A generalization of Solovay's Σ-construction with application to intermediate models[*]

Vladimir Kanovei

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Abstract

A Σ-construction of Solovay is extended to the case of intermediate sets which are not necessarily subsets of the ground model, with a more transparent description of the resulting forcing notion than in the classical paper of Grigorieff. As an application, we prove that, for a given name t (not necessarily a name of a subset of the ground model), the set of all sets of the form $t[G]$ (the G-interpretation of t), G being generic over the ground model, is Borel. This result was first established by Zapletal by a descriptive set theoretic argument.

1 Introduction

A famous Σ-construction by Solovay [2] shows that if $\mathbb{P} \in \mathfrak{M}$ is a forcing notion in a countable transitive model $\mathfrak{M}$, $t \in \mathfrak{M}$ is a $\mathbb{P}$-name, and $X \subseteq \mathfrak{M}$ is any set (*e.g.*, a real), then there is a set $\Sigma(X, t) \subseteq \mathbb{P}$ such that

(I) the inequality $\Sigma(X, t) \neq \varnothing$ is necessary and sufficient for there to exist a set $G \subseteq \mathbb{P}$, $\mathbb{P}$-generic over $\mathfrak{M}$ and satisfying $X = t[G]$;

(II) if a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ and $t[G] = X$ then $G \subseteq \Sigma(X, t)$, $\Sigma(X, t) \in \mathfrak{M}[X]$, and G is $\Sigma(X, t)$-generic over $\mathfrak{M}[X]$;

(III) therefore, in (II), $\mathfrak{M}[G]$ is a $\Sigma(X, t)$-generic extension of $\mathfrak{M}[X]$;

(IV) in addition, in (II), if a set $G \subseteq \Sigma(X, t)$ is $\Sigma(X, t)$-generic over $\mathfrak{M}$ then still $t[G] = X$.

One may ask whether the Σ-construction of Solovay can be generalized to **arbitrary** sets X, not necessarily those satisfying $X \subseteq \mathfrak{M}$. Following common practice, we'll rather write $\mathfrak{M}(X)$ in this case.

This paper is devoted to this question, and the main goal will be to define such a generalization, although on the base of a somewhat more complicated auxiliary forcing $\Sigma^+(X, t)$ which consists of "superconditions", *i.e.*, pairs of the form $\langle p, a \rangle$, where still $p \in \mathbb{P}$ while a is a finite map associating elements of X with their names. The generalization (Theorem 11) will be more direct w.r.t. (I) and (III), and rather partial w.r.t. (II) and (IV).

Remark 1. Note in passing by that if the axiom of choice holds in $\mathfrak{M}(X)$ then a set $x \subseteq \mathfrak{M}$ can be easily defined in such a way that $\mathfrak{M}[x] = \mathfrak{M}(X)$, effectively reducing the problem to the case $X \subseteq \mathfrak{M}$ already considered by Solovay; therefore our results below will make sense only in the case when $\mathfrak{M}(X)$ is a choiceless model. $\qquad\square$

We'll approach the question from a slightly different technical standpoint than in the classical paper of Grigorieff [1] where a base for such generalizations was made. This will allow us to obtain more pointed generalizations. For instance, (III) is obtained in [1] by a trick which involves a collapse forcing on the top of $\mathbb{P}$ (see Section 8), so that the resulting forcing notion in [1] has a much less transparent nature than $\Sigma^+(X, t)$ of this paper.

As an application, we prove in Section 9 that, for a given $\mathbb{P}$-name t, the set of all sets $t[G]$, $G \subseteq \mathbb{P}$ being generic over $\mathfrak{M}$, is Borel (in terms of an appropriate coding of hereditarily countable sets by reals). Immediately, this set is only analytic, of course. This result was first established by Zapletal [3] by a totally different and much less straightforward argument.

2 Basic assumptions and definitions

Definition 2. During the course of the paper, we suppose that:

- $\mathfrak{M}$ is a countable transitive model of **ZFC**,

- $\mathbb{P} \in \mathfrak{M}$ is a forcing, and $p \leq q$ means that p is a *stronger* condition,

- $t \in \mathfrak{M}$ is a $\mathbb{P}$-name of a transitive set (so $\mathbb{P}$ forces "t is transitive"),

- X is a finite or countable transitive set (not necessarily satisfying conditions $X \in \mathfrak{M}$ or $X \subseteq \mathfrak{M}$). $\qquad\square$

The assumption of transitivity of X does not reduce generality since any set X is effectively coded by the transitive set $\{X\} \cup \mathrm{TC}(X)$.

Let $\Vdash$ be $\Vdash_{\mathbb{P}}^{\mathfrak{M}}$, the $\mathbb{P}$-forcing relation over the ground model $\mathfrak{M}$.

We also assume that a reasonable **ramified** system of names for elements of $\mathbb{P}$-generic extensions of $\mathfrak{M}$ is fixed, $\dot{x}$ is a canonical name for any $x \in \mathfrak{M}$.

and $\underline{G}$ is a canonical name for G, the generic set. If t is a name and $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ then let $t[G]$ be the G-interpretation of t, so that

$$\mathfrak{M}[G] = \{t[G] : t \in \mathfrak{M} \text{ is a name}\}.$$

Definition 3. In this case, if $X = t[G]$ then $\mathfrak{M}(X)$ is the least transitive model of **ZF** (not necessarily of **ZFC**) containing X (and all sets in the transitive closure of X) and all sets in the ground model $\mathfrak{M}$. Obviously $\mathfrak{M} \subseteq \mathfrak{M}(X) \subseteq \mathfrak{M}[G]$. $\qquad\square$

For any $\mathbb{P}$-names s, t, we let $s \prec t$ mean that s occurs in t as a name of a potential element of $t[G]$. Then the set $\mathrm{PE}_t = \{s : s \prec t\}$ (of all "potential elements" of t) belongs to $\mathfrak{M}$ and if $G \subseteq \mathbb{P}$ is generic over $\mathfrak{M}$ then

$$t[G] = \{s[G] : s \in \mathrm{PE}_t \wedge \exists\, p \in G\, (p \Vdash s \in t)\}.$$

If $d \subseteq \mathrm{PE}_t$ then a condition $p \in \mathbb{P}$ is called *d-complete* iff

1) $p \Vdash s \in t$ for all $s \in d$, and

2) p decides all formulas $s \in s'$ and $s = s'$, where $s, s' \in d$.

If d is infinite then d-complete conditions do not necessarily exist.

3 Superconditions and the set Σ^+

The following definitions introduce the main technical instrument used in this paper: superconditions.

Definition 4. $\mathbb{P}^+(X, t)$ is the set of all pairs $\langle p, a \rangle$ such that $p \in \mathbb{P}$, a is a finite partial map, $\mathrm{dom}\, a \subseteq \mathrm{PE}_t$, $\mathrm{ran}\, a \subseteq X$, p is $(\mathrm{dom}\, a)$-complete, and in addition $a(\dot{x}) = x$ for any $x \in \mathfrak{M}$ such that $\dot{x} \in \mathrm{dom}\, a$.

We order $\mathbb{P}^+(X, t)$ so that $\langle p, a \rangle \leq \langle p', a' \rangle$ ($\langle p, a \rangle$ is stronger) iff $p \leq p'$ in $\mathbb{P}$ (p is stronger in $\mathbb{P}$) and a extends a' as a function. $\qquad\square$

In particular if $p \in \mathbb{P}$ then $\langle p, \varnothing \rangle \in \mathbb{P}^+(X, t)$.

Pairs in $\mathbb{P}^+(X, t)$ will be called *superconditions*.[1] Given a supercondition $\langle p, a \rangle \in \mathbb{P}^+(X, t)$, we'll call p its *condition*, and a its *assignment* — because a assigns sets to (some) names forced by p to be elements of $t[G]$.

Note: generally speaking, superconditions are not members of $\mathfrak{M}$.

We can also observe that the forcing $\mathbb{P}^+(X, t)$ just defined belongs to $\mathfrak{M}(X)$ and is a subset of the product forcing $\mathbb{P} \times \mathrm{Coll}(\mathrm{PE}_t, X) \in \mathfrak{M}(X)$.

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[1] The upper index $+$ will typically denote things related to superconditions as opposite to just conditions in $\mathbb{P}$.

Lemma 5. *If $\langle p, a \rangle \in \mathbb{P}^+(X, t)$, $q \in \mathbb{P}$, $q \leq p$, then $\langle q, a \rangle \in \mathbb{P}^+(X, t)$.* $\quad\square$

Now we define, following Solovay [2], a set $\Sigma^+(X, t)$ of all superconditions $\langle p, a \rangle$ which, informally speaking, force nothing really incompatible with the assumption that there is a set $G \subseteq \mathbb{P}$ generic over $\mathfrak{M}$ and such that $X = t[G]$ and $a(s) = s[G]$ for all $s \in \operatorname{dom} a$.

Definition 6. We define a set $\Sigma^+_\gamma(X, t) \subseteq \mathbb{P}^+(X, t)$ by transfinite induction on $\gamma \in \mathbf{Ord}$. The dependence on $\mathbb{P}$ in the definition is suppressed.

- $\Sigma^+_0(X, t)$ consists of all superconditions $\langle p, a \rangle \in \mathbb{P}^+(X, t)$ such that if $s, s' \in \operatorname{dom} a$ then $p \parallel\!\!- s \in (\text{or} =) s'$ iff $a(s) \in (\text{resp.}, =) a(s')$.

- If $\gamma \in \mathbf{Ord}$ then the set $\Sigma^+_{\gamma+1}(X, t)$ consists of all superconditions $\langle p, a \rangle \in \Sigma^+_\gamma(X, t)$ such that

  – for any set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$, dense in $\mathbb{P}$,

  – and any name $s \in \operatorname{PE}_t$ and any element $x \in X$,

  there is a stronger supercondition $\langle q, b \rangle \in \Sigma^+_\gamma(X, t)$ satisfying:

  a) $\langle q, b \rangle \leq \langle p, a \rangle$ and $q \in D$,

  b) $x \in \operatorname{ran} b$, and either $s \in \operatorname{dom} b$ or $q \parallel\!\!- s \notin t$.

- Finally if λ is a limit ordinal then $\Sigma^+_\lambda(X, t) = \bigcap_{\gamma < \lambda} \Sigma^+_\gamma(X, t)$.

The sequence of sets $\Sigma^+_\gamma(X, t)$ is decreasing, so that there is an ordinal $\lambda = \lambda(X, t)$ such that $\Sigma^+_{\lambda+1}(X, t) = \Sigma^+_\lambda(X, t)$; let $\Sigma^+(X, t) = \Sigma^+_\lambda(X, t)$. $\quad\square$

Lemma 7. *If $\langle p, a \rangle \in \Sigma^+(X, t)$, a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$ is dense in $\mathbb{P}$, and $s \in \operatorname{PE}_t$, $x \in X$, then there is a pair $\langle q, b \rangle \in \Sigma^+(X, t)$ satisfying: $\langle q, b \rangle \leq \langle p, a \rangle$, $q \in D$, $x \in \operatorname{ran} b$, and either $s \in \operatorname{dom} b$ or $q \parallel\!\!- s \notin t$.*

Proof. This holds by definition, as $\Sigma^+(X, t) = \Sigma^+_\lambda(X, t) = \Sigma^+_{\lambda+1}(X, t)$. $\quad\square$

The next lemma shows that the set $\Sigma^+(X, t)$ is closed under weakening.

Lemma 8. *Assume that $\langle p, a \rangle \in \Sigma^+(X, t)$. Then*

(i) *if $\langle q, b \rangle \in \Sigma^+_0(X, t)$ and $\langle p, a \rangle \leq \langle q, b \rangle$ then $\langle q, b \rangle \in \Sigma^+(X, t)$;*

(ii) *if $q \in \mathbb{P}$, $q \geq p$, but still $\langle q, a \rangle \in \mathbb{P}^+(X, t)$, then $\langle q, a \rangle \in \Sigma^+(X, t)$.*

Proof. (i) Prove that $\langle q, b \rangle \in \Sigma_\gamma^+(X, t)$ by induction on γ. The case $\gamma = 0$ and the limit case are rather obvious. Consider the step $\gamma \to \gamma + 1$. By the inductive hypothesis, $\langle q, b \rangle \in \Sigma_\gamma^+(X, t)$. Let $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$ be dense in $\mathbb{P}$, $s \in \mathrm{PE}_t$, and $x \in X$. As $\langle p, a \rangle \in \Sigma_{\gamma+1}^+(X, t)$, by definition there is a stronger supercondition $\langle r, c \rangle \in \Sigma_\gamma^+(X, t)$ satisfying: $\langle r, c \rangle \leq \langle p, a \rangle$, $r \in D$, $x \in \mathtt{ran}\, c$, and either $s \in \mathtt{dom}\, b$ or $r \Vdash s \notin t$. But then $\langle r, c \rangle \leq \langle q, b \rangle$ as well, and hence $\langle r, c \rangle$ witnesses $\langle q, b \rangle \in \Sigma_{\gamma+1}^+(X, t)$.

(ii) It follows from $\langle q, a \rangle \in \mathbb{P}^+(X, t)$ that $\langle q, a \rangle$ belongs to $\Sigma_0^+(X, t)$ together with $\langle p, a \rangle$. It remains to refer to (i). $\qquad\square$

We do not claim that if $\langle p, a \rangle \in \Sigma^+(X, t)$ and $q \in \mathbb{P}$, $q \leq p$ is a stronger condition then, similarly to Lemma 5, $\langle q, a \rangle \in \Sigma^+(X, t)$. In fact this hardly can be expected, as q may strengthen p in wrong way, that is, by forcing about t something incompatible with the assignment a. Nevertheless, appropriate extensions of superconditions are always possible by Lemma 7.

4 The main result

To formulate the main result, we need one more definition.

Definition 9. If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic set over $\mathfrak{M}$ then let $a[G]$ be the function defined on the set $\mathtt{dom}\, a[G] = \mathrm{PE}_t[G] = \{s \in \mathrm{PE}_t : s[G] \in t[G]\}$ so that $a[G](s) = s[G]$ for all $s \in \mathrm{PE}_t[G]$.

If $\Gamma \subseteq \mathbb{P}^+(X, t)$ then let

$$
\begin{aligned}
\Gamma\!\downarrow \;&=\; \{p \in \mathbb{P} : \exists a\, (\langle p, a \rangle \in \Gamma)\} \quad &&\text{(the \textit{projection} of } \Gamma \text{ onto } \mathbb{P});\\
A[\Gamma] \;&=\; \{a : \exists p\, (\langle p, a \rangle \in \Gamma)\} \quad &&\text{(all assignments which occur in } \Gamma);\\
a[\Gamma] \;&=\; \bigcup A[\Gamma] \quad &&\text{(the union of assignments in } \Gamma). \;\;\square
\end{aligned}
$$

Lemma 10. *If $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ then $\mathtt{ran}\, a[G] = t[G]$.* $\qquad\square$

Now the main theorem follows; we prove it in the next two sections.

Theorem 11 (compare with claims (I), (II), (III), (IV) in the introduction)**.** *In the assumptions of Definition 2 the following holds*:

(i) *the inequality $\Sigma^+(X, t) \neq \varnothing$ is necessary and sufficient for there to exist a set $G \subseteq \mathbb{P}$, $\mathbb{P}$-generic over $\mathfrak{M}$ and satisfying $X = t[G]$* ;

(ii) *if a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ and $t[G] = X$ then the set*

$$
G^+ = \{\langle p, a \rangle \in \Sigma^+(X, t) : p \in G \wedge a \subset a[G]\}
$$

is $\Sigma^+(X, t)$-generic over $\mathfrak{M}(X)$, and $G = G^+\!\downarrow$;

(iii) *hence, in* (ii), $\mathfrak{M}[G]$ *is a* $\Sigma^+(X,t)$-*generic extension of* $\mathfrak{M}(X)$;

(iv) *if a set* $\Gamma \subseteq \Sigma^+(X,t)$ *is* $\Sigma^+(X,t)$-*generic over* $\mathfrak{M}(X)$ *then the set* $H = \Gamma{\downarrow} \subseteq \mathbb{P}$ *is* $\mathbb{P}$-*generic over* $\mathfrak{M}$, $t[H] = X$, *and* $a[\Gamma] = a[H]$.

5 The bounding lemma

Here we prove claim (i) of Theorem 11. We'll show, in particular, that if indeed $X = t[G]$ for a generic set $G \subseteq \mathbb{P}$ then the essential length of the construction of Definition 6 is an ordinal in $\mathfrak{M}$ (Lemma 14).

We continue to argue in the assumptions of Definition 2.

The next lemma needs some work.

Lemma 12. *Assume that* $G \subseteq \mathbb{P}$ *is a* $\mathbb{P}$-*generic set over* $\mathfrak{M}$, *and* $t[G] = X$. *If* $\langle p, a \rangle \in \mathbb{P}^+(X,t)$, $a \subseteq a[G]$, *and* $p \in G$, *then* $\langle p, a \rangle \in \Sigma^+(X,t)$.
In particular, if $p \in G$ *then* $\langle p, \varnothing \rangle \in \Sigma^+(X,t)$.

Proof. Prove $\langle p, a \rangle \in \Sigma_\gamma^+(X,t)$ by induction on γ.

Assume that $\gamma = 0$. By the $(\text{dom}\,a)$-completeness, if $s, s' \in \text{dom}\,a$ then p decides $s \in s'$. If $p \Vdash s \in s'$ then $s[G] \in s'[G]$, therefore, as $a \subseteq a[G]$, we have $a(s) \in a(s')$. Similarly, if $p \Vdash s \notin s'$ then $a(s) \notin a(s')$.

The step $\gamma \to \gamma + 1$. Suppose, towards the contrary, that $\langle p, a \rangle \notin \Sigma_{\gamma+1}^+(X,t)$ but $p \in \Sigma_\gamma^+(X,t)$ by the inductive hypothesis. By definition, there exist: a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$, dense in $\mathbb{P}$, and elements $s \in \text{PE}_t$, $x \in X$, such that no supercondition $\langle q, b \rangle \in \Sigma_\gamma^+(X,t)$ satisfies all of

$$\langle q, b \rangle \le \langle p, a \rangle, \quad q \in D, \quad x \in \text{ran}\,b, \text{ and either } s \in \text{dom}\,b \text{ or } q \Vdash s \notin t.$$

By the genericity, there is a condition $q \in G \cap D$, $q \le p$. As $t[G] = X$, there is a finite assignment $b : (\text{dom}\,b \subseteq \text{PE}_t) \to X$ such that

$a \subseteq b$, $x \in \text{ran}\,b$, $r[G] \in t[G]$ and $b(r) = r[G]$ for every name $r \in \text{dom}\,b$, and either $s[G] \notin t[G]$ or $s \in \text{dom}\,b$.

There is a stronger condition $q' \in G \cap D$ such that if in fact $s[G] \notin t[G]$ then $q' \Vdash s \notin t$, and even more, q' is $(\text{dom}\,b)$-complete. Then $\langle q', b \rangle \in \Sigma_\gamma^+(X,t)$ by the inductive hypothesis, a contradiction.

The limit step is obvious. $\square$

Lemma 13. *If* $\langle p, a \rangle \in \Sigma^+(X,t)$ *then there is a set* $G \subseteq \mathbb{P}$, $\mathbb{P}$-*generic over* $\mathfrak{M}$, *and such that* $p \in G$ *and* $t[G] = X$.

Proof. Both the model $\mathfrak{M}$ and the set X are countable; therefore Lemma 7 allows to define a decreasing sequence of superconditions $\langle p_n, a_n \rangle \in \Sigma^+(X, t)$,

$$\langle p, u \rangle = \langle p_0, a_0 \rangle \geq \langle p_1, a_1 \rangle \geq \langle p_2, a_2 \rangle \geq \dots ,$$

such that the sequence $\{p_n\}_{n \in \omega}$ intersects every set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$, dense in $\mathbb{P}$ — hence it extends to a generic set $G = \{p \in \mathbb{P} : \exists n \, (p_n \leq p)\}$, and in addition, the union $\varphi = \bigcup_n a_n : \operatorname{dom} \varphi \to X$ of all assignments a_n satisfies:

(1) $\operatorname{ran} \varphi = X$, $\operatorname{dom} \varphi \subseteq \operatorname{PE}_t$, and

(2) for any $s \in \operatorname{PE}_t$:

   either $s \in \operatorname{dom} \varphi$ — then $s[G] \in t[G]$,

   or $q \Vdash s \notin t$ for some $q \in G$ — then $s[G] \notin t[G]$.

Due to the transitivity of both sets $t[G] = \{s[G] : s \in \operatorname{dom} \varphi\}$ and $X = \operatorname{ran} \varphi$, to prove that $t[G] = X$, it suffices to check that $\varphi(s) \in \varphi(s')$ iff $s[G] \in s'[G]$, for all names $s, s' \in \operatorname{dom} \varphi$. By the construction of φ, there is an index n such that $s, s' \in \operatorname{dom} a_n$. By definition, condition $p_n \in G$ is $(\operatorname{dom} a_n)$-complete, so p_n decides $s \in s'$.

   If $p_n \Vdash s \in s'$ then $s[G] \in s'[G]$, and on the other hand, as $\langle p_n, a_n \rangle \in \Sigma_0^+(X, t)$, we have $\varphi(s) = a_n(s) \in a_n(s') = \varphi(s')$.

   Similarly, if $q \Vdash s \notin s'$ then $s[G] \notin s'[G]$ and $\varphi(s) \notin \varphi(s')$. $\qquad\square$

   The next lemma shows that the ordinals $\lambda(X, t)$ as in Definition 6 are bounded in $\mathfrak{M}$ whenever $\Sigma^+(X, t) \neq \varnothing$.

Lemma 14 (the bounding lemma). *There is an ordinal $\lambda^*(t) \in \mathfrak{M}$ such that $\lambda(t[G], t) < \lambda^*(t)$ for every set $G \subseteq \mathbb{P}$, $\mathbb{P}$-generic over $\mathfrak{M}$. Therefore if $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ then $\Sigma^+(t[G], t) \in \mathfrak{M}$.*

Proof. Assume that a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$. Then $X = t[G] \in \mathfrak{M}[G]$, and hence $\lambda(X, t)$ is an ordinal in $\mathfrak{M}$, and its value is forced, over $\mathfrak{M}$ by a condition $p \in G$, to be equal to a certain ordinal $\lambda_p(t) \in \mathfrak{M}$. We let $\lambda^*(t) = \sup_{p \in \mathbb{P}} \lambda_p(t)$. The second part of the lemma follows from the first claim since $\Sigma^+(X, t)$ is the result of a straightforward absolute inductive construction of length $\lambda^*(t) \in \mathfrak{M}$. $\qquad\square$

Corollary 15 (= claim (i) of Theorem 11). *Tfae* :

(i) *there is a set $G \subseteq \mathbb{P}$, $\mathbb{P}$-generic over $\mathfrak{M}$, such that $t[G] = X$;*

(ii) $\Sigma^+(X, t) \neq \varnothing$;

(iii) $\Sigma^+_{\lambda^*(t)}(X, t) = \Sigma^+_{\lambda^*(t)+1}(X, t) \neq \varnothing$.

Proof. Use Lemmas 12, 13, 14. $\qquad\square$

6   Intermediate extensions: proof of the main theorem

In continuation of the proof of Theorem 11, we prove here claims (ii), (iii), (iv) of the theorem. We continue to argue in the assumptions of Definition 2.

Lemma 16 (= claim (iv) of Theorem 11)**.** *If a set* $\Gamma \subseteq \Sigma^+(X,t)$ *is* $\Sigma^+(X,t)$*-generic over* $\mathfrak{M}(X)$ *then the set* $H = \Gamma{\downarrow} \subseteq \mathbb{P}$ *is* $\mathbb{P}$*-generic over* $\mathfrak{M}$, $t[H] = X$, *and* $a[\Gamma] = a[H]$.

Proof. By Lemma 7, if a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$, is dense in $\mathbb{P}$ then the set $D^* = \{\langle p,a\rangle \in \Sigma^+(X,t) : p \in D\}$ is dense in $\Sigma^+(X,t)$ and belongs to $\mathfrak{M}(X)$. It follows that H is indeed generic.

Further, if $\langle p,a\rangle \in \Gamma \subseteq \Sigma^+(X,t)$ then by definition $\mathtt{dom}\,a \subseteq \mathrm{PE}_t$ is a finite set and if $s \in \mathtt{dom}\,a$ then $p \Vdash s \in t$ — hence, as $p \in H$, we have $s[H] \in t[H]$, that is, $s \in \mathrm{PE}_t[H]$. On the other hand, if $s \in \mathrm{PE}_t[H]$ and $x \in X$ then by Lemma 7 there is a supercondition $\langle q,b\rangle \in \Gamma$ such that $s \in \mathtt{dom}\,b$ and $x \in \mathtt{ran}\,b$. Therefore $a[\Gamma]$ maps $\mathrm{PE}_t[H]$ onto X.

Still by definition, if $\langle p,a\rangle \in \Gamma$ and $s,s' \in \mathtt{dom}\,a$, then p decides both formulas $s \in s'$ and $s = s'$, and $p \Vdash s \in s'$ iff $a(s) \in a(s')$, and the same for $=$. Therefore, if $s,s' \in \mathrm{PE}_t[H]$ then we have $s[H] = s'[H]$ if and only if $a[\Gamma](s) = a[\Gamma](s')$. We conclude that $a[\Gamma] = a[H]$.

Finally, $t[H] = \mathtt{ran}\,a[H] = \mathtt{ran}\,a[\Gamma] = X$. □

Lemma 17 (= claim (ii) of Theorem 11)**.** *If a set* $G \subseteq \mathbb{P}$ *is* $\mathbb{P}$*-generic over* $\mathfrak{M}$ *and* $X = t[G]$ *then the set*

$$G^+ = \{\langle p,a\rangle \in \Sigma^+(X,t) : p \in G \wedge a \subset a[G]\}$$

is $\Sigma^+(X,t)$*-generic over* $\mathfrak{M}(X)$, *and* $G = G^+{\downarrow}$.

Proof (lemma)**.** Otherwise there is a condition $p_0 \in G$ forcing the opposite, so that for any set $H \subseteq \mathbb{P}$, $\mathbb{P}$-generic over $\mathfrak{M}$, if $X = t[H]$ and $p_0 \in H$ then H^+ is not $\Sigma^+(X,t)$-generic over $\mathfrak{M}(X)$. By Lemma 12, $\langle p_0, \varnothing\rangle \in \Sigma^+(X,t)$.

Consider any set $\Gamma \subseteq \Sigma^+(X,t)$, $\Sigma^+(X,t)$-generic over $\mathfrak{M}(X)$ and containing $\langle p_0, \varnothing\rangle$. Then $H = (\Gamma){\downarrow}$ is $\mathbb{P}$-generic over $\mathfrak{M}(X)$ and $t[H] = X$ by Lemma 16. It remains to prove that $\Gamma = H^+$, that is, given a supercondition $\langle p,a\rangle \in \Sigma^+(X,t)$, we have $\langle p,a\rangle \in \Gamma$ iff $p \in H$ and $a \subset a[H]$.

If $\langle p,a\rangle \in \Gamma$ then by definition $p \in H = \Gamma{\downarrow}$ and $a \subseteq a[\Gamma]$, but $a[\Gamma] = a[H]$ by Lemma 16.

To prove the converse, let $\langle p,a\rangle \in \Sigma^+(X,t)$, $p \in H$, and $a \subset a[H] = a[\Gamma]$. We claim that $\langle p,a\rangle \in \Gamma$. If $s \in \mathtt{dom}\,a$ then $a \in \mathtt{dom}\,a[\Gamma]$, therefore by definition there is a condition $\langle p_s, a_s\rangle \in \Gamma$ satisfying $a \in \mathtt{dom}\,a_s$. It easily follows that there is a supercondition $\langle q,b\rangle \in \Gamma$ satisfying $q \le p$ and

$\mathtt{dom}\, a \subseteq \mathtt{dom}\, b$. Then in fact $a \subset b$ because $a, b \subset a[\Gamma]$. Therefore the supercondition $\langle q, b \rangle \in \Gamma$ is stronger than $\langle p, a \rangle \in \Sigma^+(X, t)$. We conclude that $\langle p, a \rangle$ belongs to Γ, too. $\qquad\square$

Lemma 18 (= claim (iii) of Theorem 11). *If a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$, and $X = t[G]$, then $\mathfrak{M}[G]$ is a $\Sigma^+(X, t)$-generic extension of $\mathfrak{M}(X)$.*

Proof. Note that $\mathfrak{M}[G] = \mathfrak{M}(X)[G^+]$ in the assumptions of Lemma 17. $\qquad\square$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$ (Theorem 11)

7   An example

We still argue in the assumptions of Definition 2. Consider the set

$$\Sigma(X, t) = \Sigma^+(X, t)\!\downarrow = \{p \in \mathbb{P} : \langle p, \varnothing \rangle \in \Sigma^+(X, t)\}\,;$$

thus $\Sigma(X, t) \subseteq \mathbb{P}$, and if a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ and $X = t[G]$ then $G \subseteq \Sigma(X, t)$ by Lemma 12. Is it true that, similarly to the Solovay claim (II) (Introduction), the set G is $\Sigma(X, t)$-generic over $\mathfrak{M}(X)$?

The following example easily yields a *negative* answer.

Example 19. Let $\mathbb{P}$ be the finite-support product of the Cohen forcing; a typical condition p in $\mathbb{P}$ is a map, $\mathtt{dom}\, p \subseteq \omega \times \omega$ is a finite set, and $\mathtt{ran}\, p \subseteq \omega$. Any generic set $G \subseteq \mathbb{P}$ forces reals $x_n[G]$ such that $x_n[G](i) = r$ iff there is $p \in G$ such that $p(n, i) = r$. We let $\dot{x}_n$ be the canonical name for the real $x_n[G] = \dot{x}_n[G]$, and let t be the name of the set $t[G] = \{\dot{x}_n[G] : n \in \omega\}$. In other words, $\mathfrak{M}(t[G])$ is a well-known symmetric generic extension in which AC fails and $t[G]$ is an infinite Dedekind-finite set of reals.

Sets of the form $t[G]$ are non-transitive, hence, to be in compliance with Definition 2, we define the transitive closure $U(X) = X \cup U$, where

$$U = \omega \cup \{\{m, n\} : m, n \in \omega\} \cup \{\langle m, n \rangle : m, n \in \omega\}$$

of any $X \subseteq \omega^\omega$, and accordingly let t' be the canonical name of the transitive set $t'[G] = \{\dot{x}_n[G] : n \in \omega\} \cup U$.

As sets in U belong to $\mathfrak{M}$, it will be not harmful to identify each $u \in U$ with its own canonical name $\dot{u}$. Then $\mathrm{PE}_{t'} = \{\dot{x}_n : n \in \omega\} \cup U$. $\qquad\square$

Lemma 20 (obvious). *If $p \in \mathbb{P}$ and $n, k, r \in \omega$ then $p \Vdash \dot{x}_n[G](k) = r$ iff $\langle n, k \rangle \in \mathtt{dom}\, p$ and $p(n, k) = r$.* $\qquad\square$

If $X \subseteq \omega^\omega$ then the set $\mathbb{P}^+(X \cup U, t')$ of superconditions (Definition 4) consists of all pairs $\langle p, a \rangle$ such that $p \in \mathbb{P}$, a is a map, $\mathtt{dom}\, a \subseteq \{\dot{x}_n : n \in \omega\} \cup U$ is a finite set, $\mathtt{ran}\, a \subseteq X \cup U$, $a(u) = u$ for all $u \in U \cap \mathtt{dom}\, a$, $a(\dot{x}_n) \in X$ for all $\dot{x}_n \in \mathtt{dom}\, a$, and (the completeness of Definition 4!) if a name $\dot{x}_n$ and a pair $\langle k, r \rangle$ $(n, k, r \in \omega)$ belong to $\mathtt{dom}\, a$ then p decides the formula "$\dot{x}_n[G](k) = r$", or equivalently, $\langle n, k \rangle \in \mathtt{dom}\, p$.

Note that a is a bijection for any supercondition $\langle p, a \rangle$ since $\mathbb{P}$ obviously forces any names $s \neq s'$ in $\mathrm{PE}_{t'}$ to denote different sets.

By Definition 9, if $G \subseteq \mathbb{P}$ is a generic set over $\mathfrak{M}$ then a map

$$ a[G] : \{\dot{x}_n : n \in \omega\} \cup U \xrightarrow{\text{onto}} X \cup U $$

is defined by $a[G](u) = u$ for all $u \in U$ and $a[G](\dot{x}_n) = \dot{x}_n[G]$ for all n.

Recall that $\Sigma(X \cup U, t') = \{p \in \mathbb{P} : \langle p, \varnothing \rangle \in \Sigma^+(X \cup U, t')\}$.

Lemma 21. *In the case considered, if a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ and $X = t[G]$ then*

(i) $\Sigma(X \cup U, t') = \mathbb{P}$, *and*

(ii) $\Sigma^+(X \cup U, t')$ *consists of all superconditions $\langle p, a \rangle \in \mathbb{P}^+(X \cup U, t')$ such that if both a name $\dot{x}_n$ and a pair $\langle k, r \rangle$ belong to $\mathtt{dom}\, a$ then $\langle n, k \rangle \in \mathtt{dom}\, p$, and $p(n, k) = r$ iff $a(\dot{x}_n)(k) = r$.*

Proof. (i) Let $p \in \mathbb{P}$. To prove $p \in \Sigma(X \cup U, t')$, it suffices, by Lemma 12, to define a $\mathbb{P}$-generic set $G' \subseteq \mathbb{P}$ such that still $t[G'] = X$ and $p \in G'$.

Let $N = \{n : \exists k\, (\langle n, k \rangle \in \mathtt{dom}\, p)\}$. The set $t[G] = \{\dot{x}_m[G] : m \in \omega\}$ is topologically dense in ω^ω, therefore there is a bijection $\pi : N \to \omega$ such that if $\langle n, k \rangle \in \mathtt{dom}\, p$ (hence $n \in N$) then $\dot{x}_{\pi(n)}[G](k) = p(n, k)$.

Using the permutation invariance of $\mathbb{P}$, we obtain a generic set $G' \subseteq \mathbb{P}$ such that $\dot{x}_{\pi(n)}[G] = \dot{x}_n[G']$ for all $n \in N$, still $t[G'] = t[G] = X$, and even $x_m[G] = x_m[G']$ for all but finite $m \in \omega$. Then $p \in G'$, as required.

(ii) The proof is similar. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Thus by (i) the forcing $\Sigma(X \cup U, t')$ coincides with the given forcing $\mathbb{P}$ in this case. But the set G cannot be $\mathbb{P}$-generic over $\mathfrak{M}(X)$, basically even over any smaller model $\mathfrak{M}[\dot{x}_n[G]]$, as $X = t[G] = \{\dot{x}_n[G] : n \in \omega\}$. This answers in the negative the question above in this section.

Using (ii), we can prove that $\Sigma^+(X \cup U, t')$ contains a coinitial subset in $\mathfrak{M}(X)$, order isomorphic to $\mathrm{BColl}(\{\dot{x}_n : n \in \omega\}, X)$, the bijective collapse forcing which consists of all finite partial bijections $\{\dot{x}_n : n \in \omega\} \to X$.

Corollary 22. *In the case considered in this section, the whole model $\mathfrak{M}[G]$ is a $\mathrm{BColl}(\{\dot{x}_n : n \in \omega\}, X)$-generic extension of $\mathfrak{M}(X)$.* $\qquad\square$

Most likely this result has been known since early period of forcing, although we are unable to nail a suitable reference.

8   Grigorieff's argument

To compare our approach with the basic technique of intermediate models introduced in [1], we present Grigorieff's proof of the following more abstract version of Lemma 18.

Theorem 23. *In the assumptions of Definition 2, if a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathfrak{M}$ and $X = t[G]$, then $\mathfrak{M}[G]$ is a generic extension of $\mathfrak{M}(X)$.*

Proof. Let $\alpha \in \mathbf{Ord} \cap \mathfrak{M}$ be greater than the von-Neumann rank of X. We put $Y = V_\alpha \cap \mathfrak{M}(X)$ (then $X \subseteq Y$) and let $H \subseteq \mathbb{C} = \mathrm{Coll}(\omega, Y)$ be generic over $\mathfrak{M}[G]$.[2] Then $\mathfrak{M}[G][H]$ is a generic extension of $\mathfrak{M}$ by the two-step iterated forcing theorem, and easily there is a real r such that $\mathfrak{M}(X)[H] = \mathfrak{M}[r]$.

Applying Solovay's result (III) (Introduction) we conclude that the whole model $\mathfrak{M}[G][H]$ is a generic extension of $\mathfrak{M}[r]$. But $\mathfrak{M}[r] = \mathfrak{M}(X)[H]$ is a generic extension of $\mathfrak{M}(X)$, hence $\mathfrak{M}[G][H]$ is a generic extension of $\mathfrak{M}(X)$ by the two-step iterated forcing theorem.

Now, $G \subseteq \mathfrak{M}(X)$ and $\mathfrak{M}(X) \subseteq \mathfrak{M}(X)[G] = \mathfrak{M}[G] \subseteq \mathfrak{M}[G][H]$. In other words, $\mathfrak{M}[G]$ is an intermediate model between $\mathfrak{M}(X)$ as the ground model and $\mathfrak{M}[G][H]$ as a generic extension of $\mathfrak{M}(X)$ by the choice of H. To finish the argument, Grigorieff makes use of the following result (a part of Theorem 2 in [1, 2.14], granted to Solovay), with quite a nontrivial proof.

Lemma 24. *Let $\mathbb{P}$ be a forcing in $\mathfrak{M}$, and let $G \subseteq \mathbb{P}$ be generic over $\mathfrak{M}$. If $x \in \mathfrak{M}[G]$ and $x \subseteq \mathfrak{M}$, then $\mathfrak{M}[x]$ is a generic extension of $\mathfrak{M}$.*   □

Now it suffices to apply the lemma for the models $\mathfrak{M}(X) \subseteq \mathfrak{M}[G] \subseteq \mathfrak{M}[G][H]$ in the role of the models $\mathfrak{M} \subseteq \mathfrak{M}[x] \subseteq \mathfrak{M}[G]$ in the lemma.   □

It would be interesting, of course, to track down in detail all forcing transformations in this proof, to see how the resulting forcing is related to the forcing directly given by Lemma 18. The case considered in Section 7 would be the most elementary one.

9   The property of being generic-generated is Borel

Another consequence of Lemma 14 and other results above claims that, in the assumptions of Definition 2, the set of all sets of the form $t[G]$, $G \subseteq \mathbb{P}$

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[2] It seems that we can define $Y = \mathrm{TC}(X)$ without any harm for the ensuing arguments.

being generic over $\mathfrak{M}$, is Borel in terms of an appropriate coding, of all (hereditarily countable) sets of this form, by reals. This result was first established by Zapletal (Lemma 2.4.4 in [3]) by a totally different argument using advanced technique of descriptive set theory.

In order to avoid dealing with coding in general setting, we present this result only in the simplest nontrivial (= not directly covered by Solovay's original result) case when t is a name of a set $t[G]$ which is a set of reals, by necessity at most countable.

For a real $y \in \omega^\omega$, we let $R_y = \{(y)_n : n \in \omega\} \smallsetminus \{(y)_0\}$, where $(y)_n \in \omega^\omega$ and $(y)_n(k) = y(2^n(2k+1) - 1)$ for all n and k. Thus $\{R_y : y \in \omega^\omega\}$ is the set of all at most countable sets $R \subseteq \omega^\omega$ (including the empty set).

Theorem 25. *In the assumptions of Definition 2, if $\mathbb{P}$ forces that $t[G]$ is a subset of ω^ω then the set W of all reals $y \in \omega^\omega$, such that $R_y = t[G]$ for a set $G \subseteq \mathbb{P}$ generic over $\mathfrak{M}$, is Borel.*

Proof. Let ϑ be the least ordinal not in $\mathfrak{M}$. By Corollary 15, for a real y to belong to W each of the two following conditions is necessary and sufficient:

(A) there exist an ordinal $\lambda < \vartheta$ and a sequence of sets $\Sigma_\gamma^+(X,t)$, $\gamma \leq \lambda + 1$, where $X = R_y$, satisfying Definition 6 and such that $\Sigma_\lambda^+(X,t) = \Sigma_{\lambda+1}^+(X,t) \neq \varnothing$;

(B) for any ordinal $\lambda < \vartheta$ and any sequence of sets $\Sigma_\gamma^+(X,t)$, $\gamma \leq \lambda + 1$, where $X = R_y$, satisfying Definition 6, if $\Sigma_\lambda^+(X,t) = \Sigma_{\lambda+1}^+(X,t)$ then $\Sigma_\lambda^+(X,t) \neq \varnothing$.

Condition (A) provides a $\mathbf{\Sigma}_1^1$ definition of the set W while condition (B) provides a $\mathbf{\Pi}_1^1$ definition of W, both relative to a real parameter coding the $\in$-structure of $\mathfrak{M}$. $\qquad\square$

References

[1] S. Grigorieff. Intermediate submodels and generic extensions of set theory. *Ann. Math.*, 1975, 101, pp. 447–490.

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[3] J. Zapletal. *Forcing Borel reducibility invariants*. A book in preparation, `http://people.clas.ufl.edu/zapletal/files/t3.pdf`. November 13, 2013.