction}\mathbb{B} = \vec{\mathbb{\pi}}(\lambda){\restriction}\mathbb{B}$, and if $\xi \in |\vec{\boldsymbol{\pi}}| \smallsetminus \mathbb{B}$ and $\boldsymbol{\tau} \in \mathbf{spMF}$, $\boldsymbol{\varphi} \sqsubset \boldsymbol{\tau}$, then $D_\xi[\boldsymbol{\tau}]$ is dense in $\mathbf{MT}(\boldsymbol{\tau})$ below $\mathbb{p}$.

By Lemma 23.1, there is a refinement $\boldsymbol{\chi} \in \mathbf{spMF}$ $\boldsymbol{\varphi} \sqsubset \boldsymbol{\chi}$, and ordinals $\nu > \lambda$, and $\mathbb{\theta}$, $\mathbb{B} < \mathbb{\theta} < \omega_1$, such that $|\boldsymbol{\chi}| \subseteq \mathbb{\theta}$, $\boldsymbol{\chi}{\restriction}\mathbb{B} = \vec{\mathbb{\pi}}(\nu){\restriction}\mathbb{B}$, and the permutation $\boldsymbol{h}_1[\mathbb{\theta}]$ (see Section 23) acts so that the shifted multiforcing $\boldsymbol{\chi}_* = \boldsymbol{h}_1[\mathbb{\theta}]\cdot\boldsymbol{\chi}$ satisfies $\boldsymbol{\chi}_* \subseteq \vec{\mathbb{\pi}}(\nu)$. We may note that

(C) $\boldsymbol{h}_1[\mathbb{\theta}]\mathbb{p} = \mathbb{p}$ and $\boldsymbol{h}_1[\mathbb{\theta}]\mathbb{c} = \mathbb{c}$ by the choice of $\mathbb{B}$ in Section 17 and by the fact that $\boldsymbol{h}_1[\mathbb{\theta}]$ is the identity on the domain $[0, \mathbb{B})$.

Accordingly let $\boldsymbol{\varphi}_* = \boldsymbol{h}_1[\mathbb{\theta}]\cdot\boldsymbol{\varphi}$, $\vec{\boldsymbol{\varphi}_*} = \boldsymbol{h}_1[\mathbb{\theta}]\cdot\vec{\boldsymbol{\varphi}}$, $\vec{\boldsymbol{\pi}_*} = \boldsymbol{h}_1[\mathbb{\theta}]\cdot\vec{\boldsymbol{\pi}}$, so that $\vec{\boldsymbol{\varphi}_*}$ is an extension of $\vec{\boldsymbol{\pi}_*}$ by $\boldsymbol{\varphi}_*$ as the top element. Note that $\vec{\boldsymbol{\chi}_*}(\nu) = \boldsymbol{\chi}_*$.

Lemma 27.1 *(in* $\mathbf{L}$*). If* $\xi' \in |\vec{\boldsymbol{\pi}_*}| \smallsetminus \mathbb{B}$ *and* $\boldsymbol{\tau}_* \in \mathbf{spMF}$, $\boldsymbol{\chi}_* \sqsubset \boldsymbol{\tau}_*$, *then*

$$D_{\xi'}[\boldsymbol{\tau}_*] = \{\boldsymbol{p}' \in \mathbf{MT}(\boldsymbol{\tau}_*) : \exists\, \eta' \in |\vec{\boldsymbol{\pi}_*}| \smallsetminus \mathbb{B}\,\big(\boldsymbol{p}'\,\mathtt{forc}_{\boldsymbol{\tau}_*}\,\Phi(\dot{\boldsymbol{x}}_{\eta'}, \dot{\boldsymbol{x}}_{\xi'}, \mathbb{c})\big)\}$$

is a dense set in $\mathbf{MT}(\boldsymbol{\tau}_*)$ *below* $\mathbb{p}$.

Proof. We argue in L. To prove the lemma we carefully explore the action of $\boldsymbol{h}_1[\mathbb{\theta}]$ on (B) above. The key argument will be

(D) $D_{\xi'}[\boldsymbol{\tau}_*] = \boldsymbol{h}_1[\mathbb{\theta}]\,''D_\xi[\boldsymbol{\tau}] = \{\boldsymbol{h}_1[\mathbb{\theta}]\cdot\boldsymbol{p} : \boldsymbol{p} \in D_\xi[\boldsymbol{\tau}]\}$.

To prove (D) suppose that $\xi \in |\vec{\boldsymbol{\pi}}| \smallsetminus \mathbb{B}$ and $\boldsymbol{p} \in D_\xi[\boldsymbol{\tau}]$ and show that $\boldsymbol{p}' = \boldsymbol{h}_1[\mathbb{\theta}]\cdot\boldsymbol{p}$ belongs to $D_{\xi'}[\boldsymbol{\tau}_*]$, where accordingly $\xi' = \boldsymbol{h}_1[\mathbb{\theta}](\xi)$. (The inverse implication is similar.) By definition there is an ordinal

$\eta \in |\boldsymbol{\tau}| \smallsetminus \mathbb{b}$ such that $\boldsymbol{p}\, \mathtt{forc}_{\boldsymbol{\tau}}\, \Phi(\dot{\boldsymbol{x}}_\eta, \dot{\boldsymbol{x}}_\xi, \mathbb{c})$. Then we have $\boldsymbol{p}'\, \mathtt{forc}_{\boldsymbol{\tau}_*}\, \Phi(\dot{\boldsymbol{x}}_{\eta'}, \dot{\boldsymbol{x}}_{\xi'}, \mathbb{c})$ by Theorem 21.1 and (C), where $\eta' = \boldsymbol{h}_1[\theta](\eta)$. This completes the proof of (D).

Now to prove the lemma assume that $\boldsymbol{q}' \in \mathbf{MT}(\boldsymbol{\tau}_*)$ and $\boldsymbol{q}' \leqslant \mathbb{p}$. Note that $\boldsymbol{q}' = \boldsymbol{h}_1[\theta]\cdot\boldsymbol{q}$, where $\boldsymbol{q} = \boldsymbol{h}_1[\theta]\cdot\boldsymbol{q}' \in \mathbf{MT}(\boldsymbol{\tau})$, because $\boldsymbol{h}_1[\theta] = \boldsymbol{h}_1[\theta]^{-1}$. In addition, $\boldsymbol{q} \leqslant \mathbb{p}$ by (C). Therefore by (B) there exists $\boldsymbol{p} \in D_\xi[\boldsymbol{\tau}] \subseteq \mathbf{MT}(\boldsymbol{\tau})$, satisfying $\boldsymbol{p} \leqslant \boldsymbol{q}$. Finally we put $\boldsymbol{p}' = \boldsymbol{h}_1[\theta]\cdot\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\tau}_*)$, so that $\boldsymbol{p}' \leqslant \boldsymbol{q}'$, and in addition $\boldsymbol{p}' \in D_{\xi'}[\boldsymbol{\tau}_*]$ by (D). □

The next lemma transfers the pre-density result to the key sequence $\vec{\pi}$.

Recall that $\boldsymbol{p}\, \mathtt{forc}_\infty\, \varphi$ means that $\boldsymbol{p}\, \mathtt{forc}_{\vec{\pi}\restriction\alpha}\, \varphi$ holds for some $\alpha < \omega_1$. (See just before Proposition 22.3.)

Lemma 27.2 *(in* $\mathbf{L}$*). If* $\xi' \in |\overrightarrow{\boldsymbol{\pi}_*}| \smallsetminus \mathbb{b}$ *then the set*

$$D_{\xi'} = \{\boldsymbol{s} \in \mathbf{MT}(\vec{\pi}) : \exists\, \eta' \in |\overrightarrow{\boldsymbol{\pi}_*}| \smallsetminus \mathbb{b}\, \big(\boldsymbol{s}\, \mathtt{forc}_\infty\, \Phi(\dot{\boldsymbol{x}}_{\eta'}, \dot{\boldsymbol{x}}_{\xi'}, \mathbb{c})\big)\}$$

is open dense in $\mathbb{P} = \mathbf{MT}(\vec{\pi})$ *below* $\mathbb{p}$.

Proof. We argue in L. The openness follows from Lemma 18.2. To prove the density, consider any $\boldsymbol{p}_0 \in \mathbb{P}$, $\boldsymbol{p}_0 \leqslant \mathbb{p}$. We have to find an element $\eta' \in |\overrightarrow{\boldsymbol{\pi}_*}| \smallsetminus \mathbb{b}$, and a condition $\boldsymbol{s} \in \mathbf{MT}(\vec{\pi}) = \mathbb{P}$ such that $\boldsymbol{s} \leqslant \boldsymbol{p}_0$ and $\boldsymbol{s}\, \mathtt{forc}_\infty\, \Phi(\dot{\boldsymbol{x}}_{\eta'}, \dot{\boldsymbol{x}}_{\xi'}, \mathbb{c})$.

Since $\vec{\pi}$ is $\sqsubset$-increasing, there is an ordinal $\gamma > \nu$ and a stronger condition $\boldsymbol{p}_1 \in \mathbf{MT}(\vec{\pi}(\gamma))$, $\boldsymbol{p}_1 \leqslant \boldsymbol{p}_0$. Let $\boldsymbol{\tau}_* = \vec{\pi}(\gamma)$. Then $\boldsymbol{\chi}_* \subseteq \vec{\pi}(\nu) \sqsubset \vec{\pi}(\gamma) = \boldsymbol{\tau}_*$, hence $\boldsymbol{\chi}_* \sqsubset \boldsymbol{\tau}_*$ by Lemma 5.4. But $\boldsymbol{p}_1 \in \mathbf{MT}(\boldsymbol{\tau}_*)$. By Lemma 27.1, there is a condition $\boldsymbol{s} \in D_{\xi'}[\boldsymbol{\tau}_*]$, $\boldsymbol{s} \leqslant \boldsymbol{p}_1$. Then we have $\boldsymbol{s}\, \mathtt{forc}_{\boldsymbol{\tau}_*}\, \Phi(\dot{\boldsymbol{x}}_{\eta'}, \dot{\boldsymbol{x}}_{\xi'}, \mathbb{c})$ for some $\eta' \in |\overrightarrow{\boldsymbol{\pi}_*}| \smallsetminus \mathbb{b}$, and then $\boldsymbol{s}\, \mathtt{forc}_{\vec{\pi}\restriction(\gamma+1)}\, \Phi(\dot{\boldsymbol{x}}_{\eta'}, \dot{\boldsymbol{x}}_{\xi'}, \mathbb{c})$ by Theorem 20.1. By definition this implies $\boldsymbol{s}\, \mathtt{forc}_\infty\, \Phi(\dot{\boldsymbol{x}}_{\eta'}, \dot{\boldsymbol{x}}_{\xi'}, \mathbb{c})$, as required. □

Let $\Vdash$ be the $\mathbb{P}$-forcing relation over $\mathbf{L}$. It essentially coincides with the relation $\mathtt{forc}_\infty$ by Theorem 22.3. Therefore the lemma implies:

Corollary 27.3 *(in* $\mathbf{L}$*). If* $\xi' \in |\overrightarrow{\boldsymbol{\pi}_*}| \smallsetminus \mathbb{b}$ *then the set*

$$\Delta_{\xi'} = \{\boldsymbol{s} \in \mathbf{MT}(\vec{\pi}) : \exists\, \eta' \in |\overrightarrow{\boldsymbol{\pi}_*}| \smallsetminus \mathbb{b}\, \big(\boldsymbol{s} \Vdash \Phi(\dot{\boldsymbol{x}}_{\eta'}[\underline{G}], \dot{\boldsymbol{x}}_{\xi'}[\underline{G}], \mathbb{c}[\underline{G}])\big)\}$$

is open dense in $\mathbb{P}$ *below* $\mathbb{p}$. □

Now we are able to easily accomplish **the proof of Theorem 17.1**. Namely, as $\mathbb{p} \in G$, Corollary 27.3 implies that the set $X' = \{x_{\xi'}[G] : \xi' \in |\overrightarrow{\boldsymbol{\pi}_*}| \smallsetminus \mathbb{b}\}$ (non-empty by Remark 26.2) has no $<_\Phi^{\mathbb{c}[G]}$-least element, which contradicts Assumption 17.2. □ (Theorem 17.1)

Combining this result with Theorem 16.1, already established above, we finalize the proof of Theorem 1.1. □ (Theorem 1.1)

28. Conclusions and problems

In this study, the technique of finite-support products of Jensen's forcing was employed to the problem of obtaining a model of **ZFC** in which, for a given $n \geq 3$, there exist good well-orderings of the reals in the lightface class $\varDelta^1_n$, but no well-ordering of the reals (not necessarily good) exists in the boldface class $\boldsymbol{\Delta}^1_{n-1}$ at the previous level. This result (Theorem 1.1 of this paper) is a significant strengthening of our previous result in [32], in which the negative part concerned only lightface $\varDelta^1_{n-1}$-good well-orderings. This theorem continues our series of resent research such as

- a $\Pi^1_{\mathfrak{n}}$ E_0-equivalence class containing no OD elements, while every countable $\Sigma^1_{\mathfrak{n}}$-set of reals contains only OD reals [27],
- a $\Pi^1_{\mathfrak{n}}$ real singleton $\{a\}$ such that a codes a cofinal map $f : \omega \to \omega_1^{\mathbf{L}}$, while every $\Sigma^1_{\mathfrak{n}}$ set $X \subseteq \omega$ is constructible and hence cannot code a cofinal map $\omega \to \omega_1^{\mathbf{L}}$, [28],
- a non-**ROD**-uniformizable $\Pi^1_{\mathfrak{n}}$ set with countable cross-sections, while all $\mathbf{\Sigma}^1_{\mathfrak{n}}$ sets with countable cross-sections are $\mathbf{\Delta}^1_{\mathfrak{n}+1}$-uniformizable [29].

Theorem 1.1 may also be a step towards solution of the following all-important problem by S.D. Friedman [11, P. 209], [12, P. 602]: assuming the consistency of an inaccessible cardinal, find a model for a given n in which all $\mathbf{\Sigma}^1_n$ sets of reals are Lebesgue measurable and have the Baire and perfect set properties, but there is a $\mathbf{\Delta}^1_{n+1}$ well-ordering of the reals.

From our study, it is concluded that the technique of *definable generic* inductive constructions of forcing notions in $\mathbf{L}$, developed for Jensen-type product forcing in our earlier papers [30,31], succeeds to solve (by our Theorem 1.1) another descriptive set theoretic problem of the same kind.

From the result of Theorem 1.1, we come to the following problems.

Problem 28.1. Prove that it holds in the model $\mathbf{L}[G]$ of Section 15 that there is no boldface $\mathbf{\Delta}^1_{\mathfrak{n}-1}$ well-ordering of a set of reals, whose domain includes uncountably many reals of the form $x_\xi[G]$, $\xi < \omega_1$. (See Definition 15.1.)

Problem 28.2. Prove a version of Theorem 1.1 with the additional requirement that the negation $2^{\aleph_0} > \aleph_1$ of the continuum hypothesis holds in the generic extension considered.

The model for Theorem 1.1 introduced in Section 15 (the key model) definitely satisfies the continuum hypothesis $2^{\aleph_0} = \aleph_1$. The problem of obtaining models of **ZFC** in which $2^{\aleph_0} > \aleph_1$ and there is a projective well-ordering of the real line, has been known since the early years of modern set theory. See, e.g., problem 3214 in an early survey [35] by Mathias. Harrington [19] solved this problem by constructing a generic model of **ZFC**, in which $2^{\aleph_0} > \aleph_1$ and there is a Δ^1_3 well-ordering of the continuum. This model involves various forcing notions like the almost-disjoint forcing [21] and a forcing notion by Jensen and Johnsbråten [23].

As a concluding remark, we expect that the methods developed for this research can also be useful in creating computational algorithmic models, of various complexity in terms of the second order Peano arithmetic, that represent the evolution of cell types and are related to the storage and processing of genomic information.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Acknowledgements

The idea of making use of finite-support products of Jensen-like forcing notions in order to obtain models of set theory with various definability effects, was communicated to us by Ali Enayat, and we thank Ali Enayat for fruitful discussions and helpful remarks.

The authors thank the anonymous reviewer for their extremely thorough analysis of the text, and highly appreciate the comments and suggestions, which significantly contributed to improving the quality of the publication.

References

[1] Uri Abraham, A minimal model for $\neg CH$: iteration of Jensen's reals, Trans. Am. Math. Soc. 281 (1984) 657–674, https://doi.org/10.2307/2000078, ISSN 0002-9947, 1088-6850/e.
[2] J.W. Addison, Some consequences of the axiom of constructibility, Fundam. Math. 46 (1959) 337–357.
[3] Joan Bagaria, W. Hugh Woodin, $\underset{\sim}{\Delta}^1_n$ sets of reals, J. Symb. Log. (ISSN 0022-4812) 62 (4) (1997) 1379–1428, https://doi.org/10.2307/2275649.
[4] Andrés E. Caicedo, Ralf Schindler, Projective well-orderings of the reals, Arch. Math. Log. 45 (7) (2006) 783–793, https://doi.org/10.1007/s00153-006-0002-6, ISSN 0933-5846; 1432-0665/e.
[5] Andrés Eduardo Caicedo, Sy-David Friedman, BPFA and projective well-orderings of the reals, J. Symb. Log. (ISSN 0022-4812) 76 (4) (2011) 1126–1136, https://doi.org/10.2178/jsl/1318338842.
[6] Andrés Eduardo Caicedo, Boban Veličković, The bounded proper forcing axiom and well orderings of the reals, Math. Res. Lett. (ISSN 1073-2780) 13 (2–3) (2006) 393–408, https://doi.org/10.4310/MRL.2006.v13.n3.a5.
[7] Ali Enayat, On the Leibniz – Mycielski axiom in set theory, Fundam. Math. 181 (3) (2004) 215–231, https://doi.org/10.4064/fm181-3-2, ISSN 0016-2736; 1730-6329/e.
[8] Vera Fischer, Sy David Friedman, Cardinal characteristics and projective wellorders, Ann. Pure Appl. Log. (ISSN 0168-0072) 161 (7) (2010) 916–922, https://doi.org/10.1016/j.apal.2009.11.003.
[9] Vera Fischer, Sy David Friedman, Lyubomyr Zdomskyy, Cardinal characteristics, projective wellorders and large continuum, Ann. Pure Appl. Log. (ISSN 0168-0072) 164 (7–8) (2013) 763–770, https://doi.org/10.1016/j.apal.2012.12.001.
[10] Vera Fischer, Sy David Friedman, Yurii Khomskii, Measure, category and projective wellorders, J. Log. Anal. 6 (2014) 1–25, ISSN 1759-9008/e.
[11] Sy D. Friedman, Fine Structure and Class Forcing, De Gruyter Series in Logic and Its Applications, vol. 3, de Gruyter, Berlin, 2000, ISBN 3-11-016777-8/hbk.
[12] Sy D. Friedman, Constructibility and class forcing, in: Handbook of Set Theory. In 3 volumes, Springer, Dordrecht, 2010, pp. 557–604, ISBN 978-1-4020-4843-2/hbk; 978-1-4020-5764-9/ebook.
[13] Sy-David Friedman, Stefan Hoffelner, A Σ^1_4 wellorder of the reals with NS_{ω_1} saturated, J. Symb. Log. 84 (4) (2019) 1466–1483, https://doi.org/10.1017/jsl.2019.43.
[14] Sy-David Friedman, Victoria Gitman, Vladimir Kanovei, A model of second-order arithmetic satisfying AC but not DC, J. Math. Log. 19 (1) (2019) 1–39, https://doi.org/10.1142/S0219061318500137, ISSN 0219-0613; 1793-6691/e. Article No 1850013.
[15] Victoria Gitman, Joel David Hamkins, Thomas A. Johnstone, What is the theory ZFC without power set?, Math. Log. Q. 62 (4–5) (2016) 391–406, ISSN 0942-5616; 1521-3870/e.
[16] Kurt Gödel, The Consistency of the Continuum Hypothesis, Annals of Mathematics Studies, vol. 3, Princeton University Press, Princeton, N. J., 1940.
[17] M. Groszek, T. Jech, Generalized iteration of forcing, Trans. Am. Math. Soc. (ISSN 0002-9947) 324 (1) (1991) 1–26.
[18] J. Hadamard, R. Baire, H. Lebesgue, E. Borel, Cinq lettres sur la théorie des ensembles, Bull. Soc. Math. Fr. (ISSN 0037-9484) 33 (1905) 261–273, https://doi.org/10.24033/bsmf.761.
[19] Leo Harrington, Long projective wellorderings, Ann. Math. Log. (ISSN 0003-4843) 12 (1977) 1–24, https://doi.org/10.1016/0003-4843(77)90004-3.
[20] Thomas Jech, Set Theory, the third millennium revised and expanded edition, Springer-Verlag, Berlin-Heidelberg-New York, 2003, ISBN 3-540-44085-2/hbk, pages xiii + 769.
[21] R.B. Jensen, R.M. Solovay, Some applications of almost disjoint sets, in: Yehoshua Bar-Hillel (Ed.), Math. Logic Found. Set Theory, Proc. Int. Colloqu., Jerusalem 1968, in: Studies in Logic and the Foundations of Mathematics, vol. 59, North-Holland, Amsterdam-London, 1970, pp. 84–104.
[22] Ronald Jensen, Definable sets of minimal degree, in: Yehoshua Bar-Hillel (Ed.), Math. Logic Found. Set Theory, Proc. Int. Colloqu., Jerusalem 1968, in: Studies in Logic and the Foundations of Mathematics, vol. 59, North-Holland, Amsterdam-London, 1970, pp. 122–128.
[23] Ronald B. Jensen, Havard Johnsbraten, A new construction of a non-constructible Δ^1_3 subset of ω, Fundam. Math. 81 (279–290) (1974), https://doi.org/10.4064/fm-81-4-279-290, ISSN 0016–2736; 1730–6329/e.
[24] Vladimir Kanovei, On non-wellfounded iterations of the perfect set forcing, J. Symb. Log. 64 (2) (1999) 551–574, https://doi.org/10.2307/2586484, ISSN 0022-4812; 1943-5886/e.
[25] Vladimir Kanovei, Vassily Lyubetsky, Counterexamples to countable-section Π^1_2 uniformization and Π^1_3 separation, Ann. Pure Appl. Log. (ISSN 0168-0072) 167 (3) (2016) 262–283, https://doi.org/10.1016/j.apal.2015.12.002.
[26] Vladimir Kanovei, Vassily Lyubetsky, A countable definable set containing no definable elements, Math. Notes 102 (3) (2017) 338–349, https://doi.org/10.1134/S0001434617090048, ISSN 0001-4346; 1573-8876/e, also arXiv:1408.3901.
[27] Vladimir Kanovei, Vassily Lyubetsky, Definable E_0 classes at arbitrary projective levels, Ann. Pure Appl. Log. (ISSN 0168-0072) 169 (9) (2018) 851–871, https://doi.org/10.1016/j.apal.2018.04.006.
[28] Vladimir Kanovei, Vassily Lyubetsky, Definable minimal collapse functions at arbitrary projective levels, J. Symb. Log. 84 (1) (2019) 266–289, https://doi.org/10.1017/jsl.2018.77, ISSN 0022-4812; 1943-5886/e.
[29] Vladimir Kanovei, Vassily Lyubetsky, Non-uniformizable sets with countable cross-sections on a given level of the projective hierarchy, Fundam. Math. 245 (2) (2019) 175–215, https://doi.org/10.4064/fm517-7-2018, ISSN 0016–2736; 1730–6329/e.

[30] Vladimir Kanovei, Vassily Lyubetsky, The full basis theorem does not imply analytic wellordering, Ann. Pure Appl. Log. (ISSN 0168-0072) 172 (4) (2021) 46, https://doi.org/10.1016/j.apal.2020.102929, Id/No 102929.

[31] Vladimir Kanovei, Vassily Lyubetsky, Models of set theory in which separation theorem fails, Izv. Math. 85 (6) (2021) 1181–1219, https://doi.org/10.1070/IM8521.

[32] Vladimir Kanovei, Vassily Lyubetsky, A model in which wellorderings of the reals appear at a given projective level, Axioms (ISSN 2075-1680) 11 (8) (2022) 122, https://doi.org/10.3390/axioms11080354.

[33] Azriel Levy, Definability in axiomatic set theory II, in: Yehoshua Bar-Hillel (Ed.), Math. Logic Found. Set Theory, Proc. Int. Colloqu., Jerusalem 1968, North-Holland, Amsterdam-London, 1970, pp. 129–145.

[34] N. Lusin, Sur la classification de M. Baire, C. R. Acad. Sci., Paris (ISSN 0001-4036) 164 (1917) 91–94.

[35] A.R.D. Mathias, Surrealist landscape with figures (a survey of recent results in set theory), Period. Math. Hung. 10 (109–175) (1979), https://doi.org/10.1007/BF02025889, ISSN 0031-5303; 1588-2829/e.

[36] Yiannis N. Moschovakis, Descriptive Set Theory, Studies in Logic and the Foundations of Mathematics, vol. 100, North-Holland, Amsterdam, New York, Oxford, 1980, pp. XII+637.

[37] Wacław Sierpiński, L'axiome de M. Zermelo et son rôle dans la théorie des ensembles et l'analyse, Bull. Acad. Sci. Cracovie (1918) 97–152.

[38] Robert M. Solovay, A model of set-theory in which every set of reals is Lebesgue measurable, Ann. Math. (2) 92 (1970) 1–56.