Linear ROD subsets of Borel partial orders are countably cofinal in Solovay's model

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Abstract

The following is true in the Solovay model.

1. If $\langle D\,;\le\rangle$ is a Borel partial order on a set D of the reals, $X\subseteq D$ is a ROD set, and $\le\restriction X$ is linear, then $\le\restriction X$ is countably cofinal.

2. If in addition every countable set $Y\subseteq D$ has a strict upper bound in $\langle D\,;\le\rangle$ then the ordering $\langle D\,;\le\rangle$ has no maximal chains that are ROD sets.

Linear orders, which typically appear in conventional mathematics, are countably cofinal. In fact *any* Borel (as a set of pairs) linear order on a subset of a Polish space is countably cofinal: see, *e.g.*, [1]. On the other hand, there is an uncountably-cofinal quasi-order of class $\boldsymbol{\Sigma}^1_1$ on $\mathbb{N}^{\mathbb{N}}$.

Example 1. Fix any recursive enumeration $\mathbb{Q}=\{q_k:k\in\mathbb{N}\}$ of the rationals. For any ordinal $\xi<\omega_1$, let X_ξ be the set of all points $x\in\mathbb{N}^{\mathbb{N}}$ such that the maximal well-ordered (in the sense of the usual order of the rationals) initial segment of the set $Q_x=\{q_k:x(k)=0\}$ has the order type ξ. Thus $\mathbb{N}^{\mathbb{N}}=\bigcup_{\xi<\omega_1}X_\xi$. For $x,y\in\mathbb{N}^{\mathbb{N}}$ define $x\preccurlyeq y$ iff $x\in X_\xi$, $y\in X_\eta$, and $\xi\le\eta$. Thus $\preccurlyeq$ is a prewellordering of length exactly ω_1. It is a routine exercise to check that $\preccurlyeq$ belongs to $\boldsymbol{\Sigma}^1_1$. ←ex

We can even slightly change the definition of $\preccurlyeq$ to obtain a true linear order. Define $x\preccurlyeq' y$ iff either $x\in X_\xi$, $y\in X_\eta$, and $\xi<\eta$, or $x,y\in X_\xi$ for one and the same ξ and $x<y$ in the sense of the lexicographical linear order on $\mathbb{N}^{\mathbb{N}}$. Clearly $\preccurlyeq'$ is a linear order of cofinality ω_1 and class $\boldsymbol{\Sigma}^1_1$. □

Yet there is a rather representative class of **ROD** (that is, real-ordinal definable) linear orderings which are consistently countably cofinal. This is the subject of the next theorem.

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Theorem 2. *The following sentence is true in the Solovay model: if $\le$ is a Borel partial quasi-order on a (Borel) set $D \subseteq \mathbb{N}^{\mathbb{N}}$, $X \subseteq D$ is a* **ROD** *set, and $\le \restriction X$ is a linear quasi-order, then $\le \restriction X$ is countably cofinal.* $\longleftarrow$ m

A *partial quasi-order*, PQO for brevity, is a binary relation $\le$ satisfying $x \le y \wedge y \le z \Longrightarrow x \le z$ and $x \le x$ on its domain. In this case, an *associated equivalence relation* $\equiv$ and an *associated strict partial order* $<$ are defined so that $x \equiv y$ iff $x \le y \wedge y \le x$, and $x < y$ iff $x \le y \wedge y \not\le x$. A PQO is *linear*, LQO for brevity, if we have $x \le y \vee y \le x$ for all x, y in its domain.

A PQO $\langle X\,; \le\rangle$ (meaning: X is the domain of $\le$) is *Borel* iff the set X is a Borel set in a suitable Polish space $\mathbb{X}$, and the relation $\le$ (as a set of pairs) is a Borel subset of $\mathbb{X} \times \mathbb{X}$.

Thus it is consistent with **ZFC** that **ROD** linear suborders of Borel PQOs are necessarily countably cofinal. Accordingly it is consistent with **ZF** + **DC** that any linear suborders of Borel PQOs are countably cofinal.

By *the Solovay model* we understand a model of **ZFC** in which all **ROD** sets of reals have some basic regularity properties, for instance, are Lebesgue measurable, have the Baire property, see [6]. We'll make use of the following two results related to the Solovay model.

Proposition 3 (Stern [7]). *It holds in the Solovay model that if $\rho < \omega_1$ then there is no* **ROD** *ω_1-sequence of pairwise different sets in $\mathbf{\Sigma}^0_\rho$.* □ $\longleftarrow$ p2

Proposition 4. *It holds in the Solovay model that if $\le$ is a* **ROD** *LQO on a set $D \subseteq \mathbb{N}^{\mathbb{N}}$ then there exist a* **ROD** *antichain $A \subseteq 2^{<\omega_1}$ and a* **ROD** *map $\vartheta : D \longrightarrow A$ such that $x \le y \Longleftrightarrow \vartheta(x) \le_{\texttt{lex}} \vartheta(y)$ for all $x, y \in D$.* □ $\longleftarrow$ p3

A few words on the notation. The set $2^{<\omega_1} = \bigcup_{\xi<\omega_1} 2^\xi$ consists of all transfinite binary sequences of length $< \omega_1$, and if $\xi < \omega_1$ then 2^ξ is the set of all binary sequences of length exactly ξ. A set $A \subseteq 2^{<\omega_1}$ is an *antichain* if we have $s \not\subset t$ for any $s, t \in A$, where $s \subset t$ means that t is a proper extension of s. By $\le_{\texttt{lex}}$ we denote the lexicographical order on $2^{<\omega_1}$, that is, if $s, t \in 2^{<\omega_1}$ then $s \le_{\texttt{lex}} t$ iff either 1) $s = t$ or 2) $s \not\subset t$, $t \not\subset s$, and the least ordinal $\xi < \texttt{dom}\, s, \texttt{dom}\, t$ such that $s(\xi) \ne t(\xi)$ satisfies $s(\xi) < t(\xi)$. Obviously $\le_{\texttt{lex}}$ linearly orders any antichain $A \subseteq 2^{<\omega_1}$.

Proposition 4 follows from Theorem 6 in [5] saying that if, in the Solovay model, $\le$ is a **ROD** PQO on a set $D \subseteq \mathbb{N}^{\mathbb{N}}$ then:

either a condition (I^s) holds, which for LQO relations $\le$ is equivalent to the existence of A and ϑ as in Proposition 4,

or a condition (II) holds, which is incompatible with $\le$ being a LQO.

Thus we obtain Proposition 4 as an immediate corollary.

The next simple fact will be used below.

Lemma 5. *If $\xi < \omega_1$ then any set $C \subseteq 2^\xi$ is countably $\le_{\mathtt{lex}}$-cofinal, that is, there is a set $C' \subseteq C$, at most countable and $\le_{\mathtt{lex}}$-cofinal in C.* $\square$ $\longleftarrow$ cc

Proof (Theorem 2)**.** *We argue in the Solovay model.* Suppose that $\le$ is a Borel PQO on a (Borel) set $D \subseteq \mathbb{N}^\mathbb{N}$, $X \subseteq D$ is a **ROD** set, and $\le \restriction X$ is a LQO. Our goal will be to show that $\le \restriction X$ is countably cofinal, that is, there is a set $Y \subseteq X$, at most countable and $\le$-cofinal in X.

The restricted order $\le\restriction X$ is **ROD**, of course, and hence, by Proposition 4, there is a **ROD** map $\vartheta : X \longrightarrow A$ onto an antichain $A \subseteq 2^{<\omega_1}$ (also obviously a **ROD** set) such that $x \le y \Longleftrightarrow \vartheta(x) \le_{\mathtt{lex}} \vartheta(y)$ for all $x, y \in X$.

If $\xi < \omega_1$ then let $A_\xi = A \cap 2^\xi$ and $X_\xi = \{x \in D : \vartheta(x) \in A_\xi\}$.

Case 1: there is an ordinal $\xi_0 < \omega_1$ such that A_{ξ_0} is $\le_{\mathtt{lex}}$-cofinal in A. However, by Lemma 5, there is a set $A' \subseteq A_{\xi_0}$, at most countable and $\le_{\mathtt{lex}}$-cofinal in A_{ξ_0}, and hence $\le_{\mathtt{lex}}$-cofinal in A as well by the choice of ξ_0. If $s \in A'$ then pick an element $x_s \in X$ such that $\vartheta(x_s) = s$. Then the set $Y = \{x_s : s \in A'\}$ is a countable subset of X, $\le$-cofinal in X, as required.

Case 2: not Case 1. That is, for any $\eta < \omega_1$ there is an ordinal $\xi < \omega_1$ and an element $s \in A_\xi$ such that $\eta < \xi$ and $t <_{\mathtt{lex}} s$ for all $t \in A_\eta$. Then the sequence of sets

$$D_\xi = \{z \in D : \exists\, x \in X\, (z \le x \wedge \vartheta(x) \in A_\xi)\}$$

is **ROD** and has uncountably many pairwise different terms.

We are going to get a contradiction. Recall that $\le$ is a Borel relation, hence it belongs to $\mathbf{\Sigma}^0_\rho$ for an ordinal $1 \le \rho < \omega_1$. Now the goal is to prove that all sets D_ξ belong to $\mathbf{\Sigma}^0_\rho$ as well — this contradicts to Proposition 3, and the contradiction accomplishes the proof of the theorem.

Consider an arbitrary ordinal $\xi < \omega_1$. By Lemma 5 there exists a countable set $A' = \{s_n : n < \omega\} \subseteq A_\xi$, $\le_{\mathtt{lex}}$-cofinal in A_ξ. If $n < \omega$ then pick an element $x_n \in X$ such that $\vartheta(x_n) = s_n$. Note that by the choice of ϑ any other element $x \in X$ with $\vartheta(x) = s_n$ satisfies $x \equiv x_n$, where $\equiv$ is the equivalence relation on D associated with $\le$. It follows that

$$D_\xi = \textstyle\bigcup_n X_n\,, \quad \text{where} \quad X_n = \{z \in D : z \le x_n\}\,,$$

so each X_n is a $\mathbf{\Sigma}^0_\rho$ set together with $\le$, and so is D_ξ as a countable union of sets in $\mathbf{\Sigma}^0_\rho$.

$\square$ (Theorem 2)

We continue with a few remarks and questions.

Problem 6. Can one strengthen Theorem 2 as follows: *the restricted relation $\le \restriction X$ has no monotone ω_1-sequences*? Lemma 5 admits such a strengthening: if $\xi < \omega_1$ then easily any $\le_{\mathtt{lex}}$-monotone sequence in 2^ξ is countable. □

Using Shoenfield's absoluteness, we obtain:

Corollary 7. *If $\le$ is a Borel PQO on a (Borel) set $D \subseteq \mathbb{N}^\mathbb{N}$, $X \subseteq D$ is a $\mathbf{\Sigma}^1_1$ set, and $\le \restriction X$ is a linear quasi-order, then $\le \restriction X$ is countably cofinal.* $\longleftarrow$ mc

Note that Corollary 7 fails for arbitrary LQOs of class $\mathbf{\Sigma}^1_1$ (that is, not necessarily linear suborders of Borel PQOs), see Example 1.

Proof. In the case considered, the property of countable cofinality of $\le \restriction X$ can be expressed by a Σ^1_2 formula. Thus it remains to consider a Solovay-type extension of the universe and refer to Theorem 2.[1] □

Yet there is a really elementary proof of Corollary 7.

Let Y be the set of all elements $y \in D$ $\le$-comparable with *every* element $x \in X$. This is a $\mathbf{\Sigma}^1_1$ set, and $X \subseteq Y$ (as $\le$ is linear on X). Therefore there is a Borel set Z such that $X \subseteq Z \subseteq Y$. Now let U be the set of all $z \in Z$ $\le$-comparable with *every* element $y \in Y$. Still this is a $\mathbf{\Sigma}^1_1$ set, and $X \subseteq U$ by the definition of Y. Therefore there is a Borel set W such that $X \subseteq W \subseteq U$. And by definition still $\le$ is linear on W. It follows that W does not have increasing ω_1-sequences, and hence neither does X.

Problem 8. Is Corollary 7 true for $\mathbf{\Pi}^1_1$ sets X?

We cannot go much higher though. Indeed, if $\le$ is, say, the eventual domination order on $\mathbb{N}^\mathbb{N}$, then the axiom of constructibility implies the existence of a $\le$-monotone ω_1-sequence of class Δ^1_2. □

Now a few words on Borel PQOs $\le$ having the following property:

($*$) if X is a countable set in the domain of $\le$ then there is an element y such that $x < y$ (in the sense of the corresponding strict ordering) for all $x \in X$.

A thoroughful study of some orderings of this type (for instance, the ordering on $\mathbb{R}^\omega$ defined so that $x \le y$ iff either $x(n) = y(n)$ for all but finite n or $x(n) < y(n)$ for all but finite n) was undertaken in early papers of Felix

[1] We'll not discuss the issue of an inaccessible cardinal on the background.

Hausdorff, *e.g.*, [2, 3] (translated to English in [4]). In particular, Hausdorff investigated the structure of *pantachies*, that is, maximal linearly ordered subsets of those partial orderings. As one of the first explicit applications of the axiom of choice, Hausdorff established the existence of a pantachy in any partial order, and made clear distinction between such an existence proof and an actual, well-defined construction of an individual pantachy (see [2], p. 110). The next result shows that the latter is hardly possible in **ZFC**, at least if we take for granted that any individual set-theoretic construction results in a **ROD** set.

Corollary 9. *The following sentence is true in the Solovay model: if* $\le$ *is a Borel partial quasi-order on a (Borel) set* $D \subseteq \mathbb{N}^{\mathbb{N}}$*, satisfying* $(*)$*, then* $\le$ *has no* **ROD** *pantachies.* ← 1

Proof. It follows from $(*)$ that any pantachy in $\langle D\,;\le\rangle$ is a set of uncountable cofinality. Now apply Theorem 2. □

A further corollary: it is impossible to prove the existence of pantachies in any Borel PQO satisfying $(*)$ in **ZF** + **DC**.

References

[1] L. A. Harrington, D. Marker, and S. Shelah, Borel orderings. *Trans. Amer. Math. Soc.*, 1988, 310, pp. 293–302.

[2] F. Hausdorff, Untersuchungen über Ordnungstypen IV, V. *Ber. über die Verhandlungen der Königlich Sächsische Gesellschaft der Wissenschaften zu Leipzig, Math.-phys. Klasse*, 1907, 59, pp. 84–159.

[3] F. Hausdorff, Die Graduirung nach dem Endverlauf. *Abhandlungen der Königlich Sächsische Gesellschaft der Wissenschaften zu Leipzig, Math.-phys. Klasse*, 1909, 31, pp. 295–334.

[4] F. Hausdorff, *Hausdorff on ordered sets*, Translated from the German, edited and with commentary by J. M. Plotkin. History of Mathematics, 25. AMS, Providence, RI and LMS, London, 2005. xviii+322 pp.

[5] Vladimir Kanovei. Linearization of definable order relations. *Annals of Pure and Applied Logic*, 2000, 102, 1-2, pp. 69–100.

[6] R.M. Solovay. A model of set-theory in which every set of reals is Lebesgue measurable. *Ann. Math. (2)*, 1970, 92, pp. 1–56.

[7] J. Stern. On Lusin's restricted continuum problem. *Ann. Math. (2)*, 1984, 120, pp. 7–37.