A generalization of Solovay's Σ-construction*

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Abstract

A Σ-construction of Solovay is partially extended to the case of intermediate sets which are not necessarily subsets of the ground model. As an application, we prove that, for a given name t, the set of all sets $t[G]$, G being generic over the ground model, is Borel. This result was first established by Zapletal by a totally different descriptive set theoretic argument.

1 Introduction

A famous Σ-construction by Solovay [1] shows that if $x = t[G]$ is a real in a $\mathbb{P}$-forcing extension $\mathfrak{M}[G]$ of a countable transitive model $\mathfrak{M}$, where $t \in \mathfrak{M}$ is a $\mathbb{P}$-name and $\mathbb{P} \in \mathfrak{M}$ is a forcing, then there is a set $\Sigma = \Sigma(t, x) \in \mathfrak{M}[x]$, $\Sigma \subseteq \mathbb{P}$, such that

(i) $G \subseteq \Sigma$,

(ii) if $G' \subseteq \Sigma$ is $\mathbb{P}$-generic over $\mathfrak{M}$ then still $t[G'] = x$,

(iii) $\mathfrak{M}[G]$ is a Σ-generic extension of $\mathfrak{M}[x]$; basically, G itself is Σ-generic over $\mathfrak{M}[x]$.

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One might ask whether the Σ-construction can be generalized to **arbitrary** sets $x \in \mathfrak{M}[G]$, not necessarily reals. We don't know the answer, even in the easiest case $x \subseteq \mathscr{P}(\omega)$ not covered by Solovay's construction.

This note is devoted to a minor result in this direction. Namely, given a countable transitive model $\mathfrak{M}$, a forcing $\mathbb{P} \in \mathfrak{M}$, a $\mathbb{P}$-name $t \in \mathfrak{M}$, and a set X of any kind, we define a set $\Sigma(X, t)$ (not a subset of $\mathbb{P}$), which

1) satisfies a property which resembles (i) (Lemma 5), and
2) is empty iff X is not equal to $t[G]$ for any $\mathbb{P}$-generic set G over $\mathfrak{M}$ (Corollary 8).

As an application, we prove in the last section that, for a given $\mathbb{P}$-name t, the set of all sets $t[G]$, $G \subseteq \mathbb{P}$ being generic over $\mathfrak{M}$, is Borel. (Immediately, it is only analytic, of course.) This result was first established by Zapletal [2] by a totally different argument.

2 Definition of Σ

Definition 1 (blanket assumptions)**.** We suppose that: $\overleftarrow{\text{bla}}$

— $\mathfrak{M}$ is a countable transitive model of **ZFC**,

— $\mathbb{P} \in \mathfrak{M}$ is a forcing (a partial quasi-order),

— $t \in \mathfrak{M}$ is a $\mathbb{P}$-name of a transitive set (so $\mathbb{P}$ forces "t is transitive").

— X is a transitive set (not necessarily $X \in \mathfrak{M}$ or $X \subseteq \mathfrak{M}$).

Let $\Vdash$ be $\Vdash^{\mathfrak{M}}_{\mathbb{P}}$, the $\mathbb{P}$-forcing relation over the ground model $\mathfrak{M}$. □

We also assume that a reasonable **ramified** system of names for elements of $\mathbb{P}$-generic extensions of $\mathfrak{M}$ is fixed. If t is a name and $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ then let $t[G]$ be the G-interpretation of t, so that we have $\mathfrak{M}[G] = \{t[G] : t \text{ is a name}\}$. For any $\mathbb{P}$-names s, t, we let $s \prec t$ mean that s occurs in t as a name of a potential element of $t[G]$. Then the set $\mathrm{PE}_t = \{s : s \prec t\}$ (of all "potential elements" of t) belongs to $\mathfrak{M}$ and

$$t[G] = \{s[G] : s \in \mathrm{PE}_t \wedge \exists p \in G\,(p \Vdash s \in t)\}\,.$$

for any set $G \subseteq \mathbb{P}$, generic over $\mathfrak{M}$.

If $d \subseteq \mathrm{PE}_t$ then a condition $p \in \mathbb{P}$ is called *d-complete* iff $p \Vdash s \in t$ holds for all $s \in d$ and p decides all formulas $s \in s'$ where $s, s' \in d$.

If d is infinite then d-complete conditions do not necessarily exist.

Definition 2. $\mathbb{P}^*(X, t)$ is the set of all pairs $\langle p, a\rangle$ such that $p \in \mathbb{P}$, u is a finite partial map, $\mathtt{dom}\, a \subseteq \mathrm{PE}_t$, $\mathtt{ran}\, a \subseteq X$, and p is $(\mathtt{dom}\, a)$-complete. $\overleftarrow{\text{p*}}$

We order $\mathbb{P}^*(X,t)$ so that $\langle p,a\rangle \le \langle p',au'\rangle$ ($\langle p,a\rangle$ is stronger) iff $p \le p'$ in $\mathbb{P}$ and a extends a' as a function. □

Pairs in $\mathbb{P}^*(X,t)$ will be called *superconditions*. Given a supercondition $\langle p,a\rangle \in \mathbb{P}^*(X,t)$, we'll call p its *condition*, and a its *assignment* — because a essentially assigns sets for (some) names which can be forced to be elements of t. Note: generally speaking, superconditions are not members of $\mathfrak{M}$.

Definition 3. Recall that X is a fixed transitive set by Definition 1. Here we define a set $\Sigma(X,t)$ of all superconditions $\langle p,a\rangle$ which, informally speaking, force nothing really incompatible with the assumption that $X = t[G]$ for a set $G \subseteq \mathbb{P}$ generic over $\mathfrak{M}$. The dependence on $\mathbb{P}$ in the definition of $\Sigma(X,t)$ is suppressed. $\longleftarrow$ defu

- $\Sigma_0(X,t)$ consists of all superconditions $\langle p,a\rangle \in \mathbb{P}^*(X,t)$ such that
  1) $\mathtt{ran}\, a \subseteq X$, and
  2) if $s,s' \in \mathtt{dom}\, a$ then $p \Vdash s \in s'$ iff $a(s) \in a(s')$.
- If $\gamma \in \mathbf{Ord}$ then the set $\Sigma_{\gamma+1}(X,t)$ consists of all superconditions $\langle p,a\rangle \in \Sigma_\gamma(X,t)$ such that
  – for any set $D \in M$, $D \subseteq \mathbb{P}$, dense in $\mathbb{P}$,
  – and any name $s \in \mathrm{PE}_t$,
  – and any element $x \in X$,

  there is a stronger supercondition $\langle q,b\rangle \in \Sigma_\gamma(X,t)$ such that:
  a) $\langle q,b\rangle \le \langle p,a\rangle$ and $q \in D$,
  b) $x \in \mathtt{ran}\, b$, and
  c) either $s \in \mathtt{dom}\, b$ or $q \Vdash s \notin t$.
- Finally if λ is a limit ordinal then $\Sigma_\lambda(X,t) = \bigcap_{\gamma<\lambda} \Sigma_\gamma(X,t)$.

The sequence of sets $\Sigma_\gamma(X,t)$ is decreasing, so that there is an ordinal $\lambda = \lambda(X,t)$ such that $\Sigma_{\lambda+1}(X,t) = \Sigma_\lambda(X,t)$; we let $\Sigma(X,t) = \Sigma_\lambda(X,t)$. □

The following is quite elementary.

Lemma 4. *If $\langle p,a\rangle \in \Sigma(X,t)$, a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$, and $s \in \mathrm{PE}_t$, $x \in X$, then there is a pair $\langle q,b\rangle \in \Sigma(X,t)$ satisfying: $\langle q,b\rangle \le \langle p,a\rangle$, $q \in D$, $x \in \mathtt{ran}\, b$, and either $s \in \mathtt{dom}\, b$ or $q \Vdash s \notin t$.* $\longleftarrow$ sgu

Proof. This holds by definition, as $\Sigma(X,t) = \Sigma_\lambda(X,t) = \Sigma_{\lambda+1}(X,t)$. □

We do not claim that if $\langle p, a \rangle \in \Sigma(X, t)$ and $q \in \mathbb{P}$, $q \leq p$ is a stronger condition then necessarily $\langle q, a \rangle \in \Sigma(X, t)$. In fact this cannot be expected to be the case: indeed q may strengthen p in wrong way, that is, by forcing something incompatible with the assignment a. Nevertheless, appropriate extensions of superconditions are always possible by Lemma 4.

3 Some results

We continue to argue in the assumptions of Definition 1.

Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic set over $\mathfrak{M}$. Say that a function (assignment) a, with $\mathtt{dom}\, a \subseteq \mathrm{PE}_t$, is *$G$-compatible* if $a(s) = s[G]$ for all $s \in \mathtt{dom}\, a$.

The next lemma needs some work.

Lemma 5. *Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic set over $\mathfrak{M}$, and $t[G] = X$. If $\langle p, a \rangle \in \mathbb{P}^*(X, t)$, a is G-compatible, and $p \in G$, then $\langle p, a \rangle \in \Sigma(X, t)$.* $\longleftarrow$ sgG

Proof. Prove $\langle p, a \rangle \in \Sigma_\gamma(X, t)$ by induction on γ.

Assume that $\gamma = 0$. By the $(\mathtt{dom}\, a)$-completeness, if $s, s' \in \mathtt{dom}\, a$ then p decides $s \in s'$. If $p \Vdash s \in s'$ then $s[G] \in s'[G]$, therefore $a(s) \in a(s')$ by the G-compatibility. Similarly, if $p \Vdash s \notin s'$ then $a(s) \notin a(s')$.

The step $\gamma \to \gamma + 1$. Suppose, towards the contrary, that $\langle p, a \rangle \notin \Sigma_{\gamma+1}(X, t)$ but $p \in \Sigma_\gamma(X, t)$ by the inductive hypothesis. By definition, there exist: a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$, dense in $\mathbb{P}$, and elements $s \in \mathrm{PE}_t$, $x \in X$, such that no supercondition $\langle q, b \rangle \in \Sigma_\gamma(X, t)$ satisfies all of

$$\langle q, b \rangle \leq \langle p, a \rangle, \quad q \in D, \quad x \in \mathtt{ran}\, b, \text{ and either } s \in \mathtt{dom}\, b \text{ or } q \Vdash s \notin t.$$

By the genericity, there is a condition $q \in G \cap D$, $q \leq p$. As $t[G] = X$, there is a finite assignment $b : (\mathtt{dom}\, b \subseteq \mathrm{PE}_t) \to X$ such that

$a \subseteq b$, $x \in \mathtt{ran}\, b$, $r[G] \in t[G]$ and $b(r) = r[G]$ for every name $r \in \mathtt{dom}\, b$, and either $s[G] \notin t[G]$ or $s \in \mathtt{dom}\, b$.

There is a stronger condition $q' \in G \cap D$ such that if in fact $s[G] \notin t[G]$ then $q' \Vdash s \notin t$, and even more, q' is $(\mathtt{dom}\, b)$-complete. Then $\langle q', b \rangle \in \Sigma_\gamma(X, t)$ by the inductive hypothesis, a contradiction.

The limit step is obvious. $\square$

Lemma 6. *If $\langle p, a \rangle \in \Sigma(X, t)$ then there is a set $G \subseteq \mathbb{P}$, $\mathbb{P}$-generic over $\mathfrak{M}$, and such that $p \in G$ and $t[G] = X$.* $\longleftarrow$ gins

Proof. As the model $\mathfrak{M}$ is countable, Lemma 4 allows to define a decreasing sequence of superconditions $\langle p_n, a_n \rangle \in \Sigma(X, t)$,

$$\langle p, u \rangle = \langle p_0, a_0 \rangle \geq \langle p_1, a_1 \rangle \geq \langle p_2, a_2 \rangle \geq \ldots ,$$

such that the sequence $\{p_n\}_{n\in\omega}$ intersects every set $D \in \mathfrak{M}, D \subseteq \mathbb{P}$, dense in $\mathbb{P}$ — hence it naturally extends to a generic set $G = \{p \in \mathbb{P} : \exists n \, (p_n \leq p)\}$, and in addition, the union $\varphi = \bigcup_n a_n : \mathtt{dom}\, \varphi \to X$ of all assignments a_n satisfies:

(1) $\mathtt{ran}\, \varphi = X$, $\mathtt{dom}\, \varphi \subseteq \mathrm{PE}_t$, and

(2) for any $s \in \mathrm{PE}_t$:

either $s \in \mathtt{dom}\, \varphi$ — then $s[G] \notin t[G]$,

or $q \Vdash s \notin t$ for some $q \in G$ — then $s[G] \notin t[G]$.

Due to the transitivity of both sets $t[G] = \{s[G] : s \in \mathtt{dom}\, \varphi\}$ and $X = \mathtt{ran}\, \varphi$, to prove that $t[G] = X$, it suffices to check that $\varphi(s) \in \varphi(s')$ iff $s[G] \in s'[G]$, for all names $s, s' \in \mathtt{dom}\, \varphi$. By the construction of φ, there is an index n such that $s, s' \in \mathtt{dom}\, a_n$. By definition, condition $p_n \in G$ is $(\mathtt{dom}\, a_n)$-complete, so p_n decides $s \in s'$.

If $p_n \Vdash s \in s'$ then $s[G] \in s'[G]$, and on the other hand, as $\langle p_n, a_n \rangle \in \Sigma_0(X, t)$, we have $\varphi(s) = a_n(s) \in a_n(s') = \varphi(s')$.

Similarly, if $q \Vdash s \notin s'$ then $s[G] \notin s'[G]$ and $\varphi(s) \notin \varphi(s')$. □

The next lemma shows that the ordinals $\lambda(X, t)$ as in Definition 3 are bounded in $\mathfrak{M}$ whenever $\Sigma(X, t) \neq \varnothing$.

Lemma 7. *There is an ordinal $\lambda^*(t) \in \mathfrak{M}$ such that $\lambda(t[G], t) < \lambda^*(t)$ for every set $G \subseteq \mathbb{P}$, $\mathbb{P}$-generic over $\mathfrak{M}$.* $\longleftarrow$ bou

Proof. Assume that a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$. Then $X = t[G] \in \mathfrak{M}[G]$, and hence $\lambda(X, t)$ is an ordinal in $\mathfrak{M}$, and its value is forced, over $\mathfrak{M}$, by a condition in G. □

Corollary 8. *Let X be a transitive set. The following are equivalent*: $\longleftarrow$ sf

(1) *there is a set $G \subseteq \mathbb{P}$, $\mathbb{P}$-generic over $\mathfrak{M}$, such that $t[G] = X$* ;

(2) $\Sigma(X, t) \neq \varnothing$;

(3) $\Sigma_{\lambda^*(t)}(X, t) = \Sigma_{\lambda^*(t)+1}(X, t) \neq \varnothing$.

Proof. Use Lemmas 5, 6, 7. □

We finish with a question. Let $\Sigma^*(X, t) = \{p \in \mathbb{P} : \exists u \, (\langle p, u \rangle \in \Sigma(X, t)\}$. Is it true that if $G \subseteq \Sigma^*(X, t)$ is a $\mathbb{P}$-generic set over $\mathfrak{M}$ then $X = t[G]$?

4 The property of being generic-generated is Borel

Another consequence of Lemma 7 and other results above claims that, in the assumptions of Definition 1, the set of all sets of the form $t[G]$, $G \subseteq \mathbb{P}$ being generic over $\mathfrak{M}$, is Borel in terms of an appropriate coding, of all (hereditarily countable) sets of this form, by reals. This result was first established by Zapletal [2] by a totally different argument using advanced technique of descriptive set theory.

In order to avoid dealing with coding in general setting, we present this result only in the simplest nontrivial (= not directly covered by Solovay's original result) case when sets $t[G]$ are (by necessity countable) sets of reals.

For a real $y \in \omega^\omega$, we let $R_y = \{(y)_n : n \in \omega\} \smallsetminus \{(y)_0\}$, where $(y)_n \in \omega^\omega$ and $(y)_n(k) = y(2^n(2k+1)-1)$ for all n and k. Thus $\{R_y : y \in \omega^\omega\}$ is the set of all at most countable sets $R \subseteq \omega^\omega$ (including the empty set).

Theorem 9. *In the assumptions of Definition 1, if $\mathbb{P}$ forces that $t[G]$ is a subset of ω^ω then the set W of all reals $y \in \omega^\omega$, such that $R_y = t[G]$ for a set $G \subseteq \mathbb{P}$ generic over $\mathfrak{M}$, is Borel.* $\longleftarrow$ zapt

Proof. Let ϑ be the least ordinal not in $\mathfrak{M}$. By Corollary 8, for a real y to belong to W each of the two following conditions is necessary and sufficient:

(I) there exist an ordinal $\lambda < \vartheta$ and a sequence of sets $\Sigma_\gamma(X,t)$, $\gamma \le \lambda+1$, where $X = R_y$, satisfying Definition 3 and such that $\Sigma_\lambda(X,t) = \Sigma_{\lambda+1}(X,t) \neq \varnothing$;

(II) for any ordinal $\lambda < \vartheta$ and any sequence of sets $\Sigma_\gamma(X,t)$, $\gamma \le \lambda+1$, where $X = R_y$, satisfying Definition 3, if $\Sigma_\lambda(X,t) = \Sigma_{\lambda+1}(X,t)$ then $\Sigma_\lambda(X,t) \neq \varnothing$.

Condition (I) provides a $\mathbf{\Sigma}^1_1$ definition of the set W while condition (II) provides a $\mathbf{\Pi}^1_1$ definition of W, both relative to a real parameter coding the $\in$-structure of $\mathfrak{M}$. $\square$

References

[1] R. M. Solovay. A model of set theory in which every set of reals is Lebesgue measurable. *Ann. Math.*, 1970, 92, pp. 1–56.

[2] J. Zapletal. *Forcing Borel reducibility invariants*. August, 2013.