Varia
Ideals and Equivalence Relations

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Аннотация

A selection of basic results on Borel reducibility of ideals and ERs, especially those with comparably short proofs. This is an unfinished text as yet. Some proofs have missing parts and loose ends. [1]

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Contents

1 Reducibility

There are several reasonable ways to compare ERs, usually formalized in terms of existence of a *reduction*, *i.e.*, a map of certain kind which allows to derive one of the ERs from the other one. *Borel reducibility* $\le_{\rm B}$ is the key one, yet there are several special types of $\le_{\rm B}$, in particular, those induced by a low-level maps, useful in many cases. Generally, the most of research on reducibility of Borel ERs or ideals is concentrated around the following notions of reducibility.

1.a Borel reducibility

If E and F are ERs on Polish spaces resp. $\mathbb{X}$, $\mathbb{Y}$, then

* $\mathsf{E} \le_{\rm B} \mathsf{F}$ (*Borel reducibility*) means that there is a Borel map $\vartheta : \mathbb{X} \to \mathbb{Y}$ (called *reduction*) such that $x \mathrel{\mathsf{E}} y \Longleftrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in \mathbb{X}$;
* $\mathsf{E} \sim_{\rm B} \mathsf{F}$ iff $\mathsf{E} \le_{\rm B} \mathsf{F}$ and $\mathsf{F} \le_{\rm B} \mathsf{E}$ (*Borel bi-reducibility*);
* $\mathsf{E} <_{\rm B} \mathsf{F}$ iff $\mathsf{E} \le_{\rm B} \mathsf{F}$ but not $\mathsf{F} \le_{\rm B} \mathsf{E}$ (*strict Borel reducibility*);
* $\mathsf{E} \sqsubseteq_{\rm B} \mathsf{F}$ means that there is a Borel *embedding*, *i.e.*, a $1-1$ reduction;
* $\mathsf{E} \approx_{\rm B} \mathsf{F}$ iff $\mathsf{E} \sqsubseteq_{\rm B} \mathsf{F}$ and $\mathsf{F} \sqsubseteq_{\rm B} \mathsf{E}$ (a rare form, [18, § 0]);

* $\mathsf{E} \sqsubseteq^{\mathrm{i}}_{\mathrm{B}} \mathsf{F}$ means that there is a Borel *invariant* embedding, *i.e.*, an embedding ϑ such that $\mathtt{ran}\,\vartheta = \{\vartheta(x) : x \in \mathbb{X}\}$ is an F-*invariant* set (meaning that the F-*saturation* $[\mathtt{ran}\,\vartheta]_{\mathsf{F}} = \{y' : \exists\, x\, (y \mathrel{\mathsf{F}} \vartheta(x))\}$ equals $\mathtt{ran}\,\vartheta$);

* $\mathsf{E} \le_{\mathrm{C}} \mathsf{F}$, $\mathsf{E} \sqsubseteq_{\mathrm{C}} \mathsf{F}$, $\mathsf{E} \sqsubseteq^{\mathrm{i}}_{\mathrm{C}} \mathsf{F}$ mean that there is a continuous resp. reduction, embedding, invariant embedding.

Sometimes they write $\mathbb{X}/\mathsf{E} \le_{\mathrm{B}} \mathbb{Y}/\mathsf{F}$ instead of $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$.

***Borel reducibility of ideals*:** $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$ iff $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{B}} \mathsf{E}_{\mathscr{J}}$. Thus it is required that there is a Borel map $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$ such that $x \,\Delta\, y \in \mathscr{I}$ iff $\vartheta(x) \,\Delta\, \vartheta(y) \in \mathscr{J}$. (Here $\mathscr{I}$ is an ideal on A and $\mathscr{J}$ is an ideal on B.)

Versions $\mathscr{I} \le_{\mathrm{C}} \mathscr{J}$, $\mathscr{I} \sqsubseteq_{\mathrm{B}} \mathscr{J}$, $\mathscr{I} \sqsubseteq_{\mathrm{C}} \mathscr{J}$ have the corresponding meaning.

1.b "Algebraic" Borel reducibility

This is a more special version of Borel reducibility of ideals, characterized by the property that the reduction must respect a chosen algebraic structure. We shall be especially interested in the Boolean algebra structure and a weaker Δ-group structure of sets of the form $\mathscr{P}(A)$. Let $\mathscr{I}$, $\mathscr{J}$ be ideals on resp. A, B.

***Borel BA reducibility*:** $\mathscr{I} \le_{\mathrm{B,BA}} \mathscr{J}$ if there is a Borel $\mathscr{J}$-approximate Boolean algebra homomorphism $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$ with $x \in \mathscr{I} \iff \vartheta(x) \in \mathscr{J}$.

A version: $\mathscr{I} \le^{+}_{\mathrm{B,BA}} \mathscr{J}$ if there is a set $A \in \mathscr{J}^{+}$ with $\mathscr{I} \le_{\mathrm{B,BA}} (\mathscr{J} \restriction A)$.

Here, $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$ is an $\mathscr{J}$-*approximate* Boolean algebra homomorphism if the sets $(\vartheta(x) \cup \vartheta(y)) \,\Delta\, \vartheta(x \cup y)$ and $\vartheta(\complement x) \,\Delta\, \complement(\vartheta(x))$ always belong to $\mathscr{J}$ whenever $x, y \subseteq A$. Let further a $\mathscr{J}$-approximate Δ-*homomorphism* be any map $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$ such that $(\vartheta(x) \,\Delta\, \vartheta(y)) \,\Delta\, \vartheta(x \,\Delta\, y)$ always belongs to $\mathscr{J}$. This leads to a weaker reducibility:

***Borel Δ-reducibility*:** $\mathscr{I} <_{\mathrm{B},\Delta} \mathscr{J}$ iff there is a Borel $\mathscr{J}$-approximate Δ-homomorphism $\vartheta : \mathscr{P}(A) \to \mathscr{P}(B)$ such that $x \in \mathscr{I} \iff \vartheta(x) \in \mathscr{J}$.

1.c Borel, continuous, and Baire measurable reductions

Many properties of Borel reductions hold for a bigger family of Baire measurable (BM, for brevity) maps. Any reducibility definition in §§ 1.a, 1.b admits a weaker BM version, which claims that the reduction postulated to exist is only BM, not necessarily Borel. Such a version will be denoted with a subscript BM instead of B, for instance, $\mathsf{E} \le_{\mathrm{BM}} \mathsf{F}$ means that there is a BM reduction, *i.e.*, a BM map $\vartheta : \mathbb{X} = \mathtt{dom}\,\mathsf{E} \to \mathbb{Y} = \mathtt{dom}\,\mathsf{F}$ such that $x \mathrel{\mathsf{E}} y \iff \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in \mathbb{X}$.

On the other hand, a continuous reducibility can sometimes be derived.

Lemma 1 (Louveau ?). *If $\mathscr{I}$ is a Borel ideal on a countable A, E an equivalence relation on a Polish $\mathbb{X}$, and $\mathsf{E}_{\mathscr{I}} \le_{\rm BM} \mathsf{E}$, then $\mathsf{E}_{\mathscr{I}} \le_{\rm C} \mathsf{E} \times \mathsf{E}$* (via a continuous reduction)*, that is, there exist continuous maps $\vartheta_0,\ \vartheta_1 : \mathscr{P}(A) \to \mathbb{X}$ such that, for any $x,\ y \in \mathscr{P}(\mathbb{N})$, $x \,\Delta\, y \in \mathscr{I}$ iff both $\vartheta_0(x) \mathrel{\mathsf{E}} \vartheta_0(y)$ and $\vartheta_1(x) \mathrel{\mathsf{E}} \vartheta_1(y)$.*

Proof. We *w.l.o.g.* suppose that $A = \mathbb{N}$. Let $\vartheta : \mathscr{P}(\mathbb{N}) \to \mathbb{X}$ witness that $\mathsf{E}_{\mathscr{I}} \le_{\rm BM} \mathsf{E}$. Then ϑ is continuous on a dense $\mathbf{G}_\delta$ set $D = \bigcap_i D_i \subseteq \mathscr{P}(\mathbb{N})$, all D_i dense open and $D_{i+1} \subseteq D_i$. A sequence $0 = n_0 < n_1 < n_2 < \dots$ and, for any i, a set $u_i \subseteq [n_i, n_{i+1})$ can be easily defined, by induction on i, so that $x \cap [n_i, n_{i+1}) = u_i \Longrightarrow x \in D_i$.[2] Let

$$N_1 = \textstyle\bigcup_i [n_{2i}, n_{2i+1})\,, \quad N_2 = \bigcup_i [n_{2i+1}, n_{2i+2})\,, \quad U_1 = \bigcup_i u_{2i}\,, \quad U_2 = \bigcup_i u_{2i+1}\,.$$

Now set $\vartheta_1(x) = \vartheta((x \cap N_1) \cup U_2)$ and $\vartheta_2(x) = \vartheta((x \cap N_2) \cup U_1)$ for $x \subseteq \mathbb{N}$. □

The following question should perhaps be answered in the negative in general and be open for some particular cases.

Question 2. Suppose that $\mathsf{E} \le_{\rm B} \mathsf{F}$ are Borel ERs. Does there always exist a <u>continuous</u> reduction ? □

1.d Reducibility via maps between the underlying sets

This is an even more special kind of Borel reducibility. Let $\mathscr{I}$, $\mathscr{J}$ be ideals on resp. A, B, as above.

***Rudin–Keisler order*:** $\mathscr{I} \le_{\rm RK} \mathscr{J}$ iff there exists a function $b : \mathbb{N} \to \mathbb{N}$ (a *Rudin–Keisler* reduction) such that $x \in \mathscr{I} \Longleftrightarrow b^{-1}(x) \in \mathscr{J}$.

***Rudin–Blass order*:** $\mathscr{I} \le_{\rm RB} \mathscr{J}$ iff there is a <u>finite-to-one</u> function $b : \mathbb{N} \to \mathbb{N}$ (a *Rudin–Blass* reduction) with the same property.

A version: $\mathscr{I} \le^{+}_{\rm RB} \mathscr{J}$ allows b to be defined on a proper subset of $\mathbb{N}$, in other words, we have pairwise disjoint finite non-empty sets $w_k = b^{-1}(\{k\})$ such that $x \in \mathscr{I} \iff w_x = \bigcup_{k \in x} w_k \in \mathscr{J}$.

Another version: $\mathscr{I} \le^{++}_{\rm RB} \mathscr{J}$ requires that, in addition, the sets $w_k = b^{-1}(\{k\})$ satisfy $\mathtt{max}\, w_k < \mathtt{min}\, w_{k+1}$.

There is a "clone" of the Rudin–Blass order which applies in a much more general situation. Suppose that $X = \prod_{k \in \mathbb{N}} X_k$ and $Y = \prod_{k \in \mathbb{N}} Y_k$, $0 = n_0 < n_1 < n_2 < ...$, and $H_i : X_i \to \prod_{n_i \le k < n_{i+1}} Y_k$ for any i. Then, we can define

$$\Psi(x) = H_0(x_0) \cup H_1(x_1) \cup H_2(x_2) \cup ... \in Y$$

[2] Sets like u_i are called *stabilizers*, they are of much help in study of Borel ideals.

for each $x = \{x_i\}_{i\in\mathbb{N}} \in X$. Maps Ψ of this kind were called *additive* by Farah [7]. More generally, if, in addition, $0 = m_0 < m_1 < m_2 < ...$, and $H_i : \prod_{m_i\le j<m_{i+1}} X_j \to \prod_{n_i\le k<n_{i+1}} Y_k$ for any i, then we can define

$$\Psi(x) = H_0(x \restriction [m_0, m_1)) \cup H_1(x \restriction [m_1, m_2)) \cup H_2(x \restriction [m_2, m_3)) \cup ... \in Y$$

for each $x \in X$. Farah calls maps Ψ of this kind *asymptotically additive*. All of them are Borel functions $X \to Y$, provided all sets X_j and Y_k are finite.

Suppose now that E and F are ERs on resp. $X = \prod_k X_k$ and $Y = \prod_k Y_k$.

***Additive reducibility*:** $\mathsf{E} \le_{\mathrm{A}} \mathsf{F}$ if there is an additive reduction E to F. $\mathsf{E} \le_{\mathrm{AA}} \mathsf{F}$ if there is an asymptotically additive reduction E to F.

Lemma 3 (Farah [7])**.** *Suppose that $\mathscr{I}$ and $\mathscr{J}$ are Borel ideals on $\mathbb{N}$. Then $\mathscr{I} \le_{\mathrm{RB}}^{++} \mathscr{J}$ iff $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{A}} \mathsf{E}_{\mathscr{J}}$.*

(By definition $\mathsf{E}_{\mathscr{I}}$ and $\mathsf{E}_{\mathscr{J}}$ are ERs on $\mathscr{P}(\mathbb{N})$, yet we can consider them as ERs on $2^{\mathbb{N}} = \prod_{k\in\mathbb{N}}\{0,1\}$, as usual, which yields the intended meaning for $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{A}} \mathsf{E}_{\mathscr{J}}$.)

Proof. If $\mathscr{I} \le_{\mathrm{RB}}^{++} \mathscr{J}$ via a sequence of finite sets w_i with $\mathtt{max}\, w_i < \mathtt{min}\, w_{i+1}$ then we put $n_0 = 0$ and $n_i = \mathtt{min}\, w_i$ for $k \ge 1$, so that $w_i \subseteq [n_i, n_{i+1})$, and, for any i, put $H_i(0) = [n_i, n_{i+1}) \times \{0\}$ and let $H_i(1)$ be the characteristic function of w_i within $[n_i, n_{i+1})$. Conversely, if $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{A}} \mathsf{E}_{\mathscr{J}}$ via a sequence $0 = n_0 < n_1 < n_2 < ...$ and a family of maps $H_i : \{0,1\} \to 2^{[n_i,n_{i+1})}$ then $\mathscr{I} \le_{\mathrm{RB}}^{++} \mathscr{J}$ via the sequence of sets $w_i = \{k \in [n_i, n_{i+1}) : H_i(0)(k) \ne H_i(1)(k)\}$. $\square$

The following definition is taken from [19]. Let $\mathscr{I}$, $\mathscr{J}$ be ideals on $\mathbb{N}$.

***Reducibility via inclusion*:** $\mathscr{I} \le_{\mathrm{I}} \mathscr{J}$ if there is a map $b : \mathbb{N} \to \mathbb{N}$ such that $x \in \mathscr{I} \Longrightarrow b^{-1}(x) \in \mathscr{J}$. (Note $\Longrightarrow$ instead of $\Longleftrightarrow$!)

In particular if $\mathscr{I} \subseteq \mathscr{J}$ then $\mathscr{I} \le_{\mathrm{I}} \mathscr{J}$ via $b(k) = k$. It follows that this order is not fully compatible with $\le_{\mathrm{B}}$ because $\mathscr{S}_{\{1/n\}} \subseteq \mathscr{Z}_0$ while the summable ideal $\mathscr{S}_{\{1/n\}}$ and the density-0 ideal $\mathscr{Z}_0$ are known to be $\le_{\mathrm{B}}$-incomparable.

1.e Isomorphism

Let $\mathscr{I}$, $\mathscr{J}$ be ideals on resp. A, B. *Isomorphism* $\mathscr{I} \cong \mathscr{J}$ means that there is a bijection $\beta : A \xrightarrow{\text{onto}} B$ such that we have $x \in \mathscr{I} \iff \beta\text{"}x \in \mathscr{J}$ for all $x \subseteq A$.

Sometimes they use a weaker definition: let $\mathscr{I} \cong^* \mathscr{J}$ mean that there are sets $A' \in \mathscr{I}^{\complement}$ and $B' \in \mathscr{J}^{\complement}$ such that $\mathscr{I} \restriction A' \cong \mathscr{J} \restriction B'$. Yet this implies $\mathscr{I} \cong \mathscr{J}$ in most usual cases, the only notable exception (among nontrivial ideals), is produced by the ideals $\mathscr{I} = \mathrm{Fin}$ and $\mathscr{J} = \mathrm{Fin} \oplus \mathscr{P}(\mathbb{N}) \cong \{x \subseteq \mathbb{N} : x \cap D \in \mathrm{Fin}\}$, where D is an infinite and coinfinite set [3] : then $\mathscr{I} \cong^* \mathscr{J}$ but not $\mathscr{I} \cong \mathscr{J}$.

[3] Kechris [27] called ideals $\mathscr{J}$ of this kind *trivial variations of* Fin.

1.f Remarks

The following shows simple relationships between different reducibilities: $\leftarrow$ *check this subsection once again* $\dashv$

$$\mathscr{I} \le_{\mathrm{RB}} \mathscr{J} \Rightarrow \mathscr{I} \le_{\mathrm{RK}} \mathscr{J} \Rightarrow \mathscr{I} \le_{\mathrm{BE}} \mathscr{J} \Rightarrow \mathscr{I} \le^{+}_{\mathrm{BE}} \mathscr{J} \Rightarrow \mathscr{I} \le_{\Delta} \mathscr{J} \Rightarrow \mathscr{I} \le_{\mathrm{B}} \mathscr{J}$$

For instance if $b : \mathbb{N} \to \mathbb{N}$ witnesses $\mathscr{I} \le_{\mathrm{RK}} \mathscr{J}$ then $\vartheta_b(X) = b^{-1}(X)$ witnesses $\mathscr{I} \le_{\mathrm{BE}} \mathscr{J}$. Note that any ϑ_b is an exact Boolean algebra homomorphism $\mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$; moreover, it is known that any BM Boolean algebra homomorphism $\mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ is ϑ_b for an appropriate $b : \mathbb{N} \to \mathbb{N}$. *Approximate* homomorphisms are liftings of homomorphisms into quotients of $\mathscr{P}(\mathbb{N})$, thus, any $\mathscr{J}$-approximate $\vartheta : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ induces the map $\Theta(X) = \{\vartheta(X) \,\Delta\, Y : Y \in \mathscr{J}\}$, which is a homomorphism $\mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})/\mathscr{J}$. Farah [6], and Kanovei and Reeken [24] demonstrated that in some important cases (of "non-patological" P-ideals and, generally, for all Fatou, or Fubini, ideals) we have $\mathscr{I} \le_{\mathrm{RK}} \mathscr{J} \iff \mathscr{I} \le_{\mathrm{BE}} \mathscr{J}$. On the other hand $\mathscr{I} \le_{\mathrm{RK}} \mathscr{J} \Longleftarrow \mathscr{I} \le_{\mathrm{BE}} \mathscr{J}$ fails for rather artificial P-ideals.

The right-hand end is the most intrigueing: is there a pair of Borel ideals $\mathscr{I}, \mathscr{J}$ such that $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$ but not $\mathscr{I} \le_{\Delta} \mathscr{J}$? If we actually have the equivalence then the whole theory of Borel reducibility for Borel ideals can be greatly simplified because reduction maps which are Δ-homomorphisms are much easier to deal with.

2 Introduction to ideals

As many interesting ERs appear as $\mathsf{E}_{\mathscr{I}}$ for a Borel ideal $\mathscr{I}$, we take space to discuss a few basic items related to Borel ideals. We begin with several examples and notation, and then continue with some important types of ideals.

- $\mathtt{Fin} = \{x \subseteq \mathbb{N} : x \text{ is finite}\}$, the ideal of all finite sets;
- $\mathscr{I}_1 = \mathtt{Fin} \times 0 = \{x \subseteq \mathbb{N}^2 : \{k : (x)_k \ne \emptyset\} \in \mathtt{Fin}\}$; $\leftarrow$ *gde vvedeno* $(x)_k$ *?* $\dashv$
- $\mathscr{I}_2 = \mathscr{S}_{\{1/n\}} = \{x \subseteq \mathbb{N} : \sum_{n \in x} \frac{1}{n+1}\} < +\infty$, the *summable ideal*;
- $\mathscr{I}_3 = 0 \times \mathtt{Fin} = \{x \subseteq \mathbb{N}^2 : \forall\, k\, ((x)_k \in \mathtt{Fin})\}$;
- $\mathscr{Z}_0 = EU_{\{1\}} = \{x \subseteq \mathbb{N} : \mathtt{lim}_{n \to +\infty} \frac{\#(x \cap [0,n))}{n} = 0\}$, the *density ideal*.

2.a Notation

- For any ideal $\mathscr{I}$ on a set A, we define $\mathscr{I}^+ = \mathscr{P}(A) \smallsetminus \mathscr{I}$ ($\mathscr{I}$-*positive* sets) and $\mathscr{I}^{\complement} = \{X : \complement X \in \mathscr{I}\}$ (*the dual filter*). Clearly $\emptyset \ne \mathscr{I}^{\complement} \subseteq \mathscr{I}^+$.
- If $B \subseteq A$, then we put $\mathscr{I} \restriction B = \{x \cap B : x \in \mathscr{I}\}$.

- If $\mathscr{I}$, $\mathscr{J}$ are ideals on resp. A, B, then $\mathscr{I} \oplus \mathscr{J}$ (the *disjoint sum*) is the ideal of all sets $x \subseteq C = (\{0\} \times A) \cup (\{1\} \times B)$ with $(x)_0 \in \mathscr{I}$ and $(x)_1 \in \mathscr{J}$ (where $(x)_i = \{c : \langle i, c\rangle \in x\}$, as usual).

  If the sets A, B are disjoint then $\mathscr{I} \oplus \mathscr{J}$ can be equivalently defined as the ideal of all sets $x \subseteq A \cup B$ with $x \restriction A \in \mathscr{I}$ and $x \restriction B \in \mathscr{J}$.

- The *Fubini product* $\prod_{a \in A} \mathscr{J}_a \,/\, \mathscr{I}$ of ideals $\mathscr{J}_a$ on sets B_a, over an ideal $\mathscr{I}$ on a set A is the ideal on the set $B = \{\langle a, b\rangle : a \in A \wedge b \in B_a\}$, which consists of all sets $y \subseteq B$ such that the set $\{a : (y)_a \notin \mathscr{J}_a\}$ belongs to $\mathscr{I}$, where $(y)_a = \{b : \langle a, b\rangle \in y\}$ (the cross-section).

- In particular, the Fubini product $\mathscr{I} \otimes \mathscr{J}$ of two ideals $\mathscr{I}, \mathscr{J}$ on sets resp. A, B, is equal to $\prod_{a \in A} \mathscr{J}_a \,/\, \mathscr{I}$, where $\mathscr{J}_a = \mathscr{J}$, $\forall\, a$. Thus $\mathscr{I} \otimes \mathscr{J}$ consists of all sets $y \subseteq A \times B$ such that $\{a : (y)_a \notin \mathscr{J}\} \in \mathscr{I}$.

2.b P-ideals and submeasures

Many important Borel ideals belong to the class of P-ideals.

Definition 4. An ideal $\mathscr{I}$ on $\mathbb{N}$ is a *P-ideal* if for any sequence of sets $x_n \in \mathscr{I}$ there is a set $x \in \mathscr{I}$ such that $x_n \subseteq^* x$ (*i.e.*, $x_n \smallsetminus x \in \mathtt{Fin}$) for all n; □

For instance, the ideals $\mathtt{Fin}$, $\mathscr{I}_2$, $\mathscr{I}_3$, $\mathscr{Z}_0$ (but not $\mathscr{I}_1$!) are P-ideals.

This class admits several apparently different but equivalent characterizations, one of which is connected with submeasures.

- A *submeasure* on a set A is any map $\varphi : \mathscr{P}(A) \to [0, +\infty]$, satisfying $\varphi(\emptyset) = 0$, $\varphi(\{a\}) < +\infty$ for all a, and $\varphi(x) \le \varphi(x \cup y) \le \varphi(x) + \varphi(y)$.

- A submeasure φ on $\mathbb{N}$ is *lover semicontinuous*, or l. s. c. for brevity, if we have $\varphi(x) = \mathtt{sup}_n\, \varphi(x \cap [0, n))$ for all $x \in \mathscr{P}(\mathbb{N})$.

To be a *measure*, a submeasure φ has to satisfy, in addition, that $\varphi(x \cup y) = \varphi(x) + \varphi(y)$ whenever x, y are disjoint. Note that any σ-additive measure is l. s. c., but if φ is l. s. c. then φ_∞ is not necessarily l. s. c. itself.

Suppose that φ is a submeasure on $\mathbb{N}$. Define the *tailsubmeasure* $\varphi_\infty(x) = ||x||_\varphi = \mathtt{inf}_n(\varphi(x \cap [n, \infty)))$. The following ideals are considered:

$$
\begin{aligned}
\mathtt{Fin}_\varphi &= \{x \in \mathscr{P}(\mathbb{N}) : \varphi(x) < +\infty\} && ; \\
\mathtt{Null}_\varphi &= \{x \in \mathscr{P}(\mathbb{N}) : \varphi(x) = 0\} && ; \\
\mathtt{Exh}_\varphi &= \{x \in \mathscr{P}(\mathbb{N}) : \varphi_\infty(x) = 0\} \quad = \quad \mathtt{Null}_{\varphi_\infty} && .
\end{aligned}
$$

Example 5. $\mathtt{Fin} = \mathtt{Exh}_\varphi = \mathtt{Null}_\varphi$, where $\varphi(x) = 1$ for any $x \ne \emptyset$. We also have $0 \times \mathtt{Fin} = \mathtt{Exh}_\psi$, where $\psi(x) = \sum_k 2^{-k}\, \varphi(\{l : \langle k, l\rangle \in x\})$ is l. s. c.. □

It turns out (Solecki, see Theorem 41 below) that analytic P-ideals are the same as ideals of the form $\mathtt{Exh}_\varphi$, where φ is a l. s. c. submeasure on $\mathbb{N}$. It follows that any analytic P-ideal is $\mathbf{\Pi}^0_3$.

2.c Polishable ideals

There is one more characterization of Borel P-ideals. Let T be the ordinary Polish product topology on $\mathscr{P}(\mathbb{N})$. Then $\mathscr{P}(\mathbb{N})$ is a Polish group in the sense of T and the symmetric difference as the operation, and any ideal $\mathscr{I}$ on $\mathbb{N}$ is a subgroup of $\mathscr{P}(\mathbb{N})$.

Definition 6. An ideal $\mathscr{I}$ on $\mathbb{N}$ is *polishable* if there is a Polish group topology τ on $\mathscr{I}$ which produces the same Borel subsets of $\mathscr{I}$ as $T \restriction \mathscr{I}$. □

The same Solecki's theorem (Theorem 41) proves that, for analytic ideals, to be a P-ideal is the same as to be polishable. It follows (see Example 5) that, for instance, $\mathtt{Fin}$ and $\mathscr{I}_3 = 0 \times \mathtt{Fin}$ are polishable, but $\mathscr{I}_1 = \mathtt{Fin} \times 0$ is not. The latter will be shown directly after the next lemma.

Lemma 7. *Suppose that an ideal $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ is polishable. Then there is only one Polish group topology τ on $\mathscr{I}$. This topology refines $T \restriction \mathscr{I}$ and is metrizable by a Δ-invariant metric. If $Z \in \mathscr{I}$ then $\tau \restriction \mathscr{P}(Z)$ coincides with $T \restriction \mathscr{P}(Z)$. In addition, $\mathscr{I}$ itself is T-Borel.*

Proof. Let τ witness that $\mathscr{I}$ is polishable. The identity map $f(x) = x \colon \langle \mathscr{I} ; \tau \rangle \to \langle \mathscr{P}(\mathbb{N}) ; T \rangle$ is a Δ-homomorphism and is Borel-measurable because all $(T \restriction \mathscr{I})$-open sets are τ-Borel, hence, by the Pettis theorem (Kechris [26, ??]), f is continuous. It follows that all $(T \restriction \mathscr{I})$-open subsets of $\mathscr{I}$ are τ-open, and that $\mathscr{I}$ is T-Borel in $\mathscr{P}(\mathbb{N})$ because $1-1$ continuous images of Borel sets are Borel.

A similar "identity map" argument shows that τ is unique if exists.

It is known (Kechris [26,]) that any Polish group topology admits a left-invariant compatible metric, which, in this case, is right-invariant as well since Δ is an abelian operation.

Let $Z \in \mathscr{P}(\mathbb{N})$. Then $\mathscr{P}(Z)$ is T-closed, hence, τ-closed by the above, subgroup of $\mathscr{I}$, and $\tau \restriction \mathscr{P}(Z)$ is a Polish group topology on $\mathscr{P}(Z)$. Yet $T \restriction \mathscr{P}(Z)$ is another Polish group topology on $\mathscr{P}(Z)$, with the same Borel sets. The same "identity map" argument proves that T and τ coincide on $\mathscr{P}(Z)$. □

Example 8. $\mathscr{I}_1 = \mathtt{Fin} \times 0$ is not polishable. Indeed we have $\mathtt{Fin} \times 0 = \bigcup_n W_n$, where $W_n = \{x : x \subseteq \{0, 1, ..., n\} \times \mathbb{N}\}$. Let, on the contrary, τ be a Polish group topology on $\mathscr{I}_1$. Then τ and the ordinary topology T coincide on each set W_n by the lemma, in particular, each W_n remains τ-nowhere dense in W_{n+1}, hence, in $\mathscr{I}_1$, a contradiction with the Baire category theorem for τ. □

2.d Some $\mathbf{F}_\sigma$ ideals

Any sequence $\{r_n\}_{n \in \mathbb{N}}$ of positive reals r_n with $\sum r_n = +\infty$ defines the ideal

$$\mathscr{S}_{\{r_n\}} = \{X \subseteq \mathbb{N} : \sum_{n \in X} r_n < +\infty\} = \{X : \mu_{\{r_n\}}(X) < +\infty\},$$

where $\mu_{\{r_n\}}(X) = \sum_{n\in X} r_n$. These ideals are called *summable ideals*; all of them are $\mathbf{F}_\sigma$. References [33, 35, 6]. Any summable ideal is easily a P-ideal: indeed, $\mathscr{S}_{\{r_n\}} = \mathtt{Exh}_\varphi$, where $\varphi(X) = \sum_{n\in X} r_n$ is a σ-additive measure.

Summable ideals are perhaps the easiest to study among all P-ideals. Further entries: 1) Farah [6, § 1.12] on summable ideals under $\le_{\rm BE}$, 2) Hjorth: $\le_{\rm B}$-structure of ideals $\le_{\rm B}$-reducible to summable ideals, in [13].

Lemma 9 (Folklore ?). *Suppose that* $r_n \ge 0$, $r_n \to 0$, *and* $\sum_n r_n = +\infty$. *Then any summable ideal* $\mathscr{I}$ *satisfies* $\mathscr{I} \le^{++}_{\rm RB} \mathscr{S}_{\{r_n\}}$.

Proof. Let $I = \mathscr{S}_{\{p_n\}}$, where $p_n \ge 0$ (no other requirements !). Under the assumptions of the lemma we can associate a finite set $w_n \subseteq \mathbb{N}$ to any n so that $\mathtt{max}\, w_n < \mathtt{min}\, w_{n+1}$ and $|r_n - \sum_{j\in w_n} r_i| < 2^{-n}$. □

Farah [6, § 1.10] defines a non-summable $\mathbf{F}_\sigma$ P-ideal as follows. Let $I_k = [2^k, 2^{k+1})$ and $\psi_k(s) = k^{-2}\,\mathtt{min}\{k, \#s\}$ for all k and $s \subseteq I_k$, and then

$$\psi(X) = \sum_{k=0}^{\infty} \psi_k(X \cap I_k) \quad \text{and} \quad \mathscr{I} = \mathtt{Fin}_\psi\ ;$$

it turns out that $\mathscr{I}$ is an $\mathbf{F}_\sigma$ P-ideal, but not summable. To show that $\mathscr{I}$ distincts from any $\mathscr{S}_{\{r_n\}}$, Farah notes that there is a set X (which depends on $\{r_n\}$) such that the differences $|\mu_{\{r_n\}}(X\cap I_k) - \psi_k(X\cap I_k)|$, $k = 0,1,2,\dots$, are unbounded.

Further entry: Farah [5, 4, 7] on *Tsirelson ideals*.

2.e Erdös – Ulam and density ideals

These are other types of Borel P-ideals. Any sequence $\{r_n\}_{n\in\mathbb{N}}$ of positive reals r_n with $\sum r_n = +\infty$ defines the ideal

$$EU_{\{r_n\}} = \left\{ x \subseteq \mathbb{N} : \lim_{n\to+\infty} \frac{\sum_{i\in x\cap[0,n)} r_i}{\sum_{i\in[0,n)} r_i} = 0 \right\}.$$

These ideals are called *Erdös – Ulam* (or: EU) *ideals*. Examples: $\mathscr{Z}_0 = EU_{\{1\}}$ and $\mathscr{Z}_{\tt log} = EU_{\{1/n\}}$.

This definition can be generalized. Let $\mathtt{supp}\,\mu = \{n : \mu(\{n\}) > 0\}$, for any measure μ on $\mathbb{N}$. Measures μ, ν are *orthogonal* if we have $\mathtt{supp}\,\mu \cap \mathtt{supp}\,\nu = \emptyset$. Now suppose that $\vec\mu = \{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of pairwise orthogonal measures on $\mathbb{N}$, with finite sets $\mathtt{supp}\,\mu_i$. Define $\varphi_{\vec\mu}(X) = \mathtt{sup}_n\, \mu_n(X)$: this is a l.s.c. submeasure on $\mathbb{N}$. Let finally $\mathscr{D}_{\vec\mu} = \mathtt{Exh}(\varphi_{\vec\mu}) = \{X : ||X||_{\varphi_\mu} = 0\}$. Ideals of this form are called *density ideals* by Farah [6, § 1.13]. This class includes all EU ideals (although this is not immediately transparent), and some other ideals: for instance, $0 \times \mathsf{Fin}$ is a density but non-EU ideal. Generally density ideals are more complicated than summables. We obtain an even wider class if the requirement, that the sets $\mathtt{supp}\,\mu_n$ are finite, is dropped: this wider family includes all summmable ideals, too.

References [21], [6, § 1.13].

Further entries: 1) Farah: structure of density ideals under $\le_{\mathrm{BE}}$, 2) Farah: c_0-equalities, 3) Relation to Banach spaces: Hjorth, SuGao.

Which ideals are both summable and density ?

2.f Some transfinite sequences of Borel ideals

We consider three interesting families of Borel ideals (mainly, non-P-ideals), united by their relation to countable ordinals. Note that the underlying sets of the ideals below are countable sets different from $\mathbb{N}$.

Fréchet ideals. This family consists of ideals Fr_ξ, $\xi < \omega_1$, obtained by inductive construction using Fubini products. We put $\mathsf{Fr}_1 = \mathtt{Fin}$ and $\mathsf{Fr}_{\xi+1} = \mathtt{Fin} \otimes \mathsf{Fr}_\xi$ for all ξ. Limit steps cause a certain problem. The most natural idea would be to define $\mathsf{Fr}_\lambda = \prod_{\xi<\lambda} \mathsf{Fr}_\xi \,/\, \mathtt{Fin}_\lambda$ for any limit λ, where $\mathtt{Fin}_\lambda$ is the ideal of all finite subsets of λ, or perhaps $\mathsf{Fr}_\lambda = \prod_{\xi<\lambda} \mathsf{Fr}_\xi \,/\, \mathtt{Bou}_\lambda$, where $\mathtt{Bou}_\lambda$ is the ideal of all bounted subsets of λ, or even $\mathsf{Fr}_\lambda = \prod_{\xi<\lambda} \mathsf{Fr}_\xi \,/\, 0$, where 0 is the ideal containing only the empty set, yet this appears not to be fully satisfactory in [19], where they define $\mathsf{Fr}_\lambda = \prod_{n\in\mathbb{N}} \mathsf{Fr}_{\xi_n} \,/\, \mathtt{Fin}$, where $\{\xi_n\}$ is a once and for all fixed cofinal increasing sequence of ordinals below λ, with understanding that the result is independent of the choice of ξ_n, modulo a certain equivalence.

Indecomposable ideals. Let $\mathtt{otp}\,X$ be the order type of $X \subseteq \mathtt{Ord}$. For any ordinals ξ, $\vartheta < \omega_1$ define:

$$\mathscr{I}_\vartheta^\xi = \{A \subseteq \vartheta : \mathtt{otp}\,A < \omega^\xi\} \quad (\text{nontrivial only if } \vartheta \ge \omega^\xi)\,.$$

To see that the sets $\mathscr{I}_\vartheta^\xi$ are really ideals note that ordinals of the form ω^ξ and only those ordinals are *indecomposable*, *i.e.*, are not sums of a pair of smaller ordinals, hence, the set $\{A \subseteq \vartheta : \mathtt{otp}\,A < \gamma\}$ is an ideal iff $\gamma = \omega^\xi$ for some ξ.

Weiss ideals. Let $|X|_{\mathrm{CB}}$ be the *Cantor-Bendixson rank* of $X \subseteq \mathtt{Ord}$, *i.e.*, the least ordinal α such that $X^{(\alpha)} = \emptyset$. Here $X^{(\alpha)}$ is defined by induction on α: $X^{(0)} = X$, $X^{(\lambda)} = \bigcap_{\alpha<\lambda} X^{(\alpha)}$ at limit steps λ, and finally $X^{(\alpha+1)} = (X^{(\alpha)})'$, where A', the Cantor-Bendixson derivative, is the set of all ordinals $\gamma \in x$ which are limit points of X in the interval topology. For any ordinals ξ, $\vartheta < \omega_1$ define:

$$\mathscr{W}_\vartheta^\xi = \{A \subseteq \vartheta : |A|_{\mathrm{CB}} < \omega^\xi\} \quad (\text{nontrivial only if } \vartheta \ge \omega^{\omega^\xi})\,.$$

It is less transparent that all $\mathscr{W}_\vartheta^\xi$ are ideals (Weiss, see Farah [6, § 1.14]) while $\{A \subseteq \vartheta : |A|_{\mathrm{CB}} < \gamma\}$ is not an ideal if γ is not of the form ω^ξ.

2.g "Other" ideals

This title intends to include those interesting ideals which have not yet been subject of comprehensive study. A common method to obtain interesting ideals is to consider a countable set bearing a nontrivial structure, as the underlying set. In principle, there is no difference between different countable set as which of them is taken as the underlying set for the ideals considered. Yet if the set bears a nontrivial structure (*i.e.*, more than just countability) then this gives additional insights as which ideals are meaningful. This is already transparent for the ideals defined in § 2.f.

We give two examples.

Ideals on finite sequences. The set $\mathbb{N}^{<\omega}$ of all finite sequences of natural numbers is countable, yet its own order structure is quite different from that of $\mathbb{N}$. We can exploit this in several ways, for instance, with ideals of sets $X \subseteq \mathbb{N}^{<\omega}$ which intersect every branch in $\mathbb{N}^{<\omega}$ by a set which belongs to a given ideal on $\mathbb{N}$.

3 Introduction to equivalence relations

The structure of Borel and analytic ERs under $\le_{\mathrm{B}}$ includes key ERs which play distinguished role. The plan of this section is to define some of them and outline their properties, then introduce some classes of ERs.

3.a Basic equivalence relations

Equalities can be considered as the most elementary type of ERs. Let $\mathsf{D}(X)$ denote the equality on a set X, considered as an equivalence relation on X.

A much more diverse family is made of equivalence relations generated by ideals. Recall that for any ideal $\mathscr{I}$ on a set A, $\mathsf{E}_{\mathscr{I}}$ is an ER on $\mathscr{P}(A)$, defined so that $X \mathrel{\mathsf{E}_{\mathscr{I}}} Y$ iff $X \mathbin{\Delta} Y \in \mathscr{I}$. Equivalently, $\mathsf{E}_{\mathscr{I}}$ can be considered as an ER on 2^A defined so that $f \mathrel{\mathsf{E}_{\mathscr{I}}} g$ iff $f \mathbin{\Delta} g \in \mathscr{I}$, where $f \mathbin{\Delta} g = \{a \in A : f(a) \ne g(a)\}$. Note that $\mathsf{E}_{\mathscr{I}}$ is Borel provided so is $\mathscr{I}$.

This leads us to the following all-important ERs:

- $\mathsf{E}_0 = \mathsf{E}_{\mathtt{Fin}}$, thus, E_0 is a ER on $\mathscr{P}(\mathbb{N})$ and $x \mathrel{\mathsf{E}_0} y$ iff $x \mathbin{\Delta} y \in \mathtt{Fin}$.
- $\mathsf{E}_1 = \mathsf{E}_{\mathscr{I}_1}$, thus, E_1 is a ER on $\mathscr{P}(\mathbb{N} \times \mathbb{N})$ and $x \mathrel{\mathsf{E}_0} y$ iff $(x)_k = (y)_k$ for all but finite k, where, we recall, $(x)_k = \{n : \langle k, n\rangle \in x\}$ for $x \subseteq \mathbb{N} \times \mathbb{N}$.
- $\mathsf{E}_2 = \mathsf{E}_{\mathscr{I}_2}$, thus, E_2 is a ER on $\mathscr{P}(\mathbb{N})$ and $x \mathrel{\mathsf{E}_2} y$ iff $\sum_{k \in x \Delta y} k^{-1} < \infty$.
- $\mathsf{E}_3 = \mathsf{E}_{\mathscr{I}_3}$, thus, E_1 is a ER on $\mathscr{P}(\mathbb{N} \times \mathbb{N})$ and $x \mathrel{\mathsf{E}_3} y$ iff $(x)_k \mathrel{\mathsf{E}_0} (y)_k$, $\forall\, k$.

Alternatively, E_0 can be viewed as an equivalence relation on $2^{\mathbb{N}}$ defined as $a \mathrel{\mathsf{E}_1} b$ iff $a(k) = b(k)$ for all but finite k. Similarly, E_1 can be viewed as a ER on

$\mathscr{P}(\mathbb{N})^{\mathbb{N}}$, or even on $(2^{\mathbb{N}})^{\mathbb{N}}$, defined as $x \mathsf{E}_1 y$ iff $x(k) = y(k)$ for all but finite k, for all $x, y \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$, while E_3 can be viewed as a ER on $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$, or on $(2^{\mathbb{N}})^{\mathbb{N}}$, defined as $x \mathsf{E}_3 y$ iff $x(k) \mathsf{E}_0 y(k)$ for all k.

Relations of the form $\mathsf{E}_{\mathscr{I}}$ are special case of a wider family of ERs induced by group actions, see § 3.d below.

The main structure relation between Borel equivalence relations is $\le_{\mathrm{B}}$, Borel reducibility. Some variations (see § 1.a) are involved in special cases.

Definition 10. A Borel equivalence relation E on a space $\mathbb{X}$ is:

– *countable*, if every E-class $[x]_{\mathsf{E}} = \{y \in \mathbb{X} : x \mathsf{E} y\}$, $x \in \mathbb{X}$, is countable;

– *essentially countable*, if $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$, where F is a countable Borel ER; ←*Is any ctble $\mathbf{\Sigma}^1_1$ ER actually Borel ?*⊣

– *finite*, if every E-class $[x]_{\mathsf{E}} = \{y \in \mathbb{X} : x \mathsf{E} y\}$, $x \in \mathbb{X}$, is finite;

– *hyperfinite*, if $\mathsf{E} = \bigcup_n \mathsf{E}_n$ for an increasing sequence of Borel finite ERs E_n;

– *smooth*, if $\mathsf{E} \le_{\mathrm{B}} \mathsf{D}(2^{\mathbb{N}})$ — then E is obviously Borel;

– *hypersmooth*, if $\mathsf{E} = \bigcup_n \mathsf{E}_n$ for an increasing sequence of smooth ERs E_n. □

Countable equivalence relations form a widely studied family.

- E_∞ is the $\le_{\mathrm{B}}$-largest, or *universal countable Borel ER*.

See Theorem 31 on the existence and exact definition of E_∞.

The next group includes equivalence relations induced by actions of (the additive groups of) some Banach spaces, in particular the following ones well known from textbooks:

$$\begin{array}{rcl@{\qquad}rcl}
\boldsymbol{\ell}^p & = & \{x \in \mathbb{R}^{\mathbb{N}} : \sum_n |x_n|^p < \infty\} \ (p \ge 1); & \|x\|_p & = & (\sum_n |x_n|^p)^{\frac{1}{p}}\,; \\
\boldsymbol{\ell}^\infty & = & \{x \in \mathbb{R}^{\mathbb{N}} : \mathtt{sup}_n |x_n| < \infty\}; & \|x\|_\infty & = & \mathtt{sup}_n |x_n|\,; \\
\mathbf{c} & = & \{x \in \mathbb{R}^{\mathbb{N}} : \mathtt{lim}_n x_n < \infty \text{ exists}\}; & \|x\| & = & \mathtt{sup}_n |x_n|\,; \\
\mathbf{c}_0 & = & \{x \in \mathbb{R}^{\mathbb{N}} : \mathtt{lim}_n x_n = 0\}; & \|x\| & = & \mathtt{sup}_n |x_n|\,.
\end{array}$$

Note that $\boldsymbol{\ell}^p$, $\mathbf{c}$, $\mathbf{c}_0$ are separable while $\boldsymbol{\ell}^\infty$ is non-separable. The domain of each of the four spaces consists of infinite sequences $x = \{x_n\}_{n \in \mathbb{N}}$ of reals, and is a subgroup of the group $\mathbb{R}^{\mathbb{N}}$ (with the componentwise addition). The latter can be naturally equipped with the Polish product topology, so that $\boldsymbol{\ell}^p$, $\boldsymbol{\ell}^\infty$, $\mathbf{c}$, $\mathbf{c}_0$ are Borel subgroups of $\mathbb{R}^{\mathbb{N}}$. (But not topological subgroups since the distances are different. The metric definitions as in $\boldsymbol{\ell}^p$ or $\boldsymbol{\ell}^\infty$ do not work for $\mathbb{R}^{\mathbb{N}}$.)

Each of the four mentioned Banach spaces defines an orbit equivalence — a Borel equivalence relation on $\mathbb{R}^{\mathbb{N}}$ also denoted by, resp., $\boldsymbol{\ell}^p$, $\boldsymbol{\ell}^\infty$, $\mathbf{c}$, $\mathbf{c}_0$. For instance, $x \,\boldsymbol{\ell}^p\, y$ if and only if $\sum_k |x_k - y_k|^p < +\infty$ (for all $x, y \in \mathbb{R}^{\mathbb{N}}$). It is known (see Section 4) that $\boldsymbol{\ell}^1 \sim_{\mathrm{B}} \mathsf{E}_2$ and $\boldsymbol{\ell}^p <_{\mathrm{B}} \boldsymbol{\ell}^q$ whenever $1 \le p < q$, in particular, $\boldsymbol{\ell}^1 \sim_{\mathrm{B}} \mathsf{E}_2 <_{\mathrm{B}} \boldsymbol{\ell}^q$ for any $q > 1$. On the other hand, $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$, where Z_0 is the "density 0" equivalence relation:

- $\mathsf{Z}_0 = \mathsf{E}_{\mathscr{Z}_0}$, thus, for x, y in $\mathscr{P}(\mathbb{N})$, $x\,\mathsf{Z}_0\,y$ iff $\mathtt{lim}_{n\to\infty} \frac{\#(x\Delta y)}{n} = 0$.

Another important ER is

- T_2, often called "the equality of countable sets of reals".

There is no reasonable way to turn $\mathscr{P}_{\mathtt{ctbl}}(\mathbb{N}^{\mathbb{N}})$, the set of all at most countable subsets of $\mathbb{N}^{\mathbb{N}}$, into a Polish space, in order to directly define the equality of countable sets of reals in terms of $\mathsf{D}(\cdot)$. However, nonempty members of $\mathscr{P}_{\mathtt{ctbl}}(\mathbb{N}^{\mathbb{N}})$ can be identified with equivalence classes in $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}/\mathsf{T}_2$, where $g\,\mathsf{T}_2\,h$ iff $\mathtt{ran}\,g = \mathtt{ran}\,h$: for g, $h \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$. (See below in Section 10 on equivalence relations T_α for all $\alpha < \omega_1$.)

In addition to the families of equivalence relations introduced by Definition 10, some more complicated families will be considered below, including ERs induced by Polish group actions, turbulent ERs, ERs classifiable by countable structures, pinned ERs, and some more.

3.b Borel reducibility of basic equivalence relations

The diagram on page 16 begins, at the low end, with cardinals $1 \le n \in \mathbb{N}$, $\aleph_0$, $\mathfrak{c}$, which denote the ERs of equality on resp. finite, countable, uncountable Polish spaces. As all uncountable Polish spaces are Borel isomorphic, the equivalence relations $\mathsf{D}(\mathbb{X})$, $\mathbb{X}$ a Polish space, are characterized, modulo $\le_{\mathrm{B}}$, or even modulo Borel isomorphism between the domains, by the cardinality of the domain, which can be any finite $1 \le n < \omega$, or $\aleph_0$, or $\mathfrak{c} = 2^{\aleph_0}$.

The E_0 splitting is the key element of the diagram on page 16. That $\mathsf{D}(2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E}_0$ can be proved by a rather simple embedding while the strictness can be derived from an old result of Sierpiński [39]: any linear ordering of all E_0-classes yields a Lebesgue non-measurable set of the same descriptive complexity as the ordering. That every ER $\le_{\mathrm{B}} \mathsf{E}_0$ is $\sim_{\mathrm{B}}$ to some $n \ge 1$, $\mathsf{D}(\mathbb{N})$, $\mathsf{D}(2^{\mathbb{N}})$, or E_0 itself, is witnessed by the following two classical results:

1st dichotomy (Thm 29 below). *Any Borel, even any* $\mathbf{\Pi}^1_1$ *ER* E *<u>either</u> has at most countably many equivalence classes, formally,* $\mathsf{E} \le_{\mathrm{B}} \aleph_0 = \mathsf{D}(\mathbb{N})$, *<u>or</u> satisfies* $\mathfrak{c} = \mathsf{D}(2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E}$.

2nd dichotomy (Thm 35). *Any Borel ER* E *satisfies either* $\mathsf{E} \le_{\mathrm{B}} \mathfrak{c}$ *or* $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E}$.

The linearity breaks above E_0 : each one of the four equivalence relations E_1, E_2, E_3, E_∞ of the next level is strictly $<_{\mathrm{B}}$-bigger than E_0, and they are pairwise $\le_{\mathrm{B}}$-incomparable with each other, see § **??**.

One naturally asks what is going on in the intervals between E_0 and these four equivalence relations. The following results provide some answers.

3rd dichotomy (Thm 46). *Any ER* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_1$ *satisfies* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ *or* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_1$.

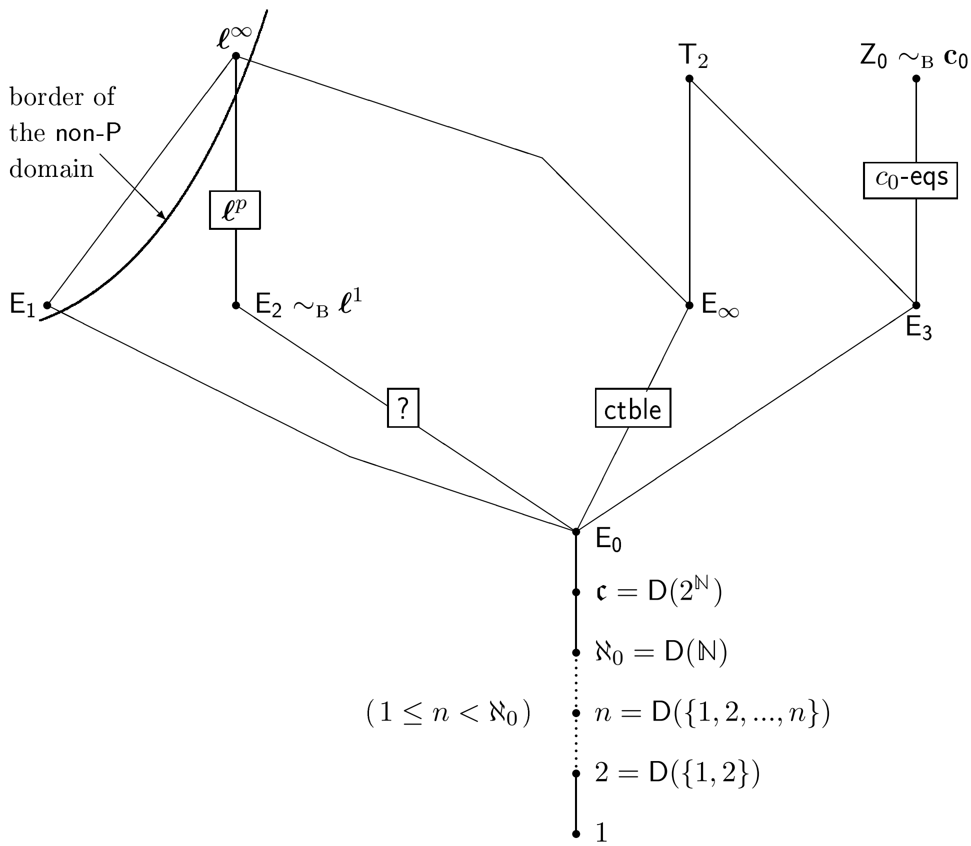

Рис. 1: Reducibility between some basic ERs
Connecting lines here indicate Borel reducibility of lower ERs to upper ones.

4th dichotomy (Thm 67). *Any ER* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_2$ *either is essentially countable or satisfies* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_2$.

See Definition 10 regarding essentially countable ERs in the 4th dichotomy. The "either" case there remains mysterious: any countable Borel ERs $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_2$ known so far are $\le_{\mathrm{B}} \mathsf{E}_0$. It is a problem whether the "either" case can be improved to $\le_{\mathrm{B}} \mathsf{E}_0$. This is marked by the framebox $\boxed{?}$ on the diagram.

The fifth dichotomy theorem is a bit more special, it will be addressed below.

6th dichotomy (Thm 64). *Any ER* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_3$ *satisfies* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ *or* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_3$.

Adams–Kechris theorem (not to be proved here). *There is continuum many pairwise* $\le_{\mathrm{B}}$*-incomparable countable Borel ERs.*

The framebox $\boxed{c_0\text{-eqs}}$ denotes c_0*-equalities*, a family of Borel ERs introduced by Farah [7], all of them are $\le_{\mathrm{B}}$-between E_3 and $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$, and there is continuum-many $\le_{\mathrm{B}}$-incomparable among them.

The non-P domain denotes the family of all ERs $\mathsf{E}_{\mathscr{I}}$, where $\mathscr{I}$ is a Borel ideal which is not a P-ideal. By Solecki [42, 43], for a Borel ideal $\mathscr{I}$ to be not a P-ideal it is necessary and sufficient that $\mathscr{I}_1 \le_{\mathrm{B}} \mathscr{I}$, or, equivalently, $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_{\mathscr{I}}$.

← E_1 and Polish grps action problem⊣

Question 1. It there any reasonable "basis" of Borel ERs above E_0? □

It was once considered [16] as a plausible hypothesis that any Borel ER which is not $\le_{\mathrm{B}} \mathsf{E}_\infty$, *i.e.*, not an essentially countable ER, satisfies $\mathsf{E}_i \le_{\mathrm{B}} \mathsf{E}$ for at least one $i = 1, 2, 3$. This turns out to be not the case: Farah [4, 5] and Velickovic [46] found an independent family of uncountable Borel ERs, based on *Tsirelson ideals*, $\le_{\mathrm{B}}$-incomparable with E_1, E_2, E_3, see below.

It is the most interesting question whether the diagram on page 16 is complete in the sense that there is no $\le_{\mathrm{B}}$-connections betwen the equivalence relations mentioned in the diagram except for those explicitly indicated by lines. Basically, one may want to prove the following non-reducibility claims:

(1) $\mathsf{E}_1 \quad \not\le_{\mathrm{B}} : \quad \mathsf{E}_2, \quad \mathsf{T}_2, \quad \mathbf{c}_0;$

(2) $\boldsymbol{\ell}^\infty \quad \not\le_{\mathrm{B}} : \quad \mathsf{E}_1, \quad \mathsf{E}_2, \quad \mathsf{T}_2, \quad \mathbf{c}_0;$

(3) $\mathsf{E}_2 \quad \not\le_{\mathrm{B}} : \quad \mathsf{E}_1, \quad \mathsf{T}_2, \quad \mathbf{c}_0;$

(4) $\mathsf{E}_\infty \quad \not\le_{\mathrm{B}} : \quad \mathsf{E}_1, \quad \mathsf{E}_2, \quad \mathbf{c}_0;$

(5) $\mathsf{E}_3 \quad \not\le_{\mathrm{B}} : \quad \boldsymbol{\ell}^\infty;$

(6) $\mathsf{T}_2 \quad \not\le_{\mathrm{B}} : \quad \boldsymbol{\ell}^\infty, \quad \mathbf{c}_0;$

(7) $\mathbf{c}_0 \quad \not\le_{\mathrm{B}} : \quad \boldsymbol{\ell}^\infty, \quad \mathsf{T}_2.$

Beginning with (1), we note that E_1 is not Borel reducible to any equivalence relation induced by a Polish action (of a Polish group) by Theorem 48 below. On the other hand, E_2, T_2, $\mathbf{c}_0$ obviously belong to this category of ERs.

(2) follows from (1) and (3) and can be omitted.

In (3), $\mathsf{E}_2 \not\le_{\mathrm{B}} \mathsf{E}_1$ can be proved by an argument rather similar to the proof of Theorem 22. Alternatively, it will follow from Theorem 40 that any Borel ideal $\mathscr{I}$ with $\mathsf{E}_{\mathscr{I}} \le_{\mathrm{B}} \mathsf{E}_1$ is isomorphic, via a bijection between the underlying sets, to $\mathscr{I}_1$ or to a trivial variation of $\mathtt{Fin}$, but $\mathscr{I}_2$ does not belong to this category. The result $\mathsf{E}_2 \not\le_{\mathrm{B}} \mathbf{c}_0$ in (3) is Theorem 22(ii).

The results $\mathsf{E}_2 \not\le_{\mathrm{B}} \mathsf{T}_2$ and $\mathbf{c}_0 \not\le_{\mathrm{B}} \mathsf{T}_2$ in (3) and (7) are proved below in Section 11 (Corollary 60); this will involve the turbulence theory.

The result of (5) is Lemma 15. It implies $\mathbf{c}_0 \not\le_{\mathrm{B}} \boldsymbol{\ell}^\infty$ in (7).

(6) will be established in Section 15.

This leaves us with (4). We don't know how to prove $\mathsf{E}_\infty \not\le_{\mathrm{B}} \mathsf{E}_1$ easily and directly. The indirect way is to use Theorem 46 below, according to which $\mathsf{E}_\infty \le_{\mathrm{B}} \mathsf{E}_1$ would imply either $\mathsf{E}_\infty \sim_{\mathrm{B}} \mathsf{E}_1$ — impossible, see above, or $\mathsf{E}_\infty \le \mathsf{E}_0$. The latter conclusion is also a contradiction since $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_\infty$ is known in the theory of countable Borel equivalence relations (see [2, p. 210]).

Question 2. Is E_∞ Borel reducible to $\mathbf{c}_0$? to $\boldsymbol{\ell}^1$ or any other $\boldsymbol{\ell}^p$? □

3.c Operations on equivalence relations

The following operations over ERs are in part parallel to the operations on ideals in § 2.a.

(o1) countable union (if it results in a ER) and countable intersection of ERs on one and the same space;

(o2) countable disjoint union $\mathsf{E} = \bigvee_k \mathsf{E}_k$ of ERs E_k on Polish spaces $\mathbb{S}_k$, that is, a ER on $\mathbb{S} = \bigcup_k (\{k\} \times \mathbb{S}_k)$ (with the topology generated by sets of the form $\{k\} \times U$, where $U \subseteq \mathbb{S}_k$ is open) defined as follows: $\langle k, x\rangle \mathrel{\mathsf{E}} \langle l, y\rangle$ iff $k = l$ and $x \mathrel{\mathsf{E}_k} y$. (If $\mathbb{S}_k$ are pairwise disjoint and open in $\mathbb{S}' = \bigcup_k \mathbb{S}_k$ then we can equivalently define $\mathsf{E} = \bigvee_k \mathsf{E}_k$ on $\mathbb{S}'$ so that $x \mathrel{\mathsf{E}} y$ iff x, y belong to the same $\mathbb{S}_k$ and $x \mathrel{\mathsf{E}_k} y$.);

(o3) product $\mathsf{E} = \prod_k \mathsf{E}_k$ of ERs E_k on spaces $\mathbb{S}_k$, that is, the ER on the product space $\prod_k \mathbb{S}_k$ defined by: $x \mathrel{\mathsf{E}} y$ iff $x_k \mathrel{\mathsf{E}_k} y_k$ for all k.

(o4) the *Fubini product* (ultraproduct) $\prod_{k\in\mathbb{N}} \mathsf{E}_k \,/\, \mathscr{I}$ of ERs E_k on spaces $\mathbb{S}_k$, modulo an ideal $\mathscr{I}$ on $\mathbb{N}$, that is, the ER on the product space $\prod_{k\in\mathbb{N}} \mathbb{S}_k$ defined as follows: $x \mathrel{\mathsf{E}} y$ iff $\{k : x_k \not\mathrel{\mathsf{E}}_k y_k\} \in \mathscr{I}$;

(o5) *countable power* ER E^∞ of a ER E on a space $\mathbb{S}$ is a ER on $\mathbb{S}^{\mathbb{N}}$ defined as follows: $x \mathrel{\mathsf{E}^\infty} y$ iff $\{[x_k]_{\mathsf{E}} : k \in \mathbb{N}\} = \{[y_k]_{\mathsf{E}} : k \in \mathbb{N}\}$, so that for any k there is l with $x_k \mathrel{\mathsf{E}} y_l$ and for any l there is k with $x_k \mathrel{\mathsf{E}} y_l$.

These operations allow us to obtain a lot of interesting ERs starting just with very primitive ones. For instance, we can define the sequence of ERs T_ξ, $\xi < \omega_1$, of H. Friedman [9] as follows [4]. Let $\mathsf{T}_0 = \mathsf{D}(\mathbb{N})$, the equality relation on $\mathbb{N}$. We put $\mathsf{T}_{\xi+1} = \mathsf{T}_\xi{}^\infty$. If $\lambda < \omega_1$ is a limit ordinal, then put $\mathsf{T}_\lambda = \bigvee_{\xi<\lambda} \mathsf{T}_\xi$.

In particular $\mathtt{dom}\,\mathsf{T}_1 = \mathbb{N}^{\mathbb{N}}$ and $x \mathrel{\mathsf{T}_1} y$ iff $\mathtt{ran}\,x = \mathtt{ran}\,y$, for $x, y \in \mathbb{N}^{\mathbb{N}}$. Thus the map $\vartheta(x) = \mathtt{ran}\,x$ witnesses that $\mathsf{T}_1 \le_{\mathrm{B}} \mathsf{D}(\mathscr{P}(\mathbb{N}))$. To show the converse, define, for any infinite $u \subseteq \mathbb{N}$, $\beta(u)$ be the increasing bijection $\mathbb{N} \xrightarrow{\text{onto}} u$, while if $u = \{k_0, ..., k_n\}$ is finite, put $\beta(u)(i) = k_i$ for $i < n$ and $\beta(u)(i) = k_n$ for $i \ge n$. Then β witnesses $\mathsf{D}(\mathscr{P}(\mathbb{N})) \le_{\mathrm{B}} \mathsf{T}_1$, thus, $\mathsf{T}_1 \sim_{\mathrm{B}} \mathsf{D}(\mathscr{P}(\mathbb{N}))$. It easily follows that $\mathsf{T}_2 \sim_{\mathrm{B}} \mathsf{D}(\mathscr{P}(\mathbb{N}))^\infty$, in fact, $\mathsf{T}_2 \sim_{\mathrm{B}} \mathsf{D}(\mathbb{X})^\infty$ for any uncountable Polish space $\mathbb{X}$ as any such $\mathbb{X}$ is Borel isomorphic to $\mathscr{P}(\mathbb{N})$ (or to $2^{\mathbb{N}}$, which is essentially the same). With $\mathbb{X} = \mathbb{N}^{\mathbb{N}}$ we obtain the definition of T_2 in § 3.a.

3.d Orbit equivalence relations of group actions

An *action* of a group $\mathbb{G}$ on a space $\mathbb{X}$ is any map $a : \mathbb{G} \times \mathbb{X} \to \mathbb{X}$, usually written as $a(g, x) = g \cdot x$, such that 1) $e \cdot x = x$, and 2) $g \cdot (h \cdot x) = (gh) \cdot x$, — then, for any $g \in \mathbb{G}$, the map $x \mapsto g \cdot x$ is a bijection $\mathbb{X}$ onto $\mathbb{X}$ with $x \mapsto g^{-1} \cdot x$ as the

[4] Hjorth [15] uses F_ξ instead of T_ξ.

inverse map. A $\mathbb{G}$-*space* is a pair $\langle \mathbb{X}\,;a\rangle$, where a is an action of $\mathbb{G}$ on $\mathbb{X}$; in this case $\mathbb{X}$ itself is also called a $\mathbb{G}$-space, and the *orbit ER*, or *ER induced by the action*, $\mathsf{E}_a^{\mathbb{X}} = \mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is defined on $\mathbb{X}$ so that $x\,\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}\,y$ iff there is $a \in \mathbb{G}$ with $y = a\cdot x$. $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$-classes are the same as $\mathbb{G}$-orbits, *i.e.*,

$$[x]_{\mathbb{G}} = [x]_{\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}} = \{y : \exists\, g \in \mathbb{G}\ (g\cdot x = y)\}\,.$$

A *homomorphism* (or $\mathbb{G}$-homomorphism) of a $\mathbb{G}$-space $\mathbb{X}$ into a $\mathbb{G}$-space $\mathbb{Y}$ is any map $F : \mathbb{X} \to \mathbb{Y}$ compatible with the actions in the sense that $F(g\cdot x) = g\cdot F(x)$ for any $x \in \mathbb{X}$ and $g \in \mathbb{G}$. A $1-1$ homomorphism is an *embedding*. An embedding $\xrightarrow{\text{onto}}$ is an *isomorphism*. Note that a homomorphism $\langle \mathbb{X}\,;a\rangle \to \langle \mathbb{Y}\,;b\rangle$ is a reduction of $\mathsf{E}_a^{\mathbb{X}}$ to $\mathsf{E}_b^{\mathbb{Y}}$, but not conversely.

A *Polish group* is a group whose underlying set is a Polish space and the operations are continuous; a *Borel group* is a group whose underlying set is a Borel set (in a Polish space) and the operations are Borel maps. A Borel group is *Polishable* if there is a Polish topology on the underlying set which 1) produces the same Borel sets as the original topology and 2) makes the group Polish.

- If both $\mathbb{X}$ and $\mathbb{G}$ are Polish and the action continuous, then $\langle \mathbb{X}\,;a\rangle$ (and also $\mathbb{X}$) is called a *Polish* $\mathbb{G}$-space. If both $\mathbb{X}$ and $\mathbb{G}$ are Borel and the action is a Borel map, then $\langle \mathbb{X}\,;a\rangle$ (and also $\mathbb{X}$) is called a *Borel* $\mathbb{G}$-space.

Example 11. (i) Any ideal $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ is a group with Δ as the operation. We cannot expect this group to be Polish in the product topology inherited from $\mathscr{P}(\mathbb{N})$ (indeed, $\mathscr{I}$ would have to be $\mathbf{G}_\delta$). However if $\mathscr{I}$ is a P-ideal then it is Polishable (see § 2.c), in other words, $\langle \mathscr{I}\,;\Delta\rangle$ is a Polish group in an appropriate Polish topology compatible with the Borel structure of $\mathscr{I}$. Given such a topology, the Δ-action of (a P-ideal) $\mathscr{I}$ on $\mathscr{P}(\mathbb{N})$ is Polish, too. $\longleftarrow$ *correct?* $\dashv$

(ii) Consider $\mathbb{G} = \mathscr{P}_{\mathtt{fin}}(\mathbb{N})$ a countable subgroup of $\langle \mathscr{P}(\mathbb{N})\,;\Delta\rangle$. Define an action of $\mathbb{G}$ on $2^{\mathbb{N}}$ as follows: $(w\cdot x)(n) = x(n)$ whenever $n \notin w$ and $(w\cdot x)(n) = 1 - x(n)$ otherwise. The orbit equivalence relation $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ of this action is obviously E_0. Note that this action is *free*: $x = w\cdot x$ implies $w = \emptyset$ (the neutral element of $\mathbb{G}$) for any $x \in 2^{\mathbb{N}}$.

Now consider any Borel pairwise E_0-inequivalent set $T \subseteq 2^{\mathbb{N}}$. Then $w\cdot T \cap T = \emptyset$ for any $w \neq \emptyset$ by the above. It easily follows that T is meager in $2^{\mathbb{N}}$. (Otherwise T is co-meager on a basic clopen set $\mathscr{O}_s(2^{\mathbb{N}}) = \{x \in 2^{\mathbb{N}} : s \subset x\}$, where $s \in 2^{<\omega}$. Put $w = \{n\}$, where $n = \mathtt{lh}\, s$. Then $w \in \mathbb{G}$ maps $T \cap \mathscr{O}_{s^\wedge 0}(2^{\mathbb{N}})$ onto $T \cap \mathscr{O}_{s^\wedge 1}(2^{\mathbb{N}})$. Thus $w\cdot T \cap T \neq \emptyset$ – contradiction.) We conclude that $\mathbb{G}\cdot T = \bigcup_{w \in \mathbb{G}} w\cdot T$ is still a meager subset of $2^{\mathbb{N}}$ in this case, and hence T cannot be a full (Borel) transversal for E_0.

(iii) The *canonical* (or *shift*) *action* of a group $\mathbb{G}$ on a set of the form $X^{\mathbb{G}}$ (X any set) is defined as follows: $g\cdot\{x_f\}_{f\in\mathbb{G}} = \{x_{g^{-1}f}\}_{f\in\mathbb{G}}$ for any element $\{x_f\}_{f\in\mathbb{G}} \in X^{\mathbb{G}}$ and any $g \in \mathbb{G}$. This is easily a Polish action provided $\mathbb{G}$ is countable, X a Polish space, and $X^{\mathbb{G}}$ given the product topology. The equivalence relation on $X^{\mathbb{G}}$ induced by this action is denoted by $\mathsf{E}(\mathbb{G}, X)$. □

The next theorem (rather difficult to be proved here) shows that the type of the group is the essential component in the difference between Polish and Borel actions: roughly, any Borel action of a Polish group $\mathbb{G}$ is a Polish action of $\mathbb{G}$.

Theorem 12 ([1, 5.2.1])**.** *Suppose that $\mathbb{G}$ is a Polish group and $\langle \mathbb{X}\,; a\rangle$ is a Borel $\mathbb{G}$-space. Then $\mathbb{X}$ admits a Polish topology which 1) produces the same Borel sets as the original topology, and 2) makes the action to be Polish.* □

If $\langle \mathbb{X}\,; a\rangle$ is a Borel $\mathbb{G}$-space (and $\mathbb{G}$ is a Borel group) then $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is easily a $\mathbf{\Sigma}^1_1$ ER on $\mathbb{X}$. Sometimes $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is even Borel: for instance, when $\mathbb{G}$ is a countable group and the action is Borel, or if $\mathbb{G} = \mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ is a Borel ideal, considered as a group with Δ as the operation, which acts on $\mathbb{X} = \mathscr{P}(\mathbb{N})$ by Δ, so that $\mathsf{E}_{\mathbb{G}}^{\mathscr{P}(\mathbb{N})} = \mathsf{E}_{\mathscr{I}}$ is Borel because $x \,\mathsf{E}_{\mathbb{G}}^{\mathscr{P}(\mathbb{N})}\, y$ iff $x \,\Delta\, y \in \mathscr{I}$. Several much less trivial cases when $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is Borel are described in [1, Chapter 7], for instance, if all $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$-classes are Borel sets of bounded rank then $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is Borel [1, 7.1.1]. Yet rather surprisingly equivalence classes generated by Borel actions are always Borel.

Theorem 13 (see [26, 15.14])**.** *If $\mathbb{G}$ is a Polish group and $\langle \mathbb{X}\,; a\rangle$ is a Borel $\mathbb{G}$-space then every equivalence class of $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is Borel.*

Proof. It can be assumed, by Theorem 12, that the action is continuous. Then for any $x \in \mathbb{X}$ the *stabilizer* $\mathbb{G}_x = \{g : g \cdot x = x\}$ is a closed subgroup of $\mathbb{G}$. [5] We can consider $\mathbb{G}_x$ as continuously acting on $\mathbb{G}$ by $g \cdot h = gh$ for all $g, h \in \mathbb{G}$. Let F denote the associated orbit ER. Then every F-class $[g]_{\mathsf{F}} = g\,\mathbb{G}_x$ is a shift of $\mathbb{G}_x$, hence, $[g]_{\mathsf{F}}$ is closed. On the other hand, the saturation $[\mathscr{O}]_{\mathsf{F}}$ of any open set $\mathscr{O} \subseteq \mathbb{G}$ is obviously open. Therefore, by Lemma 27(iv) below, F admits a Borel transversal $S \subseteq \mathbb{G}$. Yet $g \longmapsto g \cdot x$ is a Borel $1-1$ map of a Borel set S onto $[x]_{\mathsf{E}}$, hence, $[x]_{\mathsf{E}}$ is Borel by Countable-to-1 Projection. □

$\longleftarrow$ Quotient spaces ?⊣

It follows that not all $\mathbf{\Sigma}^1_1$ ERs are orbit ERs of Borel actions of Polish groups: indeed, take a non-Borel $\mathbf{\Sigma}^1_1$ set $X \subseteq \mathbb{N}^{\mathbb{N}}$, define $x \mathsf{E} y$ if either $x = y$ or $x, y \in X$, this is a $\mathbf{\Sigma}^1_1$ ER with a non-Borel class X.

$\longleftarrow$ $\mathbf{\Sigma}^1_1$ or Borel ER not induced by Borel grp ?⊣

$\longleftarrow$ Borel ER not induced by Polish grp ?⊣

[5] Kechris [26, 9.17] gives an independent proof. Both $\mathbb{G}_x$ and its topological closure, say, G' are subgroups, moreover, G' is a closed subgroup, hence, we can assume that $G' = \mathbb{G}$, in other words, that $\mathbb{G}_x$ is dense in $\mathbb{G}$, and the aim is to prove that $\mathbb{G}_x = \mathbb{G}$. By a simple argument, $\mathbb{G}_x$ is either comeager or meager in $\mathbb{G}$. But a comeager subgroup easily coincides with the whole group, hence, assume that $\mathbb{G}_x$ is meager (and dense) in $\mathbb{G}$ and draw a contradiction.

Let $\{V_n\}_{n \in \mathbb{N}}$ be a basis of the topology of $\mathbb{X}$, and $A_n = \{g \in \mathbb{G} : g \cdot x \in V_n\}$. Easily $A_n h = A_n$ for any $h \in \mathbb{G}_x$. It follows, because $\mathbb{G}_x$ is dense, that every A_n is either meager or comeager. Now, if $g \in \mathbb{G}$ then $\{g\} = \bigcap_{n \in N(g)} A_n$, where $N(g) = \{n : g \cdot x \in V_n\}$, thus, at least one of sets A_n containing g is meager. It follows that $\mathbb{G}$ is meager, contradiction.

3.e Forcings associated with pairs of equivalence relations

The range of applications of this comparably new topic is not yet clear, but at least it offers interesting technicalities.

Definition 14 (Zapletal [47])**.** Suppose that E is a Borel equivalence relation on a Polish space $\mathbb{X}$, and $\mathsf{F} <_{\mathrm{B}} \mathsf{E}$ is another Borel equivalence relation.

$\mathscr{I}_{\mathsf{E}/\mathsf{F}}$ is the collection of all Borel sets $X \subseteq \mathbb{X}$ such that $\mathsf{E} \restriction X \le_{\mathrm{B}} \mathsf{F}$. Clearly $\mathscr{I}_{\mathsf{E}/\mathsf{F}}$ is an ideal in the algebra of all Borel subsets of $\mathbb{X}$. The associated forcing $\mathbb{P}_{\mathsf{E}/\mathsf{F}}$ consists of all Borel sets $X \subseteq \mathbb{X}\tilde{t}X \notin \mathscr{I}_{\mathsf{E}/\mathsf{F}}$. □

For instance, the ideal $\mathscr{I}_{\mathsf{D}(2^{\mathbb{N}})/\mathsf{D}(\mathbb{N})}$ consists of all countable Borel sets $X \subseteq 2^{\mathbb{N}}$, therefore $\mathbb{P}_{\mathsf{D}(2^{\mathbb{N}})/\mathsf{D}(\mathbb{N})}$ contains all <u>un</u>countable Borel sets $X \subseteq 2^{\mathbb{N}}$ and is equal to the Sacks forcing. The ideal $\mathscr{I}_{\mathsf{E}_0/\mathsf{D}(2^{\mathbb{N}})}$ consists of all Borel sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_0 \restriction X$ is non-smooth (since smoothness is equivalent to being $\le_{\mathrm{B}} \mathsf{D}(2^{\mathbb{N}})$). See §7.e on the associated forcing $\mathbb{P}_{\mathsf{E}_0/\mathsf{D}(2^{\mathbb{N}})}$.

4 "Elementary" stuff

This Section gathers proofs of some reducibility/irreducibility results related to the diagram on page 16, elementary in the sense that they do not involve any special concepts. Some of them are really simple, some other quite tricky.

4.a E_3 and T_2: outcasts

These equivalence relations, together with $\mathbf{c}_0 \sim_{\mathrm{B}} \mathsf{Z}_0$, are the only non-$\mathbf{\Sigma}^0_2$ equivalences explicitly mentioned on the diagram.

Lemma 15. *E_3 is Borel irreducible to $\boldsymbol{\ell}^\infty$.*

Proof. Suppose towards the contrary that $\vartheta : 2^{\mathbb{N}\times\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ is a Borel reduction of E_3 to $\boldsymbol{\ell}^\infty$. [6] Since obviously $\boldsymbol{\ell}^\infty \sim_{\mathrm{B}} \boldsymbol{\ell}^\infty \times \boldsymbol{\ell}^\infty$, Lemma 1 reduces the general case to the case of continuous ϑ. Define $\mathbf{0}, \mathbf{1} \in 2^{\mathbb{N}}$ by $\mathbf{0}(n) = 0$, $\mathbf{1}(n) = 1$, $\forall n$. Define $\mathbb{0} \in 2^{\mathbb{N}\times\mathbb{N}}$ by $\mathbf{0}(k,n) = 0$ for all k, n, thus $(\mathbb{0})_k = \mathbf{0}$, $\forall k$. Finally, for any k define $\mathbf{z}_k \in 2^{\mathbb{N}}$ by $\mathbf{z}_k(n) = 1$ for $n < k$ and $\mathbf{z}_k(n) = 0$ for $n \ge k$.

We claim that there are increasing sequences of natural numbers $\{k_m\}$ and $\{j_m\}$ such that $|\vartheta(x)(j_m) - \vartheta(\mathbb{0})(j_m)| > m$ for any m and any $x \in 2^{\mathbb{N}\times\mathbb{N}}$ satisfying

$$(x)_k = \begin{cases} \mathbf{z}_{k_i} & \text{whenever } i < m \text{ and } k = k_i \\ \mathbf{0} & \text{for all } k < k_m \text{ not of the form } k_i. \end{cases}$$

[6] Recall that, for $x, y \in 2^{\mathbb{N}\times\mathbb{N}}$, $x \mathrel{\mathsf{E}_3} y$ means $(x)_k \mathrel{\mathsf{E}_0} (y)_k$, $\forall k$, where $(x)_k \in 2^{\mathbb{N}}$ is defined by $(x)_k(n) = x(k,n)$ for all n while $a \mathrel{\mathsf{E}_0} b$ means that $a \,\Delta\, b = \{m : a(m) \ne b(m)\}$ is finite.

To see that this implies contradiction define $x \in 2^{\mathbb{N}\times\mathbb{N}}$ so that $(x)_{k_i} = \mathbf{z}_{k_i}$, $\forall i$ and $(x)_k = \mathbf{0}$ whenever k does not have the form k_i. Then obviously $x \mathsf{E}_3 \mathbb{0}$, but $|\vartheta(x)(j_m) - \vartheta(\mathbb{0})(j_m)| > m$ for all m, hence $\vartheta(x) \boldsymbol{\ell}^\infty \vartheta(\mathbb{0})$ fails, as required.

We put $k_0 = 0$. To define j_0 and k_1, consider $x_0 \in 2^{\mathbb{N}\times\mathbb{N}}$ defined by $(x_0)_0 = \mathbf{1}$ but $(x_0)_k = \mathbf{0}$ for all $k \geq 1$. Then $x_0 \mathsf{E}_3 \mathbb{0}$ fails, and hence $\vartheta(x_0) \boldsymbol{\ell}^\infty \vartheta(\mathbb{0})$ fails either. Take any j_0 with $|\vartheta(x_0)(j_0) - \vartheta(\mathbb{0})(j_0)| > 0$. As ϑ is continuous, there is a number $k_1 > 0$ such that $|\vartheta(x)(j_0) - \vartheta(\mathbb{0})(j_0)| > 0$ holds for any $x \in 2^{\mathbb{N}\times\mathbb{N}}$ with $(x)_0 = \mathbf{z}_{k_1}$ and $(x)_k = \mathbf{0}$ for all $0 < k < k_1$.

To define j_1 and k_2, consider $x_1 \in 2^{\mathbb{N}\times\mathbb{N}}$ defined so that $(x_1)_0 = \mathbf{z}_{k_1}$, $(x_1)_k = \mathbf{0}$ whenever $0 < k < k_1$, and $(x_1)_{k_1} = \mathbf{1}$. Once again there is a number j_1 with $|\vartheta(x_1)(j_1) - \vartheta(\mathbb{0})(j_1)| > 1$, and a number $k_2 > k_1$ such that $|\vartheta(x)(j_1) - \vartheta(\mathbb{0})(j_1)| > 1$ for any $x \in 2^{\mathbb{N}\times\mathbb{N}}$ with $(x)_0 = \mathbf{z}_{k_1}$, $(x)_{k_1} = \mathbf{z}_{k_1}$, and $(x)_k = \mathbf{0}$ for all $0 < k < k_1$ and $k_1 < k < k_2$.

Et cetera. $\square$

Lemma 16. E_3 *is Borel reducible to both* T_2 *and* $\mathbf{c}_0$.

Proof. (1) If $a \in 2^{\mathbb{N}}$ and $s \in 2^{<\omega}$ then define $sx \in 2^{\mathbb{N}}$ by $(sx)(k) = x(k) +_2 s(k)$ for $k < \mathtt{lh}\, s$ and $(sx)(k) = x(k)$ for $k \geq \mathtt{lh}\, s$. If $m \in \mathbb{N}$ then $m^\wedge x \in 2^{\mathbb{N}}$ denotes the concatenation. In these terms, if $x, y \in 2^{\mathbb{N}\times\mathbb{N}}$ then obviously

$$x \mathsf{E}_3 y \iff \{m^\wedge(s(x)_m) : s \in 2^{<\omega},\ m \in \mathbb{N}\} = \{m^\wedge(s(y)_m) : s \in 2^{<\omega},\ m \in \mathbb{N}\}.$$

Now any bijection $2^{<\omega} \times \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ yields a Borel reduction of E_3 to T_2.

(2) To reduce E_3 to $\mathbf{c}_0$ consider a Borel map $\vartheta : 2^{\mathbb{N}\times\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ such that $\vartheta(x)(2^n(2k+1) - 1) = n^{-1}(x)_n(k)$. $\square$

Lemma 17. *Any countable Borel ER is Borel reducible to* T_2.

Proof. Let E be a countable Borel ER on $2^{\mathbb{N}}$. It follows from Countable-to-1 Enumeration that there is a Borel map $f : 2^{\mathbb{N}} \times \mathbb{N} \to 2^{\mathbb{N}}$ such that $[a]_{\mathsf{E}} = \{f(a,n) : n \in \mathbb{N}\}$ for all $a \in 2^{\mathbb{N}}$. The map ϑ sending any $a \in 2^{\mathbb{N}}$ to $x = \vartheta(a) \in 2^{\mathbb{N}\times\mathbb{N}}$ such that $(x)_n = f(a,n)$, $\forall n$, is a reduction required. $\square$

See further study on T_2 in Section 15, where it will be shown that T_2 is not Borel reducible to a big family of equivalence relations that includes $\mathbf{c}_0$, $\boldsymbol{\ell}^p$, $\boldsymbol{\ell}^\infty$, E_1, E_2, E_3, E_∞. On the other hand, the equivalence relations in this list, with the exception of E_3, E_∞, are not Borel reducible to T_2 — this follows from the turbulence theory presented in Section 11.

4.b Discretization and generation by ideals

Some equivalence relations on the diagram on page 16 are explicitly generated by ideals, like E_i, $i = 0, 1, 2, 3$. Some other ERs are defined differently. It will be shown below (Section 16) that **any** Borel ER E is Borel reducible to a ER of the form $\mathsf{E}_{\mathscr{I}}$, $\mathscr{I}$ a Borel ideal. On the other hand, $\mathbf{c}_0$, $\boldsymbol{\ell}^1$, $\boldsymbol{\ell}^\infty$ turn out to be Borel equivalent to some meaningful Borel ideals. Moreover, these equivalence relations admit "discretization" by means of restriction to certain subsets of $\mathbb{R}^{\mathbb{N}}$.

Definition 18. We define $\mathbb{X} = \prod_{n\in\mathbb{N}} X_n = \{x \in \mathbb{R}^{\mathbb{N}} : \forall\, n\, (x(n) \in X_n)\}$, where $X_n = \{\frac{0}{2^n}, \frac{1}{2^n}, \dots, \frac{2^n}{2^n}\}$. □

Lemma 19. $\mathbf{c}_0 \le_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$ *and* $\boldsymbol{\ell}^p \le_{\mathrm{B}} \boldsymbol{\ell}^p \restriction \mathbb{X}$ *for any* $1 \le p < \infty$.

On the other hand, $\boldsymbol{\ell}^\infty \le_{\mathrm{B}} \boldsymbol{\ell}^\infty \restriction \mathbb{Z}^{\mathbb{N}}$.

Proof. We first show that $\mathbf{c}_0 \le_{\mathrm{B}} \mathbf{c}_0 \restriction [0,1]^{\mathbb{N}}$. Let π be any bijection of $\mathbb{N}\times\mathbb{Z}$ onto $\mathbb{N}$. For $x \in \mathbb{R}^{\mathbb{N}}$, define $\vartheta(x) \in [0,1]^{\mathbb{N}}$ as follows. Suppose that $k = \pi(n,\eta)$ ($\eta \in \mathbb{Z}$). If $\eta \le x(n) < \eta+1$ then let $\vartheta(x)(k) = x(n)$. If $x(n) \ge \eta+1$ then put $\vartheta(x)(k) = 1$. If $x(n) < \eta$ then put $\vartheta(x)(k) = 0$. Then ϑ is a Borel reduction of $\mathbf{c}_0$ to $\mathbf{c}_0 \restriction [0,1]^{\mathbb{N}}$. Now we prove that $\mathbf{c}_0 \restriction [0,1]^{\mathbb{N}} \le_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$. For $x \in [0,1]^{\mathbb{N}}$ define $\psi(x) \in \mathbb{X}$ so that $\psi(x)(n)$ the largest number of the form $\frac{i}{2^n}$, $0 \le i \le 2^n$ smaller than $x(n)$. Then obviously $x \mathrel{\mathbf{c}_0} \psi(x)$ holds for any $x \in [0,1]^{\mathbb{N}}$, and hence ψ is a Borel reduction of $\mathbf{c}_0 \restriction [0,1]^{\mathbb{N}}$ to $\mathbf{c}_0 \restriction \mathbb{X}$.

Thus $\mathbf{c}_0 \le_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$, and hence in fact $\mathbf{c}_0 \sim_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$.

The argument for $\boldsymbol{\ell}^1$ is pretty similar. The result for $\boldsymbol{\ell}^\infty$ is obvious: given $x \in \mathbb{R}^{\mathbb{N}}$, replace any $x(n)$ by the largest integer value $\le x(n)$.

The version for $\boldsymbol{\ell}^p$, $1 < p < \infty$, needs some comments in the first part (reduction to $[0,1]^{\mathbb{N}}$). Note that if $\eta \in \mathbb{Z}$ and $\eta-1 \le x(n) < \eta < \zeta \le y(n) < \zeta+1$ then the value $(y(n)-x(n))^p$ in the distance $\|y-x\|_p = (\sum_n |y(n)-x(n)|^p)^{\frac{1}{p}}$ is replaced by $(\zeta-\eta)+(\eta-x(n))^p+(y(n)-\zeta)^p$ in $\|\vartheta(y)-\vartheta(x)\|_p$. Thus if this happens infinitely many times then both distances are infinite, while otherwise this case can be neglected. Further, if $\eta-1 \le x(n) < \eta \le y(n) < \eta+1$ then $(y(n)-x(n))^p$ in $\|y-x\|_p$ is replaced by $(\eta-x(n))^p+(y(n)-\eta)^p$ in $\|\vartheta(y)-\vartheta(x)\|_p$. However $(\eta-x(n))^p+(y(n)-\eta)^p \le (y(n)-x(n))^p \le 2^{p-1}((\eta-x(n))^p + (y(n)-\eta)^p)$, and hence these parts of the sums in $\|y-x\|_p$ and $\|\vartheta(y)-\vartheta(x)\|_p$ differ from each other by a factor between 1 and 2^{p-1}. Finally, if $\eta \le x(n)$, $y(n) < \eta+1$ for one and the same $\eta \in \mathbb{Z}$ then the term $(y(n)-x(n))^p$ in $\|y-x\|_p$ appears unchanged in $\|\vartheta(y)-\vartheta(x)\|_p$. Thus totally $\|y-x\|_p$ is finite iff so is $\|\vartheta(y)-\vartheta(x)\|_p$. □

Lemma 20 (Oliver [37]). $\mathbf{c}_0$ *is* $\sim_{\mathrm{B}}$ *to the ER* $\mathsf{Z}_0 = \mathsf{E}_{\mathscr{Z}_0}$.

Proof. Prove that $\mathbf{c}_0 \le_{\mathrm{B}} \mathsf{Z}_0$. It suffices, by Lemma 19, to define a Borel reduction $\mathbf{c}_0 \restriction \mathbb{X} \to \mathsf{Z}_0$, *i.e.*, a Borel map $\vartheta : \mathbb{X} \to \mathscr{P}(\mathbb{N})$ such that $x \mathrel{\mathbf{c}_0} y \iff \vartheta(x) \,\Delta\, \vartheta(y) \in \mathscr{Z}_0$ for all $x, y \in \mathbb{X}$. Let $x \in \mathbb{X}$. Then, for any n, we have $x(n) = \frac{k(n)}{2^n}$ for some natural $k(n) \le 2^n$. The value of $k(n)$ determines the intersection $\vartheta(x) \cap [2^n, 2^{n+1})$: for each $j < 2^n$, we define $2^n + j \in \vartheta(x)$ iff $j < k(n)$. Then $x(n) = \frac{\#(\vartheta(x)\cap[2^n,2^{n+1}))}{2^n}$ for any n, and moreover $|y(n)-x(n)| =$

$\frac{\#([\vartheta(x)\,\Delta\,\vartheta(y)]\cap[2^n,2^{n+1}))}{2^n}$ для всех $x,y\in\mathbb{X}$ и n. This easily implies that ϑ is as required.

To prove $\mathsf{Z}_0\le_{\rm B}\mathbf{c}_0$, we have to define a Borel map $\vartheta:\mathscr{P}(\mathbb{N})\to\mathbb{R}^{\mathbb{N}}$ such that $X\,\Delta\,Y\in\mathscr{Z}_0\iff\vartheta(X)\,\mathbf{c}_0\,\vartheta(Y)$. Most elementary ideas like $\vartheta(X)(n)=\frac{\#(X\cap[0,n))}{n}$ do not work, the right way is based on the following observation: for any sets $s,\,t\subseteq[0,n)$ to satisfy $\#(s\,\Delta\,t)\le k$ it is necessary and sufficient that $|\#(s\,\Delta\,z)-\#(t\,\Delta\,z)|\le k$ for any $z\subseteq[0,n)$. To make use of this fact, let us fix an enumeration (with repetitions) $\{z_j\}_{j\in\mathbb{N}}$ of all finite subsets of $\mathbb{N}$ such that

$$\{z_j:2^n\le j<2^{n+1}\}\quad=\quad\text{all subsets of }[0,n)$$

for every n. Define, for any $X\in\mathscr{P}(\mathbb{N})$ and $2^n\le j<2^{n+1}$, $\vartheta(X)(j)=\frac{\#(X\cap z_j)}{n}$. Then $\vartheta:\mathscr{P}(\mathbb{N})\to[0,1]^{\mathbb{N}}$ is a required reduction. □

Recall that for any sequence of reals $r_n\ge0$, $\mathsf{E}_{\{r_n\}}$ is an equivalence relation on $\mathscr{P}(\mathbb{N})$ generated by the ideal $\mathscr{S}_{\{r_n\}}=\{x\subseteq\mathbb{N}:\sum_{n\in x}r_n<+\infty\}$.

Lemma 21 (Attributed to Kechris in [13, 2.4]). *If* $r_n\ge0$, $r_n\to0$, $\sum_n r_n=+\infty$ *then* $\mathsf{E}_{\{r_n\}}\sim_{\rm B}\boldsymbol{\ell}^1$. *In particular,* $\mathsf{E}_2=\mathsf{E}_{\{1/n\}}$ *satisfies* $\mathsf{E}_2\sim_{\rm B}\boldsymbol{\ell}^1$.

Proof. To prove $\mathsf{E}_{\{r_n\}}\le_{\rm B}\boldsymbol{\ell}^1$, define $\vartheta(x)\in\mathbb{R}^{\mathbb{N}}$ for any $x\in\mathscr{P}(\mathbb{N})$ as follows: $\vartheta(x)(n)=r_n$ for any $n\in x$, and $\vartheta(x)(n)=0$ for any other n. Then $x\,\Delta\,y\in\mathscr{S}_{\{r_n\}}\iff\vartheta(x)\,\boldsymbol{\ell}^1\,\vartheta(y)$, as required.

To prove the other direction, it suffices to define a Borel reduction of $\boldsymbol{\ell}^1\restriction\mathbb{X}$ to $\mathsf{E}_{\{r_n\}}$. We can associate a (generally, infinite) set $s_{nk}\subseteq\mathbb{N}$ with any pair of n and $k<2^n$, so that the sets s_{nk} are pairwise disjoint and $\sum_{j\in s_{nk}}r_j=2^{-n}$. The map $\vartheta(x)=\bigcup_n\bigcup_{k<2^nx(n)}s_{nk}$, $x\in\mathbb{X}$, is the reduction required. □

4.c Summables irreducible to density-0

The $\le_{\rm B}$-independence of $\boldsymbol{\ell}^1$ and $\mathbf{c}_0$, two best known "Banach" equivalence relations, is quite important. In one direction it is provided by (ii) of the next theorem. The other direction actually follows from Lemma 15.

Is there any example of Borel ideals $\mathscr{I}\le_{\rm B}\mathscr{J}$ which do not satisfy $\mathscr{I}\le_{\Delta}\mathscr{J}$? Typically the reductions found to witness $\mathscr{I}\le_{\rm B}\mathscr{J}$ are Δ-homomorphisms, and even better maps. The following lemma proves that Borel reduction yields $\le_{\rm RB}^{++}$-reduction in quite a representative case. Let us say that $\mathscr{I}\le_{\rm RB}^{++}\mathscr{J}$ *holds exponentially* if there is a map $i\mapsto w_i$ withessing $\mathscr{I}\le_{\rm RB}^{++}\mathscr{J}$ [7] and in addition a sequence of natural numbers k_i with $w_i\subseteq[k_i,k_{i+1})$ and $k_{i+1}\ge2k_i$.

Theorem 22. *Suppose that* $r_n\ge0$, $r_n\to0$, $\sum_n r_n=+\infty$. *Then*

[7] Thus we have pairwise disjoint finite non-empty sets $w_k\subseteq\mathbb{N}$ (assuming $\mathscr{I},\mathscr{J}$ are ideals over $\mathbb{N}$) such that $A\in\mathscr{I}\iff w_A=\bigcup_{k\in A}w_k\in\mathscr{J}$, and $\mathtt{max}\,w_k<\mathtt{min}\,w_{k+1}$.

(i) (Farah [5, 2.1]) *If* $\mathscr{J}$ *is a Borel P-ideal and* $\mathscr{S}_{\{r_n\}} \le_{\mathrm{B}} \mathscr{J}$ *then we have* $\mathscr{S}_{\{r_n\}} \le^{++}_{\mathrm{RB}} \mathscr{J}$ *exponentially;*

(ii) (Hjorth [13]) $\mathscr{S}_{\{r_n\}}$ *is not Borel-reducible to* $\mathscr{Z}_0$.

Proof. (i) Let a Borel $\vartheta : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})$ witness $\mathscr{S}_{\{r_n\}} \le_{\mathrm{B}} \mathscr{J}$. Let, according to Theorem 41, ν be a l. s. c. submeasure on $\mathbb{N}$ with $\mathscr{J} = \mathtt{Exh}_\nu$. The construction makes use of stabilizers. Suppose that $n \in \mathbb{N}$. If $u, v \subseteq [0,n)$ then $(u \cup X) \,\Delta\, (v \cup X) \in \mathscr{S}_{\{r_n\}}$ for any $X \subseteq [n, +\infty)$, hence, $\vartheta(u \cup X) \,\Delta\, \vartheta(v \cup X) \in \mathscr{J}$. It follows, by the choice of the submeasure ν, that for any $\varepsilon > 0$ there are numbers $n' > k > n$ and a set $s \subseteq [n, n')$ such that

$$\nu\big((\vartheta(u \cup s \cup X) \,\Delta\, \vartheta(v \cup s \cup X)) \cap [k, \infty)\big) \;<\; \varepsilon$$

holds for all $u, v \subseteq [0,n)$ and all generic [8] $X \subseteq [n', \infty)$.

This allows us to define an increasing sequence of natural numbers $0 = k_0 = a_0 < b_0 < k_1 < a_1 < b_1 < k_2 < \ldots$ and, for any i, a set $s_i \subseteq [b_i, a_{i+1})$ such that, for all generic $X, Y \subseteq [a_{i+1}, \infty)$ and all $u, v \subseteq [0, b_i)$, we have

(1) $\nu\big((\vartheta(u \cup s_i \cup X) \,\Delta\, \vartheta(v \cup s_i \cup X)) \cap [k_{i+1}, \infty)\big) < 2^{-i}$;

(2) $(\vartheta(u \cup s_i \cup X) \,\Delta\, \vartheta(u \cup s_i \cup Y)) \cap [0, k_{i+1}) = \emptyset$;

(3) any $Z \subseteq \mathbb{N}$, satisfying $Z \cap [b_i, a_{i+1}) = s_i$ for infinitely many i, is generic;

(4) $k_{i+1} \ge 2k_i$ for all i;

and in addition, under the assumptions on $\{r_n\}$,

(5) there is a set $g_i \subseteq [a_i, b_i)$ such that $|r_i - \sum_{n \in g_i} r_n| < 2^{-i}$.

It follows from (5) that $A \mapsto g_A = \bigcup_{i \in A} g_i$ is a reduction of $\mathscr{S}_{\{r_n\}}$ to $\mathscr{S}_{\{r_n\}} \restriction N$, where $N = \bigcup_i [a_i, b_i)$. Let $S = \bigcup_i s_i$; note that $S \cap N = \emptyset$.

Put $\xi(Z) = \vartheta(Z \cup S) \,\Delta\, \vartheta(S)$ for any $Z \subseteq N$. Then, for any sets $X, Y \subseteq N$, $\longleftarrow$ *why* $\Delta\vartheta(S)$ *added?* $\dashv$

$$X \,\Delta\, Y \in \mathscr{S}_{\{r_n\}} \iff \vartheta(X \cup S) \,\Delta\, \vartheta(Y \cup S) \in \mathscr{J} \iff \xi(X) \,\Delta\, \xi(Y) \in \mathscr{J},$$

thus ξ reduces $\mathscr{S}_{\{r_n\}} \restriction N$ to $\mathscr{J}$. Now put $w_i = \xi(g_i) \cap [k_i, k_{i+1})$ and $w_A = \bigcup_{i \in A} w_i$. We assert that the map $i \mapsto w_i$ proves $\mathscr{S}_{\{r_n\}} \le^{++}_{\mathrm{RB}} \mathscr{J}$. In view of the above, it remains to show that $\xi(g_A) \,\Delta\, w_A \in \mathscr{J}$ for any $A \in \mathscr{P}(\mathbb{N})$.

As $\mathscr{J} = \mathtt{Exh}_\nu$, it suffices to demonstrate that $\nu(w_i \,\Delta\, (\xi(g_A) \cap [k_i, k_{i+1}))) < 2^{-i}$ for all $i \in A$ while $\nu(\xi(g_A) \cap [k_i, k_{i+1})) < 2^{-i}$ for $i \notin A$. After dropping the common term $\vartheta(S)$, it suffices to check that

(a) $\nu\big((\vartheta(g_i \cup S) \,\Delta\, \vartheta(g_A \cup S)) \cap [k_i, k_{i+1})\big) < 2^{-i}$ for all $i \in A$ while

[8] In the course of the proof, "generic" means Cohen-generic over a sufficiently large countable model of a big enough fragment of **ZFC**.

(b) $\nu\big((\vartheta(S)\,\Delta\,\vartheta(g_A\cup S))\cap[k_i,k_{i+1})\big)<2^{-i}$ for $i\notin A$.

Note that, as any set of the form $X\cup S$, where $S\subseteq N$, is generic by (3). It follows, by (2), that we can assume, in (a) and (b), that $A\subseteq[0,i]$, *i.e.*, resp. $\mathtt{max}\,A=i$ and $\mathtt{max}\,A<i$. We can finally apply (1), with $u=A\cup\bigcup_{j<i}s_j$, $X=\bigcup_{j>i}s_j$, and $v=u_i\cup\bigcup_{j<i}s_j$ if $i\in A$ while $v=\bigcup_{j<i}s_j$ if $i\notin A$.

(ii) Otherwise $\mathscr{S}_{\{r_n\}}\le^{++}_{\mathrm{RB}}\mathscr{Z}_0$ exponentially by (i). Let this be witnessed by $i\mapsto w_i$ and a sequence of numbers k_i, so that $k_{i+1}\ge 2k_i$ and $w_i\subseteq[k_i,k_{i+1})$. If $d_i=\frac{\#(w_i)}{k_{i+1}}\to 0$ then easily $\bigcup_i w_i\in\mathscr{Z}_0$ by the choice of $\{k_i\}$. Otherwise there is a set $A\in\mathscr{S}_{\{r_n\}}$ such that $d_i>\varepsilon$ for all $i\in A$ and one and the same $\varepsilon>0$, so that $w_A=\bigcup_{i\in A}w_i\notin\mathscr{Z}_0$. In both cases we have a contradiction with the assumption that the map $i\mapsto w_i$ witnesses $\mathscr{S}_{\{r_n\}}\le^{++}_{\mathrm{RB}}\mathscr{Z}_0$. □

Question 23. Farah [5] points out that Theorem 22(i) also holds for $0\times\mathtt{Fin}$ (instead of $\mathscr{S}_{\{r_n\}}$) and asks for which other ideals it is true. □

4.d The family $\boldsymbol{\ell}^p$

It follows from the next theorem that Borel reducibility between equivalence relations $\boldsymbol{\ell}^p$, $1\le p<\infty$, is fully determined by the value of p.

Theorem 24 (Dougherty – Hjorth [3]). *If* $1\le p<q<\infty$ *then* $\boldsymbol{\ell}^p<_{\mathrm{B}}\boldsymbol{\ell}^q$.

Proof. Part 1: show that $\boldsymbol{\ell}^q\not\le_{\mathrm{B}}\boldsymbol{\ell}^p$.

By Lemma 19, it suffices to prove that $\boldsymbol{\ell}^q\restriction\mathbb{X}\not\le_{\mathrm{B}}\boldsymbol{\ell}^p\restriction\mathbb{X}$. Suppose, on the contrary, that $\vartheta:\mathbb{X}\to\mathbb{X}$ is a Borel reduction of $\boldsymbol{\ell}^q\restriction\mathbb{X}$ to $\boldsymbol{\ell}^p\restriction\mathbb{X}$. Arguing as in the proof of Theorem 22, we can reduce the general case to the case when there exist increasing sequences of numbers $0=j(0)<j(1)<j(2)<\dots$ and $0=a_0<a_1<a_2<\dots$ and a map $\tau:\mathbb{Y}\to\mathbb{X}$, where $\mathbb{Y}=\prod_{n=0}^{\infty}X_{j(n)}$, which reduces $\boldsymbol{\ell}^q\restriction\mathbb{Y}$ to $\boldsymbol{\ell}^p\restriction\mathbb{X}$ and has the form $\tau(x)=\bigcup_{n\in\mathbb{N}}t_n^{x(n)}$, where $t_n^r\in\prod_{k=a_n}^{a_{n+1}-1}X_k$ for any $r\in X_{j_n}$. (See Definition 18.)

Case 1: there are $c>0$ and a number N such that $\|\tau_n^1-\tau_n^0\|_p\ge c$ for all $n\ge N$. Since $p<q$, there is a non-decreasing sequence of natural numbers $i_n\le j_n$, $n=0,1,2,\dots$, such that $\sum_n 2^{p(i_n-j_n)}$ diverges but $\sum_n 2^{q(i_n-j_n)}$ converges. (*Hint*: $i_n\approx j_n-p^{-1}\log_2 n$.)

Now consider any $n\ge N$. As $\|\tau_n^1-\tau_n^0\|_p\ge c$ and because $\|...\|_p$ is a norm, there exists a pair of rationals $u(n)<v(n)$ in X_{j_n} with $v(n)-u(n)=2^{i_n-j_n}$ and $\|\tau_n^{v(n)}-\tau_n^{u(n)}\|_p\ge c\,2^{i_n-j_n}$. In addition, put $u(n)=v(n)=0$ for $n<N$. Then the $\boldsymbol{\ell}^q$-distance between the infinite sequences $u=\{u(n)\}_{n\in\mathbb{N}}$ and $v=\{v(n)\}_{n\in\mathbb{N}}$ is equal to $\sum_{n=N}^{\infty}2^{q(i_n-j_n)}<+\infty$, while the $\boldsymbol{\ell}^p$-distance between $\tau(u)$ and $\tau(v)$ is non-smaller than $\sum_{n=N}^{\infty}c^p\,2^{p(i_n-j_n)}=\infty$. But this contradicts the assumption that τ is a reduction.

Case 2: otherwise. Then there is a strictly increasing sequence $n_0<n_1<n_2<\dots$ with $\|\tau_{n_k}^1-\tau_{n_k}^0\|_p\le 2^{-k}$ for all k. Let now $x\in\mathbb{Y}$ be the constant 0 while $y \in \mathbb{Y}$ be defined by $y(n_k) = 1$, $\forall k$ and $y(n) = 0$ for all other n. Then $x \,\boldsymbol{\ell}^q\, y$ fails ($|y(n) - x(n)| \not\to 0$) but $\tau(x) \,\boldsymbol{\ell}^p\, \tau(y)$ holds, contradiction.

Part 2: show that $\boldsymbol{\ell}^p \le_{\mathrm{B}} \boldsymbol{\ell}^q$.

It suffices to prove that $\boldsymbol{\ell}^p \restriction [0,1]^{\mathbb{N}} \le_{\mathrm{B}} \boldsymbol{\ell}^q$ (Lemma 19). We *w.l.o.g.* assume that $q < 2p$: any bigger q can be approached in several steps. For $\vec{x} = \langle x, y\rangle \in \mathbb{R}^2$, let $\|\vec{x}\|_h = (x^h + y^h)^{1/h}$.

Lemma 25. *For any* $\frac{1}{2} < \alpha < 1$ *there is a continuous map* $K_\alpha : [0,1] \to [0,1]^2$ *and positive real numbers* $m < M$ *such that for all* $x < y$ *in* $[0,1]$ *we have* $m(y-x)^\alpha \le \|K_\alpha(y) - K_\alpha(x)\|_2 \le M(y-x)^\alpha$.

Proof (Lemma)**.** The construction of such a map K can be easier described in terms of fractal geometry rather than by an analytic expression. Let $r = 4^{-\alpha}$, so that $\frac{1}{4} < r < \frac{1}{2}$ and $\alpha = -\log_4 r$. Starting with the segment $[(0,0)\,,\,(1,0)]$ of the horisontal axis of the cartesian plane, we replace it by four smaller segments of length r each (thin lines on Fig. 2, left). Each of them we replace by four segments of length r^2 (thin lines on Fig. 2, right). And so on, infinitely many steps. The resulting curve K is parametrized by giving the vertices of the polygons values equal to multiples of 4^{-n}, n being the number of the polygon. For instance, the vertices of the left polygon on Fig. 2 are given values $0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$.

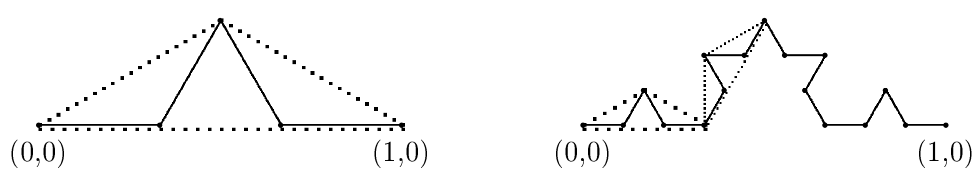

Рис. 2: $r = \frac{1}{3}$, left: step 1, right: step 2

Note that the curve $K : [0,1] \to [0,1]^2$, approximated by the polygons, is bounded by certain triangles built on the sides of the polygons. For instance, the whole curve lies inside the triangle bounded by dotted lines in Fig. 2, left. (The dotted line that follows the basic side $[(0,0)\,,\,(1,0)]$ of the triangle is drawn slightly below its true position.) Further, the parts $0 \le t \le \frac{1}{4}$ and $\frac{1}{4} \le t \le \frac{1}{2}$ of the curve lie inside the triangles bounded by (slightly different) dotted lines in Fig. 2, right. And so on. Let us call those triangles *bounding triangles*.

To prove the inequality of the lemma, consider any pair of reals $x < y \in [0,1]$. Let n be the least number such that x, y belong to non-adjacent intervals, resp., $[\frac{i-1}{4^n}\,,\,\frac{i}{4^n}]$ and $[\frac{j-1}{4^n}\,,\,\frac{j}{4^n}]$, with $j > i+1$. Then $4^{-n} \le |y - x| \le 8 \cdot 4^{-n}$.

The points $K(x)$ and $K(y)$ then belong to one and the same side or adjacent sides of the $n-1$-th polygon. Let C be a common vertice of these sides. It is quite clear geometrically that the euclidean distances from $K(x)$ and $K(y)$ to C do not exceed r^{n-1} (the length of the side), thus $\|K(x) - K(y)\|_2 \le 2\,r^{n-1}$.

Estimation from below needs more work. The points $K(x)$, $K(y)$ belong to the bounding triangles built on the segments, resp., $[K(\frac{i-1}{4^n}), K(\frac{i}{4^n})]$ and $[K(\frac{j-1}{4^n}), K(\frac{j}{4^n})]$, and obviously $i+1<j\le i+8$, so that there exist at most six bounding triangles between these two. Note that adjacent bounding triangles meet each other at only two possible angles (that depend on r but not on n), and taking it as geometrically evident that non-adjacent bounding triangles are disjoint, we conclude that there is a constant $c>0$ (that depends on r but not on n) such that the distance between two non-adjacent bounding triangles of rank n, having at most 6 bounding triangles of rank n between them, does not exceed $c\cdot r^n$. In particular, $\|K(x)-K(y)\|_2 \ge c\cdot r^n$. Combining this with the inequalities above, we conclude that $m(y-x)^\alpha \le \|K(y)-K(x)\|_2 \le M(y-x)^\alpha$, where $m=\frac{c}{8^\alpha}$ and $M=\frac{2}{r}$ (and $\alpha=-\log_4 r$). □ (*Lemma*)

Coming back to the theorem, let $\alpha = p/q$, and let K_α be as in the lemma. Let $x=\langle x_0,x_1,x_2,...\rangle \in [0,1]^{\mathbb N}$. Then $K_\alpha(x_i)=\langle x_i', x_i''\rangle \in [0,1]^2$. We put $\vartheta(x)=\langle x_0',x_0'',x_1',x_1'',x_2',x_2'',...\rangle$. Prove that ϑ reduces $\boldsymbol{\ell}^p \restriction [0,1]^{\mathbb N}$ to $\boldsymbol{\ell}^q$.

Let $x=\{x_i\}_{i\in\mathbb N}$ and $y=\{y_i\}_{i\in\mathbb N}$ belong to $[0,1]^{\mathbb N}$; we have to prove that $x\ \boldsymbol{\ell}^p\ y$ iff $\vartheta(x)\ \boldsymbol{\ell}^q\ \vartheta(y)$. To simplify the picture note the following:

$$2^{-1/2}\|w\|_2 \le \mathtt{max}\{w',w''\} \le \|w\|_q \le \|w\|_1 \le 2\|w\|_2$$

for any $w=\langle w',w''\rangle \in \mathbb R^2$. The task takes the following form:

$$\sum_i (x_i-y_i)^p<\infty \iff \sum_i \|K_\alpha(x_i)-K_\alpha(y_i)\|_2{}^q<\infty\,.$$

Furthermore, by the choice of K_α, this converts to

$$\sum_i (x_i-y_i)^p<\infty \iff \sum_i (x_i-y_i)^{\alpha q}<\infty\,,$$

which holds because $\alpha q = p$. □ (*Theorem 24*)

4.e $\boldsymbol{\ell}^\infty$: maximal $\mathbf{K}_\sigma$

Recall that $\mathbf{K}_\sigma$ denotes the class of all σ-compact sets in Polish spaces. Easy computations show that this class contains, among others, the equivalence relations E_1, E_∞, $\boldsymbol{\ell}^p$, $1\le p\le\infty$, considered as sets of pairs in corresponding Polish spaces. Note that if E a $\mathbf{K}_\sigma$ equivalence on a Polish space $\mathbb X$ then $\mathbb X$ is $\mathbf{K}_\sigma$ as well since projections of compact sets are compact. Thus $\mathbf{K}_\sigma$ ERs on Polish spaces is one and the same as $\boldsymbol{\Sigma}^0_2$ ERs on $\mathbf{K}_\sigma$ Polish spaces.

Theorem 26. *Any $\mathbf{K}_\sigma$ equivalence relation on a Polish space, in particular, E_1, E_∞, $\boldsymbol{\ell}^p$, is Borel reducible to $\boldsymbol{\ell}^\infty$.* [9]

[9] The result for $\boldsymbol{\ell}^p$ is due to Su Gao [11]. He defines $d_p(x,s)=(\sum_{k=0}^{\mathrm{lh}\,s-1}|x(k)-s(k)|^p)^{\frac1p}$ for any $x\in\mathbb R^{\mathbb N}$ and $s\in\mathbb Q^{<\omega}$ (a finite sequence of rationals). Easily the $\boldsymbol{\ell}^p$-distance $(\sum_{k=0}^\infty|x(k)-y(k)|^p)^{\frac1p}$ between any pair of $x,y\in\mathbb R^{\mathbb N}$ is finite iff there is a constant C such that $|d_p(x,s)-d_p(y,s)|<C$ for all $s\in\mathbb Q^{<\omega}$. This yields a reduction required.

Proof (from Rosendal [38]). Let $\mathbb{A}$ be the set of all $\subseteq$-increasing sequences $A = \{A_n\}_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$ — a closed subset of the Polish space $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$. Define an ER H on $\mathbb{A}$ by

$$\{A_n\} \mathrel{\mathsf{H}} \{B_n\} \quad \text{iff} \quad \exists\, N\, \forall\, m\, (A_m \subseteq B_{N+m} \wedge B_m \subseteq A_{N+m}).$$

Claim 1: $\mathsf{H} \le_{\mathrm{B}} \boldsymbol{\ell}^\infty$. This is easy. Given a sequence $A = \{A_n\}_{n\in\mathbb{N}}$, define $\vartheta(A) \in \mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ by $\vartheta(A)(n,k)$ to be the least $j \le k$ such that $n \in A_j$, or $\vartheta(A)(n,k) = k$ whenever $n \notin A_k$. Then $\{A_n\} \mathrel{\mathsf{H}} \{B_n\}$ iff there is N such that $|\vartheta(A)(n,k) - \vartheta(B)(n,k)| \le N$ for all n, k.

Claim 2: any $\mathbf{K}_\sigma$ equivalence E on a Polish space $\mathbb{X}$ is Borel reducible to H. As a $\mathbf{K}_\sigma$ set, E has the form $\mathsf{E} = \bigcup_n E_n$, where each E_n is a compact subset of $\mathbb{X} \times \mathbb{X}$ (not necessarily an ER) and $E_n \subseteq E_{n+1}$. We can *w.l.o.g.* assume that each E_n is reflexive and symmetric on its domain $D_n = \mathtt{dom}\, E_n = \mathtt{ran}\, E_n$ (a compact set), in particular, $x \in D_n \Longrightarrow \langle x, x\rangle \in E_n$. Define $P_0 = E_0$ and

$$P_{n+1} = P_n \cup E_{n+1} \cup P_n^{(2)}, \quad \text{where } P_n^{(2)} = \{\langle x,y\rangle : \exists\, z\, (\langle x,z\rangle \in P_n \wedge \langle z,y\rangle \in P_n)\},$$

by induction. Thus all P_n are still compact subsets of $\mathbb{X} \times \mathbb{X}$, moreover, of E since E is an equivalence relation, and $E_n \subseteq P_n \subseteq P_{n+1}$, therefore $\mathsf{E} = \bigcup_n P_n$.

Let $\{U_k : k \in \mathbb{N}\}$ be a basis for the topology of $\mathbb{X}$. Put, for any $x \in \mathbb{X}$, $\vartheta_n(x) = \{k : U_k \cap R_n(x) \ne \emptyset\}$, where $R_n(x) = \{y : \langle x,y\rangle \in R_n\}$. Then obviously $\vartheta_n(x) \subseteq \vartheta_{n+1}(x)$, and hence $\vartheta(x) = \{\vartheta_n(x)\}_{n\in\mathbb{N}} \in \mathbb{A}$. Then ϑ reduces E to H.

Indeed if $x \mathrel{\mathsf{E}} y$ then $\langle y,x\rangle \in P_n$ for some n, and for all m and $z \in \mathbb{X}$ we have $\langle x,z\rangle \in R_m \Longrightarrow \langle y,z\rangle \in R_{1+\mathtt{max}\{m,n\}}$. In other words, $R_m(x) \subseteq R_{1+\mathtt{max}\{m,n\}}(y)$ and hence $\vartheta_m(x) \subseteq \vartheta_{1+\mathtt{max}\{m,n\}}(y)$ hold for all m. Similarly, for some n' we have $\vartheta_m(y) \subseteq \vartheta_{1+\mathtt{max}\{m,n'\}}(y)$, $\forall\, m$. Thus $\vartheta(x) \mathrel{\mathsf{H}} \vartheta(y)$.

Conversely, suppose that $\vartheta(x) \mathrel{\mathsf{H}} \vartheta(y)$, thus, for some N, we have $R_m(x) \subseteq R_{N+m}(y)$ and $R_m(y) \subseteq R_{N+m}(x)$ for all m and y. Taking m big enough for P_m to contain $\langle x,x\rangle$, we obtain $x \in R_{N+m}(y)$, so that immediately $x \mathrel{\mathsf{E}} y$. $\square$

5 Smooth ERs and the first dichotomy

This Section is mainly related to the node $\mathfrak{c} = \mathsf{D}(2^{\mathbb{N}})$ in the diagram on page 16. After a few rather simple results on smooth ERs which admit a Borel transversal, we show that countable, and sometimes even continual unions of smooth ERs are smooth. In the end, we prove the 1st dichotomy theorem. ⟵ *where continual?*⊣

5.a Smooth and below

An important subspecies of smooth ERs consists of those having a Borel *transversal*: a set with exactly one element in every equivalence class.

Lemma 27. (i) *Any Borel ER that has a Borel transversal is smooth*;

(ii) *any Borel finite (with finite classes) ER admits a Borel transversal*;

(iii) *any Borel countable smooth ER admits a Borel transversal;*

(iv) *any Borel ER* E *on a Polish space* $\mathbb{X}$, *such that every* E*-class is closed and the saturation* $[\mathscr{O}]_{\mathsf{E}}$ *of every open set* $\mathscr{O} \subseteq \mathbb{X}$ *is Borel, admits a Borel transversal, hence, is smooth.* [10]

(v) E_0 *is not smooth.*

Proof. (i) Let T be a Borel transversal for E. The map $\vartheta(x) =$ "the only element of T E-equivalent to x" reduces E to $\mathsf{D}(T)$. [11]

(ii) Consider the set of the $<$-least elements of E-classes, where $<$ is a fixed Borel linear order on the domain of E.

(iii) Use Countable-to-1 Uniformization.

(iv) Since any uncountable Polish space is a continuous image of $\mathbb{N}^{\mathbb{N}}$, we can assume that E is a ER on $\mathbb{N}^{\mathbb{N}}$. Then, for any $x \in \mathbb{N}^{\mathbb{N}}$, $[x]_{\mathsf{E}}$ is a closed subset of $\mathbb{N}^{\mathbb{N}}$, naturally identified with a tree, say, $T_x \subseteq \mathbb{N}^{<\omega}$. Let $\vartheta(x)$ denote the leftmost branch of T_x. Then $x \mathrel{\mathsf{E}} \vartheta(x)$ and $x \mathrel{\mathsf{E}} y \implies \vartheta(x) = \vartheta(y)$, so that it remains to show that $Z = \{\vartheta(x) : x \in \mathbb{N}^{\mathbb{N}}\}$ is Borel. Note that

$$z \in Z \iff \forall m\, \forall s, t \in \mathbb{N}^m \; (s <_{\mathtt{lex}} t \wedge z \in \mathscr{O}_t \implies [z]_{\mathsf{E}} \cap \mathscr{O}_t = \emptyset),$$

where $<_{\mathtt{lex}}$ is the lexicographical order on $\mathbb{N}^m$ and $\mathscr{O}_s = \{x \in \mathbb{N}^{\mathbb{N}} : s \subset x\}$. However $[x]_{\mathsf{E}} \cap \mathscr{O}_t = \emptyset$ iff $x \notin [\mathscr{O}_t]_{\mathsf{E}}$ and $[\mathscr{O}_t]_{\mathsf{E}}$ is Borel for any t.

(v) Otherwise E_0 has a Borel transversal T by (iii), which is a contradiction, see Example 11(ii). □

[10] Srivastava [44] proved the result for ERs with $\mathbf{G}_\delta$ classes, which is the best possible as E_0 is a Borel ER, whose classes are $\mathbf{F}_\sigma$ and saturations of open sets are even open, but without any Borel transversal. See also [26, 18.20 iv)].

[11] To see that a smooth ER does not necessarily have a Borel transversal take a closed set $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with $\mathtt{dom}\, P = \mathbb{N}^{\mathbb{N}}$, not uniformizable by a Borel set, and let $\langle x, y\rangle \mathrel{\mathsf{E}} \langle x', y'\rangle$ iff both $\langle x, y\rangle$ and $\langle x', y'\rangle$ belong to P and $x = x'$.

5.b Assembling smooth equivalence relations

If E and F are smooth ERs on disjoint sets, resp., X and Y, then easily $\mathsf{E} \cup \mathsf{F}$ is a smooth ER on $X \cup Y$. The question becomes less clear when we have a Borel ER E on a Polish space $X \cup Y$ such that both $\mathsf{E} \restriction X$ and $\mathsf{E} \restriction Y$ are smooth but the sets X, Y not necessarily E-invariant in $X \cup Y$ if even disjoint; is E smooth? We answer this in the positive, even in the case of countable unions.

Theorem 28. *Let E be a Borel ER on a Borel set $X = \bigcup_k X_k$, with all X_k also Borel. Suppose that each $\mathsf{E} \restriction X_k$ is smooth. Then E is smooth.*

Proof. [12] First consider the case of a union $X = Y \cup Z$ of just two Borel sets, so that a Borel ER E is smooth on both Y and Z. We can assume that $Y \cap Z = \emptyset$. Let the smoothness be witnessed by Borel reductions $f : Y \to Q$ and $g : Z \to R$, with Q, R being disjoint Borel sets. The set

$$F = \{\langle q, r\rangle : \exists\, y \in Y\ \exists\, z \in Z\ \big(f(y) = q \wedge g(z) = r \wedge y \mathrel{\mathsf{E}} z\big)\} \subseteq Q \times R$$

is a partial $\mathbf{\Sigma}^1_1$ map $Q \to R$. Let $G : Q \to R$ be any Borel map with $F \subseteq G$, and $H : R \to Q$ be any Borel map with $F^{-1} \subseteq H$. Then $\Phi = G \cap H^{-1}$ is a $1-1$ Borel partial map $P \to Q$ with $F \subseteq \Phi$. Now the $\mathbf{\Pi}^1_1$ set

$$P = \{\langle q, r\rangle \in \Phi : \forall\, y \in Y\ \forall\, z \in Z\ \big(f(y) = q \wedge g(z) = r \Longrightarrow y \mathrel{\mathsf{E}} z\big)\},$$

satisfies $F \subseteq P \subseteq \Phi$, hence, there is a Borel function Ψ with $F \subseteq \Psi \subseteq P$. The sets $A = \mathtt{dom}\,\Psi$ and $B = \mathtt{ran}\,\Psi$ are Borel subsets of resp. Q, R, and it follows from the construction that $\Psi \cap (\mathtt{dom}\, F \times \mathtt{ran}\, F) = F$. Finally, put

$$D = \Psi \cup \{\langle q, q\rangle : q \in Q \smallsetminus A\} \cup \{\langle r, r\rangle : r \in R \smallsetminus B\},$$

then, for any $y \in Y$ there is unique $h(y) = \langle q, r\rangle \in D$ with $q = f(y)$, correspondingly, for any $z \in Z$ there is unique $h(z) = \langle q, r\rangle \in D$ with $r = g(z)$, and if $y \mathrel{\mathsf{E}} z$ then $h(y) = h(z) = \langle f(y), g(z)\rangle$, hence, h witnesses that E is smooth.

As for the general case, we can now assume that $X_k \subseteq X_{k+1}$ for all k. Then there are disjoint Borel sets W_k and Borel maps $f_k : X_k \to W_k$ which witness that $\mathsf{E} \restriction X_k$ are smooth ERs. Let $R_k = \mathtt{ran}\, f_k$ (a $\mathbf{\Sigma}^1_1$ set) and

$$F_k = \{\langle a, b\rangle \in R_k \times R_{k+1} : \exists\, x \in X_k\ (f_k(x) = a \wedge f_{k+1}(x) = b)\},$$

this is a $\mathbf{\Sigma}^1_1$ set and a $1-1$ map $R_k \to R_{k+1}$. For each k there is a Borel $1-1$ map G_k with $F_k \subseteq G_k$. Let $A_k = \mathtt{dom}\, G_k$ and $\mathtt{ran}\, G_k = B_k$: these are Borel sets with $R_k \subseteq A_k$. We can assume that $B_k \subseteq A_{k+1}$. (Otherwise G_k can be $\longleftarrow$ *nuzhno li eto assume*$\dashv$

[12] The shortest proof is to note that otherwise $\mathsf{E_0} \le_{\mathrm{B}} \mathsf{E}$ by the 2-nd dichotomy, easily leading to contradiction by a Baire category argument. Yet we prefer to give a direct proof. Note that even in the case when the sets X_k are pairwise disjoint, most obvious ideas like "to define $\vartheta(x)$ take the least k such that X_k intersects $[x]_{\mathsf{E}}$ and apply ϑ_k" do not work.

reduced in a certain iterative manner to achieve this property.) Then, for any k and $b \in A_k$ there is the least $n = n(b) \le k$ such that the application

$$h(b) = G_n^{-1}(G_{n+1}^{-1}(G_{n+2}^{-1}(...G_{k-1}^{-1}(b)...)))$$

is possible, for instance, $n(b) = k$ and $h(b) = b$ whenever $b \in A_k \smallsetminus B_{k-1}$. Then, $h(f_k(x)) = h(f_{k+1}(x))$ holds for any $x \in X_k$ because $F_k \subseteq G_k$, so that the map $g(x) = h(f_k(x))$ for $x \in X_k \smallsetminus X_{k-1}$ witnesses the smoothness of E. □

5.c The 1st dichotomy theorem.

The following result is known as the 1st dichotomy theorem.

Theorem 29 (Silver [40]). *Any* $\mathbf{\Pi}_1^1$ *ER* E *on* $\mathbb{N}^{\mathbb{N}}$ *either has at most countably many equivalence classes or admits a perfect set of pairwise* E*-inequivalent reals, in other words, either* $\mathsf{E} \le_{\mathrm{B}} \mathsf{D}(\mathbb{N})$ *or* $\mathsf{D}(2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E}$.

Proof. [13] As usual, we can suppose that E is a lightface $\mathit{\Pi}_1^1$ relation.

Case 1: any $x \in \mathbb{N}^{\mathbb{N}}$ belongs to a $\mathit{\Delta}_1^1$ E-equivalent set X (*i.e.*, all elements of X are E-equivalent to each other, in other words, the saturation $[X]_{\mathsf{E}}$ is an equivalence class). Then E has at most countably many equivalence classes.

Case 2: otherwise. Then the set H of all x, which do <u>not</u> belong to a $\mathit{\Delta}_1^1$ pairwise E-equivalent set (*the domain of nontriviality*), is non-empty.

Claim 29.1. *H is $\mathit{\Sigma}_1^1$. Any $\mathit{\Sigma}_1^1$ set $\emptyset \ne X \subseteq H$ is not pairwise E-equivalent.*

Proof. $x \in H$ iff for any $e \in \mathbb{N}$: <u>if</u> e codes a $\mathit{\Delta}_1^1$ set, say, $W_e \subseteq \mathbb{N}^{\mathbb{N}}$ and $x \in W_e$ <u>then</u> W_e is not E-equivalent. The “if” part of this characterization is $\mathit{\Pi}_1^1$ while the “then” part is $\mathit{\Sigma}_1^1$, by $\mathit{\Delta}_1^1$ Enumeration (see § A.c).

If $X \ne \emptyset$ is a pairwise E-equivalent $\mathit{\Sigma}_1^1$ set then $B = \bigcap_{x \in X} [x]_{\mathsf{E}}$ is a $\mathit{\Pi}_1^1$ E-equivalence class and $X \subseteq B$. By Separation, there is a $\mathit{\Delta}_1^1$ set C with $X \subseteq C \subseteq B$. Then, if $X \subseteq H$ then $C \subseteq H$ is a $\mathit{\Delta}_1^1$ pairwise E-equivalent set, a contradiction to the definition of H. □ (*Claim*)

Let us fix a countable transitive model $\mathfrak{M}$ of a big enought fragment of **ZFC**, and an elementary submodel of the universe w. r. t. all analytic formulas [14]. Consider $\mathbb{P} = \{X \subseteq \mathbb{N}^{\mathbb{N}} : X$ is non-empty and $\mathit{\Sigma}_1^1\}$ as a forcing to extend $\mathfrak{M}$ (smaller sets are stronger conditions), the *Gandy – Harrington forcing*. We have $\mathbb{P} \notin$ and $\not\subseteq \mathfrak{M}$, of course, but clearly $\mathbb{P}$ can be adequately coded in $\mathfrak{M}$, say, via a universal $\mathit{\Sigma}_1^1$ set.

[13] We present a forcing proof of Miller [36], with some simplifications. See [32] for another proof, based on the Gandy – Harrington topology. In fact both proofs involve essentially the same combinatorics.

[14] For instance, $\mathfrak{M}$ models **ZC** and, in addition, Replacement for Σ_{100} $\in$-formulas and the first one million of instances of Replacement overall. Being an elementary submodel is useful to guarantee that relations like the inclusion orders of $\mathbf{C}_{\mathbb{X}}$ and $\mathbf{C}_{\mathbb{G}}$ are absolute for $\mathfrak{M}$.

Corollary 29.2 (from Theorem 85). *If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic, over $\mathfrak{M}$, set, then $\bigcap G$ contains a single real, denoted x_G.* □

Reals of the form x_G, G as in the Corollary, are called $\mathbb{P}$-*generic* (over $\mathfrak{M}$). Let $\dot{x}$ be the name for x_G. Then any $A \in \mathbb{P}$ forces that $\dot{x} \in A$.

Let $\mathbb{P}^2$ consist of all "rectangles" $X \times Y$, with $X, Y \in \mathbb{P}$. It follows from the above by the product forcing lemmas that any $\mathbb{P}^2$-generic, over $\mathfrak{M}$, set $G \subseteq \mathbb{P}^2$ produces a pair of reals (a $\mathbb{P}^2$-*generic pair*), say, $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$, so that $\langle x^G_{\mathtt{left}}, x^G_{\mathtt{right}} \rangle \in W$ for any $W \in G$. Let $\dot{x}_{\mathtt{left}}$ and $\dot{x}_{\mathtt{right}}$ be their names.

Lemma 29.3. *$H \times H$ $\mathbb{P}^2$-forces $\dot{x}_{\mathtt{left}} \not\mathsf{E}\, \dot{x}_{\mathtt{right}}$.*

Proof. Otherwise a "condition" $X \times Y \in \mathbb{P}^2$ with $X \cup Y \subseteq H$ $\mathbb{P}^2$-forces $\dot{x}_{\mathtt{left}} \mathsf{E}\, \dot{x}_{\mathtt{right}}$, so that any $\mathbb{P}^2$-generic pair $\langle x, y \rangle \in X \times Y$ satisfies $x \mathsf{E} y$. By the product forcing lemmas for any pair of $\mathbb{P}$-generic x', $x'' \in X$ there is $y \in Y$ such that both $\langle x, y \rangle$ and $\langle x', y \rangle$ are $\mathbb{P}^2$-generic pairs, hence, we have

($*$) If x', $x'' \in X$ are $\mathbb{P}$-generic over $\mathfrak{M}$ then $x' \mathsf{E}\, x''$.

The set $\mathbb{P}_2$ of all non-empty Σ^1_1 subsets of $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is just a copy of $\mathbb{P}$ (not of $\mathbb{P}^2$!) as a forcing, in particular, if $G \subseteq \mathbb{P}_2$ is $\mathbb{P}_2$-generic over $\mathfrak{M}$ then there is a unique pair of reals ($\mathbb{P}_2$-*generic pair*) $\langle x^G_{\mathtt{left}}, x^G_{\mathtt{right}} \rangle$ which belongs to every W in G, and in this case, *both $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$ are $\mathbb{P}$-generic*, because if $G \subseteq \mathbb{P}_2$ is $\mathbb{P}_2$-generic then the sets G' and G'' of all projections of sets $W \in G$ to resp. 1st and 2nd co-ordinate, are easily $\mathbb{P}$-generic. Now let $G \subseteq \mathbb{P}_2$ be a $\mathbb{P}_2$-generic set, over $\mathfrak{M}$, containing the Σ^1_1 set $P = X^2 \smallsetminus \mathsf{E}$. (Note that $P \neq \emptyset$ by Lemma 29.1.) Then $\langle x^G_{\mathtt{left}}, x^G_{\mathtt{right}} \rangle \in P$, hence, $x^G_{\mathtt{left}} \not\mathsf{E}\, x^G_{\mathtt{right}}$, however, as we observed, both $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$ are $\mathbb{P}$-generic elements of X (because $P \subseteq X \times X$), which contradicts ($*$). □ (*Lemma 29.3*)

Fix enumerations $\{\mathscr{D}(n)\}_{n \in \mathbb{N}}$ and $\{\mathscr{D}^2(n)\}_{n \in \mathbb{N}}$ of all dense subsets of resp. $\mathbb{P}$ and $\mathbb{P}^2$ which are coded in $\mathfrak{M}$. Then there is a system $\{X_u\}_{u \in 2^{<\omega}}$ of sets X_u, satisfying

(i) $X_u \in \mathbb{P}$, moreover, $X_\Lambda \subseteq H$ and $X_u \in \mathscr{D}(n)$ whenever $u \in 2^n$;

(ii) $X_{u^\wedge i} \subseteq X_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$;

(iii) if $u \neq v \in 2^n$ then $X_u \times X_v \in \mathscr{D}^2(n)$.

It follows from (i) that, for any $a \in 2^{\mathbb{N}}$, the set $\{X_{a \restriction m} : m \in \mathbb{N}\}$ is $\mathbb{P}$-generic over $\mathfrak{M}$, hence, $\bigcap_m X_{a \restriction m}$ is a singleton, say, x_a, by Corollary 29.2. Moreover the map $a \mapsto x_a$ is continuous as diameters of X_u converge to 0 uniformly with $\mathtt{lh}\, u \to 0$, by (i). In addition, by (iii) and Lemma 29.3, $x_a \not\mathsf{E}\, x_b$ whenever $a \neq b$, in particular, $x_a \neq x_b$, hence, we have a perfect E-inequivalent set $Y = \{x_a : a \in 2^{\mathbb{N}}\}$.

□ (*Theorem 29*)

6 Hyperfinite and countable ERs

This Section is mainly devoted to the node E_0 in the diagram on page 16. Together with the 2nd dichotomy theorem, we present some other properties of E_0, the ideal $\mathtt{Fin}$, and hyperfinite (Borel) equivalence relations. This class of equivalence relations is a very interesting object of study even aside of pure descriptive set theory. Papers [2, 19] give a comprehensive account of most basic results, with further references.

After a rather simple theorem which shows that $\mathtt{Fin}$ is the least ideal in the sense of $\le_{\rm RB}^{++}$, $\le_{\rm RB}$, $\le_{\rm B}$, we prove the "Glimm–Effros", or second, dichotomy which asserts that $\mathsf{E}_0 = \mathsf{E}_{\mathtt{Fin}}$ is the $\le_{\rm B}$-least among all non-smooth Borel ERs. Finally, we present a characterization, in terms of the existence of transversals, of those Borel sets X for which $\mathsf{E}_0 \restriction X$ is smooth. $\longleftarrow$ *where is this?*$\dashv$

6.a Fin is the least !

The proof of the following useful result is based on a short argument involved in many other results. A somewhat more pedestrian version of the argument was used in several proofs in Section 4.

Theorem 30. (i) [20, 34, 45] *If* $\mathscr{I}$ *is a* (nontrivial) *ideal on* $\mathbb{N}$, *with the Baire property in the topology of* $\mathscr{P}(\mathbb{N})$, *then* $\mathtt{Fin} \le_{\rm RB}^{++}$ *and* $\le_{\rm RB} \mathscr{I}$;

(ii) *however* $\mathsf{D}(2^{\mathbb{N}}) <_{\rm B} \mathsf{E}_0$ *strictly, thus* $\mathsf{D}(2^{\mathbb{N}})$ *is not* $\sim_{\rm B}$*-equivalent to an equivalence relation of the form* $\mathsf{E}_{\mathscr{I}}$;

(iii) *if* $\mathscr{I} \le_{\rm RB}^{+} \mathscr{J}$ *are Borel ideals, and there is an infinite set* $Z \subseteq \mathtt{dom}\,\mathscr{I}$ *such that* $\mathscr{I} \restriction Z = \mathscr{P}_{\mathtt{fin}}(Z)$, *then* $\mathscr{I} \le_{\rm RB} \mathscr{J}$.

Proof. (i) First of all $\mathscr{I}$ must be meager in $\mathscr{P}(\mathbb{N})$. (Otherwise $\mathscr{I}$ would be comeager somewhere, easily leading to contradiction.) Thus, all $X \subseteq \mathbb{N}$ "generic" (over a certain countable family of dense open subsets of $\mathscr{P}(\mathbb{N})$) do not belong to $\mathscr{I}$. Now it suffices to define non-empty finite sets $w_i \subseteq \mathbb{N}$ with $\mathtt{max}\, w_i < \mathtt{min}\, w_{i+1}$ such that any union of infinitely many of them is "generic". Clearly the following observation yields the result: if D is an open dense subset of $\mathscr{P}(\mathbb{N})$ and $n \in \mathbb{N}$ then there is $m > n$ and a set $u \subseteq [n, m]$ with $m,\ n \in u$ such that any $x \in \mathscr{P}(\mathbb{N})$ satisfying $x \cap [n, m] = u$ belongs to D.

Thus we have $\mathtt{Fin} \le_{\rm RB}^{++} \mathscr{I}$. To derive $\mathtt{Fin} \le_{\rm RB} \mathscr{I}$ cover each w_k by a finite set u_k such that $\bigcup_{k\in\mathbb{N}} u_k = \mathbb{N}$ and still $u_k \cap u_l = \emptyset$ for $k \ne l$.

(ii) That $\mathsf{D}(2^{\mathbb{N}}) \le_{\rm B} \mathsf{E}_0$ is witnessed by any perfect set $X \subseteq 2^{\mathbb{N}}$ which is a *partial* transversal for E_0 (*i.e.*, any $x \ne y$ in X are E_0-inequivalent). On the other hand, $\mathsf{D}(2^{\mathbb{N}})$ is smooth but E_0 is non-smooth by Lemma 27(v).

(iii) Assume *w.l.o.g.* that $\mathscr{I}, \mathscr{J}$ are ideals over $\mathbb{N}$. Let pairwise disjoint finite sets $w_k \subseteq \mathbb{N}$ witness $\mathscr{I} \le_{\rm RB}^{+} \mathscr{J}$. Put $Z' = \mathbb{N} \smallsetminus Z$, $X = \bigcup_{k\in Z} w_k$, and $Y = \bigcup_{k\in Z'} w_k$. The reduction via $\{w_k\}$ reduces $\mathscr{P}_{\mathtt{fin}}(Z)$ to $\mathscr{J} \restriction X$ and $\mathscr{I} \restriction Z'$

to $\mathscr{J} \restriction Y$. Keeping the latter, replace the former by a $\le_{\rm RB}$-like reduction of $\mathscr{P}_{\tt fin}(z)$ to $\mathscr{J} \restriction Y'$, where $Y' = \mathbb{N} \smallsetminus Y$, which exists by Theorem 30. □

Despite of Theorem 30, $\mathsf{E}_0 = \mathsf{E}_{\rm Fin}$ is not the $\le_{\rm B}$-least among Borel ERs. Thus, $\mathsf{D}(2^{\mathbb{N}})$ is not a ER generated by a Borel ideal, even modulo $\sim_{\rm B}$.

6.b Countable equivalence relations

This class of equivalence relations, essentially bigger than hyperfinite (modulo $\le_{\rm B}$), is a subject of ongoing intence study. Yet we can only present here the following important theorem and a few more results below, leaving [19, 10, 30] as basic references in this domain.

Theorem 31 ([8, Thm 1], [2, 1.8]). *Any Borel countable ER* E *on a Polish space* $\mathbb{X}$:

(i) *is induced by a Polish action of a countable group* $\mathbb{G}$ *on* $\mathbb{X}$;

(ii) *satisfies* $\mathsf{E} \le_{\rm B} \mathsf{E}_\infty = \mathsf{E}(F_2, 2)$, *where* F_2 *is the free group with two generators and* $\mathsf{E}(F_2, 2)$ *is the ER induced by the shift action of* F_2 *on* 2^{F_2}.

Proof. (i) We *w.l.o.g.* assume that $\mathbb{X} = 2^{\mathbb{N}}$. According to Countable-to-1 Enumeration (in a relativized version, if necessary, see Remark 82), there is a sequence of Borel maps $f_n : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that $[a]_{\mathsf{E}} = \{f_n(a) : n \in \mathbb{N}\}$ for each $a \in 2^{\mathbb{N}}$. Put $\Gamma'_n = \{\langle a, f_n(a)\rangle : a \in \mathbb{N}\}$ (the graph of f_n) and $\Gamma_n = \Gamma'_n \smallsetminus \bigcup_{k<n} \Gamma'_k$. The sets $P_{nk} = \Gamma_n \cap {\Gamma_k}^{-1}$ form a partition of (the graph of) E onto countably many Borel injective sets. Further define $\Delta = \{\langle a, a\rangle : a \in 2^{\mathbb{N}}\}$ and let $\{D_m\}_{m\in\mathbb{N}}$ be an enumeration of all non-empty sets of the form $P_{nk} \smallsetminus \Delta$. Intersecting the sets D_m with the rectangles of the form

$$R_s = \{\langle a, b\rangle \in 2^{\mathbb{N}} \times 2^{\mathbb{N}} : s^\wedge 0 \subset a \wedge s^\wedge 1 \subset b\} \quad \text{and} \quad {R_s}^{-1},$$

we reduce the general case to the case when $\mathtt{dom}\, D_m \cap \mathtt{ran}\, D_m = \emptyset$, $\forall\, m$.

Now, for any m define $h_m(a) = b$ whenever either $\langle a, b\rangle \in D_m$ or $\langle a, b\rangle \in {D_m}^{-1}$, or $a = b \notin \mathtt{dom}\, D_m \cup \mathtt{ran}\, D_m$. Clearly h_m is a Borel bijection $2^{\mathbb{N}} \xrightarrow{\text{onto}} 2^{\mathbb{N}}$. Thus $\{h_m\}_{m\in\mathbb{N}}$ is a family of Borel automorphisms of $2^{\mathbb{N}}$ such that $[a]_{\mathsf{E}} = \{h_m(a) : m \in \mathbb{N}\}$. It does not take much effort to expand this system to a Borel action of F_ω, the free group with $\aleph_0$ generators, on $2^{\mathbb{N}}$, whose induced equivalence relation is E.

(ii) First of all, by (i), $\mathsf{E} \le_{\rm B} \mathsf{R}$, where R is induced by a Borel action $\cdot$ of F_ω on $2^{\mathbb{N}}$. The map $\vartheta(a) = \{g^{-1} \cdot a\}_{g\in F_\omega}$, $a \in 2^{\mathbb{N}}$, is a Borel reduction of R to $\mathsf{E}(F_\omega, 2^{\mathbb{N}})$. If now F_ω is a subgroup of a countable group H then $\mathsf{E}(F_\omega, 2^{\mathbb{N}}) \le_{\rm B} \mathsf{E}(H, 2^{\mathbb{N}})$ by means of the map sending any $\{a_g\}_{g\in F_\omega}$ to $\{b_h\}_{h\in H}$, where $b_g = a_g$ for $g \in F_\omega$ and b_h equal to any fixed $b' \in 2^{\mathbb{N}}$ for $h \in H \smallsetminus F_\omega$. As F_ω admits a homomorphism into F_2 [15] we conclude that $\mathsf{E} \le_{\rm B} \mathsf{E}(F_2, 2^{\mathbb{N}})$.

[15] Why ?.

It remains to transform $\mathsf{E}(F_2, 2^{\mathbb{N}})$ to $\mathsf{E}(F_2, 2)$. The inequality $\mathsf{E}(F_2, 2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E}(F_2, 2^{\mathbb{Z}\smallsetminus\{0\}})$ is clear. Further $\mathsf{E}(F_2, 2^{\mathbb{Z}\smallsetminus\{0\}}) \le_{\mathrm{B}} \mathsf{E}(F_2 \times \mathbb{Z}, 3)$, by means of the map sending any $\{a_g\}_{g\in F_2}$ ($a_g \in 2^{\mathbb{Z}\smallsetminus\{0\}}$) to $\{b_{gj}\}_{g\in F_2,\, j\in\mathbb{Z}}$, where $b_{gj} = a_g(j)$ for $j \ne 0$ and $b_{g0} = 2$. Further, for any G, $\mathsf{E}(G, 3) \le_{\mathrm{B}} \mathsf{E}(G \times \mathbb{Z}_2, 2)$ by means of the map sending any $\{a_g\}_{g\in G}$ ($a_g = 0, 1, 2$) to $\{b_{gi}\}_{g\in G,\, i\in\mathbb{Z}_2}$, where

$$b_{gi} = \begin{cases} 0, & \text{if } \ a_g = 0 \ \textbf{ or } \ a_g = 1 \text{ and } i = 0, \\ 1, & \text{if } \ a_g = 2 \ \textbf{ or } \ a_g = 1 \text{ and } i = 1. \end{cases}$$

Thus $\mathsf{E}(F_2, 2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E}(F_2 \times \mathbb{Z} \times \mathbb{Z}_2, 2)$. However, $F_2 \times \mathbb{Z} \times \mathbb{Z}_2$ admits a homomorphism into F_ω, and then into F_2 (see above), so that $\mathsf{E}(F_2, 2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E}(F_2, 2)$, as required. □

6.c Hyperfinite equivalence relations

All Borel finite ERs are smooth (see § 5.a), accordingly, all hyperfinite ERs are hypersmooth. On the other hand, any finite or hyperfinite equivalence relation is countable, of course. It follows from the next theorem that, conversely, every *hypersmooth* countable ER is hyperfinite. (But there exist countable non-hypersmooth ERs, for instance, E_∞, which are not hyperfinite.)

The theorem also shows that E_0 is a universal hyperfinite ER. (To see that E_0 is hyperfinite, let $x \mathrel{\mathsf{F}_n} y$ iff $x \,\Delta\, y \subseteq [0, n)$ for $x, y \subseteq \mathbb{N}$.)

Theorem 32 (Theorems 5.1 and, partially, 7.1 in [2] and 12.1(ii) in [19]). *The following are equivalent for a Borel ER E on a Polish space $\mathbb{X}$:*

(i) $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ *and* E *is countable;*

(ii) E *is hyperfinite;*

(iii) E *is hypersmooth and countable;*

(iv) *there is a Borel set $X \subseteq \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ such that $\mathsf{E}_1 \restriction X$ is a countable ER and E is isomorphic, via a Borel bijection of $\mathbb{X}$ onto X, to $\mathsf{E}_1 \restriction X$;*

(v) E *is induced by a Borel action of $\mathbb{Z}$, the additive group of the integers.*

(vi) *there exists a pair of Borel ERs F, R of type 2 such that $\mathsf{E} = \mathsf{F} \vee \mathsf{R}$.* [16]

Proof. (ii) $\Longrightarrow$ (iii) and (i) $\Longrightarrow$ (iii) are rather easy.

(iii) $\Longrightarrow$ (iv). Let $\mathsf{E} = \bigcup_n \mathsf{F}_n$ be a countable and hypersmooth ER on a space $\mathbb{X}$, all F_n being smooth (and countable), and $\mathsf{F}_n \subseteq \mathsf{F}_{n+1}$, $\forall n$. We may assume that $\mathbb{X} = \mathscr{P}(\mathbb{N})$ and $\mathsf{F}_0 = \mathsf{D}(\mathscr{P}(\mathbb{N}))$. Let $T_n \subseteq \mathbb{X}$ be a Borel transversal for F_n (recall Lemma 27(iii)). Now let $\vartheta_n(x)$ be the only element of T_n with

[16] An equivalence relation F is *of type n* if any F-class contains at most n elements. $\mathsf{F} \vee \mathsf{R}$ denotes the least ER which includes $\mathsf{F} \cup \mathsf{R}$.

$v \mathsf{F}_n \vartheta_n(x)$. Then $x \mapsto \{\vartheta_n(x)\}_{n\in\mathbb{N}}$ is a $1-1$ Borel map $\mathbb{X} \to \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ and $x \mathsf{E} y \Longleftrightarrow \vartheta(x) \mathsf{E}_1 \vartheta(y)$. Take X to be the image of $\mathbb{X}$.

(iv) $\Longrightarrow$ (v). Let X be as indicated. For any $\mathbb{N}$-sequence x and $n \in \mathbb{N}$, let $x\restriction_{>n} = x\restriction(n,\infty)$. It follows from (the relativized version of) Countable-to-1 Projection and Countable-to-1 Enumeration that for any n the set $X\restriction_{>n} = \{x\restriction_{>n} : x \in X\}$ is Borel and there is a countable family of Borel functions $g_i^n : X\restriction_{>n} \to X$, $i \in \mathbb{N}$, such that the set $X_\xi = \{x \in X : x\restriction_{>n} = \xi\}$ is equal to $\{g_i^n(\xi) : i \in \mathbb{N}\}$ for any $\xi \in X\restriction_{>n}$, hence, $\{g_i^n(\xi)(n) : i \in \mathbb{N}\} = \{x(n) : x \in X_\xi\}$.

For any $x \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ let $\varphi(x) = \{\varphi_n(x)\}_{n\in\mathbb{N}}$, where $\varphi_n(x)$ is the least number i such that $x(n) = f_i^n(x)(n)$; thus, $\varphi(x) \in \mathbb{N}^{\mathbb{N}}$. Let $\mu(x)$ be the sequence

$$\varphi_0(x), \varphi_0'(x), \varphi_1(x)+1, \varphi_1'(x)+1, \dots, \varphi_n(x)+n, \varphi_n'(x)+n, \dots,$$

where $\varphi_n'(x) = \mathtt{max}_{k\le n}\varphi_k(x)$. Easily if $x \ne y \in X$ satisfy $x \mathsf{E}_1 y$, *i.e.*, $x\restriction_{>n} = y\restriction_{>n}$ for some n, then $\varphi(x)\restriction_{>n} = \varphi(y)\restriction_{>n}$ but $\varphi(x) \ne \varphi(y)$, $\mu(x) \ne \mu(y)$, and $\mu(x)\restriction_{>m} = \mu(y)\restriction_{>m}$ for some $m \ge n$.

Let $<_{\mathtt{alex}}$ be the anti-lexicographical partial order on $\mathbb{N}^{\mathbb{N}}$, *i.e.*, $a <_{\mathtt{alex}} b$ iff there is n such that $a\restriction_{>n} = b\restriction_{>n}$ and $a(n) < b(n)$. For $x, y \in X$ define $x <_0 y$ iff $\mu(x) <_{\mathtt{alex}} \mu(y)$. It follows from the above that $<_0$ linearly orders every E_1-class $[x]_{\mathsf{E}_1} \cap X$ of $x \in X$. Moreover, it follows from the definition of $\mu(x)$ that any $<_{\mathtt{alex}}$-interval between some $\mu(x) <_{\mathtt{alex}} \mu(y)$ contains only finitely many elements of the form $\mu(z)$. (For φ this would not be true.) We conclude that any class $[x]_{\mathsf{E}_1} \cap X$, $x \in X$, is linearly ordered by $<_0$ similarly to a subset of $\mathbb{Z}$, the integers. That $<_0$ can be converted to a required Borel action of $\mathbb{Z}$ on X is rather easy (however the E_1-classes in X ordered similarly to $\mathbb{N}$, the inverse of $\mathbb{N}$, or finite, should be treated separately).

(v) $\Longrightarrow$ (ii). Assume *w.l.o.g.* that $\mathbb{X} = 2^{\mathbb{N}}$. An increasing sequence of ERs F_n whose union is E is defined separately on each E-class C; they "integrate" into Borel ERs F_n defined on the whole of $2^{\mathbb{N}}$ because the action allows to replace quantifiers over a E-class C by quantifiers over $\mathbb{Z}$.

Let C be any E-class of $x \in X$. Note that if an element $x_C \in C$ can be chosen in some Borel-definable way then we can define $x \mathsf{F}_n y$ iff there exist integers $j, k \in \mathbb{Z}$ with $|j| \le n$, $|k| \le n$, and $x = j\cdot x_C$, $y = k\cdot x_C$. This applies, for instance, when C is finite, thus, we can assume that C is infinite. Let $<_{\mathtt{lex}}$ be the lexicographical ordering of $2^{\mathbb{N}}$, and $<_{\mathtt{act}}$ be the partial order induced by the action, *i.e.*, $x <_{\mathtt{act}} y$ iff $y = j\cdot x$, $j > 0$. By the same reason we can assume that neither of $a = \mathtt{inf}_{<_{\mathtt{lex}}} C$ and $b = \mathtt{sup}_{<_{\mathtt{lex}}} C$ belongs to C. Let C_n be the set of all $x \in C$ with $x\restriction n \ne a\restriction n$ and $x\restriction n \ne b\restriction n$. Define $x \mathsf{F}_n y$ iff x, y belong to one and the same $<_{\mathtt{lex}}$-interval in C lying entirely within C_n, or just $x = y$. In our assumptions, any F_n has finite classes, and for any two $x, y \in C$ there is n with $x \mathsf{F}_n y$.

(v) $\Longrightarrow$ (i). This is more complicated. A preliminary step is to show that $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}(\mathbb{Z}, 2^{\mathbb{N}})$, where $\mathsf{E}(\mathbb{Z}, 2^{\mathbb{N}})$ is the orbit equivalence induced by the shift action of

$\mathbb{Z}$ on $(2^{\mathbb{N}})^{\mathbb{Z}}$: $k\cdot\{x_j\}_{j\in\mathbb{Z}} = \{x_{j-k}\}_{j\in\mathbb{Z}}$ for $k\in\mathbb{Z}$. Assuming *w.l.o.g.* that E is a ER on $2^{\mathbb{N}}$, we obtain a Borel reduction of E to $\mathsf{E}(\mathbb{Z},2^{\mathbb{N}})$ by $\vartheta(x)=\{j\cdot x\}_{j\in\mathbb{Z}}$, where $\cdot$ is a Borel action of $\mathbb{Z}$ on $2^{\mathbb{N}}$ which induces E. Then Theorem 7.1 in [2] proves that $\mathsf{E}(\mathbb{Z},2^{\mathbb{N}}) \le_{\mathrm{B}} \mathsf{E_0}$.

(vi) $\Longrightarrow$ (v). Let $\mathsf{E}=\mathsf{F}\vee\mathsf{R}$, where F, R are of type 2. For any $x\in\mathbb{X}$ (the domain of E), if $[x]_{\mathsf{F}}$ contains another element $y\ne x$ then call y the *left*, resp., *right neighbour* of x if $y<x$, resp., $y>x$, where $<$ is a fixed Borel linear ordering of $\mathbb{X}$. If the class $[x]_{\mathsf{R}}$ also contains another element, say, z, call it the neighbour of x of the opposite side w.r.t. y. The neighbour relation linearly orders any E-class similarly to a subset of $\mathbb{Z}$, which easily leads to (v). $\longleftarrow$ *wrong* $\dashv$

(v) $\Longrightarrow$ (vi). The authors of [19] present a short proof which refers to several difficult theorems on hyperfinite ERs. Here we give an elementary proof.

Let E be induced by a Borel action of $\mathbb{Z}$. We are going to define F and R on any E-class $C=[x]_{\mathsf{E}}$. If we can choose an element $x_C\in C$ in some uniform Borel-definable way then a rather easy construction is possible, which we leave to the reader. This applies, for instance, when C is finite, hence, let us assume that C is infinite. Let $<_{\mathtt{act}}$ be the linear order on C, induced by the action of $\mathbb{Z}$; it is similar to $\mathbb{Z}$. Let $<_{\mathtt{lex}}$ be the lexicographical ordering of $2^{\mathbb{N}}=\mathtt{dom}\,\mathsf{E}$.

Our goal is to define F on C so that every F-class contains exactly two (distinct) elements. The ensuing definition of R is then rather simple. (First, order pairs $\{x,y\}$ of elements of C in accordance with the $<_{\mathtt{act}}$-lexicographical ordering of pairs $\langle \mathtt{max}_{<_{\mathtt{act}}}\{x,y\}, \mathtt{min}_{<_{\mathtt{act}}}\{x,y\}\rangle$, this is still similar to $\mathbb{Z}$. Now, if $\{x,y\}$ and $\{x',y'\}$ are two F-classes, the latter being the next to the former in the sense just defined, and $x<_{\mathtt{act}} y$, $x'<_{\mathtt{act}} y'$, then define $y\,\mathsf{R}\,x'$.)

Suppose that $W\subseteq C$. An element $z\in W$ iz *lmin* (locally minimal) in W if it is $<_{\mathtt{lex}}$-smaller than both of its $<_{\mathtt{act}}$-neighbours in W. Put $W_{\mathrm{lmin}}=\{z\in W: z \text{ is lmin in } W\}$. If C_{lmin} is <u>not</u> unbounded in C in both directions then an appropriate choice of $x_C\in C$ is possible. (Take the $<_{\mathtt{act}}$-least or $<_{\mathtt{act}}$-largest point in C_{lmin}, or if $C_{\mathrm{lmin}}=\emptyset$, so that, for instance, $<_{\mathtt{act}}$ and $<_{\mathtt{lex}}$ coincide on C, we can choose something like a $<_{\mathtt{lex}}$-middest element of C.) Thus, we can assume that C_{lmin} is unbounded in C in both directions.

Let a *lmin-interval* be any $<_{\mathtt{act}}$-semi-interval $[x,x')$ between two consecutive elements $x<_{\mathtt{act}} x'$ of C_{lmin}. Let $[x,x')=\{x_0,x_1,...,x_{m-1}\}$ be the enumeration in the $<_{\mathtt{act}}$-increasing order ($x_0=x$). Define $x_{2k}\,\mathsf{F}\,x_{2k+1}$ whenever $2k+1<m$. If m is odd then x_{m-1} remains unmatched. Let C^1 be the set of all unmatched elements. Now, the nontrivial case is when C^1 is unbounded in C in both directions. We define C^1_{lmin}, as above, and repeat the same construction, extending F to a part of C^1, with, perhaps, a remainder $C^2\subseteq C^1$ where F remains indefined. *Et cetera.*

Thus, we define a decreasing sequence $C=C^0\supseteq C^1\supseteq C^2\supseteq\dots$ of subsets of C, and the equivalence relation F on each difference $C^n\smallsetminus C^{n+1}$ whose classes contain exactly two points each, and the nontrivial case is when every C^n is $<_{\mathtt{act}}$-

unbounded in C in both directions. (Otherwise there is an appropriate choice of $x_C \in C$.) If $C^\infty = \bigcap_n C^n = \emptyset$ then F is defined on C and we are done. If $C^\infty = \{x\}$ is a singleton then $x_C = x$ chooses an element in C. Finally, C^∞ cannot contain two different elements as otherwise one of C^n would contain two $<_{\mathtt{act}}$-neighbours $x <_{\mathtt{act}} y$ which survive in C^{n+1}, which is easily impossible. $\square$

6.d Non-hyperfinite countable equivalence relations

It follows from Theorem 32(i),(ii) that hyperfinite equivalence relations form an initial segment, in the sense of $\le_{\mathrm{B}}$, among all countable equivalence relations. Let us show that not all countable equivalence relations are hyperfinite.

Theorem 33. *The equivalence relation* E_∞ *is not hyperfinite.*

Proof. A clean elementary proof is given in [41]. $\square$

6.e Assembling hyperfinite equivalence relations

The following theorem shows that, similarly to the case of smooths ERs (Thm 28), hyperfinite ones possess a certain form of countable additivity.

Theorem 34. *Let* E *be a Borel ER on a Borel set* $X = \bigcup_k X_k$*, with all* X_k *also Borel. Suppose that* $\mathsf{E} \restriction X_k \le_{\mathrm{B}} \mathsf{E}_0$ *for each* k*. Then* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$.

Proof. We consider only the case when $X_k \subseteq X_{k+1}$ for all k (the result will be used below only for this particular case), the general case needs to consider separately the two–sets case, as in Theorem 28, which we leave to the reader. $\longleftarrow$ *not the best prf*$\dashv$

There are disjoint Borel sets $B_k \subseteq \mathscr{P}(\mathbb{N})$ and Borel maps $f_k : X_k \to B_k$ which witness that $\mathsf{E} \restriction X_k \le_{\mathrm{B}} \mathsf{E}_0$. We shall assume that the sets B_k are E_0-incompatible in the sense that if $k \ne n$ then $a \mathrel{\mathsf{E}_0} b$ does not hold for any $a \in B_k$ and $b \in B_n$. Let $R_k = \mathtt{ran}\, f_k$ (a $\boldsymbol{\Sigma}^1_1$ subset of B_k). Then

$$F_k = \{\langle a,b\rangle \in R_k \times R_{k+1} : \exists\, x \in X_k\ (f_k(x) = a \wedge f_{k+1}(x) = b)\}\,,$$

is a $\boldsymbol{\Sigma}^1_1$ set, $1-1$ modulo E_0 in the sense that if $\langle a,b\rangle$ and $\langle a',b'\rangle$ belong to F_k then $a \mathrel{\mathsf{E}_0} a' \iff b \mathrel{\mathsf{E}_0} b'$. As "to be $1-1$ modulo E_0" is a $\mathit{\Pi}^1_1$ property in the codes (of $\mathit{\Sigma}^1_1$ subsets of $\mathscr{P}(\mathbb{N})^2$), there is, by $\mathsf{Reflection}$, a $\mathit{\Delta}^1_1$ set F'_k with $F_k \subseteq F'_k \subseteq B_k \times B_{k+1}$ and still $1-1$ modulo E_0. The following $\mathit{\Delta}^1_1$ set

$$G_k = \{\langle a',b'\rangle : \exists\,\langle a,b\rangle \in F'_k\ (a \mathrel{\mathsf{E}_0} a' \wedge b \mathrel{\mathsf{E}_0} b')\}$$

is still $1-1$ modulo E_0, hence, both "vertical" and "horisontal" cross-sections of G_k are countable, thus, $A_k = \mathtt{dom}\, G_k$ and $B_k = \mathtt{ran}\, G_k$ are E_0-invariant Borel sets (and $R_k = \mathtt{dom}\, F_k \subseteq A_k$), and there are Borel maps $h_k : B_k \to A_k$ such that $\langle h_k(b), b\rangle \in G_k$ whenever $b \in B_k$. It follows still from the "$1-1$ modulo E_0" property that if $b \in B_k$ and $b' \mathrel{\mathsf{E}_0} b$ then $b' \in B_k$ and $h_k(b) \mathrel{\mathsf{E}_0} h_k(b')$.

We can assume that $B_{k+1} \subseteq A_k$ for all k. Then, for any k and $b \in A_k$ there is the least $n = n(b) \le k$ such that the application

$$h(b) = h_n(h_{n+1}(h_{n+2}(...h_{k-1}(b)...)))$$

is possible, for instance, $n(b) = k$ and $h(b) = b$ whenever $b \in A_k \smallsetminus B_{k-1}$. As in the proof of Theorem 28, the map $g(x) = h(f_k(x))$ for $x \in X_k \smallsetminus X_{k-1}$ witnesses $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$. □

7 The 2nd dichotomy

The following result is known as 2nd, or "Glimm–Effros", dichotomy.

Theorem 35 (Harrington, Kechris, Louveau [12]). *If* E *is a Borel ER then either* E *is smooth or* $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}$.

7.a The Gandy – Harrington closure

Beginning the proof of Theorem 35 (it will be completed in § 7.d), we suppose, as usual, that E is a lightface Δ^1_1 ER on $\mathbb{N}^{\mathbb{N}}$. Consider an auxiliary ER $x \,\overline{\mathsf{E}}\, y$ iff $x, y \in \mathbb{N}^{\mathbb{N}}$ belong to the same E-invariant Δ^1_1 sets. (A set X is E-*invariant* iff $X = [X]_{\mathsf{E}}$.) Easily $\mathsf{E} \subseteq \overline{\mathsf{E}}$. To see that $\overline{\mathsf{E}}$ is the closure of E in the Gandy – Harrington topology, prove

Lemma 35.2. *If* F *is a* Σ^1_1 *ER on* $\mathbb{N}^{\mathbb{N}}$, *and* $X, Y \subseteq \mathbb{N}^{\mathbb{N}}$ *are disjoint* F-*invariant* Σ^1_1 *sets, then there is an* F-*invariant* Δ^1_1 *set* X' *separating* X *from* Y.

Proof. By Separation, for any Σ^1_1 set A with $A \cap Y = \emptyset$ there is a Δ^1_1 set A' with $A \subseteq A'$ and $A' \cap Y = \emptyset$ — note that then $[A']_{\mathsf{F}} \cap Y = \emptyset$ because Y is F-invariant. It follows that that there is a sequence $X = A_0 \subseteq A'_0 \subseteq A_1 \subseteq A'_1 \subseteq ...$, where A'_i are Δ^1_1 sets, accordingly, $A_{i+1} = [A'_i]_{\mathsf{F}}$ are Σ^1_1 sets, and $A_i \cap Y = \emptyset$. Then $X' = \bigcup_n A_n = \bigcup_n A'_n$ and is an F-invariant Borel set which separates X from Y. To make X' Δ^1_1 we have to maintain the choice of sets A_n effectively.

Let $U \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ be a "good" universal Σ^1_1 set (see § A.c). Then there is a recursive $h : \mathbb{N} \to \mathbb{N}$ such that $[U_n]_{\mathsf{F}} = U_{h(n)}$ for each n. Moreover, applying Lemma 83 (to the complement of U as a "good" universal Π^1_1 set, and with a code for Y fixed), we obtain a pair of recursive functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for any n, if $U_n \cap Y = \emptyset$ then $U_{f(n)}, U_{g(n)}$ are complementary sets (hence, either of them is Δ^1_1) containing, resp., U_n and Y. A suitable iteration of h and f, g allows us to define a sequence $X = A_0 \subseteq A'_0 \subseteq A_1 \subseteq A'_1 \subseteq ...$ as above effectively enough for the union of those sets to be Δ^1_1. □ (*Lemma*)

Lemma 35.3. $\overline{\mathsf{E}}$ *is a* Σ^1_1 *relation.*

Proof. Let $C \subseteq \mathbb{N}$ and $W, W' \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ be as in Δ^1_1 Enumeration (§ A.c). The formula $\mathtt{inv}(e)$ saying that $e \in C$ and $W_e = W'_e$ is E-invariant, *i. e.*,

$$e \in C \wedge \forall a, b \left(a \in W_e \wedge b \notin W'_e \Longrightarrow a \not\mathrel{\mathsf{E}} b\right)$$

is obviously Π^1_1, however $x \mathrel{\overline{\mathsf{E}}} y$ iff

$$\forall e \left(\mathtt{inv}(e) \implies (x \in W_e \Longrightarrow y \in W'_e) \wedge (y \in W_e \Longrightarrow x \in W'_e)\right) \quad \square \; (\textit{Lemma})$$

Let us return to the proof of the theorem. We have two cases.

Case 1: $\mathsf{E} = \overline{\mathsf{E}}$, *i. e.*, E is Gandy – Harrington closed.

Lemma 35.4. *If* $\mathsf{E} = \overline{\mathsf{E}}$ *then there is a* Δ^1_1 *reduction of* E *to* $\mathsf{D}(2^{\mathbb{N}})$.

Proof. Let $C \subseteq \mathbb{N}$ and $W, W' \subseteq \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ be as in the Δ^1_1 Enumeration of § A.c. By Kreisel Selection there is a Δ^1_1 function $\varphi : X^2 \to C$ such that $W_{\varphi(x,y)} = W'_{\varphi(x,y)}$ is a E-invariant Δ^1_1 set containing x but not y whenever $x, y \in X$ are E-inequivalent. Then $R = \mathtt{ran}\,\varphi$ is a Σ^1_1 subset of C, hence, by Separation, there is a Δ^1_1 set N with $R \subseteq N \subseteq C$. The map $\vartheta(x) = \{n \in N : x \in D_n\}$ is a Δ^1_1 reduction of E to $\mathsf{D}(2^{\mathbb{N}})$. $\square$ (*Lemma and Case 1*)

Case 2: $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$. Then the Σ^1_1 set $H = \{x : [x]_{\mathsf{E}} \subsetneqq [x]_{\overline{\mathsf{E}_0}}\}$ (the union of all $\overline{\mathsf{E}}$-classes containing more than one E-class) is non-empty.

Lemma 35.5. *If* $X \subseteq H$ *is a* Σ^1_1 *set then* $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$ *on* X.

Proof. Suppose that $\mathsf{E} \restriction X = \overline{\mathsf{E}} \restriction X$. Then $\mathsf{E} = \overline{\mathsf{E}}$ on $Y = [X]_{\mathsf{E}}$ as well. (If $y, y' \in Y$ then there are $x, x' \in X$ such that $x \mathrel{\mathsf{E}} y$ and $x' \mathrel{\mathsf{E}} y'$, so that if $y \mathrel{\overline{\mathsf{E}}} y'$ then $x \mathrel{\overline{\mathsf{E}}} x'$ by transitivity, hence, $x \mathrel{\mathsf{E}} x'$, and $y \mathrel{\mathsf{E}} y'$ again by transitivity.) It follows that $\mathsf{E} = \overline{\mathsf{E}}$ on an even bigger set, $Z = [X]_{\overline{\mathsf{E}}}$. (Otherwise the Σ^1_1 set $Y' = Z \smallsetminus Y = \{z : \exists x \in X\,(x \mathrel{\overline{\mathsf{E}}} y \wedge x \not\mathrel{\mathsf{E}} y)\}$ is non-empty and E-invariant, together with Y, hence by Lemma 35.2 there is a E-invariant Δ^1_1 set B with $Y \subseteq B$ and $Y' \cap B = \emptyset$, which implies that no point in Y is $\overline{\mathsf{E}}$-equivalent to a point in Y', contradiction.) Then by definition $Z \cap H = \emptyset$. $\square$ (*Lemma*)

Lemma 35.6. *If* $A, B \subseteq H$ *are non-empty* Σ^1_1 *sets with* $A \mathrel{\mathsf{E}} B$ *then there exist non-empty* <u>*disjoint*</u> Σ^1_1 *sets* $A' \subseteq A$ *and* $B' \subseteq B$ *still satisfying* $A' \mathrel{\mathsf{E}} B'$.

Proof. We assert that there are points $a \in A$ and $b \in B$ with $a \neq b$ and $a \mathrel{\mathsf{E}} b$.

(Otherwise E is the equality on $X = A \cup B$. Prove that then $\mathsf{E} = \overline{\mathsf{E}}$ on X, a contradiction to Lemma 35.5. Take any $x \neq y$ in X. Let U be a clopen set containing x but not y. Then $A = [U \cap X]_{\mathsf{E}}$ and $C = [X \smallsetminus U]_{\mathsf{E}}$ are two disjoint E-invariant Σ^1_1 sets containing resp. x, y. Then $x \mathrel{\overline{\mathsf{E}}} y$ fails by Lemma 35.2.)

Thus let a, b be as indicated. Let U be a clopen set containing a but not b. Put $A' = A \cap U \cap [U^{\complement}]_{\mathsf{E}}$ and $B' = B \cap U^{\complement} \cap [U]_{\mathsf{E}}$. $\square$ (*Lemma*)

7.b Restricted product forcing

Recall that forcing notions $\mathbb{P}$ and $\mathbb{P}_2$ were introduced in § 5.c. In continuation of the proof of Theorem 35 (Case 2), let $\mathbb{P}^2\restriction\mathsf{E}$ be the collection of all sets of the form $X\times Y$, where $X, Y\subseteq\mathbb{N}^\mathbb{N}$ are non-empty $\varSigma^1_1$ sets and $X\mathrel{\mathsf{E}}Y$ (which means here that $[X]_\mathsf{E}=[Y]_\mathsf{E}$). Easily $\mathbb{P}_2\subseteq\mathbb{P}^2\restriction\mathsf{E}\subseteq\mathbb{P}^2$. The forcing [17] $\mathbb{P}^2\restriction\mathsf{E}$ is not really a product, yet if $X\times Z\in\mathbb{P}^2\restriction\mathsf{E}$ and $\emptyset\neq X'\subseteq X$ is $\varSigma^1_1$ then $Z'=Z\cap[X']_\mathsf{E}$ is $\varSigma^1_1$ and $X'\times Z'\in\mathbb{P}^2\restriction\mathsf{E}$. It follows that any $\mathbb{P}^2\restriction\mathsf{E}$-generic set $G\subseteq\mathbb{P}^2\restriction\mathsf{E}$ produces a pair of $\mathbb{P}$-generic sets $G_{\mathtt{left}}=\{\mathtt{dom}\,P:P\in G\}$ and $G_{\mathtt{right}}=\{\mathtt{ran}\,P:P\in G\}$, hence, produces a pair of $\mathbb{P}$-generic reals $x^G_{\mathtt{left}}$ and $x^G_{\mathtt{right}}$, whose names will be $\dot{x}_{\mathtt{left}}$ and $\dot{x}_{\mathtt{right}}$.

Lemma 35.2. *In the sense of the forcing $\mathbb{P}^2\restriction\mathsf{E}$, any $P=X\times Z\in\mathbb{P}^2\restriction\mathsf{E}$ forces $\langle\dot{x}_{\mathtt{left}},\dot{x}_{\mathtt{right}}\rangle\in P$ and forces $\dot{x}_{\mathtt{left}}\mathrel{\overline{\mathsf{E}}}\dot{x}_{\mathtt{right}}$, but $H\times H$ forces $\dot{x}_{\mathtt{left}}\mathrel{\not\mathsf{E}}\dot{x}_{\mathtt{right}}$.*

Proof. To see that $\dot{x}_{\mathtt{left}}\mathrel{\overline{\mathsf{E}}}\dot{x}_{\mathtt{right}}$ is forced suppose otherwise. Then, by the definition of $\overline{\mathsf{E}}$, there is a condition $P=X\times Z\in\mathbb{P}^2\restriction\mathsf{E}$ and an E-invariant $\varDelta^1_1$ set B such that P forces $\dot{x}_{\mathtt{left}}\in B$ but $\dot{x}_{\mathtt{right}}\notin B$. Then easily $X\subseteq B$ but $Z\cap B=\emptyset$, a contradiction with $[X]_\mathsf{E}=[Z]_\mathsf{E}$.

To see that $H\times H$ forces $\dot{x}_{\mathtt{left}}\mathrel{\not\mathsf{E}}\dot{x}_{\mathtt{right}}$ suppose towards the contrary that some $P=X\times Z\in\mathbb{P}^2\restriction\mathsf{E}$ with $X\cup Z\subseteq H$ forces $\dot{x}_{\mathtt{left}}\mathrel{\mathsf{E}}\dot{x}_{\mathtt{right}}$, thus,

(1) $x\mathrel{\mathsf{E}}z$ holds for every $\mathbb{P}^2\restriction\mathsf{E}$-generic pair $\langle x,z\rangle\in P$.

Claim 35.3. *If $x,y\in X$ are $\mathbb{P}$-generic over $\mathfrak{M}$, and $x\mathrel{\overline{\mathsf{E}}}y$, then $x\mathrel{\mathsf{E}}y$.*

Proof. We assert that

(2) $x\in A\iff y\in A$ holds for each E-invariant $\varSigma^1_1$ set A.

Indeed, if, say, $x\in A$ but $y\notin A$ then by the genericity of y there is a $\varSigma^1_1$ set C with $y\in C$ and $A\cap C=\emptyset$. As A is E-invariant, Lemma 35.2 yields an E-invariant $\varDelta^1_1$ set B such that $C\subseteq B$ but $A\cap B=\emptyset$. Then $x\notin B$ but $y\in B$, a contradiction to $x\mathrel{\overline{\mathsf{E}}}y$.

Let $\{\mathscr{D}_n\}_{n\in\mathbb{N}}$ be an enumeration of all dense subsets of $\mathbb{P}^2\restriction\mathsf{E}$ which are coded in $\mathfrak{M}$. We define two sequences $P_0\supseteq P_1\supseteq\ldots$ and $Q_0\supseteq Q_1\supseteq\ldots$ of conditions $P_n=X_n\times Z_n$ and $Q_n=Y_n\times Z_n$ in $\mathbb{P}^2\restriction\mathsf{E}$, so that $P_0=Q_0=P$, $x\in X_n$ and $y\in Y_n$ for any n, and finally $P_n, Q_n\in\mathscr{D}_{n-1}$ for $n\geq 1$. If this is done then we have a real z (the only element of $\bigcap_n Z_n$) such that both $\langle x,z\rangle$ and $\langle y,z\rangle$ are $\mathbb{P}^2\restriction\mathsf{E}$-generic, hence, $x\mathrel{\mathsf{E}}z$ and $y\mathrel{\mathsf{E}}z$ by (1), hence, $x\mathrel{\mathsf{E}}y$.

Suppose that P_n and Q_n have been defined. As x is generic, there is (we leave details for the reader) a condition $P'=A\times C\in\mathscr{D}_n$ and $\subseteq P_n$ such that $x\in A$. Let $B=Y_n\cap[A]_\mathsf{E}$: then $y\in B$ by (2), and easily $[B]_\mathsf{E}=[C]_\mathsf{E}=[A]_\mathsf{E}$

[17] Over a countable model $\mathfrak{M}$ chosen in accordance with the requirements in Footnote 14.

(as $[X_n]_{\mathsf{E}} = [Z_n]_{\mathsf{E}} = [Y_n]_{\mathsf{E}}$), thus, $B \times C \in \mathbb{P}^2 \restriction \mathsf{E}$, so there is a condition $Q' = V \times W \in \mathscr{D}_n$ and $\subseteq B \times C \subseteq Q_n$ such that $y \in V$. Put $Y_{n+1} = V$, $Z_{n+1} = W$, and $X_{n+1} = A \cap [W]_{\mathsf{E}}$. $\square$ (*Claim*)

It follows that $\mathsf{E} = \overline{\mathsf{E}}$ on X. (Otherwise $S = \{\langle x, y\rangle \in X^2 : x \,\overline{\mathsf{E}}\, y \wedge x \not\mathsf{E}\, y\}$ is a non-empty Σ^1_1 set, and any $\mathbb{P}_2$-generic pair $\langle x, y\rangle \in S$ implies a contradiction to Claim 35.3. Recall that $\mathbb{P}_2 =$ all non-empty Σ^1_1 subsets of $(\mathbb{N}^{\mathbb{N}})^2$.) But this implies $X \cap H = \emptyset$ by Lemma 35.5, contradiction. $\square$ (*Lemma 35.2*)

7.c Splitting system

Let us fix enumerations $\{\mathscr{D}(n)\}_{n\in\mathbb{N}}$, $\{\mathscr{D}_2(n)\}_{n\in\mathbb{N}}$, $\{\mathscr{D}^2(n)\}_{n\in\mathbb{N}}$ of all dense subsets of resp. $\mathbb{P}$, $\mathbb{P}_2$, $\mathbb{P}^2 \restriction \mathsf{E}$, which belong to $\mathfrak{M}$; we assume that $\mathscr{D}(n+1) \subseteq \mathscr{D}(n)$, $\mathscr{D}_2(n+1) \subseteq \mathscr{D}_2(n)$, and $\mathscr{D}^2(n+1) \subseteq \mathscr{D}^2(n)$. If $u, v \in 2^m$ (binary sequences of length m) have the form $u = 0^k{}^\wedge 0^\wedge w$ and $v = 0^k{}^\wedge 1^\wedge w$ for some $k < m$ and $w \in 2^{m-k-1}$ then we call $\langle u, v\rangle$ a *crucial pair*. It can be proved, *e.g.*, by induction on m, that 2^m is a connected tree (*i.e.*, a connected graph without cycles) of crucial pairs, with sequences beginning with 1 as the endpoints of the graph. We define a system of sets X_u ($u \in 2^{<\omega}$) and R_{uv}, $\langle u, v\rangle$ being a crucial pair, so that the following conditions are satisfied:

(i) $X_u \in \mathbb{P}$, moreover, $X_\Lambda \subseteq H$, and $X_u \in \mathscr{D}(n)$ for any $u \in 2^n$;

(ii) $X_{u^\wedge i} \subseteq X_u$ for all u and i;

(iii) $\mathsf{R}_{uv} \in \mathbb{P}_2$, moreover, $\mathsf{R}_{uv} \in \mathscr{D}_2(n)$ for any crucial pair $\langle u, v\rangle$ in 2^n;

(iv) $\mathsf{R}_{uv} \subseteq \mathsf{E}$ and $X_u \,\mathsf{R}_{uv}\, X_v$ for any crucial pair $\langle u, v\rangle$ in 2^n;

(v) $\mathsf{R}_{u^\wedge i,\, v^\wedge i} \subseteq \mathsf{R}_{uv}$;

(vi) if $u, v \in 2^n$ and $u(n-1) \neq v(n-1)$ then $X_u \times X_v \in \mathscr{D}^2(n)$ and also $X_u \cap X_v = \emptyset$.

Note that (iv) implies that $X_u \,\mathsf{E}\, X_v$ for any crucial pair $\langle u, v\rangle$, hence, also for any pair in 2^n because any $u, v \in 2^n$ are connected by a unique chain of crucial pairs. It follows that $X_u \times X_v \in \mathbb{P}^2 \restriction \mathsf{E}$ for any pair of $u, v \in 2^n$, for any n.

Assume that such a system has been defined. Then for any $a \in 2^{\mathbb{N}}$ the sequence $\{X_{a\restriction n}\}_{n\in\mathbb{N}}$ is $\mathbb{P}$-generic over $\mathfrak{M}$, hence, $\bigcap_n X_{a\restriction n} = \{x_a\}$, where x_a is $\mathbb{P}$-generic, and the map $a \mapsto x_a$ is continuous since diameters of X_u converge to 0 uniformly with $\mathtt{lh}\, u \to 0$ by (i), and is $1-1$ by the last condition of (vi).

Let $a, b \in 2^{\mathbb{N}}$. If $a \not\mathsf{E}_0\, b$ then, by (vi), $\langle x_a, x_b\rangle$ is a $\mathbb{P}^2 \restriction \mathsf{E}$-generic pair, hence, $x_a \not\mathsf{E}\, x_b$ by Lemma 35.2. Now suppose that $a \,\mathsf{E}_0\, b$, prove that then $x_a \,\mathsf{E}\, x_b$. We can suppose that $a = w^\wedge 0^\wedge c$ and $b = w^\wedge 0^\wedge c$, where $w \in 2^{<\omega}$ and $c \in 2^{\mathbb{N}}$ (indeed if $a \,\mathsf{E}_0\, b$ then a, b can be connected by a finite chain of such special pairs). Then $\langle x_a, x_b\rangle$ is $\mathbb{P}_2$-generic, actually, the only member of the intersection

$\bigcap_n \mathsf{R}_{w^\wedge 0^\wedge(c\restriction n)\,,\,w^\wedge 1^\wedge(c\restriction n)}$ by (iii) and (iv), in particular, $x_a \mathsf{E}\, x_b$ because we have $R_{uv} \subseteq \mathsf{E}$ for all u, v.

Thus we have a continuous $1-1$ reduction of E_0 to E.

□ (*Case 2 in Theorem 35 modulo the construction*)

7.d Construction of a splitting system

Let X_Λ be any member of $\mathscr{D}(0)$ satisfying $X_\Lambda \subseteq H$. Now suppose that X_s and R_{st} have been defined for all $s \in 2^n$ and all crucial pairs in 2^n, and extend the construction on 2^{n+1}. Temporarily, define $X_{s^\wedge i} = X_s$ and $\mathsf{R}_{s^\wedge i\,,\,t^\wedge i} = \mathsf{R}_{st}$: this leaves $\mathsf{R}_{0^n{}^\wedge 0\,,\,0^n{}^\wedge 1}$ still undefined, so we put $\mathsf{R}_{0^n{}^\wedge 0\,,\,0^n{}^\wedge 1} = \mathsf{E} \cap X_{0^n} \times X_{0^n}$. Note that the such defined system of sets X_u and relations R_{uv} at level $n+1$ satisfies all requirements of (i) – (vi) except for the requirement of membership in the dense sets – say in this case that the system is "coherent". It remains to produce a still "coherent" system of smaller sets and relations which also satisfies the membership in the dense sets. This will be achieved in several steps.

Step 1: achieve that $X_u \in \mathscr{D}(n+1)$ for any $u \in 2^{n+1}$. Take any particular $u_0 \in 2^{n+1}$. There is, by the density, $X'_{u_0} \in \mathscr{D}(n+1)$ and $\subseteq X_{u_0}$. Suppose that $\langle u_0, v\rangle$ is a crucial pair. Put $\mathsf{R}'_{u_0,v} = \{\langle x,y\rangle \in \mathsf{R}_{u_0,v} : x \in X'_{u_0}\}$ and $X'_v = \mathtt{ran}\,\mathsf{R}'_{u_0,v}$. This shows how the change spreads along the whole set 2^{n+1} viewed as the tree of crucial pairs. Finally we obtain a coherent system with the additional requirement that $X'_{u_0} \in \mathscr{D}(n+1)$. Do this consecutively for all $u_0 \in 2^{n+1}$. The total result – we re-denote it as still X_u and R_{uv} – is a "coherent" system with $X_u \in \mathscr{D}(n+1)$ for all u. Note that still $X_{0^n{}^\wedge 0} = X_{0^n{}^\wedge 1}$ and

$$\mathsf{R}_{0^n{}^\wedge 0\,,\,0^n{}^\wedge 1} = \mathsf{E} \cap (X_{0^n{}^\wedge 0} \times X_{0^n{}^\wedge 1})\,. \qquad (*)$$

Step 2: achieve that $X_{s^\wedge 0} \times X_{t^\wedge 1} \in \mathscr{D}^2(n+1)$ for all $s,\, t \in 2^{n+1}$. Consider a pair of $u_0 = s_0{}^\wedge 0$ and $v_0 = t_0{}^\wedge 1$ in 2^{n+1}. By the density there is a set $X'_{u_0} \times X'_{v_0} \in \mathscr{D}^2(n+1)$ and $\subseteq X_{u_0} \times X_{v_0}$. By definition we have $X'_{u_0} \mathsf{E}\, X'_{v_0}$, but, due to Lemma 35.6 we can maintain that $X'_{u_0} \cap X'_{v_0} = \emptyset$. The two "shockwaves", from the changes at u_0 and v_0, as in Step 1, meet only at the pair $0^m{}^\wedge 0$, $0^m{}^\wedge 1$, where the new sets satisfy $X'_{0^m{}^\wedge 0} \mathsf{E}\, X'_{0^m{}^\wedge 1}$ just because E-equivalence is everywhere kept and preserved though the changes. Now, in view of $(*)$, we can define $\mathsf{R}'_{0^n{}^\wedge 0\,,\,0^n{}^\wedge 1} = \mathsf{E} \cap (X'_{0^n{}^\wedge 0} \times X'_{0^n{}^\wedge 1})$, preserving $(*)$ as well. All pairs considered, we will be left with a coherent system of sets and relations, re-denoted as X_u and R_{uv}, which satisfies the $\mathscr{D}(n+1)$-requirements in (i) and (vi).

Step 3: achieve that $\mathsf{R}_{uv} \in \mathscr{D}_2(n+1)$ for any crucial pair at level $n+1$, and also that $X'_{0^n{}^\wedge 0} \cap X'_{0^n{}^\wedge 1} = \emptyset$. Consider any crucial pair $\langle u_0, v_0\rangle$. If this is not $\langle 0^n{}^\wedge 0, 0^n{}^\wedge 1p\rangle$ then let $\mathsf{R}'_{u_0 v_0} \subseteq \mathsf{R}_{u_0 v_0}$ be any set in $\mathscr{D}_2(n+1)$. If this is $u_0 = 0^n{}^\wedge 0$ and $v_0 = 0^n{}^\wedge 1$ then first we choose (Lemma 35.6) disjoint non-empty Σ^1_1 sets $U \subseteq X_{0^n{}^\wedge 0}$ and $V \subseteq X_{0^n{}^\wedge 1}$ still with $U \mathsf{E} V$, and only then a set $\mathsf{R}'_{u_0 v_0} \subseteq \mathsf{E} \cap (U \times V)$ which belongs to $\in \mathscr{D}_2(n+1)$. In both cases, put $X'_{u_0} = \mathtt{dom}\,\mathsf{R}'_{u_0 v_0}$

and $X'_{v_0} = \mathtt{ran}\,\mathsf{R}'_{u_0v_0}$. It remains to spread the changes, along the chain of crucial pairs, to the left of u_0 and to the right of v_0, exactly as in Case 1. Executing such a reduction for all crucial pairs $\langle u_0, v_0\rangle$ at level $n+1$ one by one, we end up with a system of sets fully satisfying (i) – (vi).

□ (*Theorem 35*)

7.e A forcing notion associated with E_0

We here consider the forcing notion $\mathbb{P}_{\mathsf{E}_0/\mathsf{D}(2^{\mathbb{N}})}$ (see §3.e), that will be denoted by $\mathbb{P}_{\mathsf{E}_0}$ below. Thus by definition $\mathbb{P}_{\mathsf{E}_0}$ consists of all Borel sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_0 \restriction X$ is non-smooth while the related ideal $\mathscr{I}_{\mathsf{E}_0} = \mathscr{I}_{\mathsf{E}_0/\mathsf{D}(2^{\mathbb{N}})}$ consists of all Borel sets $X \subseteq 2^{\mathbb{N}}$ such that $\mathsf{E}_0 \restriction X$ is smooth.

Lemma 36. (i) *$\mathscr{I}_{\mathsf{E}_0}$ is a σ-additive ideal. Let $X \subseteq 2^{\mathbb{N}}$ be a Borel set.*

(ii) *X belongs to $\mathbb{P}_{\mathsf{E}_0}$ iff $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$ (by a continuous injection).*

(iii) *X belongs to $\mathscr{I}_{\mathsf{E}_0}$ iff $\mathsf{E}_0 \restriction X$ admits a Borel transversal.*

Proof. (i) immediately follows from Theorem 28. In (ii), if $X \in \mathbb{P}_{\mathsf{E}_0}$ then $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$ by Theorem 35, while if $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$ then $\mathsf{E}_0 \restriction X$ is not smooth since E_0 itself is not smooth by Lemma 27(v). In (iii), if $\mathsf{E}_0 \restriction X$ admits a Borel transversal then it is smooth by Lemma 27(i) and hence X belongs to $\mathscr{I}_{\mathsf{E}_0}$. To prove the converse apply Lemma 27(iii). □

Note that any $X \in \mathbb{P}_{\mathsf{E}_0}$ contains a closed subset $Y \subseteq X$ also in $\mathbb{P}_{\mathsf{E}_0}$ by Theorem 35. (Apply the theorem for $\mathsf{E} = \mathsf{E}_0 \restriction X$. As $\mathsf{E}_0 \restriction X$ is not smooth, we have $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction X$, by a continuous reduction ϑ. Take as Y the full image of ϑ. Y is compact, hence closed.) Such sets Y can be chosen in a special family.

Definition 37 (Zapletal [47])**.** Suppose that two binary sequences $u^0_n \neq u^1_n \in 2^{<\omega}$ of equal length $\mathtt{lh}\,u^0_n = \mathtt{lh}\,u^1_n \geq 1$ are chosen for each n, together with one more sequence $u_0 \in 2^{<\omega}$. Define $\vartheta(a) = u_0{}^\wedge u_0^{a(0)}{}^\wedge u_1^{a(1)}{}^\wedge \dots$ for any $a \in 2^{\mathbb{N}}$. Easily ϑ is a continuous injection $2^{\mathbb{N}} \to 2^{\mathbb{N}}$, $Y = \mathtt{ran}\,\vartheta$ is a closed set in $2^{\mathbb{N}}$, ϑ witnesses $\mathsf{E}_0 \sqsubseteq_{\mathrm{C}} \mathsf{E}_0 \restriction Y$, and hence $Y \in \mathbb{P}_{\mathsf{E}_0}$.

Let $\mathbb{P}'_{\mathsf{E}_0}$ denote the collection of all sets Y definable in such a form. □

Theorem 38 (Zapletal [47])**.** *$\mathbb{P}'_{\mathsf{E}_0}$ is a dense subset of $\mathbb{P}_{\mathsf{E}_0}$: for any $X \in \mathbb{P}'_{\mathsf{E}_0}$ there exists $Y \in \mathbb{P}_{\mathsf{E}_0}$, $Y \subseteq X$. In addition, $\mathbb{P}_{\mathsf{E}_0}$ forces that the "old" continuum $\mathfrak{c}$ remains uncountable.*

Proof. The proof employs splitting technique for the forcing $\mathbb{P}_{\mathsf{E}_0}$. This technique somewhat differs from the splittings used in the proof of Theorem 35. First of all, as mentioned above, we can consider only closed sets in $\mathbb{P}_{\mathsf{E}_0}$, that enables us to replace the Gandy – Harrington stuff by a simple compactness argument. Second, the equivalence relation considered has the form $\mathsf{E}_0 \restriction X$.

For any sequences $r, w \in 2^{<\omega}$ with $\mathtt{lh}\, r \le \mathtt{lh}\, w$, define $rw \in 2^{<\omega}$ (the r-shift of w) so that $\mathtt{lh}\, rw = \mathtt{lh}\, w$ and $(rw)(k) = 1 - w(k)$ whenever $k < \mathtt{lh}\, r$ and $r(k) = 1$, and $(rw)(k) = w(k)$ otherwise. Clearly $r(rw) = w$. Similarly define $ra \in 2^{\mathbb{N}}$ for $a \in 2^{\mathbb{N}}$, and $rX = \{ra : a \in X\}$ for any set $X \subseteq 2^{\mathbb{N}}$.

We are going to define sequences $u \in 2^{<\omega}$ and $u_n^0 \ne u_n^1 \in 2^{<\omega}$ ($n \in \mathbb{N}$) such that $\mathtt{lh}\, u_n^0 = \mathtt{lh}\, u_n^1$, as in Definition 37, and also a system of closed sets $X_s \in \mathbb{P}_{\mathsf{E_0}}$ ($s \in 2^{<\omega}$) satisfying the following:

(i) $X_\Lambda \subseteq X$ and $X_{s^\wedge i} \subseteq X_s$;

(ii) $X_s \subseteq \mathscr{O}_{w_s}$, where $w_s = u_0{}^\wedge u_0^{s(0)}{}^\wedge u_1^{s(1)}{}^\wedge \dots {}^\wedge u_{k-1}^{s(k-1)} \in 2^{<\omega}$, $k = \mathtt{lh}\, s$, and $\mathscr{O}_w = \{a \in 2^{\mathbb{N}} : w \subset a\}$ for $w \in 2^{<\omega}$;

(iii) if $s, t \in 2^n$ for some n then $X_t = w_t w_s X_s$.

Then define the map ϑ as in Definition 37. The set $Y = \mathtt{ran}\, \vartheta = \bigcap_n \bigcup_{s \in 2^n} X_s \subseteq X$ belongs to $\mathbb{P}'_{\mathsf{E_0}}$, proving the density claim of the theorem.

Step 0. We put $X_\Lambda = X$ and let $u_0 \in 2^{<\omega}$ be the largest sequence such that $X_\Lambda \subseteq \mathscr{O}_{u_0}$. Let $\ell_0 = \mathtt{lh}\, u_0$.

Step 1. Here we define u_0^i and $X_{\langle i \rangle}$ for $i = 0, 1$. Let R be the set of all sequences $r \in 2^{<\omega}$ containing at least one term equal to 1 (and hence $ra \ne a$ for any a). Consider the union $Z = \bigcup_{r \in R} Z_r$ of all sets $Z_r = \{a \in X_\Lambda : ra \in X_\Lambda\}$; each Z_r is closed. The difference $D = X_\Lambda \smallsetminus Z$ is pairwise $\mathsf{E_0}$-inequivalent, hence $D \in \mathscr{I}_{\mathsf{E_0}}$ by Lemma 36. Thus at least one of Z_r, $r \in R$, belongs to $\mathbb{P}_{\mathsf{E_0}}$ by Lemma 36. Let r_1 be any $r \in R$ of this sort. Put $\ell_1 = \mathtt{lh}\, r_1$; clearly $\mathtt{lh}\, u_0 = \ell_0 < \ell_1$ and $r_1 \restriction \ell_0$ consists only of terms equal to 0.

There is a sequence $w_{\langle 0 \rangle} \in 2^{<\omega}$ such that $\mathtt{lh}\, w_{\langle 0 \rangle} = \ell_1$ and the set $X_{\langle 0 \rangle} = Z_{r_1} \cap \mathscr{O}_{w_{\langle 0 \rangle}}$ still belongs to $\mathbb{P}_{\mathsf{E_0}}$. Put $w_{\langle 1 \rangle} = r_1 w_{\langle 0 \rangle}$. Then the set $X_{\langle 1 \rangle} = r_1 X_{\langle 0 \rangle} = \{r_1 a : a \in X_{\langle 0 \rangle}\} = Z_{r_1} \cap \mathscr{O}_{w_{\langle 1 \rangle}}$ belongs to $\mathbb{P}_{\mathsf{E_0}}$ together with $X_{\langle 0 \rangle}$. Note that $u_0 \subset w_{\langle i \rangle}$, and hence there exist sequences $u_0^0 \ne u_0^1 \in 2^{<\omega}$ of length $\ell_1 - \ell_0$ such that $w_{\langle 0 \rangle} = u_0{}^\wedge u_0^0$ and $w_{\langle 1 \rangle} = u_0{}^\wedge u_0^1$. It follows from the construction that $w_{\langle 0 \rangle} w_{\langle 1 \rangle} = r_1$, therefore $X_{\langle 1 \rangle} = w_{\langle 0 \rangle} w_{\langle 1 \rangle} X_{\langle 1 \rangle}$, and (iii) holds.

Step 2. Here we define u_1^i for $i = 0, 1$ and X_s for $s \in 2^{<\omega}$ with $\mathtt{lh}\, s = 2$. Once again there is a sequence $r_2 \in R$ such that the (closed) set $Z_{r_2} = \{a \in X_{\langle 0 \rangle} : ra \in X_{\langle 0 \rangle}\}$ still belongs to $\mathbb{P}_{\mathsf{E_0}}$. Put $\ell_2 = \mathtt{lh}\, r_2$; then $\mathtt{lh}\, r_1 = \ell_1 < \ell_2$ and $r_2 \restriction \ell_1$ consists only of terms equal to 0. Once again there is a sequence $w_{\langle 0,0 \rangle} \in 2^{<\omega}$ such that $\mathtt{lh}\, w_{\langle 0,0 \rangle} = \ell_2$ and the set $X_{\langle 0,0 \rangle} = Z_{r_2} \cap \mathscr{O}_{w_{\langle 0,0 \rangle}}$ belongs to $\mathbb{P}_{\mathsf{E_0}}$. Put $w_{\langle 0,1 \rangle} = r_2 w_{\langle 0,0 \rangle}$. Then the set $X_{\langle 0,1 \rangle} = r_2 X_{\langle 0,0 \rangle} = Z_{r_2} \cap \mathscr{O}_{w_{\langle 0,1 \rangle}}$ belongs to $\mathbb{P}_{\mathsf{E_0}}$ together with $X_{\langle 0,0 \rangle}$. Also, put $w_{\langle 1,i \rangle} = r_1 w_{\langle 0,i \rangle}$ and $X_{\langle 1,i \rangle} = r_1 X_{\langle 0,i \rangle} = Z_{r_2} \cap \mathscr{O}_{w_{\langle 1,i \rangle}}$ for $i = 0, 1$ — these sets also belong to $\mathbb{P}_{\mathsf{E_0}}$. As for (iii) at this level, take, for instance, $s = \langle 0, 1 \rangle$ and $t = \langle 1, 0 \rangle$. By definition $X_{\langle 1,0 \rangle} = r_1 X_{\langle 0,0 \rangle} = r_2 r_1 X_{\langle 0,1 \rangle}$, on the other hand, $w_{\langle 1,0 \rangle} = r_2 r_1 w_{\langle 0,1 \rangle}$, too.

Finally, there exist sequences $u_1^0 \ne u_1^1 \in 2^{<\omega}$ of length $\ell_2 - \ell_1$ such that $w_{\langle i,j \rangle} = u_0{}^\wedge u_0^i{}^\wedge u_1^j$ for $i, j = 0, 1$.

Steps ≥ 3. Et cetera. The construction results in a system of sets and sequences satisfying requirements (i), (ii), (iii), as required.

To prove the additional claim of the theorem, the splitting construction has to be modified so that for any n the sets X_s, $s \in 2^n$, belong to the n-th dense subset of $\mathbb{P}_{\mathsf{E}_0}$, in the sense of a given countable sequence of dense sets. $\square$

We observe that $\mathbb{P}_{\mathsf{E}_0}$ as a forcing is somewhat closer to Silver rather than Sacks forcing. The property of minimality of the generic real, common to both Sacks and Silver, holds for $\mathbb{P}_{\mathsf{E}_0}$ as well, the proof resembles known arguments, but in addition the following is applied: if $X \in \mathbb{P}_{\mathsf{E}_0}$ and $f : X \to 2^{\mathbb{N}}$ is a Borel E_0-invariant map (that is, $x \mathrel{\mathsf{E}_0} y \Longrightarrow f(x) = f(y)$) then f is constant on a set $Y \in \mathbb{P}_{\mathsf{E}_0}$, $Y \subseteq X$. [18]

8 Ideal $\mathscr{I}_1$ and P-ideals

By definition the ideal $\mathsf{Fin} \times 0 = \mathscr{I}_1$ consists of all sets $x \subseteq \mathscr{P}(\mathbb{N} \times \mathbb{N})$ such that all, except for finitely many, cross-sections $(x)_n = \{k : \langle n, k \rangle \in x\}$ are empty.

8.a Ideals below $\mathscr{I}_1$

It turns out that there exist only three different ideals Borel reducible to $\mathscr{I}_1$, they are Fin, the disjoint sum $\mathsf{Fin} \oplus \mathscr{P}(\mathbb{N})$, and $\mathscr{I}_1$ itself.

Definition 39. An ideal $\mathscr{I}$ is a *trivial variation* of $\mathscr{J}$ if there is an infinite set D such that $I \restriction D \cong \mathscr{J}$ [19] while $\mathscr{I} \restriction \complement D = \mathscr{P}(\complement D)$. (The last condition is equivalent to $\mathscr{I} = \{x : x \cap D \in \mathscr{I} \restriction D\}$.) $\square$

Theorem 40 (Kechris [27])**.** *If $\mathscr{I} \leq_{\mathrm{B}} \mathscr{I}_1$ is a Borel (nontrivial) ideal on $\mathbb{N}$ then either $\mathscr{I} \cong \mathscr{I}_1$ or $\mathscr{I}$ is a trivial variation of Fin.*

Exercise 40.1. Prove that any trivial variation of $\mathscr{I}_1$ is isomorphic to $\mathscr{I}_1$ while any trivial variation of Fin is isomorphic either to Fin or to the disjoint sum $\mathsf{Fin} \oplus \mathscr{P}(\mathbb{N})$, *e.g.*, realized in the form of $\{x \subseteq \mathbb{N} : x \cap \mathrm{ODD} \in \mathsf{Fin}\}$. $\square$

Proof (Theorem)**.** We begin with another version of the method used in the proof of Theorem 30. Suppose that $\{\mathscr{B}_k\}_{k\in\mathbb{N}}$ is a fixed system of Borel subsets of $\mathscr{P}(\mathbb{N})$. (It will be specified later.) Then there exists an increasing sequence of integers $0 = n_0 < n_1 < n_2 < \dots$ and sets $s_k \subseteq [n_k, n_{k+1})$ such that

[18] Suppose, for the sake of brevity, that $X = 2^{\mathbb{N}}$. For any n, the set $Y_n^0 = \{a : f(a)(n) = 0\}$ is Borel and E_0-invariant. It follows that Y_n^0 is either meager or comeager. Put $b(n) = 0$ iff Y_n^0 is comeager. Then $D = \{a : f(a) = b\}$ is comeager. A splitting construction as in the proof of Theorem 38 yields a set $Y \in \mathbb{P}_{\mathsf{E}_0}$, $Y \subseteq D$.

[19] Recall that $\mathscr{I} \cong \mathscr{J}$ means isomorphism via a bijection between the underlying sets.

(1) any $x \subseteq \mathbb{N}$ with $\forall^\infty k\,(x \cap [n_k, n_{k+1}) = s_k)$ is "generic" [20] ;

(2) if $k' \ge k$ and $u \subseteq [0, n_{k'})$ then $u \cup s_{k'}$ decides $\mathscr{B}_k$ in the sense that either any "generic" $x \in \mathscr{P}(\mathbb{N})$ with $x \cap [0, n_{k'+1}) = u \cup s_{k'}$ belongs to $\mathscr{B}_k$ or any "generic" x with $x \cap [0, n_{k'+1}) = u \cup s_{k'}$ does not belong to $\mathscr{B}_k$.

Now put $\mathscr{D}_0 = \{x \cup S_1 : x \subseteq Z_0\}$ and $\mathscr{D}_1 = \{x \cup S_0 : x \subseteq Z_1\}$, where

$$S_0 = \bigcup_k s_{2k} \subseteq Z_0 = \bigcup_k [n_{2k}, n_{2k+1})\,, \quad S_1 = \bigcup_k s_{2k+1} \subseteq Z_1 = \bigcup_k [n_{2k+1}, n_{2k+2}).$$

Clearly any $x \in \mathscr{D}_0 \cup \mathscr{D}_1$ is "generic" by (1), hence, by (2),

(3) each $\mathscr{B}_k$ is clopen on both $\mathscr{D}_0$ and $\mathscr{D}_1$.

As $\mathscr{I} \le_{\mathrm{B}} \mathscr{I}_1$, it follows from Lemma 1 (and the trivial fact that $\mathscr{I}_1 \oplus \mathscr{I}_1 \cong \mathscr{I}_1$) that there exists a *continuous* reduction $\vartheta : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N} \times \mathbb{N})$ of $\mathscr{I}$ to $\mathscr{I}_1$. Thus $\mathsf{E}_{\mathscr{I}}$ is the union of an increasing sequence of (topologically) closed ERs $\mathsf{R}_m \subseteq \mathsf{E}_{\mathscr{I}}$ just because $\mathscr{I}_1$ admits such a form. We now require that $\{\mathscr{B}_k\}$ includes all sets $B^m_l = \{x \in \mathscr{P}(\mathbb{N}) : \forall\, s \subseteq [0, l)\; x \mathrel{\mathsf{R}_m} (x \,\Delta\, s)\}$. Then by (3) and the compactness of $\mathscr{D}_i$ for any l there is $m(l) \ge l$ satisfying

(4) $\forall\, x \in \mathscr{D}_0 \cup \mathscr{D}_1\, \forall\, s \subseteq [0, l)\, \big(x \mathrel{\mathsf{R}_{m(l)}} (x \,\Delta\, s)\big)$.

To prove the theorem it suffices to obtain a sequence $x_0 \subseteq x_1 \subseteq x_2 \subseteq \ldots$ of sets $x_k \in \mathscr{I}$ with $\mathscr{I} = \bigcup_n \mathscr{P}(x_n)$: that in this case $\mathscr{I}$ is as required is an easy exercise. As any topologically closed ideal is easily $\mathscr{P}(x)$ for some $x \subseteq \mathbb{N}$, it suffices to show that $\mathscr{I}$ is a union of a countable sequence of closed subideals. It suffices to demonstrate this fact separately for $\mathscr{I} \restriction Z_0$ and $\mathscr{I} \restriction Z_1$. Prove that $\mathscr{I} \restriction Z_0$ is a countable union of closed subideals, ending the proof of the theorem.

If $m \in \mathbb{N}$ and $s \subseteq u \subseteq Z_0$ are finite then let

$$I^m_{us} = \{A \subseteq Z_0 : \forall\, x \in \mathscr{D}_0\, (x \cap u = s \Longrightarrow \big(x \cup (A \smallsetminus u)\big) \mathrel{\mathsf{R}_m} x)\}\,.$$

Lemma 40.2. *Sets I^m_{us} are closed topologically and under $\cup$, and $I^m_{us} \subseteq \mathscr{I}$.*

Proof. I^m_{us} are topologically closed because so are R_m.

Suppose that $A, B \in I^m_{us}$. To prove that $A \cup B \in I^m_{us}$, let $x \in \mathscr{D}_0$ satisfy $x \cap u = s$. Then $x' = x \cup (A \smallsetminus u) \in \mathscr{D}_0$ satisfies $x' \cap u = s$, too, hence, as $B \in I^m_{us}$, we have $\big(x' \cup (B \smallsetminus u)\big) \mathrel{\mathsf{R}_m} x'$, thus, $\big(x \cup ((A \cup B) \smallsetminus u)\big) \mathrel{\mathsf{R}_m} x'$. However $x' \mathrel{\mathsf{R}_m} x$ just because $A \in I^m_{us}$. It remains to recall that R_m is a ER.

To prove that any $A \in I^m_{us}$ belongs to $\mathscr{I}$ take $x = s \cup S_1$. Then we have $x \cup (A \smallsetminus u) \mathrel{\mathsf{R}_m} x$, thus, $A \in \mathscr{I}$ as s is finite and $\mathsf{R}_m \subseteq \mathsf{E}_{\mathscr{I}}$. □ (*Lemma*)

Lemma 40.3. $\mathscr{I} \restriction Z_0 = \bigcup_{m,\,u,\,s} I^m_{us}$.

[20] We mean, Cohen generic over a certain fixed countable transitive model $\mathfrak{M}$ of a big enough fragment of **ZFC**, which contains Borel codes for all sets $\mathscr{B}_k$.

Proof. Let $A \in \mathscr{I}$, $A \subseteq Z_0$. The sets $Q_m = \{x \in \mathscr{D}_0 : (x \cup A)\, \mathsf{R}_m\, x\}$ are closed and satisfy $\mathscr{D}_0 = \bigcup_m Q_m$. It follows that one of them has a non-empty interior in $\mathscr{D}_0$, thus, there exist finite sets $s \subseteq u \subseteq Z_0$ and some m_0 with

$$\forall\, x \in \mathscr{D}_0\, (x \cap u = s \Longrightarrow (x \cup A)\, \mathsf{R}_{m_0}\, x)\,.$$

This is not exactly what we need, however, by (4), there exists a number $m = \mathtt{max}\{m_0, m(\mathtt{sup}\, u)\}$ big enough for

$$\forall\, x \in \mathscr{D}_0 : (x \cup A)\, \mathsf{R}_m\, (x \cup (A \smallsetminus u))\,.$$

It follows that $A \in I^m_{su}$, as required. □ (*Lemma*)

Let J^m_{su} be the hereditary hull of I^m_{su} (all subsets of sets in I^m_{su}). It follows from Lemma 40.2 that any J^m_{su} is a topologically closed subideal of $\mathscr{I} \restriction Z_0$, however, $\mathscr{I} \restriction Z_0$ is the union of those ideals by Lemma 40.3, as required. □

8.b $\mathscr{I}_1$ and P-ideals

Thus $\mathscr{I}_1$ is a $\le_{\mathrm{B}}$-minimal ideal over Fin : we have $\mathsf{Fin} <_{\mathrm{B}} \mathscr{I}_1$ and the $<_{\mathrm{B}}$-interval $(\mathsf{Fin}, \mathscr{I}_1)$ is empty. Although $\mathscr{I}_1$ is not *the least* over Fin, still it turns out that $\mathscr{I}_1$ is the least among all Borel ideals which are not P-ideals.

The next theorem is of great importance for the whole theory of Borel ideals.

Theorem 41 (Solecki [42, 43])**.** *The following families of ideals on* $\mathbb{N}$ *coincide*:

(i) *ideals of the form* $\mathtt{Exh}_\varphi$, *where* φ *is a l. s. c. submeasure on* $\mathbb{N}$;

(ii) *polishable ideals.*

(iii) *analytic P-ideals*;

(iv) *analytic ideals* $\mathscr{I}$ *with* $\mathscr{I}_1 \not\le_{\mathrm{RB}} \mathscr{I}$;

(v) *analytic ideals* $\mathscr{I}$ *such that all countable unions of* $\mathscr{I}$*-small sets are* $\mathscr{I}$*-small,* where a set $X \subseteq \mathscr{P}(\mathbb{N})$ is *$\mathscr{I}$-small* if there is $A \in \mathscr{I}$ such that $X \restriction A = \{x \cap A : x \in X\} \subseteq \mathscr{P}(A)$ is meager in $\mathscr{P}(A)$.

It follows that all analytic P-ideals actually belong to $\mathbf{\Pi}^0_3$, just because any ideal of type (i) is easily $\mathbf{\Pi}^0_3$.

Proof. The formal scheme of the proof is: (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) $\Longrightarrow$ (i). The hard part will be (v) $\Longrightarrow$ (i), the rest is rather elementary but tricky in some points. The elementary part of the proof is organized so that the proofs that (i) $\Longleftrightarrow$ (ii) and (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) and that the first group implies the second, are obtained independently of the hard part.

(i) $\Longrightarrow$ (ii) If $\varphi(\{n\}) > 0$ for all n then the required metric on $\mathscr{I} = \mathtt{Exh}_\varphi$ can be defined by $d_\varphi(x, y) = \varphi(x \,\Delta\, y)$. Then any set $U \subseteq \mathscr{I}$ open in the sense

$\leftarrow$ ***Give corollaries of Thm 41*** $\dashv$

of the ordinary topology (the one inherited from $\mathscr{P}(\mathbb{N})$) is d_φ-open, while any d_φ-open set is Borel in the ordinary sense. In the general case we assemble the required metric of d_φ on the domain $\{n : \varphi(\{n\}) > 0\}$ and the ordinary Polish metric on $\mathscr{P}(\mathbb{N})$ on the complementary domain.

(ii) $\Longrightarrow$ (i) Let τ be a Polish group topology on $\mathscr{I}$, generated by a Δ-invariant compatible metric d. It can be shown (Solecki [43, p. 60]) that $\varphi(x) = \sup_{y\in\mathscr{I},\, y\subseteq x} d(\emptyset, x)$ is a l.s.c. submeasure with $\mathscr{I} = \mathtt{Exh}_\varphi$. The key observation is that for any $x \in \mathscr{I}$ the sequence $\{x \cap [0,n)\}_{n\in\mathbb{N}}$ d-converges to x by the last statement of Lemma 7, which implies both that φ is l.s.c. (because the supremum above can be restricted to finite sets y) and that $\mathscr{I} = \mathtt{Exh}_\varphi$ (where the inclusion $\supseteq$ needs another "identity map" argument).

(i) $\Longrightarrow$ (iii) That any $\mathscr{I} = \mathtt{Exh}_\varphi$, φ being l.s.c., is a P-ideal, is an easy exercise: if $x_1, x_2, x_3, \dots \in \mathscr{I}$ then define an increasing sequence of numbers $n_i \in x_i$ with $\varphi(x \cap [n_i, \infty)) \le 2^{-n}$ and put $x = \bigcup_i (x \cap [n_i, \infty))$.

Any of (iii), (i), (ii), (v) $\Longrightarrow$ (iv) This is because $\mathscr{I}_1$ easily does not satisfy any of the four properties indicated. For the formal purpose to complete the proof of Theorem 41, we need here only the implication (iii) $\Longrightarrow$ (iv).

(iv) $\Longrightarrow$ (v) Suppose that sets $X_n \subseteq \mathscr{P}(\mathbb{N})$ are $\mathscr{I}$-small, so that $X_n \restriction A_n$ is meager in $\mathscr{P}(A_n)$ for some $A_n \in \mathscr{I}$, but $X = \bigcup_n X_n$ is not $\mathscr{I}$-small, and prove $\mathscr{I}_1 \le_{\mathrm{RB}} \mathscr{I}$. Arguing as in the proof of Theorem 30, we use the meagerness to find, for any n, a sequence of pairwise disjoint non-empty finite sets $w^n_k \subseteq x_n$, $k \in \mathbb{N}$, and subsets $u^n_k \subseteq w^n_k$, such that

(a) if $x \subseteq \mathbb{N}$ and $\exists^\infty k\, (x \cap w^n_k = u^n_k)$ then $x \notin X_n$.

Dropping some sets w^n_k away and reenumerating the rest, we can strengthen the disjointness to the following: $w^n_k \cap w^m_l = \emptyset$ unless both $n = m$ and $k = l$.

Now put $w^n_{ij} = w^n_{2^i(2j+1)-1}$. The sets $\overline{w}_{ij} = \bigcup_{n\le i} w^n_{ij}$ are still pairwise disjoint, and satisfy the following two properties:

(b) $\bigcup_j \overline{w}_{ij} \subseteq x_n$, hence, $\in \mathscr{I}$, for any i;

(c) if a set $Z \subseteq \mathbb{N}\times\mathbb{N}$ does not belong to $\mathscr{I}_1$, i.e., $\exists^\infty i\, \exists j\, (\langle i,j\rangle \in Z)$, then $\forall n\, \exists^\infty k\, (w^n_k \subseteq \overline{w}_Z)$, where $\overline{w}_Z = \bigcup_{\langle i,j\rangle\in K} \overline{w}_{ij})$.

We assert that the map $\langle i,j\rangle \mapsto \overline{w}_{ij}$ witnesses $\mathscr{I}_1 \le^+_{\mathrm{RB}} \mathscr{I}$. (Then a simple argument, as in the proof of Theorem 30, gives $\mathscr{I}_1 \le_{\mathrm{RB}} \mathscr{I}$.)

Indeed if $Z \subseteq \mathbb{N}\times\mathbb{N}$ belongs to $\mathscr{I}_1$ then $\overline{w}_Z \in \mathscr{I}$ by (b). Suppose that $Z \notin \mathscr{I}_1$. It suffices to show that $X_n \restriction \overline{w}_Z$ is meager in $\mathscr{P}(\overline{w}_Z)$ for any n. Note that by (c) the set $K = \{k : w^n_k \subseteq \overline{w}_Z\}$ is infinite and in fact $\overline{w}_Z \cap x_n = \bigcup_{k\in K} w^n_k$. Therefore, any $x \subseteq \overline{w}_Z$ satisfying $x \cap w^n_k = u^n_k$ for infinitely many $k \in K$, does not belong to X_n by (a). Now the meagerness of $X_n \restriction \overline{w}_Z$ is clear.

(v) $\Longrightarrow$ (iii) This also is quite easy: if a sequence of sets $Z_n \in \mathscr{I}$ witnesses that $\mathscr{I}$ is not a P-ideal, then the union of $\mathscr{I}$-small sets $\mathscr{P}(Z_n)$ is not $\mathscr{I}$-small.

8.c The hard part

We prove (v) $\Longrightarrow$ (i), the hard part of Theorem 41. A couple of definitions before the key lemma.

- Let $C(\mathscr{I})$ be the collection of all hereditary (*i.e.*, $y \subseteq x \in K \Longrightarrow y \in K$) compact $\mathscr{I}$-large sets $K \subseteq \mathscr{P}(\mathbb{N})$.
- Given sets $A, B \subseteq \mathscr{P}(\mathbb{N})$, let $A + B = \{x \cup y : x \in A \wedge y \in B\}$.

Lemma 42. *Assuming that $\mathscr{I}$ is of type* (v)*, there is a countable sequence of sets $K_m \in C(\mathscr{I})$ such that for any set $K \in C(\mathscr{I})$ there are m, n with $K_m + K_n \subseteq K$.*

Proof. Fix a continuous map $f : \mathbb{N}^{\mathbb{N}} \xrightarrow{\text{onto}} \mathscr{I}$. For any $s \in \mathbb{N}^{<\omega}$, we define

$$N_s = \{a \in \mathbb{N}^{\mathbb{N}} : s \subset a\} \quad \text{and} \quad B_s = f\text{"}N_s \quad (\text{the } f\text{-image of } N_s)\,.$$

Consider the set $T = \{s : B_s \text{ is } \mathscr{I}\text{-large}\}$. As $\mathscr{I}$ itself is clearly $\mathscr{I}$-large, $\Lambda \in T$. On the other hand, the assumption (v) easily implies that T has no endpoints and no isolated branches, hence, $P = \{a \in \mathbb{N}^{\mathbb{N}} : \forall\, n\, (a \restriction n \in T)\}$ is a perfect set. Moreover, $A_s = f\text{"}(P \cap N_s)$ is $\mathscr{I}$-large for any $s \in T$ because $B_s \smallsetminus A_s$ is a countable union of $\mathscr{I}$-small sets.

Now consider any set $K \in C(\mathscr{I})$. By definition, if $x, y \in \mathscr{I}$ then $Z = x \cup y \in \mathscr{I}$, thus, $K \restriction Z$ is not meager in $\mathscr{P}(Z)$, hence, by the compactness, $K \restriction Z$ includes a basic nbhd of $\mathscr{P}(Z)$, hence, by the hereditarity, there is a number n such that $Z \cap [n, \infty) \in K$. We conclude that $P^2 = \bigcup_n Q_n$, where each $Q_n = \{\langle a, b\rangle \in P^2 : (f(a) \cup f(b)) \cap [n, \infty) \in K\}$ is closed in P because so is K and f is continuous. Thus, there are $s, t \in T$ such that $P^2 \cap (N_s \times N_t) \subseteq Q_n$, in other words, $(A_s + A_t) \restriction [n, \infty) \subseteq K$, hence, $(\overline{A_s} + \overline{A_t}) \restriction [n, \infty) \subseteq K$, where $\overline{\ldots}$ denotes the topological closure of the hereditary hull. Thus we can take, as $\{K_m\}$, all sets of the form $K_{sn} = \overline{A_s} \restriction n$. $\square$

Using the fact that $C(\mathscr{I})$ is a filter (as easy exercise which makes main use if the hereditarity), we can define (still in the assumption that $\mathscr{I}$ is of type (v)) a $\subseteq$-decreasing sequence of sets $K_n \in C(\mathscr{I})$ such that

(1) for any $K \in C(\mathscr{I})$ there is n with $K_n \subseteq K$,

and $K_{n+1} + K_{n+1} \subseteq K_n$ for any n. Taking any other term of the sequence, we can sharpen the latter requirement to

(2) for any $n :\ K_{n+1} + K_{n+1} + K_{n+1} \subseteq K_n$.

This is the starting point for the construction of a l. s. c. submeasure φ with $\mathscr{I} = \mathtt{Exh}_\varphi$. Assuming that, in addition, $K_0 = \mathscr{P}(\mathbb{N})$, let, for any $x \in \mathscr{P}_{\mathtt{fin}}(\mathbb{N})$,

$$\begin{aligned} \varphi_1(x) &= \mathtt{inf}\,\{\, 2^{-n} : x \in K_n \,\} && \text{, and} \\ \varphi_2(x) &= \mathtt{inf}\,\{\, \textstyle\sum_{i=1}^m \varphi_1(x_i) : m \ge 1 \wedge x_i \in \mathscr{P}_{\mathtt{fin}}(\mathbb{N}) \wedge x \subseteq \bigcup_{i=1}^m x_i \,\} && . \end{aligned}$$

Then set $\varphi(x) = \sup_n \varphi_2(x \cap [0,n))$ for any $x \subseteq \mathbb{N}$. A routine verification shows that φ submeasure and that $\mathscr{I} = \mathtt{Exh}_\varphi$. (See Solecki [43]. To check that any $x \in \mathtt{Exh}_\varphi$ belongs to $\mathscr{I}$ we use the following observation: $x \in \mathscr{I}$ iff for any $K \in C(\mathscr{I})$ there is n such that $x \cap [0,n) \in K$.)

□ (*Theorem 41*)

Corollary 43. *Suppose that $\mathscr{J}$ is an analytic P-ideal. Then any ideal $\mathscr{I} \le_{\mathrm{B}} \mathscr{J}$ is an analytic P-ideal, too.*

Proof. Use equivalence (iv) $\Longleftrightarrow$ (iii) of the theorem. (The result can be obtained via a more direct argument, of course.) □

9 Equivalence relation E_1

The ideal $\mathscr{I}_1$ naturally defines the ER $\mathsf{E}_1 = \mathsf{E}_{\mathscr{I}_1}$ on $\mathscr{P}(\mathbb{N} \times \mathbb{N})$ so that $x \mathrel{\mathsf{E}_1} y$ iff $x \,\Delta\, y \in \mathscr{I}_1$. We can as well consider E_1 as an ER on $\mathbb{X}^{\mathbb{N}}$ for any uncountable Polish space $\mathbb{X}$, defined as $x \mathrel{\mathsf{E}_1} y$ iff $x(k) = y(k)$ for all but finite k.

9.a E_1 and hypersmoothness

The following notation will be rather useful in our study of subsets of $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$ or $(2^{\mathbb{N}})^{\mathbb{N}}$. If x is a function defined on $\mathbb{N}$ then, for any n, let

$$x \restriction_{<n} = x \restriction [0,n)\,, \quad x \restriction_{\le n} = x \restriction [0,n]\,, \quad x \restriction_{>n} = x \restriction (n,\infty)\,, \quad x \restriction_{\ge n} = x \restriction [n,\infty)\,.$$

For any set X of $\mathbb{N}$-sequences, let $X \restriction_{<n} = \{x \restriction_{<n} : x \in X\}$, and similarly for $\le$, $>$, $\ge$. If $\xi \in X \restriction_{>n}$ then let $\mathbf{S}_X(\xi) = \{x(n) : x \in X \wedge x \restriction_{>n} = \xi\}$.

Recall that a hypersmooth ER is a countable increasing union of Borel smooth ERs. The following lemma shows that E_1 is universal in this class.

Lemma 44. *For a Borel ER E to be hypersmooth it is necessary and sufficient that $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_1$.*

Proof. Let $\mathbb{X}$ be the domain of E. Assume that E is hypersmooth, *i.e.*, $\mathsf{E} = \bigcup_n \mathsf{E}_n$, where $x \mathrel{\mathsf{E}_n} y$ iff $\vartheta_n(x) = \vartheta_n(y)$, each $\vartheta_n : \mathbb{X} \to \mathscr{P}(\mathbb{N})$ is Borel, and $\mathsf{E}_n \subseteq \mathsf{E}_{n+1}$, $\forall n$. Then $\vartheta(x) = \{\vartheta_n(x)\}_{n \in \mathbb{N}}$ witnesses $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_1$. Conversely, if $\vartheta : \mathbb{X} \to \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ is a Borel reduction of E to E_1 then the sequence of ERs $x \mathrel{\mathsf{E}_n} y$ iff $\vartheta(x) \restriction_{\ge n} = \vartheta(y) \restriction_{\ge n}$ witnesses that E is hypersmooth. □

This Subsection contains a couple of results which describe the relationships between hypersmooth and countable ERs. The following result is given in [29] with a reference to earlier papers.

Lemma 45. (i) E_1 *is not essentially countable, i.e., there is no Borel countable* (that is, with at most countable classes) *ER E such that $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}$.*

(ii) $\mathsf{E}_0 <_{\mathrm{B}} \mathsf{E}_1$, *in other words,* $\mathtt{Fin} <_{\mathrm{B}} \mathscr{I}_1$.

Proof. (i) (A version of the argument in [29], 1.4 and 1.5.) Let $\mathbb{X}$ be the domain of E, and $\vartheta : \mathscr{P}(\mathbb{N})^{\mathbb{N}} \to \mathbb{X}$ a Borel map satisfying $x \mathrel{\mathsf{E}_1} y \implies \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$. Then ϑ is continuous on a dense $\mathbf{G}_\delta$ set $D \subseteq \mathscr{P}(\mathbb{N})^{\mathbb{N}}$. We begin with a few definitions. Let "generic" mean Cohen generic over a certain fixed countable transitive model $\mathfrak{M}$ of a big enough fragment of **ZFC**, which contains codes for D, $\vartheta \restriction D$, $\mathbb{X}$.

We are going to define, for any k, a pair of $x_k \neq y_k \in \mathscr{P}(\mathbb{N})$, a number $\ell(k)$ and a tuple $\zeta_k \in \mathscr{P}(\mathbb{N})^{\ell(k)}$ such that

(1) both $x = \langle x_0\rangle^\wedge\zeta_0{}^\wedge\langle x_1\rangle^\wedge\zeta_1{}^\wedge...$ and $y = \langle y_0\rangle^\wedge\zeta_0{}^\wedge\langle y_1\rangle^\wedge\zeta_1{}^\wedge...$ are "generic" elements of $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$;

(2) for any k, $\zeta_{\le k} = \langle x_0, y_0\rangle^\wedge\zeta_0{}^\wedge\langle x_1, y_1\rangle^\wedge\zeta_1{}^\wedge...^\wedge\langle x_k, y_k\rangle^\wedge\zeta_k$ is "generic", hence, so are $\xi_{\le k} = \langle x_0\rangle^\wedge\zeta_0...^\wedge\langle x_k\rangle^\wedge\zeta_k$ and $\eta_{\le k} = \langle y_0\rangle^\wedge\zeta_0...^\wedge\langle y_k\rangle^\wedge\zeta_k$;

(3) for any k and any $z \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ such that $\zeta_{\le k}{}^\wedge z$ is "generic" we have $\vartheta(\xi_{\le k}{}^\wedge z) = \vartheta(\eta_{\le k}{}^\wedge z)$.

If this is done then we can choose, using (2), a point $z_{(k)} \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ for any k so that $\zeta_{\le k}{}^\wedge z_{(k)} \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ is "generic", hence, by (3), for $x_{(k)} = \xi_{\le k}{}^\wedge z_{(k)})$ and $y_{(k)} = \eta_{\le k}{}^\wedge z_{(k)})$ we have $\vartheta(x_{(k)}) = \vartheta(y_{(k)})$. Note that $x_{(k)} \to x$ and $y_{(k)} \to y$, and on the other hand, all of $x_{(k)}$, x, $y_{(k)}$, y belong to D because all are "generic". It follows that $\vartheta(x) = \vartheta(y)$ by the choice of D. However obviously $\neg\, x \mathrel{\mathsf{E}_1} y$, so that ϑ is not a reduction, as required.

To define x_0, y_0, ζ_0 note that, by an ordinary splitting argument, there is a set $X \subseteq \mathscr{P}(\mathbb{N})$ of cardinality $\mathfrak{c}$ and $z \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ such that $\langle a, b\rangle^\wedge z$ is "generic" for any two $a \neq b \in X$. In particular, all $\langle a\rangle^\wedge z$, $a \in X$, are "generic". But all of them are pairwise E_1-equivalent, hence, ϑ sends all of them into one and the same F-class, which is a countable set by the choice of F. It follows that there is a pair of $a \neq b$ in X such that $\vartheta(\langle a\rangle^\wedge z) \neq \vartheta(\langle b\rangle^\wedge z)$. This equality is a property of the "generic" object $\langle a, b\rangle^\wedge z$, hence, it is forced in the sense that there is a number ℓ such that $\vartheta(\langle a\rangle^\wedge z') \neq \vartheta(\langle b\rangle^\wedge z')$ whenever $\langle a, b\rangle^\wedge z'$ is "generic" with $z' \restriction \ell = z \restriction \ell$. Put $x_0 = a$, $y_0 = b$, $\zeta_0 = z \restriction \ell$.

The induction step is carried out by the same argument.

(ii) That $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E}_1$ is witnessed by the map $f(x) = \{\langle 0, n\rangle : n \in x\}$. □

While E_1 is not countable, the conjunction of hypersmootheness and countability characterizes the essentially more primitive class of hyperfinite ERs.

9.b The 3rd dichotomy

The following major result is called the 3rd dichotomy theorem.

Theorem 46 (Kechris and Louveau [29]). *Suppose that* E *is a Borel ER on some Polish space, and* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_1$. *Then either* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ *or* $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}$.

Proof. Starting the proof, we may assume that E is a Δ^1_1 ER on $\mathscr{P}(\mathbb{N})$, and that there is a reduction ρ of E to E_1, of class Δ^1_1. Then $R = \mathtt{ran}\,\rho$ is a Σ^1_1 subset of $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$. The idea behind the proof is to show that the set R is either small enough for $\mathsf{E}_1 \restriction R$ to be Borel reducible to E_0, or otherwise it is big enough to contain a closed subset X such that $\mathsf{E}_1 \restriction X$ is Borel isomorphic to E_1.

Relations $\prec$ and $\preccurlyeq$ will denote the inverse order relations on $\mathbb{N}$, *i.e.*, $m \preccurlyeq n$ iff $n \le m$, and $m \prec n$ iff $n < m$. If $x \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ then $x \restriction_{\preccurlyeq n}$ denotes the restriction of x (a function defined on $\mathbb{N}$) on the domain $\preccurlyeq n$, *i.e.*, $[n, \infty)$. If $X \subseteq \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ then let $X \restriction_{\preccurlyeq n} = \{x \restriction_{\preccurlyeq n} : x \in X\}$. Define $x \restriction_{\prec n}$ and $X \restriction_{\prec n}$ similarly. In particular, $\mathscr{P}(\mathbb{N})^{\mathbb{N}} \restriction_{\preccurlyeq n} = \mathscr{P}(\mathbb{N})^{\preccurlyeq n} = \mathscr{P}(\mathbb{N})^{[n,\infty)}$.

For a sequence $x \in \mathscr{P}(\mathbb{N})^{\preccurlyeq n}$, let $\mathtt{dep}\,x$ (the *depth* of x) be the number (finite or ∞) of elements of the set $\nabla(x) = \{j \preccurlyeq n : x(j) \notin \Delta^1_1(x \restriction_{\prec j})\}$. The formula $\mathtt{dep}\,x \ge d$ (of two variables, d running over $\mathbb{N} \cup \{\infty\}$) is obviously Σ^1_1.

We have two cases:

***Case 1*:** all $x \in R = \mathtt{ran}\,\rho$ satisfy $\mathtt{dep}\,x < \infty$.

***Case 2*:** there exist $x \in R$ with $\mathtt{dep}\,x = \infty$.

Case 1 is the easier case. First of all we observe that R, a Σ^1_1 set, is a subset of the Π^1_1 set $Z = \{x : \mathtt{dep}\,x < \infty\}$, hence, there is a Δ^1_1 set Y with $\mathtt{ran}\,\rho \subseteq Y \subseteq Z$. The following lemma ends the argument.

Lemma 46.1. *Suppose that* $X \subseteq \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ *is a* Δ^1_1 *set and any* $x \in X$ *satisfies* $\mathtt{dep}\,x < \infty$. *Then* $\mathsf{E}_1 \restriction X \le_{\mathrm{B}} \mathsf{E}_0$.

Proof. By the choice of X for any $x \in X$ there is a number n such that $\forall\, m \preccurlyeq n\, \big(x(m) \in \Delta^1_1(x \restriction_{\prec m})\big)$. As the relation between x and n here is clearly Π^1_1, the "Kreisel selection" theorem yields a Δ^1_1 map $\nu : X \to \mathbb{N}$ such that $x(m) \in \Delta^1_1(x \restriction_{\prec n})$ holds whenever $x \in X$ and $m \preccurlyeq \nu(x)$. Now define, for each $x \in X$, $\vartheta(x) \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ as follows: $\vartheta(x) \restriction_{\preccurlyeq \nu(x)} = x \restriction_{\preccurlyeq \nu(x)}$, but $\vartheta(x)(j) = \emptyset$ for all $j < \nu(x)$. Note that $x \,\mathsf{E}_1\, \vartheta(x)$ for any $x \in X$.

The other important thing is that $\mathtt{ran}\,\vartheta \subseteq Z = \{x \in \mathscr{P}(\mathbb{N})^{\mathbb{N}} : \mathtt{dep}\,x = 0\}$, where Z is a Π^1_1 set, hence, there is a Δ^1_1 set Y with $\mathtt{ran}\,\vartheta \subseteq Y \subseteq Z$. In particular ϑ reduces $\mathsf{E}_1 \restriction X$ to $\mathsf{E}_1 \restriction Y$. We observe that $\mathsf{E}_1 \restriction Y$ is a countable ER: any E_1-class in $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$ intersects Y by an at most countable set (as so is the property of Z, a bigger set). Thus, $\mathsf{E}_1 \restriction Y$ is hyperfinite by Theorem 32. □

9.c Case 2

Since $\mathtt{dep}\,x = \infty$ is a Σ^1_1 formula, it suffices to show that for any non-empty Σ^1_1 set $R \subseteq \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ with $\forall\, x \in R\,(\mathtt{dep}\,x = \infty)$ we have a Δ^1_1 subset $X \subseteq R$ with $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_1 \restriction X$. Fix a set R, as indicated, for the course of the proof. The subset X of R will be defined with the help of a splitting construction developed in [23] for the study of "ill"founded Sacks iterations.

We shall define a map $\varphi : \mathbb{N} \to \mathbb{N}$, which assumes infinitely many values and assumes each its value infinitely many times (but $\mathtt{ran}\,\varphi$ may be a proper subset of $\mathbb{N}$), and, for each $u \in 2^{<\omega}$, a non-empty Σ^1_1 subset $X_u \subseteq R$, which satisfy a quite long list of properties. First of all, if φ is already defined at least on $[0, n)$ and $u \neq v \in 2^{<\omega}$ then let $\nu_\varphi[u, v] = \mathtt{min}_{\preccurlyeq}\{\varphi(k) : k < n \wedge u(k) \neq v(k)\}$. (Note that the minimum is taken in the sense of $\preccurlyeq$, hence, it is $\mathtt{max}$ in the sense of $\leq$, the usual order). Separately, put $\varphi[u, u] = -1$ for any u.

Now we give the list of requirements.

(i) if $\varphi(n) \notin \{\varphi(k) : k < n\}$ then $\varphi(n) \prec \varphi(k)$ for any $k < n$;

(ii) every X_u is a non-empty Σ^1_1 subset of R;

(iii) if $u \in 2^n$, $x \in X_u$, and $k < n$, then $\varphi(k) \in \nabla(x)$;

(iv) if $u, v \in 2^n$ then $X_u \restriction_{\prec\nu_\varphi[u,v]} = X_v \restriction_{\prec\nu_\varphi[u,v]}$;

(v) if $u, v \in 2^n$ then $X_u \restriction_{\preccurlyeq\nu_\varphi[u,v]} \cap X_v \restriction_{\preccurlyeq\nu_\varphi[u,v]} = \emptyset$;

(vi) $X_{u^\wedge i} \subseteq X_u$ for all $u \in 2^{<\omega}$ and $i = 0, 1$;

(vii) $\mathtt{max}_{u\in 2^n}\,\mathtt{diam}\,X_u \to 0$ as $n \to \infty$ (a reasonable Polish metric on $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$ is assumed to be fixed);

(viii) a certain condition, in terms of the Choquet game, which connects each $X_{u^\wedge i}$ with X_u so that, as a consequence, $\bigcap_n X_{a\restriction n} \neq \emptyset$ for any $a \in 2^{\mathbb{N}}$.

Let us demonstrate how such a system of sets and a function φ accomplish Case 2. According to (vii) and (viii), for any $a \in 2^{\mathbb{N}}$ the intersection $\bigcap_n X_{a\restriction n}$ contains a single point, let it be $F(a)$, and F is continuous and $1-1$.

Put $J = \mathtt{ran}\,\varphi = \{j_m : m \in \mathbb{N}\}$, in the $<$-increasing order; $J \subseteq \mathbb{N}$ is infinite. Let $n \in \mathbb{N}$. Then $\varphi(n) = j_m$ for some (unique) m : we put $\psi(n) = m$. Thus $\psi : \mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$ and the preimage $\psi^{-1}(m) = \varphi^{-1}(j_m)$ is an infinite subset of $\mathbb{N}$ for any m. This allows us to define a parallel system of sets Y_u, $u \in 2^{<\omega}$, as follows. Put $Y_\Lambda = \mathscr{P}(\mathbb{N})^{\mathbb{N}}$. Suppose that Y_u has been defined, $u \in 2^n$. Put $j = \varphi(n) = j_{\psi(n)}$. Let K be the number of all indices $k < n$ still satisfying $\varphi(k) = j$, perhaps $K = 0$. Put $Y_{u^\wedge i} = \{x \in Y_u : x(j)(K) = i\}$ for $i = 0, 1$.

Each of Y_u is clearly a basic clopen set in $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$, and one easily verifies that conditions (i) – (vii), except for (iii), are satisfied for the sets Y_u (instead of X_u) and the map ψ (instead of φ), in particular, for any $a \in 2^{\mathbb{N}}$, $\bigcap_n Y_{a\restriction n} = \{G(a)\}$ is a singleton, and the map G is continuous and $1-1$. (We can, of course, define G explicitly: $G(a)(m)(l) = a(n)$, where $n \in \mathbb{N}$ is chosen so that $\psi(n) = m$ and there is exactly l numbers $k < n$ with $\psi(k) = m$.) Note finally that $\{G(a) : a \in 2^{\mathbb{N}}\} = \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ since by definition $Y_{u^\wedge 1} \cup Y_{u^\wedge 0} = Y_u$ for all u.

We conclude that the map $\vartheta(x) = F(G^{-1}(x))$ is a continuous bijection (hence, in this case, a homeomorphism by compactness) $\mathscr{P}(\mathbb{N})^{\mathbb{N}} \xrightarrow{\text{onto}} X$. We further assert that ϑ satisfying the following: for each $y,\, y' \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ and m,

$$y \restriction_{\preccurlyeq m} = y' \restriction_{\preccurlyeq m} \quad \text{iff} \quad \vartheta(y) \restriction_{\preccurlyeq j_m} = \vartheta(y') \restriction_{\preccurlyeq j_m} . \tag{$*$}$$

Indeed, let $y = G(a)$ and $x = F(a) = \vartheta(y)$, and similarly $y' = G(a')$ and $x' = F(a') = \vartheta(y')$, where $a,\, a' \in 2^{\mathbb{N}}$. Suppose that $y \restriction_{\preccurlyeq m} = y' \restriction_{\preccurlyeq m}$. According to (v) for ψ and the sets Y_u, we then have $m \prec \nu_\psi[a \restriction n, a' \restriction n]$ for any n. It follows, by the definition of ψ, that $j_m \prec \nu_\varphi[a \restriction n, a' \restriction n]$ for any n, hence, $X_{a\restriction n} \restriction_{\preccurlyeq j_m} = X_{a\restriction n} \restriction_{\preccurlyeq j_m}$ for any n by (iv). Assuming now that Polish metrics on all spaces $\mathscr{P}(\mathbb{N})^{\preccurlyeq j}$ are chosen so that $\mathtt{diam}\, Z \ge \mathtt{diam}\,(Z \restriction_{\preccurlyeq j})$ for all $Z \subseteq \mathscr{P}(\mathbb{N})$ and j, we easily obtain that $x \restriction_{\preccurlyeq j_m} = x' \restriction_{\preccurlyeq j_m}$, *i.e.*, the right-hand side of $(*)$. The inverse implication in $(*)$ is proved similarly.

Thus we have $(*)$, but this means that ϑ is a continuous reduction of E_1 to $\mathsf{E}_1 \restriction X$, thus, $\mathsf{E}_1 \le_{\mathrm{B}} \mathsf{E}_1 \restriction X$, as required.

□ (*Theorem 46 modulo the construction* (i) – (viii))

9.d The construction

Recall that $R \subseteq \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ is a fixed non-empty $\varSigma^1_1$ set such that $\mathtt{dep}\, x = \infty$ for each $x \in R$. Set $X_\Lambda = R$.

Now suppose that the sets $X_u \subseteq R$ with $u \in 2^n$ have been defined and satisfy the applicable part of (i) – (viii).

Step 1. Our 1st task is to choose $\varphi(n)$. Let $\{j_1 < ... < j_m\} = \{\varphi(k) : k < n\}$. For any $1 \le p \le m$, let N_p be the number of all $k < n$ with $\varphi(k) = j_p$.

Case 1a. If some numbers N_p are $< m$ then choose $\varphi(n)$ among j_p with the least N_p, and among them the least one.

Case 1b: $N_p \ge m$ (then actually $N_p = m$) for all $p \le m$. It follows from our assumptions, in particular (iv), that $X_u \restriction_{\prec j_m} = X_v \restriction_{\prec j_m}$ for all $u,\, v \in 2^n$. Let $Y = X_u \restriction_{\prec j_m}$ for any such u. Take any $y \in Y$. Then $\nabla(y)$ is infinite, hence, there is some $j \in \nabla(y)$ with $j \prec j_m$. Put $\varphi(n) = j$.

We have something else to do in this case. Let $X'_u = \{x \in X_u : j \in \nabla(y)\}$ for any $u \in 2^m$. Then we easily have $X'_u = \{x \in X_u : x \restriction_{\prec j_m} \in Y'\}$, where $Y' = \{y \in Y : j \in \nabla(y)\}$ is a non-empty $\varSigma^1_1$ set, so that the sets $X'_u \subseteq X_u$ are non-empty $\varSigma^1_1$. Moreover, as j_m is the $\preccurlyeq$-least in $\{\varphi(k) : k < n\}$, we can easily show that the system of sets X'_u still satisfies (iv). This allows us to assume, without any loss of generality, that, in Case 1b, $X'_u = X_u$ for all u, or, in other words, that any $x \in X_u$ for any $u \in 2^n$ satisfies $j = \varphi(n) \in \nabla(x)$. (This is true in Case 1a, of course, because then $\varphi(n) = \varphi(k)$ for some $k < n$.)

Note that this manner to choose $\varphi(n)$ implies (i) and also implies that φ takes infinitely many values and takes each its value infinitely many times.

The continuation of the construction requires the following

Lemma 46.2. *If* $u_0 \in 2^n$ *and* $X' \subseteq X_{u_0}$ *is a non-empty* Σ^1_1 *set then there is a system of* Σ^1_1 *sets* $\emptyset \neq X'_u \subseteq X_u$ *with* $X'_{u_0} = X'$*, which still satisfies* (iv).

Proof. For any $u \in 2^n$, let $X'_u = \{x \in X_u : x \restriction_{\prec n(u)} \in X' \restriction_{\prec n(u)}\}$, where $n(u) = \nu_\varphi[u, u_0]$. In particular, this gives $X'_{u_0} = X'$, because $\nu_\varphi[u_0, u_0] = -1$. The sets X'_u are as required, via a routine verification. □ (*Lemma*)

Step 2. First of all put $j = \varphi(n)$ and $Y_u = X_u \restriction_{\prec j}$. (All Y_u are equal to Y in Case 1b, but the argument pretends to make no difference between 1a and 1b). Take any $u_1 \in 2^n$. By the construction any element $x \in X_{u_1}$ satisfies $j \in \nabla(x)$, so that $x(j) \notin \Delta^1_1(x \restriction_{\prec j})$. As X_{u_1} is a Σ^1_1 set, it follows that $\{x'(j) : x' \in X_{u_1} \wedge x' \restriction_{\prec j} = x \restriction_{\prec j}\}$ is not a singleton, in fact is uncountable. It follows that there is a number l_{u_1} having the property that the Σ^1_1 set

$$Y'_{u_1} \;=\; \{y \in Y_{u_1} : \exists\, x, x' \in X_{u_1}\, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_1} \in x(j) \wedge l_{u_1} \notin x'(j))\}$$

is non-empty. We now put $X' = \{x \in X_{u_1} : x \restriction_{\prec j} \in Y'_{u_1}\}$ and define Σ^1_1 sets $\emptyset \neq X'_u \subseteq X_u$ as in the lemma, in particular, $X'_{u_1} = X'$, $X'_{u_1} \restriction_{\prec j} = Y'_{u_1}$, still (iv) is satisfied, and in addition

$$\forall\, y \in X'_{u_1} \restriction_{\prec j}\, \exists\, x, x' \in X'_{u_1}\, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_1} \in x(j) \wedge l_{u_1} \notin x'(j)) \quad (1)$$

Now take some other $u_2 \in 2^n$. Let $\nu = \nu_\varphi[u_1, u_2]$. If $j \prec \nu$ then $X_{u_1} \restriction_{\prec j} = X_{u_2} \restriction_{\prec j}$, so that we already have, for $l_{u_2} = l_{u_1}$, that

$$\forall\, y \in X'_{u_2} \restriction_{\prec j}\, \exists\, x, x' \in X'_{u_2}\, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_2} \in x(j) \wedge l_{u_2} \notin x'(j))\,, \quad (2)$$

and can pass to some $u_3 \in 2^n$. Suppose that $\nu \preccurlyeq j$. Now things are somewhat nastier. As above there is a number l_{u_2} such that

$$Y'_{u_2} \;=\; \{y \in Y_{u_2} : \exists\, x, x' \in X_{u_2}\, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_{u_2} \in x(j) \wedge l_{u_2} \notin x'(j))\}$$

is a non-empty Σ^1_1 set, thus, we can define $X'' = \{x \in X_{u_1} : x \restriction_{\prec j} \in Y'_{u_1}\}$ and maintain the construction of Lemma 46.2, getting non-empty Σ^1_1 sets $X''_u \subseteq X'_u$ still satisfying (iv) and $X''_{u_2} = X''$, therefore, we still have (2) for the set X''_{u_2}.

Yet it is most important in this case that (1) is preserved, *i.e.*, it still holds for the set X''_{u_1} instead of X'_{u_1}! Why is this ? Indeed, according to the construction in the proof of Lemma 46.2, we have $X''_{u_1} = \{x \in X'_{u_1} : x \restriction_{\prec \nu} \in X'' \restriction_{\prec \nu}\}$. Thus, although, in principle, X''_{u_1} is smaller than X'_{u_1}, for any $y \in X''_{u_1} \restriction_{\prec j}$ we have

$$\{x \in X''_{u_1} : x \restriction_{\prec j} = y\} \;=\; \{x \in X'_{u_1} : x \restriction_{\prec j} = y\}\,,$$

simply because now we assume that $\nu \preccurlyeq j$. This implies that (1) still holds.

Iterating this construction so that each $u \in 2^n$ is eventually encountered, we obtain, in the end, a system of non-empty Σ^1_1 sets, let us call them "new" X_u, but they are subsets of the "original" X_u, still satisfying (iv), still satisfying that

$\varphi(n) \in \nabla(x)$ for each $x \in \bigcap_{u\in 2^n} X_u$, and, in addition, for any $u \in 2^n$ there is a number l_u such that $j \prec \nu_\varphi[u,v] \Longrightarrow l_u = l_v$ and

$$\forall\, y \in X_u \restriction_{\prec j} \exists\, x, x' \in X_u \, (x' \restriction_{\prec j} = x \restriction_{\prec j} = y \wedge l_u \in x(j) \wedge l_u \notin x'(j)) \,. \qquad (*)$$

Step 3. We define the $(n+1)$-th level of sets by $X_{u^\wedge 0} = \{x \in X_u : l_u \in x(j)\}$ and $X_{u^\wedge 1} = \{x \in X_u : l_u \notin x(j)\}$ for all $u \in 2^n$, where still $j = \varphi(n)$. It follows from $(*)$ that all these Σ^1_1 sets are non-empty.

Lemma 46.3. *The just defined system of sets X_s, $s \in 2^{n+1}$, satisfies* (iv), (v).

Proof. Let $s = u^\wedge i$ and $t = v^\wedge i'$ belong to 2^{n+1}, so that $u, v \in 2^n$ and $i, i' \in \{0,1\}$. Let $\nu = \nu_\varphi[u,v]$ and $\nu' = \nu_\varphi[s,t]$.

Case 3a: $\nu \preccurlyeq j = \varphi(n)$. Then easily $\nu = \nu'$, so that (v) immediately follows from (v) at level n for X_u and X_v. As for (iv), we have $X_s \restriction_{\prec\nu} = X_u \restriction_{\prec\nu}$ (because by definition $X_s \restriction_{\prec j} = X_u \restriction_{\prec j}$), and similarly $X_t \restriction_{\prec\nu} = X_v \restriction_{\prec\nu}$, therefore, $X_t \restriction_{\prec\nu'} = X_s \restriction_{\prec\nu'}$ since $X_u \restriction_{\prec\nu} = X_v \restriction_{\prec\nu}$ by (iv) at level n.

Case 3b: $j \prec \nu$ and $i = i'$. Then still $\nu = \nu'$, thus we have (v). Further, $X_u \restriction_{\prec\nu} = X_v \restriction_{\prec\nu}$ by (iv) at level n, hence, $X_u \restriction_{\preccurlyeq j} = X_v \restriction_{\preccurlyeq j}$, hence, $l_u = l_v$ (see above). Now, assuming that, say, $i = i' = 1$ and $l_u = l_v = l$, we conclude that

$$X_s \restriction_{\prec\nu'} = \{y \in X_u \restriction_{\prec\nu} : l \in y(j)\} = \{y \in X_v \restriction_{\prec\nu} : l \in y(j)\} = X_t \restriction_{\prec\nu'} \,.$$

Case 3c: $j \prec \nu$ and $i \neq i'$, say, $i = 0$ and $i' = 1$. Now $\nu' = j$. Yet by definition $X_s \restriction_{\prec j} = X_u \restriction_{\prec j}$ and $X_t \restriction_{\prec j} = X_v \restriction_{\prec j}$, so it remains to apply (iv) for level n. As for (v), note that by definition $l \notin x(j)$ for any $x \in X_s = X_{u^\wedge 0}$ while $l \in x(j)$ for any $x \in X_t = X_{v^\wedge 1}$, where $l = l_u = l_v$. $\square$ (*Lemma*)

Step 4. In addition to (iv) and (v), we already have (i), (ii), (iii), (vi) at level $n+1$. To achieve the remaining properties (vii) and (viii), it suffices to consider, one by one, all elements $s \in 2^{n+1}$, finding, at each such a substep, a non-empty Σ^1_1 subset of X_s which is consistent with the requirements of (vii) and (viii) (for instance, for (vii), just take it so the diameter is $\leq 2^{-n}$), and then reducing all other sets X_t by Lemma 46.2 at level $n+1$.

$\square$ (*Construction and Theorem 46*)

9.e Above E_1

Recall that an embedding is a $1-1$ reduction, and an invariant embedding is an embedding ϑ such that its range is an invariant set, see Subsection 1.d above.

Theorem 47 (Kechris and Louveau [29]). *Suppose that $\mathsf{E}_1 \leq_{\mathrm{B}} \mathsf{F}$, where F is an analytic ER on a Polish space $\mathbb{Y}$. Then both $\mathsf{E}_1 \sqsubseteq_{\mathrm{C}} \mathsf{F}$ and $\mathsf{E}_1 \sqsubseteq^{\mathrm{i}}_{\mathrm{B}} \mathsf{F}$.*

Proof. To prove the first statement, let $\preccurlyeq$ be the inverted order on $\mathbb{N}$, *i.e.*, $m \preccurlyeq n$ iff $n \le m$. Let $\mathfrak{P}$ be the collection of all sets $P \subseteq \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ such that there is a continuous $1-1$ map $\eta : \mathscr{P}(\mathbb{N})^{\mathbb{N}} \xrightarrow{\text{onto}} P$ such that we have

$$x \restriction_{\preccurlyeq n} = y \restriction_{\preccurlyeq n} \iff \eta(x) \restriction_{\preccurlyeq n} = \eta(y) \restriction_{\preccurlyeq n}$$

for all n and $x, y \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$, where $x \restriction_{\preccurlyeq n} = \{x_i\}_{i \preccurlyeq n}$ for any $x = \{x_i\} \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$. Clearly any such a map is a continuous embedding of E_1 into itself.

This set $\mathfrak{P}$ is a forcing notion to extend the universe by a sequence of reals x_i so that each x_n is Sacks–generic over $\{x_i\}_{i \precnsim n}$, an example of iterated Sacks extensions with an ill-founded "skeleton" of iteration, which we defined in [23]. Here, the "skeleton" is $\mathbb{N}$ with the inverted order $\preccurlyeq$.

The method of [23] contains a study of continuous and Borel functions on sets in $\mathfrak{P}$. In particular it is shown there that Borel maps admit the following *cofinal classification* on sets in $\mathfrak{P}$: if $\mathbb{Y}$ is Polish, $P' \in \mathfrak{P}$, and $\vartheta : P' \to \mathbb{Y}$ is Borel then there is a set $P \in \mathfrak{P}$, $P \subseteq P'$, on which ϑ is continuous, and either a constant or, for some n, $1-1$ on $P \restriction_{\preccurlyeq n}$ in the sense that,

$$\text{for all } x, y \in P : \quad x \restriction_{\preccurlyeq n} = y \restriction_{\preccurlyeq n} \iff \vartheta(x) = \vartheta(y)\,. \tag{$*$}$$

We apply this to a Borel map $\vartheta : \mathscr{P}(\mathbb{N})^{\mathbb{N}} \to \mathbb{Y}$ which reduces E_1 to F. We begin with $P' = \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ and find a set $P \in \mathfrak{P}$ as indicated. Since ϑ cannot be a constant on P (indeed, any $P \in \mathfrak{P}$ contains many pairwise E_1-inequivalent elements), we have $(*)$ for some n. In other words, there is a $1-1$ continuous map $f : P \restriction_{\preccurlyeq n} \to \mathbb{Y}$ (where $P \restriction_{\preccurlyeq n} = \{x \restriction_{\preccurlyeq n} : x \in P\}$) such that $\vartheta(x) = f(x \restriction_{\preccurlyeq n})$ for all $x \in P$. Now, let $x = \{x_i\}_{i \in \mathbb{N}} \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$. Define $\zeta(x) = z = \{z_i\}_{i \in \mathbb{N}}$ so that $z_i = \emptyset$ for $i < n$ and $z_{n+i} = x_i$ for all i. Finally set $\vartheta'(x) = f(\eta(\zeta(x)) \restriction_{\preccurlyeq n})$ for all $x \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$: this is a continuous embedding of E_1 in F.

Now we prove the second claim. We can assume that $\mathbb{Y} = \mathscr{P}(\mathbb{N})$ and that $\vartheta : \mathscr{P}(\mathbb{N})^{\mathbb{N}} \to \mathscr{P}(\mathbb{N})$ is already a continuous embedding E_1 into F. Let $Y = \mathtt{ran}\,\vartheta$ and $Z = [Y]_{\mathsf{F}}$. Normally Y, Z are analytic, but in this case they are even Borel. Indeed Z is the projection of $P = \{\langle z, x\rangle : z \mathrel{\mathsf{F}} \vartheta(x)\}$, a Borel subset of $\mathscr{P}(\mathbb{N}) \times \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ whose all cross-sections are E_1-equivalence classes, *i.e.*, σ-compact sets. It is known that in this case Z is Borel and, moreover, there is a Borel map $f : Z \to \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ such that $f(z) \mathrel{\mathsf{E}_1} x$ whenever $z \mathrel{\mathsf{F}} \vartheta(x)$. $\longleftarrow$ *reference* $\dashv$

We can convert f to a $1-1$ map $g : \mathscr{P}(\mathbb{N}) \to \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ with the same properties: $g(z)_n = f(z)_n$ for $n \ge 1$, but $g(z)_0 = z$. Then $f : \mathscr{P}(\mathbb{N})^{\mathbb{N}} \to Z \subseteq \mathscr{P}(\mathbb{N})$ and $g : Z \to \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ are Borel $1-1$ maps (ϑ is even continuous, but this does not matter now), and, for any $x \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$, ϑ maps $[x]_{\mathsf{E}_1}$ into $[\vartheta(x)]_{\mathsf{F}} \subseteq Z$, and g maps $[\vartheta(x)]_{\mathsf{F}}$ back into $[x]_{\mathsf{E}_1}$. It remains to apply the construction from the Cantor – Bendixson theorem, to get a Borel embedding, say, F of E_1 into F with $\mathtt{ran}\,F = Z$, *i.e.*, an invariant embedding. □

The following theorem shows that orbit equivalence relations of Polish group actions cannot reduce E_1.

Theorem 48 (Kechris and Louveau [29]). *Suppose that $\mathbb{G}$ is a Polish group and $\mathbb{X}$ is a Borel $\mathbb{G}$-space. Then E_1 is* <u>*not*</u> *Borel reducible to $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$.*

Proof. Towards the contrary, let $\vartheta : \mathscr{P}(\mathbb{N})^{\mathbb{N}} \to \mathbb{X}$ be a Borel reduction of E_1 to E. We can assume, by Theorem 47, that ϑ is in fact an invariant embedding, *i.e.*, $1-1$ and $Y = \mathtt{ran}\,\vartheta$ is an E-invariant set. Define, for $g \in \mathbb{G}$ and $x \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$, $g \cdot x = \vartheta^{-1}(g \cdot \vartheta(x))$. Then this is a Borel action of $\mathbb{G}$ on $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$ such that the induced relation $\mathsf{E}_{\mathbb{G}}^{\mathscr{P}(\mathbb{N})^{\mathbb{N}}}$ coincides with E_1.

Let us fix $x \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$.

Consider any $y = \{y_n\}_n \in [x]_{\mathsf{E}_1}$. Then $[x]_{\mathsf{E}_1} = \bigcup_n C_n(y)$, where each set $C_n(y) = \{y' \in \mathscr{P}(\mathbb{N})^{\mathbb{N}} : \forall\, m \ge n\ (y_n = y'_n)\}$ is Borel (even compact). It follows that $\mathbb{G} = \bigcup_n G_n(y)$, where each $G_n(y) = \{g \in \mathbb{G} : g(x) \in C_n(y)\}$ is Borel. Thus, as $\mathbb{G}$ is Polish, there is a number n such that $G_n(y)$ is not meager in $\mathbb{G}$ (then this will hold for all $n' \ge n$, of course). Let $n(y)$ be the least such an n.

We assert that for any n the set $Y_n(x) = \{y \restriction [n,\infty) : y \in [x]_{\mathsf{E}_1} \wedge n(x) = n\}$ is at most countable. Indeed suppose that $Y_n(x)$ is not countable. Note that if y_1 and y_2 in $[x]_{\mathsf{E}_1}$ have different restrictions $y_i \restriction [n,\infty)$ then the sets $C_n(y_1)$ and $C_n(y_2)$ are disjoint, therefore, the sets $G_n(y_1)$ and $G_n(y_2)$ are disjoint, so we would have uncountably many pairwise disjoint non-meager sets in $\mathbb{G}$, contradiction. Thus all sets $Y_n(x)$ are countable.

It is most important that $Y_n(x)$ depends on $[x]_{\mathsf{E}_1}$ rather than x itself, more exactly, if $x' \in [x]_{\mathsf{E}_1}$ then $Y_n(x) = Y_n(x')$: this is because any set $G_n(y)$ in the sense of x' is just a shift, within $\mathbb{G}$, of $G_n(y)$ in the sense of x. Therefore, putting $Y(x) = \bigcup_n \{\bar u : u \in Y_n(x)\}$, where, for $u \in \mathscr{P}(\mathbb{N})^{[n,\infty)}$, $\bar u \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ is defined by $\bar u \restriction [n,\infty) = u$ and $\bar u(k) = \emptyset$ for $k < n$, we have the set $Y = \bigcup_{x \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}} Y(x)$ with the property that $Y \cap [x]_{\mathsf{E}_1}$ is non-empty and at most countable for any $x \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$.

The other important fact is that the relation $y \in Y(x)$ is Borel: this is because it is assembled from Borel relations via the Vaught quantifier "there exists nonmeager-many", known to preserve the Borelness. It follows that ←*reference*⊣

$$Y = \{y : \exists\, x\ (y \in Y_x)\} = \{y : \forall\, x\ (x \in [y]_{\mathsf{E}_1} \Longrightarrow y \in Y(x)\}$$

is a Borel subset of $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$. By the uniformization theorem for Borel sets with countable sections, there is a Borel map f defined on $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$ so that $f(x) \in Y(x)$ for any x, which implies $\mathsf{E}_1 \le_{\mathsf{B}} \mathsf{E}_1 \restriction Y$. On the other hand, $\mathsf{E}_1 \restriction Y$ is a countable ER by the above, which is a contradiction to Lemma 45. □ ←*reference*⊣

10 Actions of the infinite symmetric group

This Section is connected with the next one (on turbulence). We concentrate on a main result in this area, due to Hjorth, that turbulent ERs are not reducible to those induced by actions of S_∞. In particular, we shall prove the following:

I. Lopez-Escobar: any invariant Borel set of countable models is the truth domain of a formula of $\mathscr{L}_{\omega_1\omega}$.

II. Any orbit ER of a Polish action of a closed subgroup of S_∞ is classifiable by countable structures (up to isomorphism).

III. Any ER, classifiable by countable structures, is Borel reducible to isomorphism of countable ordered graphs.

IV. Any Borel ER, classifiable by countable structures, is Borel reducible to one of ERs T_ξ.

V. Any ER, classifiable by countable structures and induced by a Polish action (of a Polish group), is Borel reducible to one of ERs T_ξ on a comeager set.

VI. Any "turbulent" ER E is generically T_ξ-ergodic for any $\xi < \omega_1$, in particular, E is not Borel reducible to T_ξ.

VII. Any "turbulent" ER is not classifiable by countable structures: a corollary of VI and V.

VIII. A generalization of VII: any "turbulent" ER is not Borel reducible to a ER which can be obtained from $\mathsf{D}(\mathbb{N})$ using operations defined in § 3.c.

Scott's analysis, involved in proofs of IV and V, appears only in a rather mild and self-contained version.

10.a Infinite symmetric group S_∞

Let S_∞ be the group of all permutations (*i.e.*, 1–1 maps $\mathbb{N} \xrightarrow{\text{onto}} \mathbb{N}$) of $\mathbb{N}$, with the superposition as the group operation. Clearly S_∞ is a $\mathbf{G}_\delta$ subset of $\mathbb{N}^\mathbb{N}$, hence, a Polish group. A compatible complete metric on S_∞ can be defined by $D(x,y) = d(x,y) + d(x^{-1}, y^{-1})$, where d is the ordinary complete metric of $\mathbb{N}^\mathbb{N}$, *i.e.*, $d(x,y) = 2^{-m-1}$, where m is the least such that $x(m) \neq y(m)$. Yet S_∞ admits no compatible left-invariant complete metric [1, 1.5]. $\longleftarrow$ *Proof of S_∞ not* CLI ?⊣

For instance isomorphism relations of various kinds of countable structures are orbit ERs induced by S_∞. Indeed, suppose that $\mathscr{L} = \{R_i\}_{i\in I}$ is a countable relational language, *i.e.*, $0 < \mathtt{card}\, I \le \aleph_0$ and each R_i is an m_i-ary relational symbol. We put [21] $\mathtt{Mod}_\mathscr{L} = \prod_{i\in I} \mathscr{P}(\mathbb{N}^{m_i})$, the space of (coded) *$\mathscr{L}$-structures* on $\mathbb{N}$. The *logic action* $j_\mathscr{L}$ of S_∞ on $\mathtt{Mod}_\mathscr{L}$ is defined as follows: if $x = \{x_i\}_{i\in I} \in \mathtt{Mod}_\mathscr{L}$ and $g \in S_\infty$ then $y = j_\mathscr{L}(g,x) = g\cdot x = \{y_i\}_{i\in I} \in \mathtt{Mod}_\mathscr{L}$, where we have

$$\langle k_1, ..., k_{m_i}\rangle \in x_i \iff \langle g(k_1), ..., g(k_{m_i})\rangle \in y_i$$

for all $i \in I$ and $\langle k_1, ..., k_{m_i}\rangle \in \mathbb{N}^{m_i}$. Then $\langle \mathtt{Mod}_\mathscr{L}\,; j_\mathscr{L}\rangle$ is a Polish S_∞-space and $j_\mathscr{L}$-orbits in $\mathtt{Mod}_\mathscr{L}$ are exactly the isomorphism classes of $\mathscr{L}$-structures, which is a reason to denote the associated equivalence relation $\mathsf{E}_{j_\mathscr{L}}^{\mathtt{Mod}_\mathscr{L}}$ as $\cong_\mathscr{L}$.

[21] $X_\mathscr{L}$ is often used to denote $\mathtt{Mod}_\mathscr{L}$.

If G is a subgroup of S_∞ then $j_{\mathscr{L}}$ restricted to G is still an action of G on $\mathtt{Mod}_{\mathscr{L}}$, whose orbit ER will be denoted by $\cong^G_{\mathscr{L}}$, *i.e.*, $x \cong^G_{\mathscr{L}} y$ iff $\exists\, g \in G\,(g\cdot x = y)$.

10.b Borel invariant sets

A set $M \subseteq \mathtt{Mod}_{\mathscr{L}}$ is *invariant* if $[M]_{\cong_{\mathscr{L}}} = M$. There is a convenient characterization of *Borel* invariant sets, in terms of $\mathscr{L}_{\omega_1\omega}$, an infinitary extension of $\mathscr{L} = \{R_i\}_{i\in I}$ by countable conjunctions and disjunctions. To be more exact,

1) any $R_i(v_0, ..., v_{m_i-1})$ is an atomic formula of $\mathscr{L}_{\omega_1\omega}$ (all v_i being variables over $\mathbb{N}$ and m_i is the arity of R_i), and propositional connectives and quantifiers $\exists$, $\forall$ can be applied as usual;

2) if φ_i, $i \in \mathbb{N}$, are formulas of $\mathscr{L}_{\omega_1\omega}$ whose free variables are among a finite list $v_0, ..., v_n$ then $\bigvee_i \varphi_i$ and $\bigwedge_i \varphi_i$ are formulas of $\mathscr{L}_{\omega_1\omega}$.

If $x \in \mathtt{Mod}_{\mathscr{L}}$, $\varphi(v_1, ..., v_n)$ is a formula of $\mathscr{L}_{\omega_1\omega}$, and $i_1, ..., i_n \in \mathbb{N}$, then $x \models \varphi(i_1, ..., i_n)$ means that $\varphi(i_1, ..., i_n)$ is satisfied on x, in the usual sense that involves transfinite induction on the "depth" of φ, see [26, 16.C].

Theorem 49 (Lopez-Escobar, see [26, 16.8]). *A set $M \subseteq \mathtt{Mod}_{\mathscr{L}}$ is invariant and Borel iff $M = \{x \in \mathtt{Mod}_{\mathscr{L}} : x \models \varphi\}$ for a closed formula φ of $\mathscr{L}_{\omega_1\omega}$.*

Proof. To prove the nontrivial direction let $M \subseteq \mathtt{Mod}_{\mathscr{L}}$ be invariant and Borel. Put $B_s = \{g \in S_\infty : s \subset g\}$ for any injective $s \in \mathbb{N}^{<\omega}$ (*i.e.*, $s_i \ne s_j$ for $i \ne j$), this is a clopen subset of S_∞ (in the Polish topology of S_∞ inherited from $\mathbb{N}^{\mathbb{N}}$). If $A \subseteq S_\infty$ then let $s \Vdash A(\dot g)$ mean that the set $B_s \cap A$ is co-meager in B_s, *i.e.*, $g \in A$ holds for a. a. $g \in S_\infty$ with $s \subset g$. The proof consists of two parts:

(i) $M = \{x \in \mathtt{Mod}_{\mathscr{L}} : \Lambda \Vdash \dot g\cdot x \in M\}$ (where $g\cdot x = j_{\mathscr{L}}(g, x)$, see above);

(ii) For any Borel $M \subseteq \mathtt{Mod}_{\mathscr{L}}$ and any n there is a formula $\varphi^n_M(v_0, ..., v_{n-1})$ of $\mathscr{L}_{\omega_1\omega}$ such that we have, for every $x \in \mathtt{Mod}_{\mathscr{L}}$ and every injective $s \in \mathbb{N}^n$: $x \models \varphi^n_M(s_0, ..., s_{n-1})$ iff $s \Vdash \dot g^{-1}\cdot x \in M$.

(i) is clear: since M is invariant, we have $g\cdot x \in M$ for all $x \in M$ and $g \in S_\infty$, on the other hand, if $g\cdot x \in M$ for at least one $g \in S_\infty$ then $x \in M$.

To prove (ii) we argue by induction on the Borel complexity of M. Suppose, for the sake of simplicity, that $\mathscr{L}$ contains a single binary predicate, say, $R(\cdot,\cdot)$; then $\mathtt{Mod}_{\mathscr{L}} = \mathscr{P}(\mathbb{N}^2)$. If $M = \{x \subseteq \mathbb{N}^2 : \langle k, l\rangle \notin x\}$ for some $k, l \in \mathbb{N}$ then take

$$\forall u_0 ... \forall u_m\ (\textstyle\bigwedge_{i<j\le m}(u_i \ne u_j) \wedge \bigwedge_{i<n}(u_i = v_i) \implies \neg R(u_k, u_l))\,,$$

where $m = \mathtt{max}\{l, k, n\}$, as $\varphi^n_M(v_0, ..., v_{n-1})$. Further, take

$$\begin{aligned}\textstyle\bigwedge_{k\ge n} \forall u_0 ... \forall u_{k-1}\ \bigvee_{m\ge k} \exists w_0 ... \exists w_{m-1}\ (\bigwedge_{i<j<k}(u_i \ne u_j) \wedge \bigwedge_{i<n}(u_i = v_i)\\ \textstyle\implies \bigwedge_{i<j<m}(w_i \ne w_j) \wedge \bigwedge_{i<k}(w_i = v_i) \wedge \varphi^m_M(w_0, ..., w_{m-1}))\end{aligned}$$

as $\varphi^n_{\neg M}(v_0, ..., v_{n-1})$. Finally, if $M = \bigcap_j M_j$ then we take $\bigwedge_j \varphi^n_{M_j}(v_0, ..., v_{n-1})$ as $\varphi^n_M(v_0, ..., v_{n-1})$. □ (*Theorem 49*)

10.c ERs classifiable by countable structures

The classifiability by countable structures means that we can associate, in a Borel way, a countable $\mathscr{L}$-structure, say, $\vartheta(x)$ with any point $x \in \mathbb{X} = \mathtt{dom}\,\mathsf{E}$ so that $x \mathrel{\mathsf{E}} y$ iff $\vartheta(x)$ and $\vartheta(y)$ are isomorphic.

Definition 50 (Hjorth [15, 2.38])**.** An ER E is *classifiable by countable structures* if there is a countable relational language $\mathscr{L}$ such that $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathscr{L}}$. □

Remark 51. Any E classifiable by countable structures is $\mathbf{\Sigma}^1_1$, of course, and many of them are Borel. The equivalence relations T_2, E_3, all countable Borel ERs (see the diagram on page 16) are classifiable by countable structures, but E_1, E_2, Tsirelson ERs are not. □

Theorem 52 (Becker and Kechris [1])**.** *Any orbit ER of a Polish action of a closed subgroup of S_∞ is classifiable by countable structures.*

Thus all orbit ERs of Polish actions of S_∞ and its closed subgroups are Borel reducible to a very special kind of actions of S_∞.

Proof. First show that any orbit ER of a Polish action of S_∞ itself is classifiable by countable structures. Hjorth's simplified argument [15, 6.19] is as follows. Let $\mathbb{X}$ be a Polish S_∞-space with basis $\{U_l\}_{l\in\mathbb{N}}$, and let $\mathscr{L}$ be the language with relations R_{lk} where each R_{lk} has arity k. If $x \in \mathbb{X}$ then define $\vartheta(x) \in \mathtt{Mod}_{\mathscr{L}}$ by stipulation that $\vartheta(x) \models R_{lk}(s_0, ..., s_{k-1})$ iff 1) $s_i \ne s_j$ whenever $i < j < k$, and 2) $\forall\, g \in B_s\, (g^{-1}{\cdot}x \in U_l)$, where $B_s = \{g \in S_\infty : s \subset g\}$ and $s = \langle s_0, ..., s_{k-1}\rangle \in \mathbb{N}^k$. Then ϑ reduces $\mathsf{E}^{\mathbb{X}}_{S_\infty}$ to $\cong_{\mathscr{L}}$.

⟵ Hjorth requires $\in \overline{U_l}$. Why ? Also, it seems that $\forall^* g \in B_s$ extends the proof to Borel actions.⊣

To accomplish the proof of the theorem, it remains to apply the following result (an immediate corollary of Theorem 2.3.5b in [1]):

Proposition 52.1. *If $\mathbb{G}$ is a closed subgroup of a Polish group $\mathbb{H}$ and $\mathbb{X}$ is a Polish $\mathbb{G}$-space then there is a Polish $\mathbb{H}$-space $\mathbb{Y}$ such that $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}} \le_{\mathrm{B}} \mathsf{E}^{\mathbb{Y}}_{\mathbb{H}}$.*

Proof. Hjorth [15, 7.18] outlines a proof as follows. Let $Y = \mathbb{X} \times \mathbb{H}$; define $\langle x, h\rangle \approx \langle x', h'\rangle$ if $x' = g{\cdot}x$ and $h' = gh$ for some $g \in \mathbb{G}$, and consider the quotient space $\mathbb{Y} = Y/{\approx}$ with the topology induced by the Polish topology of Y via the surjection $\langle x, h\rangle \mapsto [\langle x, h\rangle]_{\approx}$, on which $\mathbb{H}$ acts by $h'{\cdot}[\langle x, h\rangle]_{\approx} = [\langle x, hh'^{-1}\rangle]_{\approx}$. Obviously $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}} \le_{\mathrm{B}} \mathsf{E}^{\mathbb{Y}}_{\mathbb{H}}$ via the map $x \mapsto [\langle x, 1\rangle]_{\approx}$, hence, it remains to prove that $\mathbb{Y}$ is a Polish $\mathbb{H}$-space, which is not really elementary — we refer the reader to [15, 7.18] or [1, 2.3.5b]. □ (*Proposition*)

To bypass 52.1 in the proof of Theorem 52, we can use a characterization of all closed subgroups of S_∞. Let $\mathscr{L}$ be a language as above, and $x \in \mathtt{Mod}_{\mathscr{L}}$. Define $\mathtt{Aut}_x = \{g \in S_\infty : g{\cdot}x = x\}$: the group of all automorphisms of x.

Proposition 52.2 (see [1, 1.5])**.** *$G \subseteq S_\infty$ is a closed subgroup of S_∞ iff there is an $\mathscr{L}$-structure $x \in \mathtt{Mod}_{\mathscr{L}}$ of a countable language $\mathscr{L}$, such that $G = \mathtt{Aut}_x$.*

Proof. For the nontrivial direction, let G be a closed subgroup of S_∞. For any $n \geq 1$, let I_n be the set of all G-orbits in $\mathbb{N}^n$, *i.e.*, equivalence classes of the ER $s \sim t$ iff $\exists g \in G\,(t = g \circ s)$, thus, I_n is an at most countable subset of $\mathscr{P}(\mathbb{N}^n)$. Let $I = \bigcup_n I_n$, and, for any $i \in I_n$, let R_i be an n-ary relational symbol, and $\mathscr{L} = \{R_i\}_{i \in I}$. Let $x \in \mathtt{Mod}_{\mathscr{L}}$ be defined as follows: if $i \in I_n$ then $x \models R_i(k_0, ..., k_{n-1})$ iff $\langle k_0, ..., k_{n-1}\rangle \in i$. Then $G = \mathtt{Aut}_x$, actually, if G is not necessarily closed subgroup then $\mathtt{Aut}_x = \overline{G}$. □ (*Proposition*)

Now come back to Theorem 52. The same argument as in the beginning of the proof shows that any orbit ER of a Polish action of G, a closed subgroup of S_∞, is $\leq_{\mathrm{B}} \cong^G_{\mathscr{L}}$ for an appropriate countable language $\mathscr{L}$. Yet, by 52.2, $G = \mathtt{Aut}_{y_0}$ where $y_0 \in \mathtt{Mod}_{\mathscr{L}'}$ and $\mathscr{L}'$ is a countable language disjoint from $\mathscr{L}$. The map $x \longmapsto \langle x, y_0\rangle$ witnesses that $\cong^G_{\mathscr{L}} \leq_{\mathrm{B}} \cong_{\mathscr{L} \cup \mathscr{L}'}$.

□ (*Theorem 52*)

10.d Reduction to countable graphs

It could be expected that the more complicated a language $\mathscr{L}$ is accordingly the more complicated isomorphism equivalence relation $\cong_{\mathscr{L}}$ it produces. However this is not the case. Let $\mathscr{G}$ be the language of (oriented binary) graphs, *i.e.*, $\mathscr{G}$ contains a single binary predicate, say $R(\cdot,\cdot)$.

Theorem 53. *If $\mathscr{L}$ is a countable relational language then $\cong_{\mathscr{L}} \leq_{\mathrm{B}} \cong_{\mathscr{G}}$. Therefore, an ER E is classifiable by countable structures iff $\mathsf{E} \leq_{\mathrm{B}} \cong_{\mathscr{G}}$. In other words, a single binary relation can code structures of any countable language.*

Becker and Kechris [1, 6.1.4] outline a proof based on coding in terms of lattices, unlike the following argument, yet it may in fact involve the same idea.

Proof. Let $\mathrm{HF}(\mathbb{N})$ be the set of all hereditarily finite sets over the set $\mathbb{N}$ considered as the set of atoms, and ε be the associated "membership" (any $n \in \mathbb{N}$ has no ε-elements, $\{0,1\}$ is different from 2, *etc.*). Let $\simeq_{\mathrm{HF}(\mathbb{N})}$ be the $\mathrm{HF}(\mathbb{N})$ version of $\cong_{\mathscr{G}}$, *i.e.*, if $P, Q \subseteq \mathrm{HF}(\mathbb{N})^2$ then $P \simeq_{\mathrm{HF}(\mathbb{N})} Q$ means that there is a bijection b of $\mathrm{HF}(\mathbb{N})$ such that $Q = b \cdot P = \{\langle b(s), b(t)\rangle : \langle s, t\rangle \in P\}$. Obviously $(\cong_{\mathscr{G}}) \sim_{\mathrm{B}} (\simeq_{\mathrm{HF}(\mathbb{N})})$, thus, we have to prove that $\cong_{\mathscr{L}} \leq_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$ for any $\mathscr{L}$.

An action of S_∞ on $\mathrm{HF}(\mathbb{N})$ is defined as follows. If $g \in S_\infty$ then $g \circ n = g(n)$ for any $n \in \mathbb{N}$, and, by ε-induction, $g \circ \{a_1, ..., a_n\} = \{g \circ a_1, ..., g \circ a_n\}$ for all $a_1, ..., a_n \in \mathrm{HF}(\mathbb{N})$. Clearly the map $a \mapsto g \circ a$ $(a \in \mathrm{HF}(\mathbb{N}))$ is an ε-isomorphism of $\mathrm{HF}(\mathbb{N})$, for any fixed $g \in S_\infty$.

Lemma 53.1. *Suppose that $X, Y \subseteq \mathrm{HF}(\mathbb{N})$ are ε-transitive subsets of $\mathrm{HF}(\mathbb{N})$, the sets $\mathbb{N} \smallsetminus X$ and $\mathbb{N} \smallsetminus Y$ are infinite, and $\varepsilon \restriction X \simeq_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction Y$. Then there is $f \in S_\infty$ such that $Y = f \circ X = \{f \circ s : s \in X\}$.*

Proof. It follows from the assumption $\varepsilon \restriction X \cong_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction Y$ that there is an ε-isomorphism $\pi : X \xrightarrow{\text{onto}} Y$. Easily $\pi \restriction (X \cap \mathbb{N})$ is a bijection of $X_0 = X \cap \mathbb{N}$ onto $Y_0 = Y \cap \mathbb{N}$, hence, there is $f \in S_\infty$ such that $f \restriction X_0 = \pi \restriction X_0$, and then we have $f \circ s = \pi(s)$ for any $s \in X$. □ (*Lemma*)

Coming back to the proof of Theorem 53, we first show that $\cong_{\mathscr{G}(m)} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$ for any $m \ge 3$, where $\mathscr{G}(m)$ is the language with a single m-ary predicate. Note that $\langle i_1, ..., i_m \rangle \in \mathrm{HF}(\mathbb{N})$ whenever $i_1, ..., i_m \in \mathbb{N}$.

Put $\Theta(x) = \{\vartheta(s) : s \in x\}$ for every element $x \in \mathtt{Mod}_{\mathscr{G}(m)} = \mathscr{P}(\mathbb{N}^m)$, where $\vartheta(s) = \mathrm{TC}_\varepsilon(\{\langle 2i_1, ..., 2i_m \rangle\})$ for each $s = \langle i_1, ..., i_m \rangle \in \mathbb{N}^m$, and finally, for $X \subseteq \mathrm{HF}(\mathbb{N})$, $\mathrm{TC}_\varepsilon(X)$ is the least ε-transitive set $T \subseteq \mathrm{HF}(\mathbb{N})$ with $X \subseteq T$. It easily follows from Lemma 53 that $x \cong_{\mathscr{G}(m)} y$ iff $\varepsilon \restriction \Theta(x) \simeq_{\mathrm{HF}(\mathbb{N})} \varepsilon \restriction \Theta(y)$. This ends the proof of $\cong_{\mathscr{G}(m)} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$.

It remains to show that $\cong_{\mathscr{L}'} \le_{\mathrm{B}} \simeq_{\mathrm{HF}(\mathbb{N})}$, where $\mathscr{L}'$ is the language with infinitely many binary predicates. In this case $\mathtt{Mod}_{\mathscr{L}'} = \mathscr{P}(\mathbb{N}^2)^{\mathbb{N}}$, so that we can assume that every $x \in \mathtt{Mod}_{\mathscr{L}'}$ has the form $x = \{x_n\}_{n \ge 1}$, with $x_n \subseteq (\mathbb{N} \smallsetminus \{0\})^2$ for all n. Let $\Theta(x) = \{s_n(k,l) : n \ge 1 \wedge \langle k,l \rangle \in x_n\}$ for any such x, where

$$s_n(k,l) = \mathrm{TC}_\varepsilon(\{\{...\{\langle k,l \rangle\}...\}\,,\,0\})\,,\ \text{ with } n+2 \text{ pairs of brackets } \{\,,\,\}\,.$$

Then Θ is a continuous reduction of $\cong_{\mathscr{L}'}$ to $\simeq_{\mathrm{HF}(\mathbb{N})}$. □ (*Theorem*)

10.e Borel countably classified ERs: reduction to T_ξ

Equivalence relations T_ξ of § 3.c offer a perfect calibration tool for those Borel ERs which admit classification by countable structures. First of all,

Proposition 54. *Every T_ξ admits classification by countable structures.*

Proof. T_0, the equality on $\mathbb{N}$, is the orbit ER of the action of S_∞ by $g \cdot x = x$ for all g, x. The operation (o2) of § 3.c (countable disjoint union) easily preserves the property of being Borel reducible to an orbit ER of continuous action of S_∞.

Now consider operation (o5) of countable power. Suppose that a ER E on a Polish space $\mathbb{X}$ is Borel reducible to F, the orbit relation of a continuous action of S_∞ on some Polish $\mathbb{Y}$. Let D be the set of all points $x = \{x_k\}_{k \in \mathbb{N}} \in \mathbb{X}^{\mathbb{N}}$ such that either $x_k \not\mathrel{\mathsf{E}} x_l$ whenewer $k \ne l$, or there is m such that $x_k \mathrel{\mathsf{E}} x_l$ iff m divides $|k-l|$. Then $\mathsf{E}^\infty \le_{\mathrm{B}} (\mathsf{E}^\infty \restriction D)$ (via a Borel map $\vartheta : \mathbb{X}^{\mathbb{N}} \to D$ such that $x \mathrel{\mathsf{E}^\infty} \vartheta(x)$ for all x). On the other hand, obviously $(\mathsf{E}^\infty \restriction D) \le_{\mathrm{B}} \mathsf{F}'$, where, for $y,\, y' \in \mathbb{Y}^{\mathbb{N}}$, $y \mathrel{\mathsf{F}'} y'$ means that there is $f \in S_\infty$ such that $y_k \mathrel{\mathsf{F}} y'_{f(k)}$ for all k. Finally, F' is the orbit ER of a continuous action of $S_\infty \times {S_\infty}^{\mathbb{N}}$, which can be realized as a closed subgroup of S_∞, so it remains to apply Theorem 52.1. □

The relations T_α are known in different versions, which reflect the same idea of coding sets of α-th cumulative level over $\mathbb{N}$, as, *e.g.*, in [18, § 1], where results similar to Proposition 54 are obtained in much more precise form.

Theorem 55. *If* E *is a Borel ER classifiable by countable structures then* $\mathsf{E} \le_{\mathrm{B}} \mathsf{T}_\xi$ *for some* $\xi < \omega_1$.

Proof. The proof (a version of the proof in [9]) is based on Scott's analysis. Define, by induction on $\alpha < \omega_1$, a family of Borel ERs $\equiv^\alpha$ on $\mathbb{N}^{<\omega} \times \mathscr{P}(\mathbb{N}^2)$:

$*$ $A \equiv^\alpha_{st} B$ means $\langle s, A\rangle \equiv^\alpha \langle t, B\rangle$;

thus, all $\equiv^\alpha_{st}$ ($s,\, t \in \mathbb{N}^{<\omega}$) are binary relations on $\mathscr{P}(\mathbb{N}^2)$, and among them all relations $\equiv^\alpha_{ss}$ are ERs;

- $A \equiv^0_{st} B$ iff $A(s_i, s_j) \Longleftrightarrow B(t_i, t_j)$ for all $i,\, j < \mathtt{lh}\, s = \mathtt{lh}\, t$;
- $A \equiv^{\alpha+1}_{st} B$ iff $\forall\, k\, \exists\, l\, (A \equiv^\alpha_{s^\wedge k\,,\, t^\wedge l} B)$ and $\forall\, l\, \exists\, k\, (A \equiv^\alpha_{s^\wedge k\,,\, t^\wedge l} B)$;
- if $\lambda < \omega_1$ is limit then: $A \equiv^\lambda_{st} B$ iff $A \equiv^\alpha_{st} B$ for all $\alpha < \lambda$.

Easily $\equiv^\beta \subseteq \equiv^\alpha$ whenever $\alpha < \beta$.

Recall that, for $A,\, B \subseteq \mathbb{N}^2$, $A \cong_{\mathscr{G}} B$ means that there is $f \in S_\infty$ with $A(k,l) \Longleftrightarrow B(f(k), f(l))$ for all $k,\, l$. Then we have $\cong_{\mathscr{G}} \subseteq \bigcap_{\alpha<\omega_1} \equiv^\alpha_{\Lambda\Lambda}$ by induction on α (in fact $=$ rather than $\subseteq$, see below), where Λ is the empty sequence. Call a set $P \subseteq \mathscr{P}(\mathbb{N}^2) \times \mathscr{P}(\mathbb{N}^2)$ *unbounded* if $P \cap \equiv^\alpha_{\Lambda\Lambda} \ne \emptyset$ for all $\alpha < \omega_1$.

Lemma 55.1. *Any unbounded* $\mathbf{\Sigma}^1_1$ *set* P *contains* $\langle A, B\rangle \in P$ *with* $A \cong_{\mathscr{G}} B$.

It follows that $A \cong_{\mathscr{G}} B$ iff $A \equiv^\alpha_{\Lambda\Lambda} B$ for all $\alpha < \omega_1$ (take $P = \{\langle A, B\rangle\}$).

Proof. Since P is $\mathbf{\Sigma}^1_1$, there is a continuous map $F : \mathbb{N}^\mathbb{N} \xrightarrow{\text{onto}} P$. For $u \in \mathbb{N}^{<\omega}$, let $P_u = \{F(a) : u \subset a \in \mathbb{N}^\mathbb{N}\}$. There is a number n_0 such that $P_{\langle n_0\rangle}$ is still unbounded. Let $k_0 = 0$. By a simple cofinality argument, there is l_0 such that $P_{\langle n_0\rangle}$ is still unbounded *over* $\langle k_0\rangle$, $\langle l_0\rangle$ in the sense that there is no ordinal $\alpha < \omega_1$ such that $P_{\langle i_0\rangle} \cap \equiv^\alpha_{\langle k_0\rangle\langle l_0\rangle} = \emptyset$. Following this idea, we can define infinite sequences of numbers $n_m,\, k_m,\, l_m$ such that both $\{k_m\}_{m\in\mathbb{N}}$ and $\{l_m\}_{m\in\mathbb{N}}$ are permutations of $\mathbb{N}$ and, for any m, the set $P_{\langle n_0,\dots,n_m\rangle}$ is still unbounded over $\langle k_0, \dots, k_m\rangle$, $\langle l_0, \dots, l_m\rangle$ in the same sense. Note that $a = \{n_m\}_{m\in\mathbb{N}} \in \mathbb{N}$ and $F(a) = \langle A, B\rangle \in P$ (both $A,\, B$ are subsets of $\mathbb{N}^2$).

Prove that the map $f(k_m) = l_m$ witnesses $A \cong_{\mathscr{G}} B$, *i.e.*, $A(k_j, k_i)$ iff $B(l_j, l_i)$ for all $j,\, i$. Take $m > \mathtt{max}\{j, i\}$ big enough for the following: if $\langle A', B'\rangle \in P_{\langle i_0,\dots,i_m\rangle}$ then $A(k_j, k_i)$ iff $A'(k_j, k_i)$, and similarly $B(l_j, l_i)$ iff $B'(l_j, l_i)$. By the construction, there is a pair $\langle A', B'\rangle \in P_{\langle i_0,\dots,i_m\rangle}$ with $A' \equiv^0_{\langle k_0,\dots,k_m\rangle\langle l_0,\dots,l_m\rangle} B'$, in particular, $A'(k_j, k_i)$ iff $B'(l_j, l_i)$, as required. □ (*Lemma*)

Corollary 55.2 (See, *e.g.*, Friedman [9]). *If* E *is a Borel ER and* $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathscr{G}}$ *then* $\mathsf{E} \le_{\mathrm{B}} \equiv^\alpha_{\Lambda\Lambda}$ *for some* $\alpha < \omega_1$.

Proof. Let ϑ be a Borel reduction of E to $\cong_{\mathscr{G}}$. Then $\{\langle\vartheta(x),\vartheta(y)\rangle : x \not\mathsf{E}\, y\}$ is a $\mathbf{\Sigma}^1_1$ subset of $\mathscr{P}(\mathbb{N}^2)\times\mathscr{P}(\mathbb{N}^2)$ which does not intersect $\cong_{\mathscr{G}}$, hence, it is bounded by Lemma 55.1. Take an ordinal $\alpha<\omega_1$ which witnesses the boundedness. $\square$

Now, if E is a Borel ER classifiable by countable structures then $\mathsf{E} \le_{\mathrm{B}} \cong_{\mathscr{G}}$ by Theorem 53, hence, it remains to establish the following:

Proposition 55.3. *Any ER $\equiv^\alpha$ is Borel reducible to some T_ξ.*

Proof. We have $\equiv^0 \le_{\mathrm{B}} \mathsf{T}_0$ since $\equiv^0$ has countably many equivalence classes, all of which are clopen sets. To carry out the step $\alpha\mapsto\alpha+1$ note that the map $\langle s,A\rangle \mapsto \{\langle s^\wedge k, A\rangle\}_{k\in\mathbb{N}}$ is a Borel reduction of $\equiv^{\alpha+1}$ to $(\equiv^\alpha)^\infty$. To carry out the limit step, let $\lambda=\{\alpha_n : n\in\mathbb{N}\}$ be a limit ordinal, and $\mathsf{R}=\bigvee_{n\in\mathbb{N}}\equiv^{\alpha_n}$, *i.e.*, R is a ER on $\mathbb{N}\times\mathbb{N}^{<\omega}\times\mathscr{P}(\mathbb{N}^2)$ defined so that $\langle m,s,A\rangle\;\mathsf{R}\;\langle n,t,B\rangle$ iff $m=n$ and $A \equiv^{\alpha_m}_{st} B$. However the map $\langle s,A\rangle\mapsto\{\langle m,s,A\rangle\}_{m\in\mathbb{N}}$ is a Borel reduction of $\equiv^\lambda$ to R^∞. $\square$ (*Proposition*)

$\square$ (*Theorem 55*)

11 Turbulent group actions

This is an entirely different class of orbit ERs, disjoint with those which admit classification by countable structures.

11.a Local orbits and turbulence

Suppose that a group $\mathbb{G}$ acts on a space $\mathbb{X}$. If $G\subseteq\mathbb{G}$ and $X\subseteq\mathbb{X}$ then let

$$\mathsf{R}^X_G=\{\langle x,y\rangle\in X^2 : \exists\, g\in G\,(x=g\cdot y)\}$$

and let $\sim^X_G$ denote the ER-hull of R^X_G, *i.e.*, the $\subseteq$-least ER on X such that $x\;\mathsf{R}^X_G\;y \Longrightarrow x\sim^X_G y$. In particular $\sim^{\mathbb{X}}_{\mathbb{G}}=\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$, but generally we have $\sim^X_G\subsetneqq\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}\restriction X$. Finally, define $\mathscr{O}(x,X,G)=[x]_{\sim^X_G}=\{y\in X : x\sim^X_G y\}$ for $x\in X$ – the *local orbit* of x. In particular, $[x]_{\mathbb{G}}=[x]_{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}}=\mathscr{O}(x,\mathbb{X},\mathbb{G})$, the full $\mathbb{G}$-orbit of $x\in\mathbb{X}$.

Definition 56 (This particular version taken from Kechris [28, § 8]). Suppose that $\mathbb{X}$ is a Polish space and $\mathbb{G}$ is a Polish group acting on $\mathbb{X}$ continuously.

(t1) A point $x\in\mathbb{X}$ is *turbulent* if for any open non-empty set $X\subseteq\mathbb{X}$ containing x and any nbhd $G\subseteq\mathbb{G}$ (not necessarily a subgroup) of $1_{\mathbb{G}}$, the local orbit $\mathscr{O}(x,X,G)$ is somewhere dense (*i.e.*, not a nowhere dense set) in $\mathbb{X}$.

(t2) An orbit $[x]_{\mathbb{G}}$ is *turbulent* if x is such (then all $y\in[x]_{\mathbb{G}}$ are turbulent).

(t3) The action (of $\mathbb{G}$ on $\mathbb{X}$) is *generically* [22], or *gen. turbulent* and $\mathbb{X}$ is a *gen. turbulent* Polish $\mathbb{G}$-space, if the union of all dense, turbulent, and meager orbits $[x]_{\mathbb{G}}$ is comeager. □

Our proof of the following theorem, based on ideas in [15, § 3.2], [28, § 12], [9], is designed so that only quite common tools of descriptive set theory are involved. It will also be shown that "turbulent" ERs are not reducible actually to a much bigger family of ERs than orbit ERs of Polish actions of S_∞.

Theorem 57 (Hjorth [15]). *Suppose that $\mathbb{G}$ is a Polish group, $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space. Then $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ is not BM reducible* [23] *to a Polish action of S_∞, hence, not classifiable by countable structures.*

We begin the proof with two rather simple technical results.

Lemma 57.1. *In the assumptions of the theorem, suppose that $\emptyset \ne X \subseteq \mathbb{X}$ is an open set, $G \subseteq \mathbb{G}$ is a nbhd of $1_{\mathbb{G}}$, and $\mathscr{O}(x, X, G)$ is dense in X for X-comeager many $x \in X$. Let $U, U' \subseteq X$ be non-empty open and $D \subseteq X$ comeager in X. Then there exist points $x \in D \cap U$ and $x' \in D \cap U'$ with $x \sim^X_G x'$.*

Proof. Under our assumptions there exist points $x_0 \in U$ and $x'_0 \in U'$ with $x_0 \sim^X_G x'_0$, *i.e.*, there are elements $g_1, ..., g_n \in G \cup G^{-1}$ such that $x'_0 = g_n \cdot g_{n-1} \cdot ... \cdot g_1 \cdot x_0$ and in addition $g_k \cdot ... \cdot g_1 \cdot x_0 \in X$ for all $k \le n$. Since the action is continuous, there is a nbhd $U_0 \subseteq U$ of x_0 such that $g_k \cdot ... \cdot g_1 \cdot x \in X$ for all k and $g_n \cdot g_{n-1} \cdot ... \cdot g_1 \cdot x \in U'$ for all $x \in U_0$. Since D is comeager, easily there is $x \in U_0 \cap D$ such that $x' = g_n \cdot g_{n-1} \cdot ... \cdot g_1 \cdot x \in U' \cap D$. □ (*Lemma*)

Lemma 57.2. *In the assumptions of the theorem, for any open non-empty $U \subseteq \mathbb{X}$ and $G \subseteq \mathbb{G}$ with $1_{\mathbb{G}} \in G$ there is an open non-empty $U' \subseteq U$ such that the local orbit $\mathscr{O}(x, U', G)$ is dense in U' for U'-comeager many $x \in U'$.*

Proof. Let $\text{INT}\,\overline{X}$ be the interior of the closure of X. If $x \in U$ and $\mathscr{O}(x, U, G)$ is somewhere dense (in U) then the set $U_x = U \cap \text{INT}\,\overline{\mathscr{O}(x, U, G)} \subseteq U$ is open and $\sim^U_G$-invariant (an observation made, *e.g.*, in [28, proof of 8.4]), moreover, $\mathscr{O}(x, U, G) \subseteq U_x$, hence, $\mathscr{O}(x, U, G) = \mathscr{O}(x, U_x, G)$. It follows from the invariance that the sets U_x are pairwise disjoint, and it follows from the turbulence that the union of them is dense in U. Take any non-empty U_x as U'. □ (*Lemma*)

11.b Ergodicity

The non-reducibility in Theorem 57 will be established in a special stronger form. Let E, F be ERs on Polish spaces resp. $\mathbb{X}$, $\mathbb{Y}$. A map $\vartheta : \mathbb{X} \to \mathbb{Y}$ is

[22] In this research direction, "generically", or, in our abbreviation, "gen." (property) intends to mean that (property) holds on a comeager domain.

[23] Reducible via a Baire measurable function. This is weaker than Borel reducibility, of course.

- E,F-*invariant* if $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all $x, y \in \mathbb{X}$;
- *gen.* E,F-invariant if $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ holds for all x, y in a comeager subset of $\mathbb{X}$;
- *gen.* F-constant if $\vartheta(x) \mathrel{\mathsf{F}} \vartheta(y)$ for all x, y in a comeager subset of $\mathbb{X}$.

Finally, following Hjorth and Kechris, say that E is *gen.* F-*ergodic* if every BM E,F-invariant map is gen. F-constant.

Proposition 57.2. *E is gen. F-ergodic if and only if every Borel gen. E,F-invariant map is gen. F-constant.*

Proof. Let E, F live in resp. $\mathbb{X}$, $\mathbb{Y}$. Suppose that $\vartheta : \mathbb{X} \to \mathbb{Y}$ is a Borel gen. E,F-invariant map. There is a Borel comeager set $D \subseteq \mathbb{X}$ on which ϑ is E,F-invariant. Then we can extend $\vartheta \restriction D$ to a BM map $\vartheta' : \mathbb{X} \to \mathbb{Y}$ which is still (everywhere) E,F-invariant. This proves implication $\Longrightarrow$ of the lemma. To prove the opposite implication, let $\vartheta : \mathbb{X} \to \mathbb{Y}$ be a BM E,F-invariant map. Then $\vartheta \restriction D$ is Borel for a suitable comeager Borel set $D \subseteq \mathbb{X}$. Let ϑ' be any Borel extension of $\vartheta \restriction D$ to the whole $\mathbb{X}$. $\square$

Proposition 57.3. *Suppose that E is gen. F-ergodic and does not have a comeager equivalence class. Then E is not Borel reducible to F.* $\square$

This is exactly how the non-reducibility is often established. [24] Our proof of Theorem 57 is of this type. It consists of two parts [25]:

Lemma 57.4. *If $\mathbb{G}$ is a Polish group, $\mathbb{X}$ a Polish $\mathbb{G}$-space, and $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ is BM reducible to a Polish action of S_∞, then there is a comeager $\mathbf{G}_\delta$ set $D \subseteq \mathbb{X}$ such that $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}} \restriction D$ is Borel reducible to one of ERs T_ξ.*

In other words, any ER, BM reducible to a Polish action of S_∞, is "generically" Borel reducible to one of T_ξ. Note that any ER Borel reducible, in proper sense, to one of T_ξ, is Borel.

Lemma 57.5. *Any ER induced by a gen. turbulent Polish action is gen. T_ξ-ergodic for every ξ.*

$\square$ (Theorem 57 modulo 57.4 and 57.5)

[24] Yet there are cases when E is neither F-ergodic nor Borel reducible to F, for instance, among the ERs of the form $\boldsymbol{\ell}^p$.

[25] There are slightly different ways to the same goal. Hjorth [15, 3.18] proves outright and with different technique, that any gen. turbulent ER is gen. ergodic w.r.t. any Polish action of S_∞. Kechris [28, § 12] proves that 1) any gen. T_2-ergodic ER is gen. ergodic w.r.t. any Polish action of S_∞, and 2) any turbulent ER is gen. T_2-ergodic.

11.c "Generical" reduction of countably classified ERs to T_ξ

Here, we prove Lemma 57.4. Suppose that $\mathbb{G}$ is a Polish group, $\mathbb{X}$ a Polish $\mathbb{G}$-space, and the orbit ER $\mathsf{E} = \mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ is BM reducible to a Polish action of S_∞. Then, according to Theorems 52 and 53, there is a BM reduction $\rho : \mathbb{X} \to \mathscr{P}(\mathbb{N}^2)$ of E to $\cong_{\mathscr{G}}$, the isomorphism of binary relations on $\mathbb{N}$. The remainder of the argument borrows notation from the proof of Theorem 55.

There is a dense $\mathbf{G}_\delta$ set $D_0 \subseteq \mathbb{X}$ such that $\vartheta = \rho \restriction D_0$ is continuous on D_0. By definition, we have $x \mathrel{\mathsf{E}} y \Longrightarrow \vartheta(x) \cong_{\mathscr{G}} \vartheta(y)$ and $x \mathrel{\not\mathsf{E}} y \Longrightarrow \vartheta(x) \not\cong_{\mathscr{G}} \vartheta(y)$ for all $x, y \in D_0$. We are mostly interested in the second implication, and the aim is to find a $\mathbf{G}_\delta$ dense set $D \subseteq D_0$ such that, for some $\alpha < \omega_1$, we have

$(*)$ implication $x \mathrel{\not\mathsf{E}} y \Longrightarrow \vartheta(x) \not\equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$ holds for all $x, y \in D$.

(Recall that $A \not\cong_{\mathscr{G}} B$ iff $\exists\, \alpha < \omega_1\; A \not\equiv^\alpha_{\Lambda\Lambda} B$, see a remark after Lemma 55.1.)

To find such an α we apply a Cohen forcing argument. Let us fix a countable transitive model $\mathfrak{M}$ of **ZFHC**, *i.e.*, **ZFC** minus the Power Set axiom but plus the axiom: "every set belongs to $\mathrm{HC} = \{x : x \text{ is hereditarily countable}\}$".

We shall assume that $\mathbb{X}$ is coded in $\mathfrak{M}$ in the sense that there is a set $D_{\mathbb{X}} \in \mathfrak{M}$ which is a dense (countable) subset of $\mathbb{X}$, and $d_{\mathbb{X}} \restriction D_{\mathbb{X}}$ (the distance function of $\mathbb{X}$ restricted to $D_{\mathbb{X}}$) also belongs to $\mathfrak{M}$. Further, $\mathbb{G}$, the action, D_0, ϑ are also assumed to be coded in $\mathfrak{M}$ in a similar sense. In this assumption, in particular, the notion of a Cohen generic, over $\mathfrak{M}$, point of $\mathbb{X}$, or of $\mathbb{G}$, makes sense, in particular, the set D of all Cohen generic, over $\mathfrak{M}$, points of $\mathbb{X}$ is a dense $\mathbf{G}_\delta$ subset of $\mathbb{X}$ and $D \subseteq D_0$. We are going to prove that D fulfills $(*)$.

Suppose that $x, y \in D$, and $\langle x, y\rangle$ is a Cohen generic, pair over $\mathfrak{M}$. If $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} y$ is false then we have $\vartheta(x) \not\cong_{\mathscr{D}} \vartheta(y)$, moreover, this fact holds in $\mathfrak{M}[x, y]$ by the Mostowski absoluteness, hence, arguing in $\mathfrak{M}[x, y]$ (which is still a model of **ZFHC**) we find an ordinal $\alpha \in \mathtt{Ord}^{\mathfrak{M}} = \mathtt{Ord}^{\mathfrak{M}[x,y]}$ with $\vartheta(x) \not\equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$. Moreover, since the Cohen forcing satisfies CCC, there is an ordinal $\alpha \in \mathfrak{M}$ such that we have $\vartheta(x) \not\equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$ for <u>every</u> Cohen generic, over $\mathfrak{M}$, pair $\langle x, y\rangle \in D^2$ such that $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} y$ is false. It remains to show that this also holds when $x, y \in D$ (are generic separately, but) do not form a pair, Cohen generic over $\mathfrak{M}$.

Let $g \in \mathbb{G}$ be Cohen generic over $\mathfrak{M}[x, y]$. [26] Then $x' = g \cdot x$ is easily Cohen generic over $\mathfrak{M}[x, y]$ (because the action is continuous), furthermore, $x' \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x$, hence, $x' \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} y$ <u>fails</u>. Yet y is generic over $\mathfrak{M}$ and x' is generic over $\mathfrak{M}[y]$, thus, $\langle x', y\rangle$ is Cohen generic over $\mathfrak{M}$, hence, we have $\vartheta(x') \not\equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$ by the choice of α. On the other hand, $\vartheta(x) \equiv^\alpha_{\Lambda\Lambda} \vartheta(x')$ holds because $x' \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x$, thus, we finally obtain $\vartheta(x') \not\equiv^\alpha_{\Lambda\Lambda} \vartheta(y)$, as required.

$\square$ *(Lemma 57.4)*

[26] In this case, we cannot, generally speaking, define $\mathfrak{M}[x, y]$ as a generic extension of $\mathfrak{M}$, hence, let $\mathfrak{M}[x, y]$ be any (countable transitive) model of **ZFHC** containing x, y, and all sets in $\mathfrak{M}$. It is not really harmful here that $\mathfrak{M}[x, y]$ can contain more ordinals than $\mathfrak{M}$.

11.d Ergodicity of turbulent actions w.r.t. T_ξ

Here, we prove Lemma 57.5. The proof involves a somewhat stronger property than gen. ergodicity in § 11.b. Suppose that F is an ER on a Polish space $\mathbb{X}$.

- An action of $\mathbb{G}$ on $\mathbb{X}$ and the induced equivalence relation $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ are *hereditarily generically* (*h. gen.*, for brevity) F-ergodic if ER $\sim_G^X$ is generically F-ergodic whenever $X \subseteq \mathbb{X}$ is a non-empty open set, $G \subseteq \mathbb{G}$ is a non-empty open set containing $1_{\mathbb{G}}$, and the local orbit $\mathscr{O}(x, X, G)$ is dense in X for comeager (in X) many $x \in X$.

This obviously implies gen. F-ergodicity of $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ provided the action is gen. turbulent. Therefore, Lemma 57.5 is a corollary of the following theorem:

Theorem 58. *Let $\mathbb{X}$ be a gen. turbulent Polish $\mathbb{G}$-space. Suppose that an ER F belongs to $\mathscr{F}_0$, the least collection of ERs containing* $\mathsf{D}(\mathbb{N})$ (the equality on $\mathbb{N}$) *and closed under the operations (o1) – (o5) of § 3.c. Then $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is h. gen. F-ergodic, in particular, is not Borel reducible to F.*

Remark 58.1. Due to the other creative operation, the Fubini product, $\mathscr{F}_0$ contains a lot of ERs very different from T_ξ, among them some Borel ERs which do not admit classification by countable structures, *e.g.*, all ERs of the form $\mathsf{E}_{\mathscr{I}}$, where $\mathscr{I}$ is one of Fréchet ideals, indecomposable ideals, or Weiss ideals of § 2.f. (In fact it is not so easy to show that ideals of the two last families produce ERs in $\mathscr{F}_0$.) In particular, it follows that *no gen. turbulent ER is Borel reducible to a Fréchet, or indecomposable, or Weiss ideal.* □

Our proof of Theorem 58 goes on by induction on the number of applications of the basic operations, in several following subsections.

Right now, we begin with the initial step: prove that, under the assumptions of the theorem, $\mathsf{E}_{\mathbb{G}}^{\mathbb{X}}$ is h. gen. $\mathsf{D}(\mathbb{N})$-ergodic. Suppose that $X \subseteq \mathbb{X}$ and $G \subseteq \mathbb{G}$ are non-empty open sets, $1_{\mathbb{G}} \in G$, and $\mathscr{O}(x, X, G)$ is dense in X for X-comeager many $x \in X$, and prove that $\sim_G^X$ is generically $\mathsf{D}(\mathbb{N})$-ergodic.

Consider, accordingly with Proposition 57.2, a Borel gen. $\sim_G^X, \mathsf{D}(\mathbb{N})$-invariant map $\vartheta : \mathbb{X} \to \mathbb{N}$. Suppose, on the contrary, that ϑ is not gen. $\mathsf{D}(\mathbb{N})$-constant. Then there exist two open non-empty sets $U_1, U_2 \subseteq X$, two numbers $\ell_1 \neq \ell_2$, and a comeager set $D \subseteq X$ such that $\vartheta(x) = \ell_1$ for all $x \in D \cap U_1$, $\vartheta(x) = \ell_2$ for all $x \in D \cap U_2$, and $\vartheta \restriction D$ is "strictly" $\sim_G^X, \mathsf{D}(\mathbb{N})$-invariant. Lemma 57.1 yields a pair of points $x_1 \in U_1 \cap D$ and $x_2 \in U_2 \cap D$ with $x_1 \sim_G^X x_2$, contradiction.

11.e Inductive step of countable power

To carry out this step in the proof of Theorem 58, suppose that

- $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space, F is a Borel ER on a Polish space $\mathbb{Y}$, and the action of $\mathbb{G}$ on $\mathbb{X}$ is h. gen. F-ergodic, and prove that the action is h. gen. F^∞-ergodic. Fix a nonempty open set $X_0 \subseteq \mathbb{X}$ and a nbhd G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$, such that $\mathscr{O}(x, X_0, G_0)$ is dense in X_0 for X_0-comeager many $x \in X_0$. Consider, accordingly to Proposition 57.2, a Borel function $\vartheta : X_0 \to \mathbb{Y}^{\mathbb{N}}$, $\sim^{X_0}_{G_0}, \mathsf{F}^\infty$-invariant on a dense $\mathbf{G}_\delta$ set $D_0 \subseteq X_0$, so that

$$x \sim^{X_0}_{G_0} x' \implies \forall\, k\, \exists\, l\, (\vartheta_k(x)\, \mathsf{F}\, \vartheta_l(x')) \quad : \qquad \text{for all } x,\, x' \in D_0\,,$$

where $\vartheta_k(x) = \vartheta(x)(k)$, $\vartheta_k : X_0 \to \mathbb{Y}$, and prove that ϑ is gen. F^∞-constant.

Below, let $\mathbf{C}_{\mathbb{X}}$ be the Cohen forcing for $\mathbb{X}$, which consists of rational balls with centers in a fixed dense countable subset of $\mathbb{X}$, and let $\mathbf{C}_{\mathbb{G}}$ be the Cohen forcing for $\mathbb{G}$ defined similarly (the dense subset is assumed to be a subgroup). Smaller sets are stronger conditions. Let us fix a countable transitive model $\mathfrak{M}$ of **ZFHC** (see above), which contains all relevant objects or their codes, in particular, codes of the topologies of $\mathbb{X}$, $\mathbb{G}$, $\mathbb{Y}$ and the Borel map ϑ.

Claim 58.2. *Suppose that $\langle x, g\rangle \in \mathbb{X} \times \mathbb{G}$ is $\mathbf{C}_{\mathbb{X}} \times \mathbf{C}_{\mathbb{G}}$-generic over $\mathfrak{M}$. Then $g \cdot x$ is $\mathbf{C}_{\mathbb{X}}$-generic over $\mathfrak{M}$.* (Because the action is continuous.) $\square$

Coming back to the theorem, fix $k \in \mathbb{N}$. Consider an open non-empty $U \subseteq U_0$. By the invariance of ϑ and Claim 58.2 there are conditions $U' \in \mathbf{C}_{\mathbb{X}}$, $U' \subseteq U$, and $Q \in \mathbf{C}_{\mathbb{G}}$, $Q \subseteq G_0$, and a number l, such that $\vartheta_k(x)\, \mathsf{F}\, \vartheta_l(g \cdot x)$ holds for any $\mathbf{C}_{\mathbb{X}} \times \mathbf{C}_{\mathbb{G}}$-generic over $\mathfrak{M}$ pair $\langle x, g\rangle \in U' \times Q$. As Q is open, there is $g_0 \in Q \cap \mathfrak{M}$ and a nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$ such that $g_0\, G \subseteq Q$.

Claim 58.3 (The key point of the turbulence)**.** *If $x, x' \in U'$ are $\mathbf{C}_{\mathbb{X}}$-generic over $\mathfrak{M}$ and $x \sim^{U'}_{G} x'$ then we have $\vartheta_k(x)\, \mathsf{F}\, \vartheta_k(x')$.*

Proof. We argue by induction on $n(x, x') =$ the least number n such that there exist $g_1, ..., g_n \in G$ satisfying

$$(*) \quad x' = g_n \cdot g_{n-1} \cdot ... \cdot g_1 \cdot x, \quad \text{and} \quad g_k \cdot ... \cdot g_1 \cdot x \in U' \quad \text{for all} \quad k \le n\,.$$

Suppose that $n(x, x') = 1$, thus, $x = h \cdot x'$ for some $h \in G$. Take any $\mathbf{C}_{\mathbb{G}}$-generic, over $\mathfrak{M}[x, x']$ (see Footnote 26) element $g \in Q \cup Q^{-1}$, close enough to g_0 for $g' = gh^{-1}$ to belong to Q. Then g is $\mathbf{C}_{\mathbb{G}}$-generic over $\mathfrak{M}[x]$, hence, $\langle x, g\rangle$ is $\mathbf{C}_{\mathbb{X}} \times \mathbf{C}_{\mathbb{G}}$-generic over $\mathfrak{M}$ by the product forcing theorem. Therefore $\vartheta_k(x)\, \mathsf{F}\, \vartheta_l(g \cdot x)$. Moreover, g' also is $\mathbf{C}_{\mathbb{G}}$-generic over $\mathfrak{M}[x']$, so that $\vartheta_k(x')\, \mathsf{F}\, \vartheta_l(g' \cdot x')$ by the same argument. Yet we have $g' \cdot x' = gh^{-1} \cdot (h \cdot x) = g \cdot x$.

As for the inductive step, suppose that $(*)$ holds for some $n \ge 2$. Take a $\mathbf{C}_{\mathbb{G}}$-generic, over $\mathfrak{M}[x]$, element $g_1' \in G$ close enough to g_1 for $g_2' = g_2\, g_1\, {g_1'}^{-1}$ to belong to G and for $x^* = g_1' \cdot x$ to belong to U'. Note that x^* is $\mathbf{C}_{\mathbb{X}}$-generic over $\mathfrak{M}$ (product forcing) and $n(x^*, x') \le n - 1$ because $g_2' \cdot x^* = g_2 \cdot g_1 \cdot x$. $\square$ (*Claim*)

To summarize, we have shown that for any k and any open $\emptyset \ne U \subseteq U_0$ there exist: an open set $\emptyset \ne U' \subseteq U$, and an open $G \subseteq G_0$ with $1_{\mathbb{G}} \in G$, such that $\vartheta_k(x)$ is gen. $\sim_G^{U'}$, F-invariant on U'. We can also assume that the orbit $\mathscr{O}(x, U', G)$ is dense in U' for U'-comeager many $x \in U'$, by Lemma 57.2. Then, by the h. gen. F-ergodicity, ϑ_k is gen. F-constant on U', that is, there is a dense $\mathbf{G}_\delta$ set $D' \subseteq U'$ and $y' \in \mathbb{Y}$ such that $\vartheta_k(x) \,\mathsf{F}\, y'$ for all $x \in D'$.

We conclude that there exist: an U_0-comeager set $D \subseteq U_0$, and a countable set $Y = \{y_j : j \in \mathbb{N}\} \subseteq \mathbb{Y}$ such that, for any k and for any $x \in D$ there is j with $\vartheta_k(x) \,\mathsf{F}\, y_j$. Let $\eta(x) = \bigcup_{k \in \mathbb{N}} \{j : \vartheta_k(x) \,\mathsf{F}\, y_j\}$. Then, for any pair $x, x' \in D$, $\vartheta(x) \,\mathsf{F}^\infty\, \vartheta(x')$ iff $\eta(x) = \eta(x')$, so that, by the invariance of ϑ, we have:

$$x \sim_{G_0}^{U_0} x' \implies \eta(x) = \eta(x') \quad : \quad \text{for all } x, x' \in D. \tag{$*$}$$

It remains to show that η is a constant on a comeager subset of D.

Suppose, on the contrary, that there exist two non-empty open sets $U_1, U_2 \subseteq U_0$, a number $j \in \mathbb{N}$, and a comeager set $D' \subseteq D$ such that $j \in \eta(x_1)$ and $j \notin \eta(x_2)$ for all $x_1 \in D' \cap U_1$ and $x_2 \in D' \cap U_2$. Now Lemma 57.1 yields a contradiction to $(*)$, as in the end of § 11.d.

□ (*Inductive step of countable power in Theorem 58*)

11.f Inductive step of the Fubini product

To carry out this step in the proof of Theorem 58, suppose that

- $\mathbb{X}$ is a gen. turbulent Polish $\mathbb{G}$-space, for any k, F_k be a Borel ER on a Polish space $\mathbb{Y}_k$, the action of $\mathbb{G}$ on $\mathbb{X}$ is h. gen. F_k-ergodic for any k, and $\mathsf{F} = \prod_k \mathsf{F}_k \,/\, \mathtt{Fin}$ is, accordingly, a Borel ER on $\mathbb{Y} = \prod_k \mathbb{Y}_k$,

and prove that the action is h. gen. F-ergodic.

Fix a nonempty open set $U_0 \subseteq \mathbb{X}$ and a nbhd G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$, such that U_0-comeager many orbits $\mathscr{O}(x, U_0, G_0)$ with $x \in U_0$ are dense in U_0. Consider a Borel function $\vartheta : U_0 \to \mathbb{Y}$, $\sim_{G_0}^{U_0}$, F-invariant on a dense $\mathbf{G}_\delta$ set $D_0 \subseteq U_0$, *i.e.*,

$$x \sim_{G_0}^{U_0} y \implies \exists\, k_0\, \forall\, k \geq k_0\, (\vartheta_k(x) \,\mathsf{F}_k\, \vartheta_k(y)) \quad : \quad \text{for all } x, y \in D_0,$$

where $\vartheta_k(x) = \vartheta(x)(k)$, and prove that ϑ is gen. F-constant.

Consider an open non-empty set $U \subseteq U_0$. By the invariance of ϑ and Claim 58.2 there are conditions $U' \in \mathbf{C}_{\mathbb{X}}$, $U' \subseteq U$, and $Q \in \mathbf{C}_{\mathbb{G}}$, $Q \subseteq G_0$, and a number k_0, such that $\vartheta_k(x) \,\mathsf{F}_k\, \vartheta_k(g \cdot x)$ holds for all $k \geq k_0$ and for any $\mathbf{C}_{\mathbb{X}} \times \mathbf{C}_{\mathbb{G}}$-generic over $\mathfrak{M}$ pair $\langle x, g \rangle$ of $x \in U'$ and $g \in Q$. As Q is open, there is $g_0 \in Q \cap \mathfrak{M}$ and a symmetric nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$ such that $g_0\, G \subseteq Q$.

Claim 58.2. *If $k \geq k_0$ and points $x, y \in U'$ are $\mathbf{C}_{\mathbb{X}}$-generic over $\mathfrak{M}$ and $x \sim_G^{U'} y$ then $\vartheta_k(x) \,\mathsf{F}_k\, \vartheta_k(y)$.* (Similarly to Claim 58.3.) □

Thus, for any open non-empty $U \subseteq U_0$ there exist: a number k_0, an open non-empty $U' \subseteq U$, and a nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$, such that $\vartheta_k(x)$ is gen. $\sim_G^{U'}, \mathsf{F}_k$-invariant on U' for all $k \geq k_0$. We can assume that U'-comeager many orbits $\mathscr{O}(x, U', G)$ are dense in U', by Lemma 57.2. Now, by the h. gen. F_k-ergodicity, any ϑ_k with $k \geq k_0$ is gen. F_k-constant on such a set U', hence, ϑ itself is gen. F-constant on U' since $\mathsf{F} = \prod_k \mathsf{F}_k \,/\, \mathtt{Fin}$. It remains to show that these constants are F-equivalent to each other.

Suppose, on the contrary, that there exist two non-empty open sets U_1, $U_2 \subseteq U_0$ and a pair of $y \not\mathsf{F}\, y'$ in $\mathbb{Y}$ such that $\vartheta(x) \,\mathsf{F}\, y$ and $\vartheta(x') \,\mathsf{F}\, y'$ for comeager many $x \in U_1$ and $x' \in U_2$. Contradiction follows as in the end of § 11.e.

$\square$ (*Inductive step of Fubini product in Theorem 58*)

11.g Other inductive steps

Here, we accomplish the proof of Theorem 58, by carrying out induction steps, related to operations (o1), (o2), (o3) of § 3.c.

Countable union. Suppose that F_1, F_2, F_3, ... are Borel ERs on a Polish space $\mathbb{Y}$, and $\mathsf{F} = \bigcup_k \mathsf{F}_k$ is still a ER, and the Polish and gen. turbulent action of $\mathbb{G}$ on $\mathbb{X}$ is h. gen. F_k-ergodic for any k, and prove that it remains h. gen. F-ergodic.

Fix a nonempty open set $U_0 \subseteq \mathbb{X}$ and a nbhd G_0 of $1_{\mathbb{G}}$ in $\mathbb{G}$, such that U_0-comeager many orbits $\mathscr{O}(x, U_0, G_0)$ with $x \in U_0$ are dense in U_0. Consider a Borel function $\vartheta : U_0 \to \mathbb{Y}$, $\sim_{G_0}^{U_0}, \mathsf{F}$-invariant on a dense $\mathbf{G}_\delta$ set $D_0 \subseteq U_0$. It follows from the invariance that for any open $\emptyset \neq U \subseteq U_0$ there exist: a number k and open non-empty sets $U' \subseteq U$ and $Q \subseteq G_0$ such that $\vartheta(x) \;\mathsf{F}_k\; \vartheta(g \cdot x)$ holds for any $\mathbf{C}_{\mathbb{X}} \times \mathbf{C}_{\mathbb{G}}$-generic, over $\mathfrak{M}$, pair $\langle x, g \rangle \in U' \times Q$. We can find, as above, $g_0 \in Q \cap \mathfrak{M}$ and a nbhd $G \subseteq G_0$ of $1_{\mathbb{G}}$ such that $g_0 \, G \subseteq Q$. Similarly to Claims 58.3 and 58.2, we have $\vartheta(x) \,\mathsf{F}_k\, \vartheta(x')$ for any pair of $\mathbf{C}_{\mathbb{X}}$-generic, over $\mathfrak{M}$, elements x, $x' \in U'$, satisfying $x \sim_G^{U'} x'$. It follows, by the ergodicity, that ϑ is F_k-constant, hence, F-constant, on a comeager subset of U'. It remains to show that these F-constants are F-equivalent to each other, which is demonstrated exactly as in the end of § 11.e.

Disjoint union. Let F_k be Borel ERs on Polish spaces $\mathbb{Y}_k$, $k = 0, 1, 2, \dots$. By definition, $\bigvee_k \mathsf{F}_k = \bigcup_k \mathsf{F}'_k$, where each F'_k is a Borel ER defined on the space $\mathbb{Y} = \bigcup_k \{k\} \times \mathbb{Y}_k$ as follows: $\langle l, y \rangle \;\mathsf{F}'_k\; \langle l', y' \rangle$ iff either $l = l'$ and $y = y'$ or $l = l' = k$ and $y \,\mathsf{F}_k\, y'$.

Countable product. Let F_k be ERs on a Polish spaces $\mathbb{Y}_k$. Then $\mathsf{F} = \prod_k \mathsf{F}_k$ is a ER on the space $\mathbb{Y} = \prod_k \mathbb{Y}_k$. For any map $\vartheta : \mathbb{X} \to \mathbb{Y}$, to be gen. E, F-invariant (where E is any ER on $\mathbb{X}$) it is necessary and sufficient that every co-ordinate map $\vartheta_k(x) = \vartheta(x)(k)$ is gen. E, F_k-invariant. This allows to easily accomplish this induction step.

□ (*Theorem 58, Lemma 57.5, Theorem 57*)

11.h An application to the shift actions of ideals

Say that a Borel ideal $\mathscr{Z} \subseteq \mathscr{P}(\mathbb{N})$ is *special* if there is a sequence of reals $r_n > 0$ with $\{r_n\} \to 0$, such that $\mathscr{S}_{\{r_n\}} \subseteq \mathscr{Z}$. *Nontrivial* in the next theorem means: containing no cofinite sets.

Theorem 59. *Suppose that $\mathscr{Z}$ is a nontrivial Borel special ideal, and F belongs to the family $\mathscr{F}_0$ of Theorem 58. Then $\mathsf{E}_{\mathscr{Z}}$ is generically F-ergodic, hence, is not Borel reducible to F.*

Proof. The "hence" statement follows because by the nontriviality all $\mathsf{E}_{\mathscr{Z}}$-equivalence classes are meager subsets of $\mathscr{P}(\mathbb{N})$.

As $\mathscr{Z}$ is special, let $\{r_k\} \to 0$ be a sequence of positive reals such that $\mathscr{S}_{\{r_n\}} \subseteq \mathscr{Z}$. It obviously suffices to prove that $\mathsf{E}_{\{r_n\}} = \mathsf{E}_{\mathscr{S}_{\{r_n\}}}$ is generically F-ergodic. Further, by Theorem 58, it suffices to prove that the shift action of $\mathscr{S}_{\{r_n\}}$ on $\mathscr{P}(\mathbb{N})$ is Polish and gen. turbulent.

The ideal $\mathscr{S}_{\{r_n\}}$ is easily a P-ideal, hence, a polishable group (with Δ as the operation). For instance, $\mathscr{S}_{\{r_n\}}$ is a Polish group in the topology generated by the metric $d_{\{r_n\}}(a,b) = \varphi_{\{r_n\}}(a \,\Delta\, b)$ on $\mathscr{S}_{\{r_n\}}$, where

- $\varphi_{\{r_n\}}(x) = \sum_{n \in x} r_n$ for $x \in \mathscr{P}(\mathbb{N})$, so that $\mathscr{S}_{\{r_n\}} = \{x : \varphi_{\{r_n\}}(x) < +\infty\}$.

The shift action of $\mathscr{S}_{\{r_n\}}$ by $x \cdot y = x \,\Delta\, y$ on $\mathscr{P}(\mathbb{N})$ (considered in the product topology; $\mathscr{P}(\mathbb{N})$ is here identified with $2^{\mathbb{N}}$) is then continuous. It remains to verify the turbulence.

Let $x \in \mathscr{P}(\mathbb{N})$. The orbit $[x]_{\mathscr{S}_{\{r_n\}}} = \mathscr{S}_{\{r_n\}} \,\Delta\, x$ is easily dense and meager, hence, it suffices to prove that x is a turbulent point of the action. Consider an open set $X \subseteq \mathscr{P}(\mathbb{N})$ containing x, and a $d_{\{r_n\}}$-hbhd G of $\emptyset$ (the neutral element of $\mathscr{S}_{\{r_n\}}$); we may assume that, for some k, $X = \{y \in \mathscr{P}(\mathbb{N}) : y \cap [0,k) = u\}$, where $u = x \cap [0,k)$, and $G = \{g \in \mathscr{S}_{\{r_n\}} : \varphi(g) < \varepsilon\}$ for some $\varepsilon > 0$. Prove that the local orbit $\mathscr{O}(x,X,G)$ is somewhere dense in X.

Let $l \geq k$ be big enough for $r_n < \varepsilon$ for all $n \geq l$. Put $v = x \cap [0,l)$ and prove that $\mathscr{O}(x,X,G)$ is dense in $Y = \{y \in \mathscr{P}(\mathbb{N}) : y \cap [0,l) = v\}$. Consider an open set $Z = \{z \in Y : z \cap [l,j) = w\}$, where $j \geq l$, $w \subseteq [l,j)$. Let z be the only element of Z with $z \cap [j,+\infty) = x \cap [j,+\infty)$, thus, $x \,\Delta\, z = \{l_1, ..., l_m\} \subseteq [l,j)$. Each $g_i = \{l_i\}$ belongs to G by the choice of l (indeed, $l_i \geq l$). Moreover, easily $x_i = g_i \,\Delta\, g_{i-1} \,\Delta\, ... \,\Delta\, g_1 \,\Delta\, x = \{l_1, ..., l_i\} \,\Delta\, x$ belongs to X for any $i = 1, ..., m$, and $x_m = z$, thus, $z \in \mathscr{O}(x,X,G)$, as required. □

The next corollary returns us to the discussion in the end of § 3.b.

Corollary 60. *The equivalence relations $\mathbf{c}_0$ and E_2 are not Borel reducible to any ideal F in the family $\mathscr{F}_0$ of Theorem 58, in particular, are not Borel reducible to T_2.*

Proof. According to lemmas 20 and 21, it suffices to prove that the ideals $\mathscr{Z}_0$ (density 0) and $\mathscr{S}_{\{1/n\}}$ are special. The latter is special by definition. As for the former, see ??? (that $\mathscr{S}_{\{1/n\}} \subseteq \mathscr{Z}_0$). $\square$

12 Ideal $\mathscr{I}_3$ and the equivalence relation E_3

The ideal $0 \times \mathtt{Fin}$ is traditionally denoted by $\mathscr{I}_3$. It consists of all sets $x \subseteq \mathscr{P}(\mathbb{N} \times \mathbb{N})$ such that all cross-sections $(x)_n = \{k : \langle n, k\rangle \in x\}$ are finite. It defines the ER $\mathsf{E}_3 = \mathsf{E}_{\mathscr{I}_3}$ on $\mathscr{P}(\mathbb{N} \times \mathbb{N})$ by $x \mathrel{\mathsf{E}_3} y$ iff $x \,\Delta\, y \in \mathscr{I}_3$. But we rather consider E_3 as an ER on $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$ defined by $x \mathrel{\mathsf{E}_3} y$ iff $x(n) \mathrel{\mathsf{E}_0} y(n)$ for all n : here x, y belong to $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$.

12.a Ideals below $\mathscr{I}_3$

Lemma 61. $\mathtt{Fin} <_{\mathrm{B}} \mathscr{I}_3$. *$\mathscr{I}_3$ and $\mathscr{I}_1$ are $\le_{\mathrm{B}}$-incomparable.*

Proof. To see that $\mathtt{Fin} <_{\mathrm{B}} \mathscr{I}_3$ take $\vartheta(x) = \{\langle n, 0\rangle : n \in x\}$. That $\mathscr{I}_3 \not\le_{\mathrm{B}} \mathscr{I}_1$ can be shown as follows: otherwise by Theorem 40 $\mathscr{I}_3$ would be isomorphic either to one of $\mathtt{Fin}$, $\mathscr{I}_1$, or to a trivial variation of $\mathtt{Fin}$, which can be easily shown to be not the case. To see that $\mathscr{I}_1 \not\le_{\mathrm{B}} \mathscr{I}_3$ recall that $\mathscr{I}_3 = 0 \times \mathtt{Fin}$ is of the form $\mathtt{Exh}_\psi$ for a l.s.c. submeasure ψ (Example 5) and apply Theorem 41. □

The following theorem is analogous to Theorem 40, yet the method of its proof is absolutely different.

Theorem 62 (Kechris [27]). *If $\mathscr{I} \le_{\mathrm{B}} \mathscr{I}_3$ is a Borel (nontrivial) ideal on $\mathbb{N}$ then either $\mathscr{I} \cong \mathscr{I}_3$ or $\mathscr{I}$ is a trivial variation of $\mathtt{Fin}$.*

Proof. First of all we make use of Theorem 41: $\mathscr{I}_1 \not\le_{\mathrm{B}} \mathscr{I}$ according to Lemma 61, therefore, $\mathscr{I} = \mathtt{Exh}_\varphi$ for a l.s.c. submeasure φ on $\mathbb{N}$. We can *w.l.o.g.* suppose that $\varphi(x) \le 1$ for any $x \in \mathscr{P}(\mathbb{N})$. Now put $U_n = \{k : \varphi(\{k\}) \le \frac{1}{n}\}$.

We assert that $\mathtt{lim}_{n\to\infty} \varphi(U_n) = 0$. Indeed, otherwise $\varphi(U_n) > \varepsilon$ for some $\varepsilon > 0$ and all n. As φ is l.s.c. we can choose a sequence of numbers $n_1 < n_2 < n_3 < \dots$ and for any l a finite set $w_l \subseteq U_{n_l} \smallsetminus U_{n_{l+1}}$ with $\varphi(w_l) > \varepsilon$. Then $W = \bigcup_l w_l \notin \mathscr{I}$ and obviously $\{\varphi(\{k\})\}_{k\in W} \to 0$. Note that the Borel ideal $\mathscr{Z} = \mathscr{I} \restriction W$ satisfies $\mathscr{Z} \le_{\mathrm{B}} \mathscr{I}$ (via the identity map), because $W \notin \mathscr{I}$. On the other hand, $\mathscr{Z}$ is isomorphic to a special ideal (see §11.h) via the order preserving bijection of W onto $\mathbb{N}$. It follows from Theorem 59 that $\mathsf{E}_{\mathscr{Z}}$ is not Borel reducible to any equivalence relation in $\mathscr{F}_0$, hence, neither is $\mathsf{E}_{\mathscr{I}}$. But $\mathsf{E}_{\mathscr{I}_3} = \mathsf{E}_3$ obviously belongs to $\mathscr{F}_0$, which is a contradiction because $\mathscr{I} \le_{\mathrm{B}} \mathscr{I}_3$.

Thus $\varphi(U_n) \to 0$. Then clearly a set $x \in \mathscr{P}(\mathbb{N})$ belongs to $\mathscr{I}$ iff $x \cap (U_n \smallsetminus U_{n+1})$ is finite for any m, which easily implies that $\mathscr{I}$ is as required. □ ⟵ *check the proof*⊣

12.b Assembling equivalence relations

The next theorem, similar to a couple of results above, will be used in the proof of a dichotomy theorem related to E_3. ⟵ *give ref*⊣

Theorem 63. *Suppose that $\mathbb{X}$, $\mathbb{Y}$ are Polish spaces, $P \subseteq \mathbb{X} \times \mathbb{Y}$ is a Borel set, E is a Borel ER on P, and $\mathbb{G}$ is a countable group acting on $\mathbb{X}$ in a Borel way so that $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ implies $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x'$. Finally, assume that $\mathsf{E} \restriction P(x)$ is smooth for each $x \in \mathbb{X}$, where $P(x) = \{\langle x',y\rangle \in P : x' = x\}$. Then E is Borel-reducible to a Borel action of $\mathbb{G}$.*

Proof. We can assume that $\mathbb{X} = \mathbb{Y} = 2^{\mathbb{N}}$ and both P and E are $\varDelta^1_1$. We can also assume that the action of $\mathbb{G}$ (a countable group) is $\varDelta^1_1$. Then clearly $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x' \Longrightarrow \varDelta^1_1(x) = \varDelta^1_1(x')$. Define $P^*(x) = \bigcup_{a\in\mathbb{G}} P(a{\cdot}x)$ for $x \in \mathbb{X}$.

Claim 63.1. *Suppose that $\langle x,y\rangle$ and $\langle x',y'\rangle$ belong to P and $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x'$. Then $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ iff the equivalence $\langle x,y\rangle \in U \Longleftrightarrow \langle x',y'\rangle \in U$ holds for any $\mathsf{E} \restriction P^*(x)$--invariant $\varDelta^1_1(x)$ set $U \subseteq P^*(x)$.*

Proof. Note that $\mathsf{E} \restriction P^*(x)$ is still smooth by Theorem 28 because $\mathbb{G}$ is countable. In addition $\mathsf{E} \restriction P^*(x)$ is $\varDelta^1_1(x)$. This observation yields the result, because otherwise, *i.e.*, if the ER, defined om $P^*(x)$ by intersections with $\mathsf{E} \restriction P^*(x)$-invariant $\varDelta^1_1(x)$ sets, is coarser than $\mathsf{E} \restriction P^*(x)$, then it is known from the proof of the 2nd dichotomy theorem (Theorem 35) that we would have $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E} \restriction P^*(x)$, a contradiction with the smoothness. □ (*Claim*)

For any $x \in \mathbb{X}$ let $E(x)$ be the set of all $e \in \mathbb{N}$ which code a $\varDelta^1_1(x)$ subset of P, and, for $e \in E(x)$, let $\mathbf{W}^e_x$ be the $\varDelta^1_1(x)$ subset of P coded by e. (It is known that $\{\langle x,e\rangle : e \in E(x)\}$ is $\varPi^1_1$.) Let $\mathtt{inv}(x,e)$ be the formula

$$x \in \mathbb{X} \wedge e \in E(x) \wedge \mathbf{W}^e_x \subseteq P^*(x) \wedge \mathbf{W}^e_x \text{ is } \mathsf{E} \restriction P^*(x)\text{--invariant}\,.$$

Corollary 63.2. *Let $\langle x,y\rangle$, $\langle x',y'\rangle$ be as in Claim 63.1. Then $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ iff $\langle x,y\rangle \in \mathbf{W}^e_x \Longleftrightarrow \langle x',y'\rangle \in \mathbf{W}^e_x$ holds for any e with $\mathtt{inv}(x,e)$.* □

Implication $\Longleftarrow$ of the “iff” in this Corollary can be considered as a property of the $\varPi^1_1$ set $C = \{\langle x,e\rangle : \mathtt{inv}(x,e)\}$, *i.e.*, the property that

- for all pairs $\langle x,y\rangle$ and $\langle x',y'\rangle$ in P with $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x'$, we have: if $\forall\,\langle x,e\rangle \in C\ (\langle x,y\rangle \in \mathbf{W}^e_x \Leftrightarrow \langle x',y'\rangle \in \mathbf{W}^e_x)$ then $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$.

This is easily a $\varPi^1_1$ property in the codes, hence, by the $\varPi^1_1$ Reflection, there is a $\varDelta^1_1$ set $B \subseteq C$ satisfying the same property, that is, we have

Corollary 63.3. *Let $\langle x,y\rangle$, $\langle x',y'\rangle$ be as in Claim 63.1. Then $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ iff $\langle x,y\rangle \in \mathbf{W}^e_x \Longleftrightarrow \langle x',y'\rangle \in \mathbf{W}^e_x$ holds for any e with $\langle x,e\rangle \in B$.* □

To continue the proof of the theorem, define, for any $\langle x,y\rangle \in P$,

$$D_{xy} = \{\langle a,e\rangle : a \in \mathbb{G} \wedge \langle a{\cdot}x,e\rangle \in B \wedge \langle x,y\rangle \in \mathbf{W}^e_{a\cdot x}\}\,.$$

Clearly $\langle x,y\rangle \mapsto D_{x,y}$ is a $\varDelta^1_1$ map $P \to \mathscr{P}(\mathbb{G}\times\mathbb{N})$.

If $D \subseteq \mathbb{G}\times\mathbb{N}$ and $b \in \mathbb{G}$ then put $b \circ D = \{\langle ab^{-1},e\rangle : \langle a,e\rangle \in D\}$.

Claim 63.4. *Suppose that* $\langle x,y\rangle$ *and* $\langle x',y'\rangle$ *belong to* P, $b\in\mathbb{G}$, *and* $x'=b\cdot x$. *Then* $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ *iff* $b\circ D_{xy}=D_{x'y'}$.

Proof. Assume that $b\circ D_{xy}=D_{x'y'}$. According to Corollary 63.3, to prove $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ it suffices to show that $\langle x,y\rangle\in\mathbf{W}^e_x \iff \langle x',y'\rangle\in\mathbf{W}^e_x$ holds whenever $\langle x,e\rangle\in B$. We have

$$\langle x,y\rangle\in\mathbf{W}^e_x \Leftrightarrow \langle\Lambda,e\rangle\in D_{xy} \Leftrightarrow \langle b^{-1},e\rangle\in D_{x'y'} \Leftrightarrow \langle x',y'\rangle\in\mathbf{W}^e_{b^{-1}\cdot x'}=\mathbf{W}^e_x\,,$$

as required. Conversely, let $\langle x,y\rangle \mathsf{E} \langle x',y'\rangle$. If $\langle a,e\rangle\in D_{xy}$ then $\langle a\cdot x,e\rangle\in B$ and $\langle x,y\rangle\in\mathbf{W}^e_{a\cdot x}$, hence, $\langle x',y'\rangle\in\mathbf{W}^e_{a\cdot x}$, too, because the set $\mathbf{W}^e_{a\cdot x}$ is invariant and $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$. Yet $a\cdot x=ab^{-1}\cdot x'$, therefore, by definition, $\langle ab^{-1},e\rangle\in D_{x'y'}$. The same argument can be carried out in the opposite direction, so that $\langle a,e\rangle\in D_{xy}$ iff $\langle ab^{-1},e\rangle\in D_{x'y'}$, that means $b\circ D_{xy}=D_{x'y'}$. □ (*Claim*)

To end the proof of the theorem, consider $\mathbb{Z}=\mathbb{X}\times\mathscr{P}(\mathbb{G}\times\mathbb{N})$, a Polish space. Define a Borel action $b\cdot\langle x,D\rangle=\langle b\cdot x,b\circ D\rangle$ of $\mathbb{G}$ on $\mathbb{Z}$. We assert that $\vartheta(x,y)=\langle x,D_{xy}\rangle$ is a Borel reduction of $\mathsf{E}\restriction P$ to the action $\mathsf{E}^{\mathbb{Z}}_{\mathbb{G}}$. Indeed, let $\langle x,y\rangle$ and $\langle x',y'\rangle$ belong to P. Suppose that $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$. Then $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x'$, so that $x'=b\cdot x$ for some $b\in\mathbb{G}$. Moreover, $b\circ D_{xy}=D_{x'y'}$ by Claim 63.4, hence, $\vartheta(x',y')=b\cdot\vartheta(x,y)$. Let, conversely, $\vartheta(x',y')=b\cdot\vartheta(x,y)$, so that $x'=b\cdot x$ and $D_{x'y'}=b\circ D_{xy}$. Then $\langle x,y\rangle \mathrel{\mathsf{E}} \langle x',y'\rangle$ by Claim 63.4, as required. □

12.c The 6th dichotomy

Theorem 64 (Hjorth and Kechris [16, 17]). *If* $\mathsf{E}\le_{\mathrm{B}}\mathsf{E}_3$ *is a Borel ER then either* $\mathsf{E}\le_{\mathrm{B}}\mathsf{E}_0$ *or* $\mathsf{E}\sim_{\mathrm{B}}\mathsf{E}_3$.

Proof (a modification of the proof in [17]). We may assume that E is a $\varDelta^1_1$ ER on a recursively presented Polish space $\mathbb{X}$, and there is a $\varDelta^1_1$ reduction $\vartheta:\mathbb{X}\to\mathscr{P}(\mathbb{N})^{\mathbb{N}}$ of E to E_3. Let $Q=\mathtt{ran}\,\vartheta$, a $\varSigma^1_1$ subset of $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$.

For $x,\,y\in\mathscr{P}(\mathbb{N})^{\mathbb{N}}$ and $n\in\mathbb{N}$, define $x\equiv_n y$ iff $x\mathrel{\mathsf{E}_3}y$ and $x\restriction_{<n}=y\restriction_{<n}$ (the latter requirement means $x_k=y_k$ for all $k<n$). For $n,k,p\in\mathbb{N}$ put [27]

$$\mathscr{A}_{nkp}=\{A\subseteq\mathscr{P}(\mathbb{N})^{\mathbb{N}}:A \text{ is } \varSigma^1_1\wedge\forall\,x,y\in A\;(x\equiv_n y\Longrightarrow x_k\,\Delta\,y_k\subseteq[0,p))\}\,.$$

Claim 64.1. *If* $A\in\mathscr{A}_{nkp}$ *then there is a* $\varDelta^1_1$ *set* $B\in\mathscr{A}_{nkp}$ *with* $A\subseteq B$.

Proof. (Reflection) □ (*Claim*)

Put $A_{nkp}=\bigcup\{A:A\in\mathscr{A}_{nkp}\}$ and $\widehat{A}=\bigcup_n\bigcap_{k\ge n}\bigcup_p A_{nkp}$

***Case 1*:** $Q\subseteq\widehat{A}$. ***Case 2*:** otherwise.

[27] Hjorth and Kechris [17] define $\mathscr{A}_{nkp}$ with $\forall\,x,y\in Q\cap A$ instead of $\forall\,x,y\in A$. Let us use $\mathscr{A}'_{nkp}$ to denote their version, thus, $\mathscr{A}_{nkp}\subseteq\mathscr{A}'_{nkp}$. However if Case 1 holds in the sense of $\mathscr{A}'_{nkp}$ then it also holds in the sense of $\mathscr{A}_{nkp}$ because $A\in\mathscr{A}'_{nkp}$ iff $A\cap Q\in\mathscr{A}_{nkp}$.

12.d Case 1

We are going to prove that in this case $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$.

As easily $\widehat{A}$ is Π^1_1 by Claim 64.1 and a standard computation, there is a Δ^1_1 set R such that $Q \subseteq R \subseteq \widehat{A}$. Thus, for $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_0$ it suffices now to prove

Lemma 65. $\mathsf{E}_3 \restriction R \le_{\mathrm{B}} \mathsf{E}_0$ *for any* Δ^1_1 *set* $R \subseteq \widehat{A}$.

Proof. By Kreisel Selection there exists a Δ^1_1 map $\nu : R \to \mathbb{N}$ such that

$$\forall\, k \ge \nu(x)\ \exists\, p\ \exists\, B \in \mathscr{A}_{\nu(x),k,p}\ (x \in B \in \Delta^1_1)$$

for any $x \in R$. Let $R_n = \{x \in R : \nu(x) \le n\}$, these are increasing Δ^1_1 subsets of R, and $R = \bigcup_n R_n$. According to Theorem 34, it suffices to prove that $\mathsf{E}_3 \restriction R_n \le_{\mathrm{B}} \mathsf{E}_0$ for any n. Thus let us fix n. By definition we have

$$\forall\, x \in R_n\ \forall\, k \ge n\ \exists\, p\ \exists\, B \in \mathscr{A}_{nkp}\ (x \in B \in \Delta^1_1)\,. \qquad (*)$$

Recall that $\mathbf{C}$ is the least class of sets containing all open sets and closed under the A-operation and the complement. A map f is called $\mathbf{C}$-*measurable* iff all f-preimages of open sets belong to $\mathbf{C}$.

Claim 65.1. *For any* n *there is a* $\mathbf{C}$*-measurable map* $f : R_n \to \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ *such that* $f(x) = f(y) \equiv_n x$ *whenever* $x, y \in R_n$ *satisfy* $x \equiv_n y$.

Proof. Let $E \subseteq \mathbb{N}$ be the Π^1_1 set of all codes of Δ^1_1 subsets of $\mathscr{P}(\mathbb{N})^{\mathbb{N}}$, and let $W_e \subseteq \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ be the Δ^1_1 set coded by $e \in E$. We have, by $(*)$,

$$\forall\, x \in R_n\ \forall\, k \ge n\ \exists\, p\ \exists\, e \in E\ (x \in W_e \in \mathscr{A}_{nkp})\,,$$

and an ordinary application of the Kreisel selection yields a pair of Δ^1_1 maps $\pi, \varepsilon : R_n \times \mathbb{N} \to \mathbb{N}$ such that $\varepsilon(x,k) \in E$ and $x \in W_{\varepsilon(x,k)} \in \mathscr{A}_{n,k,\pi(x,k)}$ hold whenever $x \in R_n$ and $k \ge n$. Let $\tilde\pi(x,k)$ and $\tilde\varepsilon(x,k)$ to be the least, in the sense of any fixed recursive ω-long wellordering of $\mathbb{N} \times \mathbb{N}$, of all possible pairs $\pi(x',k)$ and $\varepsilon(x',k)$ with $x' \in R_n \cap [x]_{\equiv_n}$. Then $\tilde\pi$ and $\tilde\varepsilon$ are $\equiv_n$-invariant in the 1st argument. In addition, we have $W_{\tilde\varepsilon(x,k)} \in \mathscr{A}_{n,k,\tilde\pi(x,k)}$ and the set $Z_{xk} = R_n \cap [x]_{\equiv_n} \cap W_{\tilde\varepsilon(x,k)}$ is nonempty, whenever $x \in R_n$ and $k \ge n$.

Let $x \in R_n$. For any $k \ge n$, the set $Y_{xk} = \{y_k : y \in Z_{xk}\} \subseteq \mathscr{P}(\mathbb{N})$ is finite (and nonempty) by the definition of $\mathscr{A}_{nkp}$, thus, let $f_k(x)$ be the least member of Y_{xk} in the sense of the lexicographical order of $\mathscr{P}(\mathbb{N})$. Define $f(x) \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ so that $f(x)_k = x_k$ for $k < n$ and $f(x)_k = f_k(x)$ for $k \ge n$.

That $f(x) = f(y)$ whenever $x \equiv_n y$ follows from the invariance of ε and π. To see that $f(x) \equiv_n x$ note that by definition $f_k(x)\ \mathsf{E}_0\ x_k$ for $k \ge n$: indeed, $f_k(x) = y_k$ for some $y \in [x]_{\equiv_n}$, but $x \equiv_n y$ implies $x_k\ \mathsf{E}_0\ y_k$ for all k. Finally, the $\mathbf{C}$-measurability needs a routine check. $\square$ (*Claim*)

For any $u \in \mathscr{P}(\mathbb{N})^n$ let $R_n(u) = \{x \in R_n : x \restriction_{<n} = u\}$.

Claim 65.2. *If* $u \in \mathscr{P}(\mathbb{N})^n$ *then* $\mathsf{E}_3 \restriction R_n(u)$ *is smooth.*

Proof. As E_3 and $\equiv_n$ coincide on $R_n(u)$, the relation $\mathsf{E}_3 \restriction R_n(u)$ is smooth via a **C**-measurable, hence, a Baire-measurable map. Suppose, towards the contrary, that it is not really smooth, *i.e.*, via a Borel map. Then, by the 2-nd dichotomy theorem, we have $\mathsf{E}_0 \le_{\mathrm{B}} \mathsf{E}_3 \restriction R_n(u)$, hence, E_0 turns out to be smooth via a Baire-measurable map, which is easily impossible. □ (*Claim*)

To complete the proof of the lemma, let $\mathbb{G} = \mathscr{P}_{\mathtt{fin}}(\mathbb{N})^n$, acting on $\mathbb{X} = \mathscr{P}(\mathbb{N})^n$ componentwise and by Δ at each of the n co-ordinates, so that, for $u, v \in \mathbb{X}$, we have $u \mathsf{E}_{\mathbb{G}}^{\mathbb{X}} v$ iff $u_k \mathsf{E}_0 v_k$ for all $k < n$. Let us apply Theorem 63 with $\mathbb{G}$ and $\mathbb{X}$ as indicated, and $P = R_n$ and $\mathsf{E} = \mathsf{E}_3 \restriction R_n$, Claim 65.2 witnesses the principal requirement. We obtain: $\mathsf{E}_3 \restriction R_n$ is Borel reducible to a ER induced by a Borel action of $\mathbb{G}$. Yet $\mathbb{G}$ is the increasing union of a countable sequence of its finite subgroups, hence, any ER induced by a Borel action of $\mathbb{G}$ is hyperfinite, hence, Borel reducible to E_0.

□ (*Lemma 65 and Case 1 in Theorem 64*)

12.e Case 2

Then the Σ^1_1 set $H = Q \smallsetminus \widehat{A}$ is non-empty. Our idea will be to define a Borel subset X of H such that $\mathsf{E}_3 \restriction X \sim_{\mathrm{B}} \mathsf{E}_3$, the "or" case of Theorem 64.

By definition, $H = \bigcap_n \bigcup_{k>n} H_{nk}$, where $H_{nk} = H \smallsetminus \bigcup_p A_{nkp}$. Note that

$$H_{nk} = \{x \in H : \forall\, p \,\forall\, A \in \Delta^1_1\, (x \in A \Longrightarrow A \notin \mathscr{A}_{nkp})\}$$

by Claim 64.1, and hence H_{nk} is Σ^1_1 by rather elementary computation.

Let b be any recursive bijection $\mathbb{N}^2 \xrightarrow{\text{onto}} \mathbb{N}$, increasing in each argument. Put $L(n) = \mathtt{max}\{r : b(r,0) \le n\}$ – thus for any $\ell > L(n)$ we have $b(\ell, j) > n$, $\forall\, j$.

The splitting system used here will contain non-empty Σ^1_1 sets $X_s \subseteq \mathscr{P}(\mathbb{N})^{\mathbb{N}}$, $s \in 2^{<\omega}$, numbers k_m, $m \in \mathbb{N}$, and elements $g_s \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$, $s \in 2^{<\omega}$, satisfying the following requirements (i) – (vi):

(i) $X_\Lambda \subseteq H$, $X_{s^\wedge i} \subseteq X_s$, $\mathtt{diam}\, X_s \le 2^{-\mathtt{lh}\, s}$, and a certain condition, in terms of the Choquet game, holds, connecting each $X_{s^\wedge i}$ with X_s so that, as a consequence, $\bigcap_n X_{a \restriction n} \ne \emptyset$ for any $a \in 2^{\mathbb{N}}$.

(ii) $0 < k_0 < k_1 < \dots$ and $X_{0^{n+1}} \subseteq \bigcap_{r<L(n)} H_{r,k_r}$. [28]

(iii) If $s \in 2^{n+1}$ then $g_s(i)$ is finite for all i and $= \emptyset$ for all $i > k_{L(n)}$; in addition, $g_{0^{n+1}}(i) = \emptyset$ for all i.

[28] Recall that 0^m is a sequence of m zeros.

(iv) For any $s \in 2^{n+1}$, we have $\forall x \in X_{0^{n+1}} \, \exists y \in X_s \, (y \equiv_{k_{L(n)}} g_s \cdot x)$; [29]

(v)

(vi)

$\square$ *(Theorem 64)*

13 Summable ideals

Farah [6, § 1.12] gives the following classification of summable ideals $\mathscr{S}_{\{r_n\}}$, based on the distribution of numbers r_n:

(S1) *Atomic* ideals: there is $\varepsilon > 0$ such that the set $A_\varepsilon = \{n : r_n \ge \varepsilon\}$ is infinite and satisfies $\mu_{\{r_n\}}(\complement A_\varepsilon) < +\infty$. In this case $\mathscr{S}_{\{r_n\}} = \{X : X \cap A_\varepsilon \in \mathtt{Fin}\}$; Kechris [27] called such ideals *trivial variations of* $\mathtt{Fin}$.

(S2) *Dense* (summable) ideals: $r_n \to 0$.

(S3) There is a decreasing sequence of positive reals $\varepsilon_n \to 0$ sich that all sets $D_n = A_{\varepsilon_{n+1}} \smallsetminus A_{\varepsilon_n}$ are infinite.

(S4) Ideals of the form $\mathtt{Fin} \oplus \text{dense}$: there is a real $\varepsilon > 0$ such that the set A_ε is infinite, $\mu_{\{r_n\}}(\complement A_\varepsilon) = +\infty$, and $\mathtt{lim}_{n\to\infty,\, n \in \complement A_\varepsilon} r_n = 0$. ←— *define ⊕ somewhere*⊣

In the sense of $\le_{\mathrm{B}}$, all ideals of types (S2), (S3), (S4) are equivalent to each other, and all ideals of type (S1) are equivalent to each other, so that we have just 2 summable ideals modulo $\sim_{\mathrm{B}}$, namely, $\mathtt{Fin}$ and $\mathscr{S}_{\{1/n\}}$. The structure under $\le_{\mathrm{RB}}$ or $\le_{\mathrm{BE}}$ is much more complicated (Farah ?).

13.a A useful lemma

Lemma 66 (Attributed to Kechris in [13]). *Suppose that A, X are Borel sets, E is a Borel ER on A, and $\rho : A \to X$ is a Borel map satisfying the following: first, the ρ-image of any E-class is at most countable, secong, ρ-images of any different E-classes are disjoint. Then E is an essentially countable ER.*

Proof. The relation: $x \,\mathsf{R}\, y$ iff $x, y \in Y$ belong to the ρ-image of one and the same E-class in A, is a $\mathbf{\Sigma}^1_1$ ER on the set $Y = \mathtt{ran}\,\vartheta$, moreover,

$$\mathsf{R} \subseteq P = \{\langle x, y\rangle : \neg\, \exists a, b \in A\, (a \not\mathsf{E}\, b \wedge x = \rho(a) \wedge y = \rho(b))\},$$

where P is $\mathbf{\Pi}^1_1$, hence, there is a Borel set U with $\mathsf{R} \subseteq U \subseteq P$, in particular, $U \cap (Y \times Y) = \mathsf{R}$. As all R-equivalence classes are at most countable, we can assume that all cross-sections of U are at most countable, too.

[29] For $g, x \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$, $g \cdot x = y \in \mathscr{P}(\mathbb{N})^{\mathbb{N}}$ is defined by $y(n) = g(n) \,\Delta\, x(n)$, $\forall n$.

Now it suffices to find a Borel ER F with $\mathsf{R} \subseteq \mathsf{F} \subseteq U$. Say that a set $Z \subseteq X$ is "stable" if $U \cap (Z \times Z)$ is a ER, for example, Y is "stable". We observe that the set $D_0 = \{y : Y \cup \{y\}$ is "stable"$\}$ is $\mathbf{\Pi}^1_1$ and satisfies $Y \subseteq D_0$, hence, there is a Borel set Z_1 with $Y \subseteq Z_1 \subseteq D_0$. Similarly,

$$D_1 = \{y' \in Z_1 : Y \cup \{y, y'\} \text{ is "stable" for any } y \in Z_1\}$$

is $\mathbf{\Pi}^1_1$ and satisfies $Y \subseteq D_1$ by the definition of Z_1, so that there is a Borel set Z_2 with $Y \subseteq Z_2 \subseteq D_1$. Generally, we define

$$D_n = \{y' \in Z_n : Y \cup \{y_1, ..., y_n, y'\} \text{ is "stable" for all } y_1, ..., y_n \in Z_n\}$$

find that $Y \subseteq D_n$, and choose a Borel set Z_n with $Y \subseteq Z_n \subseteq D_n$. Then, by the construction, $Y \subseteq Z = \bigcap_n Z_n$, and, for any finite $Z' \subseteq Z$, the set $Y \cup Z'$ is "stable", so that Z itself is "stable", and we can take $\mathsf{F} = U \cap (Z \times Z)$. □

13.b Under the summable ideal

Subsets of $\mathbb{N}$ will be systematically identified with their characteristic functions.

For $a, b \in 2^{\mathbb{N}}$ put $a \,\Delta\, b = \{n : a(n) \neq b(n)\}$ (identified with the function $c(n) = 1$ iff $a(n) \neq b(n)$) and $\Sigma(a,b) = \sum_{n \in a \Delta b} \frac{1}{n+1}$. (This can be a nonnegative real or $+\infty$.) Generally, we define $\Sigma_k^m(a,b) = \sum_{n \in a\Delta b,\, k \le n \le m} \frac{1}{n+1}$, and accordingly $\Sigma_k^\infty(a,b) = \sum_{n \in a\Delta b,\, k \le n < \infty} \frac{1}{n+1}$. Define $\Sigma(a) = \sum_{\{n : a(n)=1\}} \frac{1}{n+1}$ and similarly $\Sigma_k^m(a)$ and $\Sigma_k^\infty(a)$.

Recall that the *summable ideal* is defined as

$$\mathscr{S}_{\{1/n\}} = \{a \in 2^{\mathbb{N}} : \Sigma(a) < +\infty\}.$$

(The notation $\mathscr{I}_2$ and $\mathscr{I}_0$ is also used.) $\mathsf{E}_{\{1/n\}}$ will denote the associated Borel ER on $2^{\mathbb{N}}$, *i.e.*, $a \,\mathsf{E}_{\{1/n\}}\, b$ iff $\Sigma(a,b) < +\infty$.

Theorem 67. *Let* E *be a Borel ER on a Polish space* $\mathbb{X}$, *and* $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_{\{1/n\}}$. *Then either* $\mathsf{E} \sim_{\mathrm{B}} \mathsf{E}_{\{1/n\}}$ *or* E *is essentially countable.*

Proof. This is a long proof. Let $\vartheta : \mathbb{X} \to 2^{\mathbb{N}}$ be a Borel reduction E to $\mathsf{E}_{\{1/n\}}$. We can assume that ϑ is in fact continuous: indeed it is known that there is a stronger Polish topology on $\mathbb{X}$ which makes ϑ continuous but does not add new Borel subsets of $\mathbb{X}$. Now, as any Polish $\mathbb{X}$ is a $1-1$ continuous image of a closed subset of $\mathbb{N}^{\mathbb{N}}$, we can assume that $\mathbb{X} = \mathbb{N}^{\mathbb{N}}$.

Finally, we can assume that ϑ is Δ^1_1, not merely Borel.

If $a \in A \subseteq 2^{\mathbb{N}}$ and $q \in \mathbb{Q}^+$ then let $\mathtt{Gal}^q_A(a)$ be the set of all $b \in A$ such that there is a finite chain $a = a_0, a_1, ..., a_n = b$ of reals $a_i \in A$ such that $\Sigma(a_i, a_{i+1}) < q$ for all i, the *q-galaxy of a in A*.

Definition 67.1. A set $A \subseteq 2^{\mathbb{N}}$ is *q-"grainy"*, where $q \in \mathbb{Q}^+$, iff $\Sigma(a,b) < 1$ for all $a \in A$ and $b \in \mathtt{Gal}^q_A(a)$. A set A is *"grainy"* if it is q-"grainy" for some $q \in \mathbb{Q}^+$. (In other words it is required that the galaxies are rather small.) □

Claim 67.2. *Any q-"grainy" Σ^1_1 set $A \subseteq 2^{\mathbb{N}}$ is covered by a q-"grainy" Δ^1_1 set.*

Proof. [30] The set $D_0 = \{b \in 2^{\mathbb{N}} : A \cup \{b\}$ is q-"grainy"$\}$ is Π^1_1 and $A \subseteq D_0$, hence, there is a Δ^1_1 set B_1 with $A \subseteq B_1 \subseteq D_0$. Note that $A \cup \{a\}$ is q-"grainy" for any $a \in B_1$. It follows that the Π^1_1 set

$$D_1 = \{b \in B_1 : A \cup \{a,b\} \text{ is } q\text{-"grainy" for any } a \in B_1\}$$

still contains A, hence, there is a Δ^1_1 set B_2 with $A \subseteq B_2 \subseteq D_1 \subseteq B_1$. Note that $A \cup \{a_1, a_2\}$ is q-"grainy" for any $a_1,\, a_2 \in B_2$. In general, as soon as we have got a Δ^1_1 set B_n with $A \subseteq B_n$ and such that $A \cup \{a_1, ..., a_n\}$ is q-"grainy" for any $a_1, ..., a_n \in B_n$, then the Π^1_1 set

$$D_n = \{b \in B_n : A \cup \{a_1, ..., a_n, b\} \text{ is } q\text{-"grainy" for any } a_1, ..., a_n \in B_n\}$$

contains A, hence, there is a Δ^1_1 set B_{n+1} with $A \subseteq B_{n+1} \subseteq D_n \subseteq B_n$.

As usual in similar cases, the choice of the sets B_n can be made effective enough for the set $B = \bigcap_n B_n$ to be still Δ^1_1, not merely Borel. On the other hand, $A \subseteq B$ and B is q-"grainy". □ (*Claim*)

Coming back to the proof of the theorem, let C be the union of all "grainy" Δ^1_1 sets. An ordinary computation shows that C is Π^1_1. We have two cases.

***Case 1*:** $\mathtt{ran}\,\vartheta \subseteq C$. ***Case 2*:** otherwise.

13.c Case 1

We are going to prove that, in this case, E is essentially countable. First note that, by Separation, there is a Δ^1_1 set $H^* \subseteq 2^{\mathbb{N}}$ with $\mathtt{ran}\,\vartheta \subseteq H^* \subseteq C$.

Fix a standard enumeration $\{W_e\}_{e \in E}$ of all Δ^1_1 subsets of $2^{\mathbb{N}}$, where, as usual, $E \subseteq \mathbb{N}$ is a Π^1_1 set. By Kreisel Selection, there exist Δ^1_1 functions $a \longmapsto e(a)$ and $a \longmapsto q(a)$, defined on H^*, such that for any $a \in H^*$ the Δ^1_1 set $W(a) = W_{e(a)}$ contains a and is $q(a)$-"grainy". The final point of our argument will be an application of Lemma 66, where ρ will be a derivate of the function $G(a) = \mathtt{Gal}^{q(a)}_{W(a)}(a)$. We prove

Claim 67.2. *If $a \in H^*$ then $\gamma_a = \{G(b) : b \in [a]_{\mathsf{E}_{\{1/n\}}} \cap H^*\}$ is at most countable.*

[30] The result can be achieved as a routine application of a reflection principle, yet we would like to show how it works with a low level technique.

Proof. Otherwise there is a pair of $e \in E$ and $q \in \mathbb{Q}^+$ and an uncountable set $B \subseteq [a]_{\mathsf{E}_{\{1/n\}}} \cap H^*$ such that $q(b) = q$ and $e(b) = e$ for any $b \in B$ and $G(b') \neq G(b)$ for any two different $b, b' \in B$. Note that any $G(b)$, $b \in B$, is a q-galaxy in one and the same set $W(a) = W(b) = W_e$, therefore, if $b \neq b' \in B$ then $b' \notin G(b)$ and $\Sigma(b, b') \geq q$. On the other hand, as $B \subseteq [a]_{\mathsf{E}_{\{1/n\}}}$, we have $\Sigma(a, b) < +\infty$ for all $b \in B$, hence, there is m and a still uncountable set $B' \subseteq B$ such that $\Sigma_m^\infty(a, b) < q/2$ for all $b \in B'$. Now take a pair of $b \neq b' \in B'$ with $b \restriction [0, m) = b' \restriction [0, m)$: then $\Sigma(b, b') < q$, contradiction. □ (*Claim*)

It follows that $x \mapsto G(\vartheta(x))$ maps any E-class into a countable set of galaxies $G(a)$. To code the galaxies by single points, let $S(a) = \bigcup_m \{b \restriction m : b \in G(a)\}$. Thus $S(a) \subseteq 2^{<\omega}$ codes the Polish topological closure of the galaxy $G(a)$.

Claim 67.3. *If $a, b \in H^*$ and $\neg\, a \,\mathsf{E}_{\{1/n\}}\, b$ then b does not belong to the (topological) closure of $G(a)$, in particular, $b \restriction m \notin S(a)$ for some m.*

Proof. Take m big enough for $\Sigma_0^{m-1}(a, b) \geq 2$. Then $s = b \restriction m$ does not belong to $S(a)$ because any $a' \in G(a)$ satisfies $\Sigma(a, a') < 1$. □ (*Claim*)

Elementary computation shows that the sets

$$\mathbf{G} = \{\langle a, b\rangle : a \in H^* \wedge b \in G(a)\} \quad \text{and} \quad \mathbf{S} = \{\langle a, s\rangle : a \in H^* \wedge s \in S(a)\}.$$

belong to $\varSigma_1^1$, but this is not enough to claim that $a \mapsto S(a)$ is a Borel map. Yet we can change it appropriately to get a Borel map with similar properties. First of all define the following $\varSigma_1^1$ ER on H^*:

$$a \,\mathsf{F}\, b \qquad \text{iff} \qquad e(a) = e(b) \wedge q(a) = q(b) \wedge G(a) = G(b).$$

(To see that F is $\varSigma_1^1$ note that here $G(a) = G(b)$ is equivalent to $b \in G(a)$, and that $\mathbf{G}$ is $\varSigma_1^1$.) It follows from Claim 67.3 and Kreisel Selection that there is a $\varDelta_1^1$ function $\mu : H^* \times H^* \to \mathbb{N}$ such that for any pair of $a, b \in H^*$ with $a \not\mathsf{E}_{\{1/n\}}\, b$ we have $b \restriction \mu(a, b) \notin S(a)$. Then the set

$$R(a) = \{b \restriction \mu(a', b) : a', b \in H^* \wedge a \,\mathsf{F}\, a' \wedge a' \not\mathsf{E}_{\{1/n\}}\, b)\} \subseteq 2^{<\omega}$$

does not intersect $S(a)$, for any $a \in H^*$, hence, the $\varSigma_1^1$ set

$$\mathbf{R} = \{\langle a, s\rangle : a \in H^* \wedge s \in R(a)\}$$

does not intersect $\mathbf{S}$. Note that by definition $\mathbf{R}$ is F-invariant w.r.t. the 1st argument, *i.e.*, if $a, a' \in H^*$ satisfy $a \,\mathsf{F}\, a'$ then $R(a) = R(a')$. It follows from Lemma 35.2 that there is a $\varDelta_1^1$ set $\mathbf{Q} \subseteq H^* \times 2^{<\omega}$ with $\mathbf{S} \subseteq \mathbf{Q}$ but $\mathbf{R} \cap \mathbf{Q} = \emptyset$, F-invariant in the same sense. Then the map $a \mapsto Q(a) = \{s : \mathbf{Q}(a, s)\}$ is $\varDelta_1^1$.

Claim 67.4. *Suppose that $a, b \in H^*$. Then: $a \,\mathsf{F}\, b$ implies $Q(a) = Q(b)$ and $a \not\mathsf{E}_{\{1/n\}}\, b$ implies $Q(a) \neq Q(b)$.*

Proof. The first statement holds just because Q is F-invariant. Now suppose that $a \not\mathsf{E}_{\{1/n\}} b$. Then by definition $s = b \restriction \mu(a,b) \in R(a)$, hence, $s \notin Q(a)$. On the other hand, $s \in S(b) \subseteq Q(b)$. $\square$ (*Claim*)

Define $\tau(x) = Q(\vartheta(x))$ for $x \in \mathbb{N}^{\mathbb{N}}$, so that τ is a Δ^1_1 map $\mathbb{N}^{\mathbb{N}} \to \mathscr{P}(2^{<\omega})$.

Claim 67.5. *If* $x \in \mathbb{N}^{\mathbb{N}}$ *then* $T_a = \{\tau(y) : y \in [x]_{\mathsf{E}}\}$ *is at most countable.*

Proof. Suppose that $y,\ z \in [x]_{\mathsf{E}}$. Then $a = \vartheta(x)$, $b = \vartheta(y)$, and $c = \vartheta(z)$ belong to H^*, and $b,\ c \in [a]_{\mathsf{E}_{\{1/n\}}}$. It follows from Claim 67.4 that if $G(b) = G(c)$, $e(b) = e(c)$, and $q(b) = q(c)$, then $Q(b) = Q(c)$. It remains to note that G takes only countably many values on $H^* \cap [a]_{\mathsf{E}_{\{1/n\}}}$ by Claim 67.2. $\square$ (*Claim*)

Finally note that, if $x \not\mathsf{E}\, y \in \mathbb{N}^{\mathbb{N}}$ then $\vartheta(x)$, $\vartheta(y)$ belong to H^* and satisfy $\vartheta(x) \not\mathsf{E}_{\{1/n\}} \vartheta(y)$, hence, $\tau(x) \neq \tau(y)$ by Claim 67.4. Thus, the Borel map τ witnesses that the given ER E is essentially countable by Lemma 66.

13.d Case 2

Thus we suppose that the Σ^1_1 set $B^* = \mathtt{ran}\,\vartheta \smallsetminus C$ is non-empty. Note that, by Claim 67.2, there is no non-empty Σ^1_1 "grainy" set $A \subseteq B^*$.

Let $\mathscr{B}_s = \{a \in 2^{\mathbb{N}} : s \subset a\}$ for $s \in 2^{<\omega}$ and $\mathscr{N}_u = \{x \in \mathbb{N}^{\mathbb{N}} : u \subset x\}$ for $u \in \mathbb{N}^{<\omega}$ (basic open nbhds in $2^{\mathbb{N}}$ and $\mathbb{N}^{\mathbb{N}}$).

If $A,\ B \subseteq 2^{\mathbb{N}}$ and $m,\ k \in \mathbb{N}$, then $A\,\mathsf{R}^m_{\geq k}\,B$ will mean that for any $a \in A$ there is $b \in B$ with $\Sigma^{\infty}_k(a,b) < 2^{-m}$, and conversely, for any $b \in B$ there is $a \in A$ with $\Sigma^{\infty}_k(a,b) < 2^{-m}$. This is not a ER, of course, yet the conjunction of $A\,\mathsf{R}^m_{\geq k}\,B$ and $B\,\mathsf{R}^m_{\geq k}\,C$ implies $A\,\mathsf{R}^{m-1}_{\geq k}\,C$.

0^m will denote the sequence of m zeros.

To prove that $\mathsf{E}_{\{1/n\}} \leq_{\mathrm{B}} \mathsf{E}$ in Case 2, we define an increasing sequence of natural numbers $0 = k_0 < k_1 < k_2 < \ldots$, and also objects A_s, g_s, v_s for any $s \in 2^{<\omega}$, which satisfy the following list of requirements (i) – (viii).

(i) if $s \in 2^m$ then $g_s \in 2^{k_m}$, and $s \subset t \Longrightarrow g_s \subset g_t$;

(ii) $\emptyset \neq A_s \subseteq B^* \cap \mathscr{B}_{g_s}$, A_s is Σ^1_1, and $s \subset t \Longrightarrow A_t \subseteq A_s$.

(iii) if $s \in 2^n$ then $A_{0^n}\,\mathsf{R}^{n+2}_{\geq k_n}\,A_s$;

(iv) if $s \in 2^n$, $m < n$, $s(m) = 0$, then $\Sigma^{k_{m+1}-1}_{k_m}(g_s, g_{0^m}) < 2^{-m-1}$;

(v) if $s \in 2^n$, $m < n$, $s(m) = 1$, then $|\Sigma^{k_{m+1}-1}_{k_m}(g_s, g_{0^m}) - \frac{1}{m+1}| < 2^{-m-1}$;

(vi) if $s, t \in 2^n$, $m < n$, $s(m) = t(m)$, then $|\Sigma^{k_{m+1}-1}_{k_m}(g_s, g_t)| < 2^{-m}$;

(vii) if $s \in 2^n$ then $v_s \in \mathbb{N}^n$, and $s \subset t \Longrightarrow v_s \subset v_t$;

(viii) $A_s \subseteq \{a \in B^* : \vartheta^{-1}(a) \cap \mathscr{N}_{v_s} \neq \emptyset\}$.

We can now accomplish Case 2 as follows. For any $a \in 2^{\mathbb{N}}$ define $F(a) = \bigcup_n g_{a\restriction n} \in 2^{\mathbb{N}}$ (the only element satisfying $g_{a\restriction n} \subset F(a)$ for all n) and $\rho(a) = \bigcup_n v_{a\restriction n} \in \mathbb{N}^{\mathbb{N}}$. It follows, by (viii) and the continuity of ϑ, that $F(a) = \vartheta(\rho(a))$ for any $a \in 2^{\mathbb{N}}$. Thus the next claim proves that ρ is a Borel (in fact, here continuous) reduction $\mathsf{E}_{\{1/n\}}$ to E and ends Case 2.

Claim 67.2. *The map F reduces $\mathsf{E}_{\{1/n\}}$ to $\mathsf{E}_{\{1/n\}}$, that is, the equivalence $a \mathbin{\mathsf{E}_{\{1/n\}}} b \Longleftrightarrow F(a) \mathbin{\mathsf{E}_{\{1/n\}}} F(b)$ holds for all $a, b \in 2^{\mathbb{N}}$.*

Proof. By definition $\Sigma(F(a), F(b)) = \mathtt{lim}_{n\to\infty} \Sigma_0^{k_n-1}(g_{a\restriction n}, g_{b\restriction n})$. However it follows from (iv), (v), (vi) that

$$|\Sigma_0^{k_n-1}(g_{a\restriction n}, g_{b\restriction n}) - \Sigma_0^{n-1}(a \restriction n, b \restriction n)| \le \textstyle\sum_{m<n} 2^{-m} < 2\,.$$

We conclude that $|\Sigma(F(a), F(b)) - \Sigma(a, b)| \le 2$, as required. □ (*Claim*)

13.e Construction

The construction goes on by induction. To begin with we set $k_0 = 0$, $g_\Lambda = \Lambda$ and $A_\Lambda = B^*$. Suppose that, for some n, we have the objects as required for all $n' \le n$, and extend the construction on the level $n+1$.

As A_{0^n} is not “grainy” (see above), there is a pair of elements $a^0, a^1 \in A_{0^n}$ such that $|\Sigma(a^0, a^1) - \frac{1}{n+1}| < 2^{-n-2}$. Note that $a^0 \restriction k_n = a^1 \restriction k_n$ by (i) and (ii), hence, there is $k_{n+1} > k_n$ such that $|\Sigma_{k_n}^{k_{n+1}-1}(a^0, a^1) - \frac{1}{n+1}| < 2^{-n-2}$. According to (iii), for any $s \in 2^n$ there exist $b_s^0, b_s^1 \in A_s$ such that and $\Sigma_{k_n}^{\infty}(a^i, b_s^i) < 2^{-n-2}$ for $i = 0, 1$; we can, of course, assume that $b_{0^n}^i = a^i$. Moreover, the number k_{n+1} can be chosen big enough for the following to hold:

$$\Sigma_{k_{n+1}}^{\infty}(b_s^i, a^0) < 2^{-n-3} \quad \text{— for all} \quad s \in 2^n \quad \text{and} \quad i = 0, 1. \tag{1}$$

We let $g_{s^\wedge i} = b_s^i \restriction k_{n+1}$ for all $s^\wedge i \in 2^{n+1}$. This definition preserves (i). To check (iv) for $s' = s^\wedge 0 \in 2^{n+1}$ and $m = n$, note that

$$\Sigma_{k_n}^{k_{n+1}-1}(g_{s'}, g_{0^{n+1}}) = \Sigma_{k_n}^{k_{n+1}-1}(b_s^0, a^0) < 2^{-n-2}.$$

To check (v) for $s' = s^\wedge 1 \in 2^{n+1}$ and $m = n$, note that

$$|\Sigma_{k_n}^{k_{n+1}-1}(g_{s'}, g_{0^{n+1}}) - \tfrac{1}{n+1}| \le \Sigma_{k_n}^{k_{n+1}-1}(b_s^1, a^1) + |\Sigma_{k_n}^{k_{n+1}-1}(a^0, a^1) - \tfrac{1}{n+1}| < 2^{-n-1}.$$

To fulfill (vii), choose, for any $s^\wedge i \in 2^{n+1}$, a sequence $v_{s^\wedge i} \in \mathbb{N}^{n+1}$ so that $v_s \subset v_{s^\wedge i}$ and there is $\mathscr{N}_{v_{s^\wedge i}} \cap \vartheta^{-1}(b_s^i) \ne \emptyset$.

Let us finally define the sets $A_{s'} \subseteq A_s$, for all $s' = s^\wedge i \in 2^{n+1}$ (so that $s \in 2^n$ and $i = 0, 1$). To fulfill (ii) and (viii), we begin with

$$A'_{s^\wedge i} = \{a \in A_s \cap \mathscr{B}_{g_{s^\wedge i}} : \vartheta^{-1}(a) \cap \mathscr{N}_{v_{s^\wedge i}} \ne \emptyset\}\,.$$

This is a Σ^1_1 subset of A_s, containing b^i_s. To fulfill (iii), we define $A_{0^{n+1}}$ to be the set of all $a \in A'_{0^{n+1}}$ such that

$$\forall\, s' = s^\wedge i \in 2^{n+1}\ \exists\, b \in A'_{s'}\ \big(\Sigma^\infty_{k_{n+1}}(a,b) < 2^{-n-3}\big)\ ;$$

this is still a Σ^1_1 set containing $b^0_{0^n} = a^0$ by (1). It remains to define, for any $s^\wedge i \neq 0^{n+1}$, $A_{s^\wedge i}$ to be the set of all $b \in A'_{s^\wedge i}$ such that

$$\exists\, b \in A_{0^{n+1}}\ \big(\Sigma^\infty_{k_{n+1}}(a,b) < 2^{-n-3}\big)\ .$$

This ends the definition for the level $n+1$.

□ (*Construction and Theorem 67*)

14 c_0-equalities

Suppose that $\langle X_k\,; d_k\rangle$ is a finite metric space for each $k \in \mathbb{N}$. Farah [7] defines an equivalence relation $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ on $\mathbb{X} = \prod_{k\in\mathbb{N}} X_k$ as follows:

$$x\,\mathsf{D}\,y \quad \text{iff} \quad \lim_{k\to\infty} d_k(x_k, y_k) = 0\,.$$

ERs of this form are called c_0-*equalities*. In addition, $\mathsf{D}(\langle X_k\,; d_k\rangle)$ is *nontrivial* if $\limsup_{k\to\infty} \mathtt{diam}(X_k) > 0$ (otherwise $\mathsf{D}(\langle X_k\,; d_k\rangle)$ makes everything equivalent). Every c_0-equality is easily a Borel ER, more exactly, of class $\mathbf{\Pi}^0_3$.

14.a Some examples and simple results

Example 68. (1) Let $X_k = \{0,1\}$ with $d_k(0,1) = 1$ for all k. Then clearly the relation $\mathsf{D}(\langle X_k\,; d_k\rangle)$ on $2^{\mathbb{N}} = \prod_k \{0,1\}$ is just E_0.

(2) Let $X_{kl} = \{0,1\}$ with $d_{kl}(0,1) = k^{-1}$ for all $k,\, l \in \mathbb{N}$. Then the relation $\mathsf{D}(\langle X_{kl}\,; d_{kl}\rangle)$ on $2^{\mathbb{N}\times\mathbb{N}} = \prod_{k,l}\{0,1\}$ is just $\mathsf{E}_3 = \mathsf{E}_{0\times\mathtt{Fin}}$.

(3) Generally, if $0 = n_0 < n_1 < n_2 < \ldots$ and φ_i is a submeasure on $[n_i, n_{i+1})$, then let $X_i = \mathscr{P}([n_i, n_{i+1}))$ and $d_i(u,v) = \varphi_i(u\,\Delta\,v)$ for $u,\, v \subseteq [n_i, n_{i+1})$. Then $\mathsf{D}(\langle X_i\,; d_i\rangle)$ is clearly isomorphic to $\mathsf{E}_{\mathscr{I}}$, where

$$\mathscr{I} = \mathtt{Exh}(\varphi) = \{x \subseteq \mathbb{N} : \lim_{n\to\infty} \varphi(x \cap [n,\infty)) = 0\}$$

and $\varphi(x) = \mathtt{sup}_i\, \varphi_i(x \cap [n_i, n_{i+1}))$.

(4) Let $\mathsf{D}_{\mathtt{max}} = \mathsf{D}(\langle X_k\,; d_k\rangle)$, where $X_k = \{0, \frac{1}{k}, \frac{2}{k}, \ldots, 1\}$ and d_k is the distance on X_k inherited from $\mathbb{R}$. □

Proposition 69 (Farah [7] with a reference to Oliver**).** (i) $\mathsf{D}_{\mathtt{max}} \sim_{\mathrm{B}} \mathsf{Z}_0$;

(ii) *if* D *is a* c_0*-equality then* $\mathsf{D} \leq_{\mathrm{B}} \mathsf{D}_{\mathtt{max}}$*, moreover,* $\mathsf{D} \leq_{\mathrm{A}} \mathsf{D}_{\mathtt{max}}$.

Thus $\mathsf{D}_{\mathtt{max}}$ is a maximal, in a sense, among c_0-equalities.

Proof. (i) It is clear that $\mathsf{D}_{\mathtt{max}}$ is the same as $\mathbf{c}_0 \restriction \mathbb{X}$, where $\mathbb{X} \subseteq \mathbb{R}^{\mathbb{N}}$ is defined as in the proof of Lemma 20, where it is also shown that $\mathbf{c}_0 \sim_{\mathrm{B}} \mathbf{c}_0 \restriction \mathbb{X}$.

(ii) To prove $\mathsf{D} \le_{\mathrm{B}} \mathsf{D}_{\mathtt{max}}$, it suffices, by (i) and Lemma 20, to show that $\mathsf{D} \le_{\mathrm{B}} \mathbf{c}_0$. The proof is based on the following:

Claim 69.1. *Any finite n-element metric space $\langle X\,; d\rangle$ is isometric to an n-element subset of $\langle \mathbb{R}^n\,; \rho_n\rangle$, where ρ_n be the distance on $\mathbb{R}^n$ defined by $\rho_n(x,y) = \mathtt{max}_{i<n} |x_i - y_i|$.*

Proof of the claim. Let $X = \{x_1, \dots, x_n\}$. It suffices to prove that for any $k \ne l$ there is a set of reals $\{r_1, \dots, r_n\}$ such that $|r_k - r_l| = d(x_k, x_l)$ and

$(*)$ $|r_i - r_j| \le d_{ij} = d(x_i, x_j)$ for all i, j.

We can assume that $k = 1$ and $l = n$.

Step 1. There is a least number $h_1 \ge 0$ such that $(*)$ holds for the numbers $\{\underbrace{0, 0, \dots, 0}_{n-1 \text{ times}}, h\}$ for any $0 \le h \le h_1$. Then, for some k, $1 \le k < n$, we have $h_1 - 0 = d_{kn}$ exactly. Suppose that $k \ne 1$; then it can be assumed that $k = n-1$.

Step 2. Similarly, there is a least number $h_2 \ge 0$ such that $(*)$ holds for the numbers $\{\underbrace{0, 0, \dots, 0}_{n-2 \text{ times}}, h, h_1 + h\}$ for any $0 \le h \le h_2$. Then, for some k, ν, $1 \le k < n-1 \le \nu \le n$, we have $h_2 - 0 = d_{k\nu}$ exactly. Suppose that $k \ne 1$; then it can be assumed that $k = n-2$.

Step 3. Similarly, there is a least number $h_3 \ge 0$ such that $(*)$ holds for the numbers $\{\underbrace{0, 0, \dots, 0}_{n-3 \text{ times}}, h, h_2 + h, h_1 + h_2 + h\}$ for any $0 \le h \le h_3$. Then again, for some k, ν, $1 \le k < n-2 \le \nu \le n$, we have $h_3 - 0 = d_{k\nu}$ exactly. Suppose that $k \ne 1$; then it can be assumed that $k = n-3$.

Et cetera.

This process ends, after a number m $(m < n)$ steps, in such a way that the index k obtained at the final step is equal to 1. Then $(*)$ holds for the numbers $\{\underbrace{0, 0, \dots, 0}_{n-m \text{ times}}, r_{n-m+1}, r_{n-m+1}, \dots, r_n\}$, where $r_{n-m+j} = h_m + h_{m-1} + \dots + h_{m-j+1}$ for each $j = 1, \dots m$. Moreover it follows from the construction that there is a decreasing sequence $n = k_0 > k_1 > k_2 > \dots > k_\mu = 1$ $(\mu \le m)$ such that $r_{k_i} - r_{k_{i+1}} = d_{k_{i+1}, k_i}$ exactly for any i. Then $d_{1n} \le \sum_i r_{k_i} - r_{k_{i+1}}$ by the triangle inequality. But the right-hand side is a part of the sum $r_n = h_1 + \dots + h_m$, and hence $r_n \ge d_{1n}$. It follows that, cutting the construction at an appropriate step $m' \le m$) (and taking an appropriate value of $h \le h_{m'}$), we obtain a sequence of numbers $r_1 = 0 \le r_2 \le \dots \le r_{n-1} \le r_n$ still satisfying $(*)$ and satisfying $r_n = r_n - r_0 = d_{1n}$. This ends the proof. $\square$ (*Claim*)

Now, to carry out the proof of $\mathsf{D} \le_{\mathrm{B}} \mathbf{c}_0$, suppose that $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ is an equivalence relation on $\mathbb{X} = \prod_{k\in\mathbb{N}} X_k$, where each $\langle X_k\,; d_k\rangle$ is a finite metric space. Let n_k be the number of elements in X_k. Let, by the claim, $\eta_k : X_k \to \mathbb{R}^{n_k}$ be an isometric embedding of $\langle X_k\,; d_k\rangle$ into $\langle \mathbb{R}^{n_k}\,; \rho_{n_k}\rangle$. The map $\vartheta(x) = \eta_0(x_0)^\wedge\eta_1(x_1)^\wedge\eta_2(x_2)^\wedge\dots$ (from $\mathbb{X}$ to $\mathbb{R}^{\mathbb{N}}$) reduces D to $\mathbf{c}_0$. $\square$

The structure of c_0-equalities tend to be connected more with the additive reducibility $\le_{\mathrm{A}}$ (see § 1.d on $\le_{\mathrm{A}}$ and the associated relations $<_{\mathrm{A}}$ and $\sim_{\mathrm{A}}$) than with the general Borel reducibility. In particular, we have

Lemma 70. *For any c_0-equality $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$, if D' is a Borel ER on a set $\prod_k X'_k$ (with finite nonempty X'_k) and $\mathsf{D}' \le_{\mathrm{A}} \mathsf{D}$ then D' is a c_0-equality.*

Proof. Let a sequence $0 = n_0 < n_1 < n_2 < \dots$ and a collection of maps $H_i : X'_i \to \prod_{n_i\le k<n_{i+1}} X_k$ witness $\mathsf{D}' \le_{\mathrm{A}} \mathsf{D}$. For $x',\, y' \in X'_i$ put

$$d'_i(x',y') = \max_{n_i\le k<n_{i+1}} d_k(H_i(x')_k, H_i(y')_k)\,.$$

Then easily $\mathsf{D}' = \mathsf{D}(\langle X'_k\,; d'_k\rangle)$. $\square$

Lemma 71 (Farah [7] with a reference to Hjorth**).** *Every c_0-equality $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ is induced by a continuous action of a Polish group.*

(The domain $\mathbb{X} = \prod_k X_k$ of D is considered with the product topology.)

Proof. (sketch) For any k let S_k be the (finite) group of all permutations of X_k, with the distance $\rho_k(s,t) = \max_{x\in X_k} d_k(s(x), t(x))$. Then

$$\mathbb{G} = \{g \in \textstyle\prod_k S_k : \lim\limits_{k\to\infty} \rho_k(g_k, e_k) = 0\}\,, \quad \text{where } e_k \in S_k \text{ is the identity}\,,$$

is easily a subgroup of $\prod_k S_k$, moreover, the distance $d(g,h) = \sup_k \rho_k(g_k,h_k)$ converts $\mathbb{G}$ into a Polish group, the natural action of which on $\mathbb{X}$ (*i.e.*, $(g\cdot x)_k = g_k(x_k)$, $\forall\, k$) is continuous and induces D. $\square$

14.b Classification

Recall that for a metric space $\langle A; d\rangle$, a rational $q > 0$, and $a \in A$, $\mathtt{Gal}^q_A(a)$ is the set of all $b \in A$ which can be connected with a by a finite chain $a = a_0, a_1, \dots, a_n = b$ with $d(a_i, a_{i+1}) < q$ for all i. Farah defines, for $r > 0$,

$$\delta(r, A) = \inf\,\{q \in \mathbb{Q}^+ : \exists\, a \in A\ (\mathtt{diam}(\mathtt{Gal}^q_A(a)) \ge r)\}$$

(with the understanding that here $\inf\emptyset = +\infty$), and

$$\Delta(A) = \{d(a,b) : a \ne b \in A\}\,, \quad \text{so that} \quad \mathtt{diam}\, A = \sup(\Delta(A) \cup \{0\})\,.$$

Now let $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ be a c_0-equality on $\mathbb{X} = \prod_{k\in\mathbb{N}} X_k$. The basic properties of D are determined by the following two conditions:

(co1) $\liminf_{k\to\infty} \delta(r, X_k) = 0$ for some $r > 0$.

(co2) $\forall\, \varepsilon > 0\, \exists\, \varepsilon' \in (0, \varepsilon)\, \exists^\infty k\, \big(\Delta(X_k) \cap [\varepsilon', \varepsilon) \neq \emptyset\big)$.

Easily (co1) implies both the nontriviality of $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ and (co2).

Theorem 72 (Farah [7]**).** *Let* $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ *be a nontrivial* c_0*-equality. Then*

$\overleftarrow{\text{Comment upon turbulent in (iii).}}\dashv$

(i) *If* (co2)*, hence,* (co1) *fail then* $\mathsf{D} \sim_{\mathrm{A}} \mathsf{E_0}$*, hence,* $\mathsf{D} \sim_{\mathrm{B}} \mathsf{E_0}$;

(ii) *If* (co1) *fails but* (co2) *holds then* $\mathsf{D} \sim_{\mathrm{A}} \mathsf{E_3}$*, hence,* $\mathsf{D} \sim_{\mathrm{B}} \mathsf{E_3}$;

(iii) *If* (co1)*, hence,* (co2) *hold then* $\mathsf{E_0} <_{\mathrm{A}} \mathsf{D}$ *and* $\mathsf{D_1} \le_{\mathrm{A}} \mathsf{D}$ *for a turbulent* c_0*-equality* $\mathsf{D_1}$ *satisfying* $\mathsf{E_3} \le_{\mathrm{A}} \mathsf{D_1}$.

Proof. (i) To show that $\mathsf{E_0} \le_{\mathrm{A}} \mathsf{D}$ note that, by the nontriviality of D, there exist: a number $p > 0$, an increasing sequence $0 = n_0 < n_1 < n_2 < \dots$, and, for any i, a pair of points $x_{n_i}, y_{n_i} \in X_{n_i}$ with $d_{n_i}(x_{n_i}, y_{n_i}) \ge p$. For n not of the form n_i fix an arbitrary $x_n \in X_n$. Now, if $a \in 2^{\mathbb{N}}$, then define $\vartheta(a) \in \prod_k X_k$ so that $\vartheta(a)_n = z_n$ for n not of the form n_i, while $\vartheta(a)_{n_i} = x_{n_i}$ or $= y_{n_i}$ if resp. $a_i = 0$ or $= 1$. This map ϑ witnesses $\mathsf{E_0} \le_{\mathrm{A}} \mathsf{D}$.

Now prove that $\mathsf{D} \le_{\mathrm{A}} \mathsf{E_0}$. As (co2) fails, there is $\varepsilon > 0$ such that for each ε' with $0 < \varepsilon' < \varepsilon$ we have only finitely many k with the propery that $\varepsilon' \le d_k(\xi, \eta) < \varepsilon$ for some $\xi, \eta \in X_k$. Let G_k be the (finite) set of all $\frac{\varepsilon}{2}$-galaxies in X_k, and let $\vartheta : \mathbb{X} = \prod_k X_k \to G = \prod_k G_k$ be defined as follows: $\vartheta(x)_k$ is that galaxy in G_k to which x_k belongs. Let E be the G-version of $\mathsf{E_0}$, *i.e.*, if $g, h \in G$ then $g \mathrel{\mathsf{E}} h$ iff $g_k = h_k$ for all but finite k. As easily $\mathsf{E} \le_{\mathrm{A}} \mathsf{E_0}$, it suffices to demonstrate that $\mathsf{D} \le_{\mathrm{A}} \mathsf{E}$ via ϑ. Suppose that $x, y \in \mathbb{X}$ and $\vartheta(x) \mathrel{\mathsf{E}} \vartheta(y)$ and prove $x \mathrel{\mathsf{D}} y$ (the nontrivial direction). Let, on the contrary, $x \not\mathrel{\mathsf{D}} y$, so that there is a number $p > 0$ with $d_k(x_k, y_k) > p$ for infinitely many k. We can assume that $p < \frac{\varepsilon}{2}$. On the other hand, as $\vartheta(x) \mathrel{\mathsf{E}} \vartheta(y)$, there is k_0 such that x_k and y_k belong to one and the same $\frac{\varepsilon}{2}$-galaxy in X_k for all $k > k_0$. Then, for any $k > k_0$ with $d_k(x_k, y_k) > p$ (*i.e.*, for infinitely many values of k) there exists an element $z_k \in X_k$ in the same galaxy such that $p < d_k(x_k, z_k) < \varepsilon$, but this is a contradiction to the choice of ε (indeed, take $\varepsilon' = p$).

(ii) Let us show first that if (co2) holds then $\mathsf{E_3} \le_{\mathrm{A}} \mathsf{D}$ (independently of (co1)). It follows from (co2) that there exist: an infinite sequence $\varepsilon_1 > \varepsilon_2 > \varepsilon_3 > \dots > 0$, for any i an infinite set J_i, and for any $j \in J_i$ a pair of elements $x_{ij}, y_{ij} \in X_j$ with $d_j(x_{ij}, y_{ij}) \in [\varepsilon_{i+1}, \varepsilon_i)$. We may assume that the sets J_i are pairwise disjoint. Then the c_0-equality $\mathsf{D}' = \mathsf{D}(\langle\{x_{ij}, y_{ij}\}\,; d_j\rangle_{i\in\mathbb{N},\, j\in J_i})$ satisfies both $\mathsf{D}' \le_{\mathrm{A}} \mathsf{D}$ and $\mathsf{D}' \cong \mathsf{E_3}$ (via a bijection between the underlying sets).

Now, assuming that, in addition, (co1) fails, we show that $\mathsf{D} \le_{\mathrm{A}} \mathsf{E_3}$. For all $k, n \in \mathbb{N}$ let G_{kn} be the (finite) set of all $\frac{1}{n}$-galaxies in X_k. For any $x \in \mathbb{X} = \prod_i X_i$ define $\vartheta(x) \in G = \prod_{k,n} G_{kn}$ so that $\vartheta(x)_{kn}$ is that $\frac{1}{n}$-galaxy in

G_{kn} to which x_k belongs (for all k, n). The ER E on G, defined so that $g \mathrel{\mathsf{E}} h$ iff $\forall n \ \forall^\infty k \ (g_{kn} = h_{kn}) \ (g, h \in G)$ is easily $\le_{\mathrm{A}} \mathsf{E}_3$, so it suffices to show that $\mathsf{D} \le_{\mathrm{A}} \mathsf{E}$ via ϑ. Suppose that $x, y \in \mathbb{X}$ and $\vartheta(x) \mathrel{\mathsf{E}} \vartheta(y)$ and prove $x \mathrel{\mathsf{D}} y$ (the nontrivial direction). Otherwise there is some $r > 0$ with $d_k(x_k, y_k) > r$ for infinitely many k. As (co1) fails for this r, there is n big enough for $\delta(r, X_k) > \frac{1}{n}$ to hold for almost all k. Then, by the choice of r, we have $\vartheta(x)_{kn} \ne \vartheta(y)_{kn}$ for infinitely many k, hence, $\vartheta(x) \not\mathrel{\mathsf{E}} \vartheta(y)$, contradiction.

(iii) Fix $r > 0$ with $\mathtt{lim\,inf}_{k\to\infty} \delta(r, X_k) = 0$. As for any increasing sequence $n_0 < n_1 < n_2 < \dots$ we have $\mathsf{D}(\langle X_{n_i} ; d_{n_i}\rangle) \le_{\mathrm{A}} \mathsf{D}$, it can be assumed that $\mathtt{lim}_k \, \delta(r, X_k) = 0$, and further that $\delta(r, X_k) < \frac{1}{k}$ for all k. Then every X_k contains a $\frac{1}{k}$-galaxy $Y_k \subseteq X_k$ of $\mathtt{diam}\, Y_k \ge r$. As easily $\mathsf{D}(\langle Y_k ; d_k\rangle) \le_{\mathrm{A}} \mathsf{D}$, the following lemma suffices to prove (iii).

Lemma 72.1. *Suppose that $r > 0$ and each X_k is a single $\frac{1}{k}$-galaxy in itself with $\mathtt{diam}(X_k) \ge r$. Then $\mathsf{D} = \mathsf{D}(\langle X_k ; d_k\rangle)$ is turbulent and $\mathsf{E}_3 \le_{\mathrm{A}} \mathsf{D}$.*

Proof. We know from the proof of (iii) above that $\mathsf{E}_3 \le_{\mathrm{A}} \mathsf{D}$. Now prove that the natural action of the Polish group $\mathbb{G}$ defined as in the proof of Lemma 71 is turbulent under the assumptions of the lemma.

That every D-class is dense in $\mathbb{X} = \prod_k X_k$ (with the product topology on $\mathbb{X}$) is an easy exercise. To see that every D-class $[x]_{\mathsf{D}}$ also is meager in $\mathbb{X}$, note that by the assumptions of the lemma any X_k contains a pair of elements x'_k, x''_k with $d_k(x'_k, x''_k) \ge r$. Let y_k be one of x'_k, x''_k which is d_k-fahrer than $\frac{r}{2}$ from x_k. Now the set $Z = \{z \in \mathbb{X} : \exists^\infty k \ (z_k = y_k)\}$ is comeager in $\mathbb{X}$ and disjoint from $[x]_{\mathsf{D}}$. It remains to prove that local orbits are somewhere dense.

Let G be an open nbhd of the identity in $\mathbb{G}$ and $\emptyset \ne X \subseteq \mathbb{X}$ be open in $\mathbb{X}$. We can assume that, for some n, G is the $\frac{1}{n}$-ball around the identity in $\mathbb{G}$ while $X = \{x \in \mathbb{X} : \forall\, k < n \ (x_k = \xi_k)\}$, where elements $\xi_k \in X_k$, $k < n$, are fixed. It is enough to prove that all classes of the local orbit relation $\sim^G_X$ are dense in X. Consider an open set $Y = \{y \in \mathbb{X} : \forall\, k < m \ (y_k = \xi_k)\} \subseteq X$, where $m > n$ and elements $\xi_k \in X_k$, $n \le k < m$, are fixed in addition to the above.

Let $x \in X$. Then $x_k = \xi_k$ for $k < n$. Let $n \le k < m$. The elements ξ_k and x_k belong to X_k, which is a $\frac{1}{k}$-galaxy, therefore, there is a chain, of a length $\ell(k)$, of elements of X_k, which connects x_k and ξ_k so that every step within the chain has d_k-length $< \frac{1}{k}$. Then there is a permutation g_k of X_k such that $g_k^{\ell(k)}(x_k) = \xi_k$, $g_k(\xi_k) = x_k$, and $d_k(\xi, g_k(\xi)) < \frac{1}{k}$ for all $\xi \in X_k$. Let g_k be the identity on X_k whenever $k < n$ or $k \ge m$. This defines an element $g \in \mathbb{G}$ which obviously belongs to G, moreover, X is g-invariant and $g^\ell(x) \in U$, where $\ell = \prod_{n \le k < m} \ell(k)$, hence, $x \sim^G_X g(x)$, as required. □ (*Lemma*)

□ (*Theorem 72*)

Remark 73. Theorem 72 shows that any nontrivial c_0-equality D $\le_{\mathrm{A}}$-contains a turbulent c_0-equality D' with $\mathsf{E}_3 \le_{\mathrm{A}} \mathsf{D}'$ (and the turbulence of D' holds, in particular, via the natural action defined in the proof of Lemma 71), unless D is $\sim_{\mathrm{A}}$ to E_0 or E_3, and that (co1) is necessary for the turbulence of D itself and sufficient for a turbulent c_0-equality $\mathsf{D}' \le_{\mathrm{A}} \mathsf{D}$ to exist. □

14.c LV-equalities

By Farah, an LV-*equality* is a c_0-equality $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ satisfying

$$\text{(LV1)}\quad \forall\, m\, \forall\, \varepsilon > 0\, \forall^\infty k\, \forall\, x_0, ..., x_m \in X_k\ \big(d_k(x_0, x_m) \le \mathtt{max}_{j<m}\, d_k(x_j, x_{j+1}) + \varepsilon\big).$$

In other words, the metrics involved are postulated to be "asymptotically close" to ultrametrics. This sort of c_0-equalities was first considered by Louveau and Velickovic [31]. The following simple fact is analogous to Lemma 70.

Lemma 74. *For any* LV-*equality* D, *if* D' *is a Borel ER on a set* $\prod_k X'_k$ *(with finite nonempty* X'_k*) and* $\mathsf{D}' \le_{\mathrm{A}} \mathsf{D}$ *then* D' *is an* LV-*equality.* □

Example 75 (Louveau and Velickovic [31])**.** We define $X_k = \{1, 2, ..., 2^k\}$ and $d_k(m, n) = \log(|m - n| + 1)/k$ for $1 \le m, n \le 2^k$. □

Theorem 76 (Essentially, Louveau and Velickovic [31])**.** *Let* $\mathsf{D} = \mathsf{D}(\langle X_k\,; d_k\rangle)$ *be a turbulent* LV-*equality. Then we can associate, with each infinite* $A \subseteq \mathbb{N}$, *a* LV-*equality* $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}$ *such that for all* A, $B \subseteq \mathbb{N}$ *the following are equivalent*:

(i) $A \subseteq^* B$ (*i.e.*, $A \smallsetminus B$ is finite)*;*

(ii) $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}_B$;

(iii) $\mathsf{D}_A \le_{\mathrm{BM}} \mathsf{D}_B$ (*i.e.*, via a Baire measurable reduction).

This theorem was the first major application of c_0-equalities. One of its corollaries is that there exist big families of mutually irreducible Borel ERs !

Proof. As D is turbulent, the necessary turbulence condition (co1) of § 14.b holds, moreover, as in the proof of Theorem72 (case (iii)), we can assume that it takes the following special form for some $r > 0$:

(1) Each X_k is a single $\mathtt{min}\{\frac{r}{2}, \frac{1}{k+1}\}$-galaxy of $\mathtt{diam}(X_k) \ge 4r$.

The intended transformations (reduction to a certain infinite subsequence of spaces $\langle X_k\,; d_k\rangle$, and then each X_k to a suitable galaxy $Y_k \subseteq X_k$) preserve (LV1), of course, moreover, going to subsequences once again, we can assume that (LV1) holds in the following special form:

(2) $d_k(x_0, x_{m_k}) \le \mathtt{max}_{i<m_k}\, d_k(x_i, x_{i+1}) + \frac{1}{k+1}$ whenever $x_0, ..., x_{m_k} \in X_k$, where $m_k = 2^{\prod_{j=0}^{k-1} \#(X_j)}$.

We can derive the following important consequence:

(3) For any k there is a set $Y_k \subseteq X_k$ of $\#(Y_k) = m_k$ such that we have $d_k(x,y) \ge r$ for all $x \ne y$ in Y_k.

To prove this note that by (1) there is a set $\{x_0, ..., x_m\} \subseteq X_k$ such that $d_k(x_0, x_m) \ge 4r$ but $d_k(x_i, x_{i+1}) < r$ for all i. We may assume that m is the least possible length of such a sequence $\{x_i\}$. Now let us define a subsequence $\{y_0, y_1, ..., y_n\}$ of $\{x_i\}$, the number $n \le m$ will be specified in the course of the construction. Put $y_0 = x_0$. If $y_j = x_{i(j)}$ has been defined, and there is $l > i(j)$, $l \le m$, such that $d_k(y_j, x_l) \ge r$, then let $y_{j+1} = x_l$ for the least such l, otherwise put $n = j$ and stop the construction.

By definition $d_k(y_j, y_{j+1}) \ge r$ for all $j < n$, moreover, $d_k(y_{j'}, y_{j+1}) \ge r$ for any $j' < j$ by the minimality of m. Thus $Y_k = \{y_j : j \le n\}$ satisfies $d_k(x,y) \ge r$ for all $x \ne y$ in Y_k. It remains to prove that $n \ge m_k$. Indeed we have $d_k(y_j, y_{j+1}) < 2r$ by the construction, hence, if $n \le m_k$ then we would have $d_k(y_0, y_n) \le 3r$ by (2), which implies $d_k(y_n, x_m) \ge r$, a contradiction to the assumption that the construction stops with y_n,

This said, we proceed to the proof of the theorem. First note that

Lemma 76.1. (iii) *implies that* (ii) *holds at least for some (infinite)* $A' \subseteq A$.

$\longleftarrow$ *Is it true that for a pair of* $\mathbf{c}_0$*-equalities* D, D', *if* $\mathsf{D} \le_{\mathrm{B}} \mathsf{D}'$ *then* $\mathsf{D} \le_{\mathrm{A}} \mathsf{D}'$?⊣

Proof. A Borel reduction can be extracted from a Baire measurable one by a version of the "stabilizers" construction (see proofs of ... .) □ (*Lemma 76.1*)

Thus it remains only to show that (ii) implies (i), even simpler, that, for any disjoint infinite sets $A, B \subseteq \mathbb{N}$, $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}_B$ fails. Suppose, towards the contrary, that $\mathsf{D}_A \le_{\mathrm{A}} \mathsf{D}_B$ holds, and let this be witnessed by a reduction Ψ defined (as in §1.d) from an increasing sequence $\mathtt{min}\, B = n_0 < n_1 < n_2 < ...$ of numbers $n_i \in B$ and a collection of maps $H_k : X_k \to \prod_{j \in [n_i, n_{i+1}) \cap B} X_j$, $k \in A$. Let

$$f_k(\delta) = \max_{\xi, \eta \in X_k,\, d_k(\xi,\eta) < \delta} \; \max_{j \in [n_i, n_{i+1}) \cap B} \; d_j(H_k(\xi)_j, H_k(\eta)_j),$$

for $k \in \mathbb{N}$ and $\delta > 0$ (with the understanding that $\mathtt{max}\, \emptyset = 0$ if applicable). Then $f(\delta) = \mathtt{sup}_{k \in A} f_k(\delta)$ is a nondecreasing map $\mathbb{R}^+ \to [0, \infty)$.

Lemma 76.2. $\mathtt{lim}_{\delta \to 0} f(\delta) = 0$.

Proof. Otherwise there is $\varepsilon > 0$ such that $f(\delta) \ge \varepsilon$ for all δ. Then the numbers

$$\mu_k = \mathtt{min}_{\xi, \eta \in X_k,\, \xi \ne \eta} \, d_k(\xi, \eta) \quad \text{(all of them are } > 0)$$

must satisfy $\mathtt{inf}_{k \in A}\, \mu_k = 0$. This allows us to define a sequence $k_0 < k_1 < k_2 < ...$ of numbers $k_i \in A$, and, for any k_i, a pair of $\xi_i, \eta_i \in X_{k_i}$ with $d_{k_i}(\xi_i, \eta_i) \to 0$, and also $j_i \in [n_{k_i}, n_{k_i+1}) \cap B$ such that $d_{j_i}(H_{k_i}(\xi_i)_{j_i}, H_{k_i}(\eta_i)_{j_i}) \ge \varepsilon$. Let $x, y \in \prod_{k \in A} X_k$ satisfy $x_{k_i} = \xi_i$ and $y_{k_i} = \eta_i$ for all i and $x_k = y_k$ for all $k \in A$ not of the form k_i. Then easily $x \,\mathsf{D}_A\, y$ holds but $\Psi(x) \,\mathsf{D}_B\, \Psi(y)$ fails, which is a contradiction. □ (*Lemma 76.2*)

Let $k \in A$, and let $Y_k \subseteq X_k$ be as in (3). Then there exist elements $x_k \neq y_k$ in Y_k such that $H_k(x_k) \restriction k = H_k(y_k) \restriction k$. By (1) there is a chain $x_k = \xi_0, \xi_1, ..., \xi_n = y_k$ of elements $\xi_i \in X_k$ with $d_k(z_i, z_{i+1}) \leq \frac{1}{k+1}$ for all $i < n$. Now $H_k(\xi_i) \in \prod_{j \in [n_i, n_{i+1}) \cap B} X_j$ for each $i \leq n$. Let $j \in [n_i, n_{i+1}) \cap B$. If $j > k$ then the elements $y_i^j = H_k(\xi_i)_j$, $i \leq n$, satisfy $d_j(y_i^j, y_{i+1}^j) \leq f_k(\frac{1}{k+1})$. As clearly $n < m_j$, we conclude that $d_j(H_k(x_k)_j, H_k(y_k)_j) \leq f_k(\frac{1}{k+1}) + \frac{1}{j+1}$ by (2). If $j < k$ then simply $H_k(x_k)_j = H_k(y_k)_j$ by the choice of x_k, y_k. Thus totally

(4) $d_j(H_k(x_k)_j, H_k(y_k)_j) \leq f(\frac{1}{k+1}) + \frac{1}{k+1}$ for all $j \in [n_i, n_{i+1}) \cap B$.

(as $k \notin B$). Let $x = \{x_k\}_{k \in A}$ and $y = \{y_k\}_{k \in A}$, both are elements of $\prod_{k \in A} X_k$, and $x \mathsf{D}_A y$ fails because $d_k(x_k, y_k) \geq r$ for all k. On the other hand, we have $\Psi(x) \mathsf{D}_B \Psi(y)$ by (4), because $f(\delta) \to 0$ with $\delta \to 0$ by Lemma 76.2. This is a contradiction to the assumption that Ψ reduces D_A to D_B.

□ (*Theorem 76*)

15 T_2 is not reducible to ...

This section contains a theorem saying that the ER T_2 of equality of countable sets of the reals is not Borel reducible to ERs which belong to a family of *pinned* ERs, including, for instance, continuous actions of CLI groups and some ideals, not only Polishable, and is closed under the Fubini product modulo $\mathtt{Fin}$. But the *prima facie* definition of the family is based on a rather metamathematical property which we extracted from Hjorth [14].

Recall that T_2 is defined on $(\mathbb{N}^\mathbb{N})^\mathbb{N}$ as follows: $x \,\mathsf{T}_2\, y$ iff $\mathtt{ran}\, x = \mathtt{ran}\, y$.

Suppose that X is $\mathbf{\Sigma}^1_1$ or $\mathbf{\Pi}^1_1$ in the universe $\mathbb{V}$, and an extension $\mathbb{V}^+$ of $\mathbb{V}$ is considered. In this case, let $X^\#$ denote what results by the definition of X applied in $\mathbb{V}^+$. There is no ambiguity here by Shoenfield, and easily $X = X^\# \cap \mathbb{V}$.

15.a Pinned ERs do not reduce T_2

Fix a Polish space $\mathbb{X}$ and let $\{B_n\}_{n\in\mathbb{N}}$ be a base of its topology. By a *Borel code* for $\mathbb{X}$ we shall understand a pair $p = \langle T, f\rangle$ of a wellfounded tree $\emptyset \neq T = T_p \subseteq \mathtt{Ord}^{<\omega}$ (then $\Lambda \in T$) and a map $f : \mathtt{Max}\, T \to \mathbb{N}$, where $\mathtt{Max}\, T$ is the set of all $\subseteq$-maximal elements of T. We define $\mathsf{B}_p(t) \subseteq \mathbb{N}^\mathbb{N}$ for any $t \in T$ by induction on the rank of t in T, so that

- $\mathsf{B}_p(t) = \mathsf{B}_{f(t)}$ for all $t \in \mathtt{Max}\, T$, and
- $\mathsf{B}_p(t) = \complement \bigcup_{t^\wedge \xi \in T} \mathsf{B}_p(t^\wedge \xi)$ for $t \in T \smallsetminus \mathtt{Max}\, T$;
- finally, put $\mathsf{B}_p = \mathsf{B}_p(\Lambda)$.

For a Borel code $p = \langle T, F\rangle$, let $\mathtt{sup}\, p = \mathtt{sup}\, T$ be the least ordinal γ with $T \subseteq \gamma^{<\omega}$. A code p is *countable* if $\mathtt{sup}\, p < \omega_1$, in this case the coded set B_p is a Borel subset of $\mathbb{X}$.

Definition 77. A $\mathbf{\Sigma}^1_1$ ER E is *pinned* if, for any (perhaps, uncountable) Borel code p, <u>if</u> B_p is 2wise $\mathsf{E}^\#$-equivalent in any generic extension of $\mathbb{V}$ and non-empty in some generic extension of $\mathbb{V}$, <u>then</u> there is a point $x \in \mathtt{dom}\, \mathsf{E}$, "pinning" p in the sense that $\mathsf{B}_p \subseteq [x]_{\mathsf{E}^\#}$ in any extension of $\mathbb{V}$. □

Claim 77.1. T_2 *is <u>not</u> pinned.*

Proof. Consider a Borel code p for the set $\{x \in (\mathbb{N}^\mathbb{N})^\mathbb{N} : \mathtt{ran}\, x = \mathbb{N}^\mathbb{N} \cap \mathbb{V}\}$, so that $\mathsf{B}_p \subseteq (\mathfrak{c}^\mathbb{V})^{<\omega}$. Then <u>if</u> of Definition 77 holds, actually, B_p is a T_2-equivalence class in any universe where it is non-empty, but <u>then</u> fails. □

Lemma 77.2. *If* E, F *are* $\mathbf{\Sigma}^1_1$ *ERs,* $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$*, and* F *is pinned, then so is* E*.*

Proof. Suppose that, in $\mathbb{V}$, $\vartheta : \mathbb{X} \to \mathbb{Y}$ is a Borel reduction of E to F, where $\mathbb{X} = \mathtt{dom}\, \mathsf{E}$ and $\mathbb{Y} = \mathtt{dom}\, \mathsf{F}$. We can assume that $\mathbb{X}$ and $\mathbb{Y}$ are just two copies of

$2^{\mathbb{N}}$. Let r be a (countable) Borel code for ϑ as a subset of $\mathbb{X}\times\mathbb{Y}$. Let p be a Borel code satisfying <u>if</u> of Definition 77. There is perhaps no Borel code q such that $\mathsf{B}_q = \mathsf{B}_r``\mathsf{B}_p$ everywhere, but still there is a code $q$ with $\mathsf{B}_q \subseteq \mathsf{B}_r``\mathsf{B}_p$ and $B_q \neq \emptyset$ somewhere. Indeed, let, in $\mathbb{V}$, $\lambda = \mathtt{card}(\mathtt{sup}\, p)$ and $\kappa = \lambda^+$ (the next cardinal). Consider the formula $A(p,r,y)$ saying:

- $y \in \mathbb{Y}$ and there is a forcing term $\tau \in \mathbf{L}[p,r,y]$ such that the forcing $\text{Coll}(\mathbb{N},\lambda)$ forces $\tau[\underline{G}] \in \mathsf{B}_p$ and $y = \mathsf{B}_r(\tau[\underline{G}])$.

As it is known, there is a Borel code q such that $\overleftarrow{rfrnce\ ?}\dashv$

$$\mathsf{B}_q = \{y : \mathbf{L}_\kappa[p,r,y] \models A(p,r,y)\}$$

in <u>any</u> extension of $\mathbb{V}$. Then easily $\mathsf{B}_q \subseteq \mathsf{B}_r``\mathsf{B}_p$, hence, B_q is 2wise $\mathsf{F}^{\#}$-equivalent in any universe, in addition, B_q is nonempty somewhere.

As F is pinned, there is, in $\mathbb{V}$, a point $y \in \mathbb{Y}$ such that $\mathsf{B}_q \subseteq [y]_{\mathsf{F}^{\#}}$ holds, in particular, in $\text{Coll}(\mathbb{N},\lambda)$-generic extension $\mathbb{V}^+$ of $\mathbb{V}$, where $\mathsf{B}_q \neq \emptyset$, hence, there is $x \in \mathsf{B}_p \cap \mathbb{V}^+$ with $y\,\mathsf{F}^{\#}\,\mathsf{B}_r(x)$. It follows, by Shoenfield, that $y\,\mathsf{F}\,\vartheta(x')$ for some $x' \in \mathbb{X}$ in $\mathbb{V}$. Thus $x\,\mathsf{E}^{\#}\,x'$, which implies that $x' \in \mathbb{V}$ pins p, as required. □

15.b Fubini product of pinned ERs is pinned

Recall that the Fubini product $\mathsf{E} = \prod_{k\in\mathbb{N}} \mathsf{E}_k \,/\, \mathtt{Fin}$ of ERs E_k on $\mathbb{N}^{\mathbb{N}}$ modulo $\mathtt{Fin}$ is a ER on $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ defined as follows: $x\,\mathsf{E}\,y$ if $x(k)\,\mathsf{E}_k\,y(k)$ for all but finite k.

Proposition 78. *The family of all pinned* $\mathbf{\Sigma}^1_1$ *ERs is closed under Fubini products modulo* $\mathtt{Fin}$*.*

Proof. Suppose that ERs E_k on $\mathbb{N}^{\mathbb{N}}$ are pinned; prove that the Fubini product $\mathsf{E} = \prod_{k\in\mathbb{N}} \mathsf{E}_k \,/\, \mathtt{Fin}$ is pinned. Define $x\,\mathsf{F}_k\,y$ iff $x(k)\,\mathsf{E}_k\,y(k)$: F_k are $\mathbf{\Sigma}^1_1$ ERs on $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ and $x\,\mathsf{E}\,y$ iff $x\,\mathsf{F}_k\,y$ for almost all k.

Claim 78.1. *Each* F_k *is pinned.*

Proof. Consider a Borel code p for a subset of $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, satisfying <u>if</u> of Definition 77 w.r.t. F_k. By the same argument as in the proof of Lemma 77.2, there is a Borel code q for a subset of $\mathbb{N}^{\mathbb{N}}$, such that $\mathsf{B}_q \neq \emptyset$ in some extension of $\mathbb{V}$ and $\mathsf{B}_q \subseteq \{x(k) : x \in \mathsf{B}_p\}$ in any extension of $\mathbb{V}$, hence, q satisfies <u>if</u> of Definition 77 w.r.t. E_k. As E_k is pinned, there is $a \in \mathbb{N}^{\mathbb{N}}$ such that $\mathsf{B}_q \subseteq [a]_{\mathsf{E}_k^{\#}}$ in any extension, but then easily $\mathsf{B}_p \subseteq [x]_{\mathsf{F}_k^{\#}}$ in any extension, where $x \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \cap \mathbb{V}$ has only to satisfy $x(k) = a$ for the given k. □ (*Claim*)

In continuation of the proof of the proposition, consider a Borel code p for a subset of $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, satisfying <u>if</u> of Definition 77 w.r.t. E. Our plan is to find another Borel code $\bar p$ with $\mathsf{B}_{\bar p} \subseteq \mathsf{B}_p$ everywhere, which satisfies <u>if</u> of Definition 77 for almost all E_k. This involves a forcing by Borel codes.

Let, in $\mathbb{V}$, $\lambda = \mathtt{sup}\, p$ and $\kappa = \lambda^+$, thus, $\mathtt{sup}\, p < \kappa$. Let $\mathbb{P}$ be the set of all Borel codes $q \in \mathbb{V}$ for subsets of $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ such that $\mathtt{sup}\, q < \kappa$ and $\mathsf{B}_q \neq \emptyset$ in a generic extension of the universe $\mathbb{V}$. $\mathbb{P}$ is considered as a forcing, with $q \preccurlyeq p$ (q is stronger) iff $\mathsf{B}_q \subseteq \mathsf{B}_p$ in all generic extensions of $\mathbb{V}$. It is known that $\mathbb{P}$ forces a point of $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$, so that $\bigcap_{q\in G} \mathsf{B}_q = \{x_G\}$ for any $\mathbb{P}$-generic, over $\mathbb{V}$, set $G \subseteq \mathbb{P}$. Let $\dot{x}$ be the name of the generic element of $(\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$.

By the choice of p, $\langle p,p\rangle$ $\mathbb{P}\times\mathbb{P}$-forces $\dot{x}_{\mathtt{left}}\ \mathsf{E}^{\#}\ \dot{x}_{\mathtt{right}}$, hence, there are codes $q,\, r \in \mathbb{P}$ and a number k_0 such that $\langle q,r\rangle$ $\mathbb{P}\times\mathbb{P}$-forces $\dot{x}_{\mathtt{left}}\ {\mathsf{F}_k}^{\#}\ \dot{x}_{\mathtt{right}}$ for any $k \geq k_0$. By a standard argument, we have $x\ {\mathsf{F}_k}^{\#}\ y$ for all $k \geq k_0$ in any extension of $\mathbb{V}$ for any two $\mathbb{P}$-generic, over $\mathbb{V}$, elements $x,\, y \in \mathsf{B}_q$. We can straightforwardly define in $\mathbb{V}$ a Borel code $\bar p$ (perhaps, not a member of $\mathbb{P}$!) such that, in any extension of $\mathbb{V}$, $\mathsf{B}_{\bar p}$ is the set of all $\mathbb{P}$-generic, over $\mathbb{V}$, elements of B_q. Then $\bar p$ satisfies <u>if</u> of Definition 77 w.r.t. any F_k with $k \geq k_0$. Hence, by the claim, there is, in $\mathbb{V}$, a sequence of points $x_k \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}}$ such that $\mathsf{B}_{\bar p} \subseteq [x_k]_{\mathsf{F}_k^{\#}}$ in any generic extension of $\mathbb{V}$, for any $k \geq k_0$. Define $x \in (\mathbb{N}^{\mathbb{N}})^{\mathbb{N}} \cap \mathbb{V}$ so that $x(k) = x_k(k)$ for any $k \geq k_0$, then, by the definition of F_k, we have $\mathsf{B}_{\bar p} \subseteq [x]_{\mathsf{F}_k^{\#}}$ for all $k \geq k_0$ in any extension of $\mathbb{V}$. Yet $\bigcap_{k\geq k_0} [x]_{\mathsf{F}_k^{\#}} \subseteq [x]_{\mathsf{E}^{\#}}$. $\square$ (*Proposition*)

15.c Complete left-invariant actions produce pinned ERs

Recall that a Polish group $\mathbb{G}$ is *complete left-invariant*, CLI for brevity, if $\mathbb{G}$ admits a compatible left-invariant complete metric. Then easily $\mathbb{G}$ also admits a compatible <u>right</u>-invariant complete metric, which will be practically used.

Theorem 79. (Hjorth [14]) *Suppose that $\mathbb{G}$ is a Polish* CLI *group continuously acting on a Polish space $\mathbb{X}$. Then $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$ is pinned, hence, T_2 is not Borel reducible to $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$.*

Proof. Fix a Borel code $\widehat p$ satisfying <u>if</u> of Definition 77 w.r.t. $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$. Let κ be a cardinal in $\mathbb{V}$ satisfying $\mathtt{sup}\,\widehat p < \kappa$. Define forcing $\mathbb{P}$ as above, thus, $\mathbb{P}$ forces an element of $\mathbb{X}$.

Let ρ be a compatible <u>right</u>-invariant metric on $\mathbb{G}$.

For any $\varepsilon > 0$, let $G_\varepsilon = \{g \in \mathbb{G} : \rho(g, 1_{\mathbb{G}}) < \varepsilon\}$. Say that $q \in \mathbb{P}$ *is of size* $\leq \varepsilon$ if $\langle q,q\rangle$ $(\mathbb{P}\times\mathbb{P})$-forces that there is $g \in {G_\varepsilon}^{\#}$ with $\dot{x}_{\mathtt{left}} = g\cdot\dot{x}_{\mathtt{right}}$. In this case, in any generic extension of the universe, if $\langle x,y\rangle \in \mathsf{B}_q \times \mathsf{B}_q$ is a $(\mathbb{P}\times\mathbb{P})$-generic pair then there is $g \in {G_\varepsilon}^{\#}$ with $y = g\cdot x$.

Lemma 79.1. *If $q \in \mathbb{P}$, $q \preccurlyeq \widehat p$, and $\varepsilon > 0$, then there exists a condition $r \in \mathbb{P}$, $r \preccurlyeq q$, of size $\leq \varepsilon$.*

Proof. Otherwise for any $r \in \mathbb{P}$, $r \preccurlyeq q$, there is a pair of conditions $r',\, r'' \in \mathbb{P}$ stronger than r and such that $\langle r', r''\rangle$ $(\mathbb{P}\times\mathbb{P})$-forces that there is no $g \in {G_\varepsilon}^{\#}$

with $\dot{x}_{\mathtt{left}} = g \cdot \dot{x}_{\mathtt{right}}$. Applying, in a sufficiently generic extension $\mathbb{V}^+$ of $\mathbb{V}$, an ordinary splitting construction, we find a perfect set $X \subseteq \mathsf{B}_q$ such that any pair $\langle x, y \rangle \in X^2$ with $x \neq y$ is $(\mathbb{P} \times \mathbb{P})$-generic, hence, there is no $g \in G_\varepsilon{}^\#$ with $y = g \cdot x$. Fix $x_0 \in X$. As X is a pairwise $\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}$-equivalent set (together with B_q) we can associate, in $\mathbb{V}^+$, with each $x \in X$, an element $g_x \in G^\#$ such that $x = g_x \cdot x_0$, and $g_x \notin G_\varepsilon{}^\#$ by the above. Moreover, we have $g_y g_x^{-1} \cdot x = y$ for all $x,\, y \in X$, hence $g_y g_x^{-1} \notin G_\varepsilon{}^\#$ whenever $x \neq y$, which implies $\rho(g_x, g_y) \geq \varepsilon$ by the right invariance. But this contradicts the separability of G. □ (*Lemma*)

It follows that there is, in $\mathbb{V}$, a sequence of codes $q_n \in \mathbb{P}$ such that $q_0 \preccurlyeq \widehat{p}$, $q_{n+1} \preccurlyeq q_n$, q_n has size $\leq 2^{-n}$, and B_{q_n} has $\mathbb{X}$-diameter $\leq 2^{-n}$ for any n. The only limit point x of the sequence of sets B_{q_n} belongs to $\mathbb{V}$, thus, it remains to show that $\mathsf{B}_{\widehat{p}} \subseteq [x]_{(\mathsf{E}^{\mathbb{X}}_{\mathbb{G}})^\#}$ in any extension $\mathbb{V}^+$ of the universe $\mathbb{V}$.

We can assume that $\mathbb{V}^+$ is rich enough to contain, for any n, an element $x_n \in \mathsf{B}_{q_n}$ such that each pair $\langle x_n, x_{n+1} \rangle$ is $(\mathbb{P} \times \mathbb{P})$-generic (over $\mathbb{V}$). Then $\mathtt{lim}_n\, x_n = x$. Moreover, for any n, both x_n and x_{n+1} belong to B_{q_n}, hence, as q_n has size $\leq 2^{-n-1}$, there is $g_{n+1} \in \mathbb{G}^\#$ with $\rho(1, g) \leq 2^{-n}$ such that $x_{n+1} = g_{n+1} \cdot x_n$. Thus, $x_n = h_n \cdot x_0$, where $h_n = g_n ... g_1$. Note that $\rho(h_n, h_{n-1}) = \rho(g_n, 1_{\mathbb{G}}) \leq 2^{-n+1}$ by the right-invariance of the metric, thus, $\{h_n\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $\mathbb{G}^\#$. Let $h = \mathtt{lim}_{n \to \infty} h_n \in \mathbb{G}^\#$ be its limit. As the action is continuous, we have $x = \mathtt{lim}_n\, x_n = h \cdot x_0$. It follows that $x \mathrel{\mathsf{E}^{\mathbb{X}}_{\mathbb{G}}} x_0$. However $x_0 \in \mathsf{B}_{q_0} \subseteq \mathsf{B}_{\widehat{p}}$, therefore, $\mathsf{B}_{\widehat{p}} \subseteq [x]_{(\mathsf{E}^{\mathbb{X}}_{\mathbb{G}})^\#}$, as required.

□ (*Theorem 79*)

15.d All $\mathbf{F}_\sigma$ ideals are pinned

Let us say that a Borel ideal $\mathscr{I}$ is *pinned* if so is the induced ER $\mathsf{E}_{\mathscr{I}}$. It immediately follows from Theorem 79 that any polishable ideal is pinned. Yet there are pinned ideals among non-polishable ones.

Theorem 80. *Any $\mathbf{F}_\sigma$ ideal $\mathscr{I} \subseteq \mathscr{P}(\mathbb{N})$ is pinned.*

Proof. We have $\mathscr{I} = \bigcup_n F_n$, where all sets $F_n \subseteq \mathscr{P}(\mathbb{N})$ are closed. It can be assumed that $F_n \subseteq F_{n+1}$, moreover, since for any closed $F \subseteq \mathscr{P}(\mathbb{N})$ the set $\Delta F = \{X \,\Delta\, Y : x, y \in F\}$ is also closed (by the compactness of $\mathscr{P}(\mathbb{N})$), it can be assumed that $\Delta F_n \subseteq F_{n+1}$ for all n.

Let $\widehat{p}$ be a Borel code, for a subset of $\mathscr{P}(\mathbb{N})$, satisfying <u>if</u> of Definition 77 w.r.t. the induced ER $\mathsf{E}_{\mathscr{I}}$ on $\mathscr{P}(\mathbb{N})$, thus, $\widehat{p} \in \mathbb{P}$, where $\mathbb{P}$ is a forcing defined as in the proof of Proposition 78 (but now $\mathbb{P}$ forces a subset of $\mathscr{P}(\mathbb{N})$, of course). Obviously there exists a pair of conditions $q,\, r \in \mathbb{P}$ with $q,\, r \leq \widehat{p}$, and a number $\nu \in \mathbb{N}$, such that $\langle q, r \rangle$ forces that $\langle \dot{x}_{\mathtt{left}}, \dot{x}_{\mathtt{right}} \rangle \in F_\nu{}^\#$. Then $\langle q, q \rangle$ forces $\dot{x}_{\mathtt{left}} \,\Delta\, \dot{x}_{\mathtt{right}} \in F_{\nu+1}{}^\#$ because $\Delta F_\nu \subseteq F_{\nu+1}$. It follows that, in $\mathbb{V}$, there is a sequence of numbers $i_0 < i_1 < i_2 < ...$, a sequence $q \succcurlyeq p_0 \succcurlyeq p_1 \succcurlyeq p_2 \succcurlyeq ...$ of codes in $\mathbb{P}$, and, for any n, a set $u_n \subseteq [0, n)$, such that

(1) each p_n $\mathbb{P}$-forces $\dot{x} \cap [0,n) = u_n$;

(2) any $\mathbb{P}$-generic, over $\mathbb{V}$, $x, y \in \mathsf{B}_{p_n}$ satisfy $x \,\Delta\, y \in F_{\nu+1}{}^{\#}$.

Let, in $\mathbb{V}$, $a = \bigcup_n u_n$, then $a \cap [0,n) = u_n$ for all n. Prove that a pins $\mathsf{B}_{\widehat{p}}$, *i.e.*, $\mathsf{B}_{\widehat{p}} \subseteq [a]_{\mathsf{E}_{\mathscr{I}}{}^{\#}}$ in any extension of $\mathbb{V}$.

We can assume that, in the extension, for any n there is a $\mathbb{P}$-generic, over $\mathbb{V}$, element $x_n \in \mathsf{B}_{p_n}$. Then we have, by (2), $x_0 \,\Delta\, x_n \in F_{\nu+1}{}^{\#}$ for any n, thus, $x_0 \,\Delta\, a \in F_{\nu+1}{}^{\#}$ as well, because $\{x_n\} \to a$. We conclude that $x_0 \mathrel{\mathsf{E}_{\mathscr{I}}{}^{\#}} a$, and $\mathsf{B}_{\widehat{p}} \subseteq [a]_{\mathsf{E}_{\mathscr{I}}{}^{\#}}$, as required. □

15.e Another family of pinned ideals

We here present another family of pinned ideals. Suppose that $\{\varphi_i\}_{i\in\mathbb{N}}$ is a sequence of lower semicontinuous (l. s. c.) submeasures on $\mathbb{N}$. Define

$$\mathtt{Exh}_{\{\varphi_i\}} = \{X \subseteq \mathbb{N} : \varphi_\infty(X) = 0\}, \quad \text{where} \quad \varphi_\infty(X) = \limsup_{i\to\infty} \varphi_i(X).$$

the exhaustive ideal of the sequence of submeasures. By Solecki's Theorem 41 for any Borel P-ideal $\mathscr{I}$ there is a single l. s. c. submeasure φ such that $\mathscr{I} = \mathtt{Exh}_{\{\varphi_i\}} = \mathtt{Exh}_\varphi$, where $\varphi_i(x) = \varphi(x \cap [i,\infty))$, however, for example, the non-polishable ideal $\mathscr{I}_1 = \mathtt{Fin} \times 0$ also is of the form $\mathtt{Exh}_{\{\varphi_i\}}$, where for $x \subseteq \mathbb{N}^2$ we define $\varphi_i(x) = 0$ or 1 if resp. $x \subseteq$ or $\not\subseteq \{0,...,n-1\} \times \mathbb{N}$.

Theorem 81. *Any ideal of the form* $\mathtt{Exh}_{\{\varphi_i\}}$ *is pinned.*

Proof. Thus let $\mathscr{I} = \mathtt{Exh}_{\{\varphi_i\}}$, all φ_i being l. s. c. submeasures on $\mathbb{N}$. We can assume that the submeasures φ_i decrease, *i.e.*, $\varphi_{i+1}(x) \le \varphi_i(x)$ for any x, for if not consider the l. s. c. submeasures $\varphi'_i(x) = \sup_{j\ge i} \varphi_j(x)$. Let $\widehat{p}$ be a Borel code, for a subset of $\mathscr{P}(\mathbb{N})$, satisfying if of Definition 77 w. r. t. the induced ER $\mathsf{E}_{\mathscr{I}}$ on $\mathscr{P}(\mathbb{N})$, thus, $\widehat{p} \in \mathbb{P}$, where $\mathbb{P}$ is a forcing defined as in the proof of Proposition 78 ($\mathbb{P}$ forces a subset of $\mathscr{P}(\mathbb{N})$).

Using the same arguments as above, we see that for any $p \in \mathbb{P}$, $p \preccurlyeq \widehat{p}$, and $n \in \mathbb{N}$, there are $i \ge n$ and codes $q, r \in \mathbb{P}$ with $q, r \preccurlyeq p$, such that $\langle q, r\rangle$ $\mathbb{P} \times \mathbb{P}$-forces that $\varphi_i(\dot{x}_{\mathtt{left}} \,\Delta\, \dot{x}_{\mathtt{right}}) \le 2^{-n-1}$, hence, any two $\mathbb{P}$-generic, over $\mathbb{V}$, elements $x, y \in \mathsf{B}_q$ satisfy $\varphi_i(x \,\Delta\, y) \le 2^{-n}$. It follows that, in $\mathbb{V}$, there is a sequence of numbers $i_0 < i_1 < i_2 < ...$, a sequence $\widehat{p} \succcurlyeq p_0 \succcurlyeq p_1 \succcurlyeq p_2 \succcurlyeq ...$ of codes in $\mathbb{P}$, and, for any n, a set $u_n \subseteq [0,n)$, such that

(1) each p_n $\mathbb{P}$-forces $\dot{x} \cap [0,n) = u_n$;

(2) any $\mathbb{P}$-generic, over $\mathbb{V}$, $x, y \in \mathsf{B}_{p_n}$ satisfy $\varphi_{i_n}(x \,\Delta\, y) \le 2^{-n}$.

Let, in $\mathbb{V}$, $a = \bigcup_n u_n$, then $a \cap [0,n) = u_n$ for all n. Prove that a pins $\mathsf{B}_{\widehat{p}}$, *i.e.*, $\mathsf{B}_{\widehat{p}} \subseteq [a]_{\mathsf{E}^{\#}_{\mathscr{I}}}$ in any extension of $\mathbb{V}$.

We can assume that, in the extension, for any n there is a $\mathbb{P}$-generic, over $\mathbb{V}$, element $x_n \in \mathsf{B}_{p_n}$. Then we have, by (2), $\varphi_{i_n}(x_n \,\Delta\, x_m) \le 2^{-n}$ whenever $n \le m$. It follows that $\varphi_{i_n}(x_n \,\Delta\, a) \le 2^{-n}$, because $a = \mathtt{lim}_m\, x_m$ by (1). However we assume that the submeasures φ_j decrease, hence, $\varphi_\infty(x_n \,\Delta\, a) \le 2^{-n}$. On the other hand, $\varphi_\infty(x_n \,\Delta\, x_0) = 0$ because all elements of B_{p_0} are pairwise $\mathsf{E}^{\#}_{\mathscr{I}}$-equivalent. We conclude that $\varphi_\infty(x_0 \,\Delta\, a) \le 2^{-n}$ for any n, in other words, $\varphi_\infty(x_0 \,\Delta\, a) = 0$, $x_0 \mathrel{\mathsf{E}^{\#}_{\mathscr{I}}} a$, and $\mathsf{B}_{\widehat{p}} \subseteq [a]_{\mathsf{E}^{\#}_{\mathscr{I}}}$, as required. □

Question 3. Are all Borel ideals pinned ? The expected answer "yes" would show that T_2 is not Borel reducible to any Borel ideal. Moreover, is any orbit ER of a Borel action of a Borel abelian group pinned ? But even this would not fully cover Hjorth's Theorem 79. □

Question 4 (Kechris)**.** If Question 3 answers in the positive, is it true that T_2 is the $\le_{\mathrm{B}}$-least non-pinned Borel ER ? □

[47]

16 Universal analytic ERs and reduction to ideals

A Technical introduction

A.a Notation

- $\mathbb{N} = \{0, 1, 2, ...\}$: natural numbers. $\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$.
- $\mathbb{N}^{\mathbb{N}}$ is *the Baire space*. If $s \in \mathbb{N}^{<\omega}$ (a finite sequence of natural numbers) then $\mathscr{O}_s(\mathbb{N}^{\mathbb{N}}) = \{x \in \mathbb{N}^{\mathbb{N}} : s \subset x\}$, a basic clopen nbhd in $\mathbb{N}^{\mathbb{N}}$.
- $X \subseteq^* Y$ means that the difference $X \smallsetminus Y$ is finite.
- If a basic set A is fixed then $\complement X = X^{\complement} = A \smallsetminus X$ for any $X \subseteq A$.
- If $X \subseteq A \times B$ and $a \in A$ then $(X)_a = \{b : \langle a, b\rangle \in X\}$, a *cross-section*.
- $\#X = \#(X)$ is the number of elements of a finite set X.
- $f"X = \{f(x) : x \in X \cap \mathtt{dom}\, f\}$, the *$f$-image* of X.
- Δ is the symmetric difference.
- $\exists^\infty x$ ... means: "there exist infinitely many x such that ...",
  $\forall^\infty x$ ... means: "for all but finitely many x, ... holds".

- *An ideal on a set* A is, as usual, any set $\emptyset \neq \mathscr{I} \subseteq \mathscr{P}(A)$, closed under $\cup$ and satisfying $x \in \mathscr{I} \Longrightarrow y \in \mathscr{I}$ whenever $y \subseteq x \subseteq A$. Thus, any ideal contains $\emptyset$. We'll usually consider only *nontrivial* ideals, *i.e.*, those which contain all singletons $\{a\} \subseteq A$ and do not contain A, *i.e.*, $\mathscr{P}_{\mathtt{fin}}(A) \subseteq \mathscr{I} \subsetneqq \mathscr{P}(A)$.

- If $\mathscr{I}$ is an ideal on a set A then let $\mathsf{E}_{\mathscr{I}}$ be an equivalence relation (ER, for brevity) on $\mathscr{P}(A)$, defined as follows: $X \mathrel{\mathsf{E}_{\mathscr{I}}} Y$ iff $X \mathbin{\Delta} Y \in \mathscr{I}$.

- If E is an ER on a set X then $[y]_{\mathsf{E}} = \{x \in X : y \mathrel{\mathsf{E}} x\}$ for any $y \in X$ (the E-*class* of x) and $[Y]_{\mathsf{E}} = \bigcup_{y \in Y} [y]_{\mathsf{E}}$ (the E-*saturation* of Y) for $Y \subseteq X$. A set $Y \subseteq X$ is E-*invariant* if $[Y]_{\mathsf{E}} = Y$.

- If E is an ER on a set X then a set $Y \subseteq X$ is *pairwise* E-*equivalent,* resp., *pairwise* E-*inequivalent*, if $x \mathrel{\mathsf{E}} y$, resp., $x \not\mathrel{\mathsf{E}} y$ holds for all $x \neq y$ in Y.

- If X, Y are sets and E any binary relation then $X \mathrel{\mathsf{E}} Y$ means that we have both $\forall\, x \in X\, \exists\, y \in Y\, (x \mathrel{\mathsf{E}} y)$ and $\forall\, y \in Y\, \exists\, x \in X\, (x \mathrel{\mathsf{E}} y)$.

A.b Descriptive set theory

A basic knowledge of Borel and projective hierarchy, both classical and effective, in *the Baire space* $\mathbb{N}^{\mathbb{N}}$ and other (recursively presented, in the effective case) Polish spaces, is assumed.

A map f (between Borel sets in Polish spaces) is *Borel* iff its graph is a Borel set iff all f-preimages of open sets are Borel. A map f is *Baire measurable* (*BM*, for brevity) iff all f-preimages of open sets are Baire measurable sets.

A.c Trivia of "effective" descriptive set theory

Apart of the very common knowledge, the whole instrumentarium of "effective" descriptive set theory employed in the study of reducibility of ideals and ERs, can be summarized in a rather short list of key "principles". In those below, by a *recursively presented* Polish space one can understand any product space of the form $\mathbb{N}^m \times (\mathbb{N}^{\mathbb{N}})^n$ without any harm for applications below, yet in fact this notion is much wider.

Remark 82. For the sake of brevity, the results below are formulated only for the "lightface" parameter-free classes Σ^1_1, Π^1_1, Δ^1_1, but they remain true for $\Sigma^1_1(p)$, $\Pi^1_1(p)$, $\Delta^1_1(p)$ for any fixed real parameter p. □

Reduction and Separation: If X, Y are Π^1_1 sets of a recursively presented Polish space then there disjoint Π^1_1 sets $X' \subseteq X$ and $Y' \subseteq Y$ with $X' \cup Y' = X \cup Y$. The sets $\overline{X', Y'}$ are said to *reduce* the pair X, Y.

If X, Y are disjoint $\varSigma^1_1$ sets of a recursively presented Polish space then there is a $\varDelta^1_1$ set Z with $X \subseteq Z$ and $Y \cap Z = \emptyset$. The set Z is said to *separate* the X from Y.

Countable-to-1 Projection: If P is a $\varDelta^1_1$ subset of the product $\mathbb{X} \times \mathbb{Y}$ of two recursively presented Polish spaces and for any $x \in \mathbb{X}$ the cross-section $P_x = \{y : P(x,y)\}$ is at most countable then $\mathtt{dom}\, P$ is a $\varDelta^1_1$ set in $\mathbb{X}$.

It follows that images of $\varDelta^1_1$ sets via countable-to-1, in particular, 1-to-1 $\varDelta^1_1$ maps are $\varDelta^1_1$ sets, while images via arbitrary $\varDelta^1_1$ maps are, generally, $\varSigma^1_1$.

Countable-to-1 Enumeration: If P, $\mathbb{X}$, $\mathbb{Y}$ are as in Countable-to-1 Projection then there is a $\varDelta^1_1$ map $f : \mathtt{dom}\, P \times \mathbb{N} \to \mathbb{Y}$ such that $P_x = \{f(x,n) : n \in \mathbb{N}\}$ for all $x \in \mathtt{dom}\, P$.

Countable-to-1 Uniformization: If P, $\mathbb{X}$, $\mathbb{Y}$ are as in Countable-to-1 Projection then P can be uniformized by a $\varDelta^1_1$ set.

Kreisel Selection: If $\mathbb{X}$ is a recursively presented Polish space, $P \subseteq \mathbb{X} \times \mathbb{N}$ is a $\varPi^1_1$ set, and $X \subseteq \mathtt{dom}\, P$ is a $\varDelta^1_1$ set then there is a $\varDelta^1_1$ function $f : X \to \mathbb{N}$ such that $\langle x, f(x) \rangle \in P$ for al $x \in X$.

The proof is surprisingly simple. Let $Q \subseteq P$ be a $\varPi^1_1$ set which uniformizes P. For any $x \in X$ let $f(x)$ be the only n with $\langle x, n \rangle \in Q$. Immediately, (the graph of) f is $\varPi^1_1$, however, as $\mathtt{ran}\, f \subseteq \mathbb{N}$, we have $f(x) = n \iff \forall\, m \neq n\, (f(x) \neq m)$ whenever $x \in X$, which demonstrates that f is $\varSigma^1_1$ as well.

$\varDelta^1_1$ Enumeration: If $\mathbb{X}$ is a recursively presented Polish space then there exist $\varPi^1_1$ sets $C \subseteq \mathbb{N}$ and $W \subseteq \mathbb{N} \times \mathbb{X}$ and a $\varSigma^1_1$ set $W' \subseteq \mathbb{N} \times \mathbb{X}$ such that $W_e = W'_e$ for any $e \in C$ and a set $X \subseteq \mathbb{X}$ is $\varDelta^1_1$ iff there is $e \in C$ such that $X = W_e = W'_e$. (Here $W_e = \{x : W(e,x)\}$ and similarly W'_e.)

There is a generalization useful for relativised classes $\varDelta^1_1(y)$.

Relativized $\varDelta^1_1$ Enumeration: If $\mathbb{X}$, $\mathbb{Y}$ are recursively presented Polish spaces then there exist $\varPi^1_1$ sets $C \subseteq \mathbb{Y} \times \mathbb{N}$ and $W \subseteq \mathbb{Y} \times \mathbb{N} \times \mathbb{X}$ and a $\varSigma^1_1$ set $W' \subseteq \mathbb{Y} \times \mathbb{N} \times \mathbb{X}$ such that $W_{ye} = W'_{ye}$ for any $\langle y, e \rangle \in C$ and, for any $y \in \mathbb{Y}$, a set $X \subseteq \mathbb{X}$ is $\varDelta^1_1(y)$ iff there is e such that $\langle y, e \rangle \in C$ and $X = W_{ye} = W'_{ye}$. ($W_{ye} = \{x : W(y,e,x)\}$ and similarly W'_{ye}.)

Suppose that $\mathbb{X}$ is a recursively presented Polish space. A set $U \subseteq \mathbb{N} \times \mathbb{X}$, is a *a universal* $\varPi^1_1$ *set* if for any $\varPi^1_1$ set $X \subseteq \mathbb{X}$ there is an index n with $X = U_n = \{x : \langle n, x \rangle \in U\}$, and a *a "good" universal* $\varPi^1_1$ *set* if in addition for any other $\varPi^1_1$ set $V \subseteq \mathbb{N} \times \mathbb{X}$ there is a recursive function $f : \mathbb{N} \to \mathbb{N}$ such that $V_n = U_{f(n)}$ for all n.

The notions of universal and "good" universal $\varSigma^1_1$ sets are similar.

Universal Sets: For any recursively presented Polish space $\mathbb{X}$ there exist a "good" universal Π^1_1 set $U \subseteq \mathbb{N} \times \mathbb{X}$ and a "good" universal Σ^1_1 set $V \subseteq \mathbb{N} \times \mathbb{X}$. (In fact we can take $V = (\mathbb{N} \times \mathbb{X}) \smallsetminus U$.)

If a "good" universal Π^1_1 set U is fixed then a collection $\mathscr{A}$ of Π^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 *in the codes* if $\{n : U_n \in \mathscr{A}\}$ is a Π^1_1 set. Similarly, if a "good" universal Σ^1_1 set V is fixed then a collection $\mathscr{A}$ of Σ^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 *in the codes* if $\{n : V_n \in \mathscr{A}\}$ is a Π^1_1 set. These notions quite obviously do not depend on the choice of "good" universal sets.

To show how "good" universal sets work, we prove:

Proposition 83. *Let $\mathbb{X}$ be a recursively presented Polish space and $U \subseteq \mathbb{N} \times \mathbb{X}$ a "good" universal Π^1_1 set. Then for any pair of Π^1_1 sets $V, W \subseteq \mathbb{N} \times \mathbb{X}$ there are recursive functions $f, g : \mathbb{N} \to \mathbb{N}$ such that for any $m, n \in \mathbb{N}$ the pair of cross-sections $U_{f(m,n)}$, $U_{g(m,n)}$ reduces the pair V_m, W_n.*

Proof. Consider the following Π^1_1 sets in $(\mathbb{N} \times \mathbb{N}) \times \mathbb{X}$:

$$P = \{\langle m, n, x\rangle : \langle m, x\rangle \in V \wedge n \in \mathbb{N}\}, \;\; Q = \{\langle m, n, x\rangle : \langle n, x\rangle \in W \wedge m \in \mathbb{N}\}.$$

By Reduction, there is a pair of Π^1_1 sets $P' \subseteq P$ and $Q' \subseteq Q$ which reduce the given pair P, Q. Accordingly, the pair P'_{mn}, Q'_{mn} reduces P_{mn}, Q_{mn} for any m, n. Finally, by the "good" universality there are recursive functions f, g such that $P'_{mn} = U_{f(m,n)}$ and $Q'_{mn} = U_{g(m,n)}$ for all m, n. $\square$

The following principle is less elementary than the results cited above, but it is very useful because it allows to "compress" some sophisticated arguments with multiple applications of Separation and Kreisel selection.

Reflection: Assume that $\mathbb{X}$ is a recursively presented Polish space.

Π^1_1 *form*: Suppose that a collection $\mathscr{A}$ of Π^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 in the codes. (In the sense of a fixed "good" universal Π^1_1 set $U \subseteq \mathbb{N} \times \mathbb{X}$.) Then for any $X \in \mathscr{A}$ there is a Δ^1_1 set $Y \in \mathscr{A}$ with $Y \subseteq X$.

Σ^1_1 *form*: Suppose that a collection $\mathscr{A}$ of Π^1_1 sets $X \subseteq \mathbb{X}$ is Π^1_1 in the codes. Then for any $X \in \mathscr{A}$ there is a Δ^1_1 set $Y \in \mathscr{A}$ with $X \subseteq Y$.

One of (generally, irrelevant here) consequences of this principle is that the set of all codes of a properly Π^1_1 set or properly Σ^1_1 set is never Π^1_1.

A.d Polish–like families and the Gandy – Harrington topology

The following notion is similar to the Choquet property but somewhat more convenient to provide the nonemptiness of countable intersections of pointsets.

Definition 84. A family $\mathscr{F}$ is *Polish–like* if there exists a countable collection $\{\mathscr{D}_n : n \in \mathbb{N}\}$ of dense subsets $\mathscr{D}_n \subseteq \mathscr{F}$ such that we have $\bigcap_n F_n \neq \emptyset$ whenever $F_0 \supseteq F_1 \supseteq F_2 \supseteq ...$ is a decreasing sequence of sets $F_n \in \mathscr{F}$ which intersects every $\mathscr{D}_n$. (Here, a set $\mathscr{D} \subseteq \mathscr{F}$ is *dense* if $\forall F \in \mathscr{F}\, \exists D \in \mathscr{D}\, (D \subseteq F)$.) □

For instance if $\mathscr{X}$ is a Polish space then the collection of all its non-empty closed sets is Polish–like, for take $\mathscr{D}_n$ to be all closed sets of diameter $\le n^{-1}$.

Theorem 85 (Kanovei [22], Hjorth [13])**.** *The collection $\mathscr{F}$ of all non-empty $\varSigma^1_1$ subsets of $\mathbb{N}^{\mathbb{N}}$ is Polish–like.* □

Proof. For any $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ define $\mathtt{pr}\, P = \{x : \exists y\, P(x,y)\}$ (the projection). If $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ and $s,\, t \in \mathbb{N}^{<\omega}$ then let $P_{st} = \{\langle x, y\rangle \in P : s \subset x \wedge t \subset y\}$. Let $\mathscr{D}(P,s,t)$ be the collection of all $\varSigma^1_1$ sets $\emptyset \neq X \subseteq \mathbb{N}^{\mathbb{N}}$ such that either $X \cap \mathtt{pr}\, P_{st} = \emptyset$ or $X \subseteq \mathtt{pr}\, P_{s^\wedge i,\, t^\wedge j}$ for some i, j. (Note that in the "or" case i is unique but j may be not unique.) Let $\{\mathscr{D}_n : n \in \mathbb{N}\}$ be an arbitrary enumeration of all sets of the form $\mathscr{D}(P,s,t)$, where $P \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ is $\varPi^0_1$. Note that in this case all sets of the form $\mathtt{pr}\, P_{st}$ are $\varSigma^1_1$ subsets of $\mathbb{N}^{\mathbb{N}}$, therefore, $\mathscr{D}(P,s,t)$ is easily a dense subset of $\mathscr{F}$, so that all $\mathscr{D}_n \subseteq \mathscr{F}$ are dense.

Now consider a decreasing sequence $X_0 \supseteq X_1 \supseteq ...$ of non-empty $\varSigma^1_1$ sets $X_k \subseteq \mathbb{N}^{\mathbb{N}}$, which intersects every $\mathscr{D}_n$; prove that $\bigcap_n X_n \neq \emptyset$. Call a set $X \subseteq \mathbb{N}^{\mathbb{N}}$ *positive* if there is n such that $X_n \subseteq X$. For any n, fix a $\varPi^0_1$ set $P^n \subseteq \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ such that $X_n = \mathtt{pr}\, P^n$. For any $s,\, t \in \mathbb{N}^{<\omega}$, if $\mathtt{pr}\, P^n_{st}$ is positive then, by the choice of the sequence of X_n, there is a unique i and some j such that $\mathtt{pr}\, P^n_{s^\wedge i,\, t^\wedge j}$ is also positive. It follows that there is a unique $x = x_n \in \mathbb{N}^{\mathbb{N}}$ and some $y = y_n \in \mathbb{N}^{\mathbb{N}}$ (perhaps not unique) such that $\mathtt{pr}\, P^n_{x\restriction k,\, y\restriction k}$ is positive for any k. As P^n is closed, we have $P^n(x,y)$, hence, $x_n = x \in X_n$.

It remains to show that $x_m = x_n$ for $m \neq n$. To see this note that if both P_{st} and $Q_{s't'}$ are positive then either $s \subseteq s'$ or $s' \subseteq s$. □

The collection of all non-empty $\varSigma^1_1$ subsets of $\mathbb{N}^{\mathbb{N}}$ is a base of the *Gandy – Harrington topology*, which has many remarkable applications in descriptive set theory. This topology is easily not Polish, even not metrizable at all, yet it shares the following important property of Polish topologies:

Corollary 86. *The Gandy – Harrington topology is Baire, i.e., every comeager set is dense.*

Proof. This can be proved using Choquet property of the topology, see [12], however, the Polish–likeness (Theorem 85) also immediately yields the result. □

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