A GROSZEK - LAVER PAIR OF UNDISTINGUISHABLE E_0 CLASSES

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Abstract. A generic extension $\mathbf{L}[x,y]$ of $\mathbf{L}$ by reals x,y is defined, in which the union of E_0-classes of x and y is a Π^1_2 set, but neither of these two E_0-classes is separately ordinal-definable.

1. Introduction

Let a *Groszek - Laver pair* be any unordered OD (ordinal-definable) pair $\{X,Y\}$ of sets $X,Y\subseteq\omega^\omega$ such that neither of X,Y is separately OD. As demonstrated in [3], if $\langle x,y\rangle$ is a Sacks$\times$Sacks generic pair of reals over $\mathbf{L}$, the constructible universe, then their degrees of constructibility $X=[x]_{\mathbf{L}}\cap\omega^\omega$ and $Y=[y]_{\mathbf{L}}\cap\omega^\omega$ form such a pair in $\mathbf{L}[x,y]$; the set $\{X,Y\}$ is definable as the set of all $\mathbf{L}$-degrees of reals, $\mathbf{L}$-minimal over $\mathbf{L}$.

As the sets X,Y in this example are obviously uncountable, one may ask whether there can consistently exist a Groszek – Laver pair of *countable* sets. The next theorem answers this question in the positive in a rather strong way: both sets are E_0-classes in the example! (Recall that the equivalence relation E_0 is defined on 2^ω as follows: $x\mathrel{\mathsf{E}_0}y$ iff $x(n)=y(n)$ for all but finite n.)

Theorem 1.1. *It is true in a suitable generic extension $\mathbf{L}[x,y]$ of $\mathbf{L}$, by a pair of reals $x,y\in2^\omega$ that the union of E_0-equivalence classes $[x]_{\mathsf{E}_0}\cup[y]_{\mathsf{E}_0}$ is Π^1_2, but neither of the sets $[x]_{\mathsf{E}_0},[y]_{\mathsf{E}_0}$ is separately OD.*

The forcing we employ is a conditional product $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ of an "E_0-large tree" [1] version $\mathbb{P}$ of a forcing notion, introduced in [12] to define a model with a Π^1_2 E_0-class containing no OD elements. The forcing in [12] was a clone of Jensen's minimal Π^1_2 real singleton forcing [7] (see also Section 28A of [6]), but defined on the base of the Silver forcing instead of the Sacks forcing. The crucial advantage of Silver's forcing here is that it leads to a

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[1] An E_0-large tree is a perfect tree $T\subseteq2^{<\omega}$ such that $\mathsf{E}_0\restriction[T]$ is not smooth, see [9, 10.9].

Jensen-type forcing naturally closed under the 0-1 flip at any digit, so that the corresponding extension contains a $\mathit{\Pi}^1_2$ E_0-class of generic reals instead of a $\mathit{\Pi}^1_2$ generic singleton as in [7].

In another relevant note [11] it is demonstrated that a countable OD set of reals (not an E_0-class), containing no OD elements, exists in a generic extension of $\mathbf{L}$ via the countable finite-support product of Jensen's [7] forcing itself. The existence of such a set was discussed as an open question at the *Mathoverflow* website [2] and at FOM [3], and the result in [11] was conjectured by Enayat (Footnote 3) on the base of his study of finite-support products of Jensen's forcing in [2].

The remainder of the paper is organized as follows.

We introduce E_0-large perfect trees in $2^{<\omega}$ in Section 2, study their splitting properties in Section 3, and consider E_0-large-tree forcing notions in Section 4, *i.e.*, collections of E_0-large trees closed under both restriction and action of a group of transformations naturally associated with E_0.

If $\mathtt{P}$ is an E_0-large-tree forcing notion then the *conditional product forcing* $\mathtt{P}\times_{\mathsf{E}_0}\mathtt{P}$ is a part of the full forcing product $\mathtt{P}\times\mathtt{P}$ which contains all conditions $\langle T,T'\rangle$ of trees $T,T'\in\mathtt{P}$, E_0-connected in some way. This key notion, defined in Section 5, goes back to early research on the Gandy – Harrington forcing [5, 4].

The basic E_0-large-tree forcing $\mathtt{P}$ employed in the proof of Theorem 1.1 is defined, in $\mathbf{L}$, in the form $\mathtt{P}=\bigcup_{\xi<\omega_1}\mathtt{U}_\xi$ in Section 10. The model $\mathbf{L}[x,y]$ which proves the theorem is then a $(\mathtt{P}\times_{\mathsf{E}_0}\mathtt{P})$-generic extension of $\mathbf{L}$; it is studied in Section 11. The elements $\mathtt{U}_\xi$ of this inductive construction are countable E_0-large-tree forcing notions in $\mathbf{L}$.

The key issue is, given a subsequence $\{\mathtt{U}_\eta\}_{\eta<\xi}$ and accordingly the union $\mathtt{P}_{<\xi}=\bigcup_{\eta<\xi}\mathtt{U}_\eta$, to define the next level $\mathtt{U}_\xi$. We maintain this task in Section 7 with the help of a well-known splitting/fusion construction, modified so that it yields E_0-large perfect trees. Generic aspects of this construction lead to the CCC property of $\mathtt{P}$ and $\mathtt{P}\times_{\mathsf{E}_0}\mathtt{P}$ and very simple reading of real names, but most of all to the crucial property that if $\langle x,y\rangle$ is a pair of reals $(\mathtt{P}\times_{\mathsf{E}_0}\mathtt{P})$-generic over $\mathbf{L}$ then any real $z\in\mathbf{L}[x,y]$ $\mathtt{P}$-generic over $\mathbf{L}$ belongs to $[x]_{\mathsf{E}_0}\cup[y]_{\mathsf{E}_0}$. This is Lemma 11.4 proved, on the base of preliminary results in Section 9.

The final Section 12 briefly discusses some related topics.

2. E_0-large trees

Let $2^{<\omega}$ be the set of all strings (finite sequences) of numbers $0,1$, including the empty string Λ. If $t\in 2^{<\omega}$ and $i=0,1$ then $t^\frown i$ is the extension of t by i as the rightmost term. If $s,t\in 2^{<\omega}$ then $s\subseteq t$ means that t extends

[2] A question about ordinal definable real numbers. *Mathoverflow*, March 09, 2010. http://mathoverflow.net/questions/17608.

[3] Ali Enayat. Ordinal definable numbers. FOM Jul 23, 2010. http://cs.nyu.edu/pipermail/fom/2010-July/014944.html

s, $s \subset t$ means proper extension, and $s^\frown t$ is the concatenation. If $s \in 2^{<\omega}$ then $\mathtt{lh}(s)$ is the length of s, and we let $2^n = \{s \in 2^{<\omega} : \mathtt{lh}(s) = n\}$ (strings of length n).

Let any $s \in 2^{<\omega}$ **act** on 2^ω so that $(s \cdot x)(k) = x(k) + s(k) \pmod 2$ whenever $k < \mathtt{lh}(s)$ and simply $(s \cdot x)(k) = x(k)$ otherwise. If $X \subseteq 2^\omega$ and $s \in 2^{<\omega}$ then, as usual, let $s \cdot X = \{s \cdot x : x \in X\}$.

Similarly if $s, t \in 2^{<\omega}$ and $\mathtt{lh}(s) = m \le n = \mathtt{lh}(t)$, then define $s \cdot t \in 2^n$ so that $(s \cdot t)(k) = t(k) + s(k) \pmod 2$ whenever $k < m$ and $(s \cdot t)(k) = t(k)$ whenever $m \le k < n$. If $m > n$ then let simply $s \cdot t = (s \restriction n) \cdot t$. Note that $\mathtt{lh}(s \cdot t) = \mathtt{lh}(t)$ in both cases. Let $s \cdot T = \{s \cdot t : t \in T\}$ for $T \subseteq 2^{<\omega}$.

If $T \subseteq 2^{<\omega}$ is a tree and $s \in T$ then put $T \restriction_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$.

Let **PT** be the set of all *perfect* trees $\varnothing \ne T \subseteq 2^{<\omega}$ (those with no endpoints and no isolated branches). If $T \in \mathbf{PT}$ then there is a largest string $s \in T$ such that $T = T \restriction_s$; it is denoted by $s = \mathtt{stem}(T)$ (the *stem* of T); we have $s^\frown 1 \in T$ and $s^\frown 0 \in T$ in this case. If $T \in \mathbf{PT}$ then

$$[T] = \{a \in 2^\omega : \forall\, n\, (a \restriction n \in T)\} \subseteq 2^\omega$$

is the perfect set of all *paths through* T; clearly $[S] \subseteq [T]$ iff $S \subseteq T$.

Let **LT** (large trees) be the set of all *special* E_0*-large trees*: those $T \in \mathbf{PT}$ such that there is a double sequence of non-empty strings $q_n^i = q_n^i(T) \in 2^{<\omega}$, $n < \omega$ and $i = 0, 1$, such that

- $\mathtt{lh}(q_n^0) = \mathtt{lh}(q_n^1) \ge 1$ and $q_n^i(0) = i$ for all n;
- T consists of all substrings of strings of the form $r^\frown q_0^{i(0)}{}^\frown q_1^{i(1)}{}^\frown \dots {}^\frown q_n^{i(n)}$ in $2^{<\omega}$, where $r = \mathtt{stem}(T)$, $n < \omega$, and $i(0), i(1), \dots, i(n) \in \{0, 1\}$.

We let $\mathbf{spl}_0(T) = \mathtt{lh}(r)$ and then by induction $\mathbf{spl}_{n+1}(T) = \mathbf{spl}_n(T) + \mathtt{lh}(q_n^i)$, so that $\mathbf{spl}(T) = \{\mathbf{spl}_n(T) : n < \omega\} \subseteq \omega$ is the set of *splitting levels* of T. Then

$$[T] = \{a \in 2^\omega : a \restriction \mathtt{lh}(r) = r \wedge \forall\, n\, \big(a \restriction [\mathbf{spl}_n(T), \mathbf{spl}_{n+1}(T)) = q_n^0 \text{ or } q_n^1\big)\}.$$

Lemma 2.1. *Assume that $T \in \mathbf{LT}$ and $h \in \mathbf{spl}(T)$. Then*

(i) *if $u, v \in 2^h \cap T$ then $T \restriction_v = (u \cdot v) \cdot T \restriction_u$ and $(u \cdot v) \cdot T = T$;*

(ii) *if $\sigma \in 2^{<\omega}$ then $T = \sigma \cdot T$ or $T \cap (\sigma \cdot T)$ is finite.*

Proof. (ii) Suppose that $T \cap (\sigma \cdot T)$ is infinite. Then there is an infinite branch $x \in [T]$ such that $y = \sigma \cdot x \in [T]$, too. We can assume that $\mathtt{lh}(\sigma)$ is equal to some $h = \mathbf{spl}_n(T)$. (If $\mathbf{spl}_{n-1}(T) < h < \mathbf{spl}_n(T)$ then extend σ by $\mathbf{spl}_n(T) - h$ zeros.) Then $\sigma = (x \restriction h) \cdot (y \restriction h)$. It remains to apply (i). $\square$

Example 2.2. If $s \in 2^{<\omega}$ then $T[s] = \{t \in 2^{<\omega} : s \subseteq t \vee t \subset s\}$ is a tree in **LT**, $\mathtt{stem}(T[s]) = s$, and $q_n^i(T[s]) = \langle i \rangle$ for all n, i. Note that $T[\Lambda] = 2^{<\omega}$ (the full binary tree), and $T[\Lambda] \restriction_s = (2^{<\omega}) \restriction_s = T[s]$ for all $s \in 2^{<\omega}$. $\square$

3. Splitting of large trees

The *simple splitting* of a tree $T \in \mathbf{LT}$ consists of smaller trees

$$T(\to 0) = T\restriction_{\mathtt{stem}(T)^\frown 0} \quad \text{and} \quad T(\to 1) = T\restriction_{\mathtt{stem}(T)^\frown 1},$$

so that $[T(\to i)] = \{x \in [T] : x(h) = i\}$, where $h = \mathbf{spl}_0(T) = \mathtt{lh}(\mathtt{stem}(T))$. Clearly $T(\to i) \in \mathbf{LT}$ and $\mathbf{spl}(T(\to i)) = \mathbf{spl}(T) \smallsetminus \{\mathbf{spl}_0(T)\}$.

Lemma 3.1. *If $R, S, T \in \mathbf{LT}$, $S \subseteq R(\to 0)$, $T \subseteq R(\to 1)$, $\sigma \in 2^{<\omega}$, $T = \sigma \cdot S$, and $\mathtt{lh}(\sigma) \le \mathtt{lh}(\mathtt{stem}(S)) = \mathtt{lh}(\mathtt{stem}(T))$ then $U = S \cup T \in \mathbf{LT}$, $\mathtt{stem}(U) = \mathtt{stem}(R)$, and $S = U(\to 0)$, $T = U(\to 1)$.* □

The splitting can be iterated, so that if $s \in 2^n$ then we define

$$T(\to s) = T(\to s(0))(\to s(1))(\to s(2))\dots(\to s(n-1)).$$

We separately define $T(\to \Lambda) = T$, where Λ is the empty string as usual.

Lemma 3.2. *In terms of Example 2.2, $T[s] = (2^{<\omega})(\to s) = (2^{<\omega})\restriction_s$, $\forall s$. Generally if $T \in \mathbf{LT}$ and $2^n \subseteq T$ then $T(\to s) = T\restriction_s$ for all $s \in 2^n$.* □

If $T, S \in \mathbf{LT}$ and $n \in \omega$ then let $S \subseteq_n T$ (S *n-refines* T) mean that $S \subseteq T$ and $\mathbf{spl}_k(T) = \mathbf{spl}_k(S)$ for all $k < n$. In particular, $S \subseteq_0 T$ iff simply $S \subseteq T$. By definition if $S \subseteq_{n+1} T$ then $S \subseteq_n T$ (and $S \subseteq T$), too.

Lemma 3.3. *Suppose that $T \in \mathbf{LT}$, $n < \omega$, and $h = \mathbf{spl}_n(T)$. Then*

(i) *$T = \bigcup_{s \in 2^n} T(\to s)$ and $[T(\to s)] \cap [T(\to t)] = \varnothing$ for all $s \ne t$ in 2^n;*

(ii) *if $S \in \mathbf{LT}$ then $S \subseteq_n T$* **iff** *$S(\to s) \subseteq T(\to s)$ for all strings $s \in 2^{\leqslant n}$* **iff** *$S \subseteq T$ and $S \cap 2^h = T \cap 2^h$;*

(iii) *if $s \in 2^n$ then $\mathtt{lh}(\mathtt{stem}(T(\to s))) = h$ and there is a string $u[s] \in 2^h \cap T$ such that $T(\to s) = T\restriction_{u[s]}$;*

(iv) *if $u \in 2^h \cap T$ then there is a string $s[u] \in 2^n$ s.t. $T\restriction_u = T(\to s[u])$;*

(v) *if $s_0 \in 2^n$ and $S \in \mathbf{LT}$, $S \subseteq T(\to s_0)$, then there is a unique tree $T' \in \mathbf{LT}$ such that $T' \subseteq_n T$ and $T'(\to s_0) = S$.*

Proof. (iii) Define $u[s] = \mathtt{stem}(T)^\frown q_0^{s(0)}(T)^\frown q_1^{s(1)}(T)^\frown \dots {}^\frown q_{n-1}^{s(n-1)}(T)$.

(iv) Define $s = s[u] \in 2^n$ by $s(k) = u(\mathbf{spl}_k(T))$ for all $k < n$.

(v) Let $u_0 = u[s_0] \in 2^h$. Following Lemma 2.1, define T' so that $T' \cap 2^h = T \cap 2^h$, and if $u \in T \cap 2^h$ then $T'\restriction_u = (u \cdot u_0) \cdot S$; in particular $T'\restriction_{u_0} = S$. □

Lemma 3.4 (fusion)**.** *Suppose that $\dots \subseteq_5 T_4 \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0$ is an infinite decreasing sequence of trees in $\mathbf{LT}$. Then*

(i) *$T = \bigcap_n T_n \in \mathbf{LT}$;*

(ii) *if $n < \omega$ and $s \in 2^{n+1}$ then $T(\to s) = T \cap T_n(\to s) = \bigcap_{m \ge n} T_m(\to s)$.*

Proof. Both parts are clear, just note that $\mathbf{spl}(T) = \{\mathbf{spl}_n(T_n) : n < \omega\}$. □

4. Large-tree forcing notions

Let a *large-tree forcing notion* (**LTF**) be any set $\mathtt{P} \subseteq \mathbf{LT}$ such that

(4.1) if $u \in T \in \mathtt{P}$ then $T \restriction_u \in \mathtt{P}$;

(4.2) if $T \in \mathtt{P}$ and $s \in 2^{<\omega}$ then $s \cdot T \in \mathtt{P}$.

We'll typically consider **LTF**s $\mathtt{P}$ containing the full tree $2^{<\omega}$. In this case, $\mathtt{P}$ contains all trees $T[s]$ of Example 2.2 by Lemma 3.2.

Any **LTF** $\mathtt{P}$ can be viewed as a forcing notion (if $T \subseteq T'$ then T is a stronger condition), and then it adds a real in 2^ω.

If $\mathtt{P} \subseteq \mathbf{LT}$, $T \in \mathbf{LT}$, $n < \omega$, and all split trees $T(\to s)$, $s \in 2^n$, belong to $\mathtt{P}$, then we say that T is an *n-collage over* $\mathtt{P}$. Let $\mathbf{LC}_n(\mathtt{P})$ be the set of all trees $T \in \mathbf{LT}$ which are n-collages over $\mathtt{P}$, and $\mathbf{LC}(\mathtt{P}) = \bigcup_n \mathbf{LC}_n(\mathtt{P})$. Note that $\mathbf{LC}_n(\mathtt{P}) \subseteq \mathbf{LC}_{n+1}(\mathtt{P})$ by (4.1).

Lemma 4.1. *Assume that $\mathtt{P} \subseteq \mathbf{LT}$ is a **LTF** and $n < \omega$. Then*

(i) *if $T \in \mathbf{LT}$ and $s_0 \in 2^n$ then $T(\to s_0) \in \mathtt{P}$ iff $T \in \mathbf{LC}_n(\mathtt{P})$;*

(ii) *if $P \in \mathbf{LC}_n(\mathtt{P})$, $s_0 \in 2^n$, $S \in \mathtt{P}$, and $S \subseteq P(\to s_0)$, then there is a tree $Q \in \mathbf{LC}_n(\mathtt{P})$ such that $Q \subseteq_n P$ and $Q(\to s_0) = S$;*

(iii) *if $P \in \mathbf{LC}_n(\mathtt{P})$ and a set $D \subseteq \mathtt{P}$ is open dense in $\mathtt{P}$, then there is a tree $Q \in \mathbf{LC}_n(\mathtt{P})$ such that $Q \subseteq_n P$ and $Q(\to s) \in D$ for all $s \in 2^n$;*

(iv) *if $P \in \mathbf{LC}_n(\mathtt{P})$, $S, T \in \mathtt{P}$, $s, t \in 2^n$, $S \subseteq P(\to s^\frown 0)$, $T \subseteq P(\to t^\frown 1)$, $\sigma \in 2^{<\omega}$, and $T = \sigma \cdot S$, then there is a tree $Q \in \mathbf{LC}_{n+1}(\mathtt{P})$, $Q \subseteq_{n+1} P$, such that $Q(\to s^\frown 0) \subseteq S$ and $Q(\to t^\frown 1) \subseteq T$.*

Recall that a set $D \subseteq \mathtt{P}$ is *open dense* in $\mathtt{P}$ iff, 1st, if $S \in \mathtt{P}$ then there is a tree $T \in D$, $T \subseteq S$, and 2nd, if $S \in \mathtt{P}$, $T \in D$, and $S \subseteq T$, then $S \in D$, too.

Proof. (i) If $T \in \mathbf{LC}_n(\mathtt{P})$ then by definition $T(\to s_0) \in \mathtt{P}$. To prove the converse, let $h = \mathbf{spl}_n(T)$, and let $h[s] \in 2^h \cap T$ satisfy $T(\to s) = T \restriction_{u[s]}$ for all $s \in 2^n$ by Lemma 3.3(iii). If $T(\to s_0) \in \mathtt{P}$ then $T(\to s) = T \restriction_{u[s]} = (u[s] \cdot u[s_0]) \cdot T \restriction_{u[s]}$ by Lemma 2.1, so $T(\to s) \in \mathtt{P}$ by (4.2). Thus $T \in \mathbf{LC}_n(\mathtt{P})$.

(ii) By Lemma 3.3(v) there is a tree $Q \in \mathbf{LT}$ such that $Q \subseteq_n P$ and $Q(\to s_0) = S$. We observe that Q belongs to $\mathbf{LC}_n(\mathtt{P})$ by (i).

(iii) Apply (ii) consecutively 2^n times (all $s \in 2^n$).

(iv) We first consider the case when $t = s$. If $\mathtt{lh}(\sigma) \le L = \mathtt{lh}(\mathtt{stem}(S)) = \mathtt{lh}(\mathtt{stem}(T))$ then by Lemma 3.1 $U = S \cup T \in \mathbf{LT}$, $\mathtt{stem}(U) = \mathtt{stem}(P(\to s))$, and $U(\to 0) = S$, $U(\to 1) = T$. Lemma 3.3(v) yields a tree $Q \in \mathbf{LT}$ such that $Q \subseteq_n P$ and $Q(\to s) = U$, hence $\mathtt{stem}(Q(\to s)) = \mathtt{stem}(P(\to s))$ by the above. This implies $\mathbf{spl}_n(Q) = \mathbf{spl}_n(P)$ by Lemma 3.3(iii), and hence $Q \subseteq_{n+1} P$. And finally $Q \in \mathbf{LC}_{n+1}(\mathtt{P})$ by (i) since $Q(\to s^\frown 0) = S \in \mathtt{P}$.

Now suppose that $\mathtt{lh}(\sigma) > L$. Take any string $u \in S$ with $\mathtt{lh}(u) \geq \mathtt{lh}(s)$. The set $S' = S\restriction_u \subseteq S$ belongs to $\mathbb{P}$ and obviously $\mathtt{lh}(\mathtt{stem}(S')) \geq \mathtt{lh}(\sigma)$. It remains to follow the case already considered for the trees S' and $T' = \sigma \cdot S'$.

Finally consider the general case $s \neq t$. Let $h = \mathbf{spl}_n(P)$, $H = \mathbf{spl}_{n+1}(P)$. Let $u = u[s]$ and $v = u[t]$ be the strings in $P \cap 2^h$ defined by Lemma 3.3(iii) for P, so that $P\restriction_u = P(\to s)$ and $P\restriction_v = P(\to t)$, and let $U, V \in 2^H \cap P$ be defined accordingly so that $P\restriction_U = P(\to s^\frown 1)$ and $P\restriction_V = P(\to t^\frown 1)$. Let $\rho = u \cdot v$. Then $P(\to s) = \rho \cdot P(\to t)$ by Lemma 2.1. However we have $U = u^\frown \tau$ and $V = v^\frown \tau$ for one and the same string τ, see the proof of Lemma 3.3(iii). Therefore $U \cdot V = u \cdot v = \rho$ and $P(\to s^\frown 1) = \rho \cdot P(\to t^\frown 1)$ still by Lemma 2.1.

It follows that the tree $T_1 = \rho \cdot T$ satisfies $T_1 \subseteq P(\to s^\frown 1)$. Applying the result for $s = t$, we get a tree $Q \in \mathbf{LC}_{n+1}(\mathbb{P})$, $Q \subseteq_{n+1} P$, such that $Q(\to s^\frown 0) \subseteq S$ and $Q(\to s^\frown 1) \subseteq T_1$. Then by definition $\mathbf{spl}_k(P) = \mathbf{spl}_k(Q)$ for all $k \leq n$, and $Q(\to s) \subseteq P(\to s)$ for all $s \in 2^{n+1}$ by Lemma 3.3(ii). Therefore the same strings u, v satisfy $Q\restriction_u = Q(\to s)$ and $Q\restriction_v = Q(\to t)$. The same argument as above implies $Q(\to t^\frown 1) = \rho \cdot Q(\to s^\frown 1)$. We conclude that $Q(\to t^\frown 1) \subseteq \rho \cdot T_1 = T$, as required. $\square$

5. Conditional product forcing

Along with any **LTF** $\mathbb{P}$, we'll consider the **conditional product** $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, which by definition consists of all pairs $\langle T, T' \rangle$ of trees $T, T' \in \mathbb{P}$ such that there is a string $s \in 2^{<\omega}$ satisfying $s \cdot T = T'$. We order $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ componentwise so that $\langle S, S' \rangle \leq \langle T, T' \rangle$ ($\langle S, S' \rangle$ is stronger) iff $S \subseteq T$ and $S' \subseteq T'$. [4]

Remark 5.1. $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ forces a pair of $\mathbb{P}$-generic reals. Indeed if $\langle T, T' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ with $s \cdot T = T'$ and $S \in \mathbb{P}$, $S \subseteq T$, then there is a tree $S' = s \cdot S \in \mathbb{P}$ (we make use of (4.2)) such that $\langle S, S' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ and $\langle S, S' \rangle \leq \langle T, T' \rangle$. $\square$

But $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic pairs are not necessarily generic in the sense of the true forcing product $\mathbb{P} \times \mathbb{P}$. Indeed, if say $\mathbb{P} =$ Sacks (all perfect trees) then any $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$-generic pair $\langle x, y \rangle$ has the property that x, y belong to same E_0-invariant Borel sets coded in the ground universe, while for any uncountable and co-uncountable Borel set U coded in the ground universe there is a $\mathbb{P} \times \mathbb{P}$-generic pair $\langle x, y \rangle$ with $x \in U$ and $y \notin U$.

Lemma 5.2. *Assume that* $\mathbb{P}$ *is a* **LTF**, $n \geq 1$, $P \in \mathbf{LC}_n(\mathbb{P})$, *and a set* $D \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ *is open dense in* $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$. *Then there is a tree* $Q \in \mathbf{LC}_n(\mathbb{P})$ *such that* $Q \subseteq_n P$ *and* $\langle Q(\to s), Q(\to t) \rangle \in D$ *whenever* $s, t \in 2^n$ *and* $s(n-1) \neq t(n-1)$.

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[4] Conditional product forcing notions of this kind were considered in [5, 4, 8] and some other papers with respect to the Gandy – Harrington and similar forcings, and recently in [13] with respect to many forcing notions.

Proof (compare to Lemma 4.1(iii))**.** Let $s,t\in 2^n$ be any pair with $s(n-1)\ne t(n-1)$. By the density there is a condition $\langle S,T\rangle\in D$ such that $S\subseteq P(\to s)$ and $T\subseteq P(\to t)$. Note that $T=\sigma\cdot S$ for some $s\in 2^{<\omega}$ since $\langle S,T\rangle\in \mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$. Applying Lemma 4.1(iv) ($n+1$ there corresponds to n here) we obtain a tree $P'\in \mathbf{LC}_n(\mathbb{P})$ such that $P'\subseteq_n P$ and $P'(\to s)\subseteq S$, $P'(\to t)\subseteq T$. Then $\langle P'(\to s),P'(\to t)\rangle\in D$, as D is open. Consider all pairs $s,t\in 2^n$ with $s(n-1)\ne t(n-1)$ one by one. □

Lemma 5.3. *Assume that* $\mathbb{P}$ *is a* **LTF**, $\langle T,T'\rangle\in\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$, $n<\omega$, $s,t\in 2^n$. *Then* $\langle T(\to s),T'(\to t)\rangle\in\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$.

Proof. Let $\sigma\in 2^{<\omega}$ satisfy $\sigma\cdot T=T'$. Note that $\mathbf{spl}(T)=\mathbf{spl}(T')$, hence we define $h=\mathbf{spl}_n(T)=\mathbf{spl}_n(T')$. By Lemma 3.3(iii), there are strings $u\in 2^h\cap T$ and $v\in 2^h\cap T'$ such that $T(\to s)=T\restriction_u$ and $T'(\to t)=T'\restriction_v$. Then obviously $\sigma\cdot T\restriction_u=T'\restriction_{v'}$, where $v'=\sigma\cdot u$. On the other hand $T'\restriction_v=(v\cdot v')\cdot T'\restriction_{v'}$ by Lemma 2.1. It follows that $T'\restriction_v=(v\cdot v'\cdot\sigma)\cdot T\restriction_u$, as required. □

Corollary 5.4. *Assume that* $\mathbb{P}$ *is a* **LTF**. *Then* $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ *forces* $\dot{\boldsymbol{x}}_{\texttt{left}}\not\mathrel{\mathsf{E}_0}\dot{\boldsymbol{x}}_{\texttt{right}}$, *where* $\langle\dot{\boldsymbol{x}}_{\texttt{left}},\dot{\boldsymbol{x}}_{\texttt{right}}\rangle$ *is a name of the* $(\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P})$*-generic pair.*

Proof. Otherwise a condition $\langle T,T'\rangle\in\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ forces $\dot{\boldsymbol{x}}_{\texttt{right}}=\sigma\cdot\dot{\boldsymbol{x}}_{\texttt{left}}$, where $\sigma\in 2^{<\omega}$. Find n and $s,t\in 2^n$ such that $T'(\to t)\cap(\sigma\cdot T(\to s))=\varnothing$ and apply the lemma. □

6. Multitrees

Let a *multitree* be any sequence $\varphi=\{\langle\tau_k^\varphi,h_k^\varphi\rangle\}_{k<\omega}$ such that

(6.1) if $k<\omega$ then $h_k^\varphi\in\omega\cup\{-1\}$, and the set $|\varphi|=\{k:h_k^\varphi\ne -1\}$ (the *support* of φ) is finite;

(6.2) if $k\in|\varphi|$ then $\tau_k^\varphi=\langle T_k^\varphi(0),T_k^\varphi(1),\dots,T_k^\varphi(h_k^\varphi)\rangle$, where each $T_k^\varphi(n)$ is a tree in **LT** and $T_k^\varphi(n)\subseteq_n T_k^\varphi(n-1)$ whenever $1\le n\le h_k^\varphi$, while if $k\notin|\varphi|$ then simply $\tau_k^\varphi=\Lambda$ (the empty sequence).

In this context, if $n\le h_k^\varphi$ and $s\in 2^n$ then let $T_k^\varphi(s)=T_k^\varphi(n)(\to s)$.

Let φ,ψ be multitrees. Say that φ *extends* ψ, symbolically $\psi\preccurlyeq\varphi$, if $|\psi|\subseteq|\varphi|$, and, for every $k\in|\psi|$, we have $h_k^\varphi\ge h_k^\psi$ and τ_k^φ extends τ_k^ψ, so that $T_k^\varphi(n)=T_k^\psi(n)$ for all $n\le h_k^\psi$;

If $\mathbb{P}$ is a **LTF** then let $\mathbf{MT}(\mathbb{P})$ (*multitrees over* $\mathbb{P}$) be the set of all multitrees φ such that $T_k^\varphi(n)\in\mathbf{LC}_n(\mathbb{P})$ whenever $k\in|\varphi|$ and $n\le h_k^\varphi$.

7. Jensen's extension of a large-tree forcing notion

Let $\mathbf{ZFC}'$ be the subtheory of $\mathbf{ZFC}$ including all axioms except for the power set axiom, plus the axiom saying that $\mathscr{P}(\omega)$ exists. (Then ω_1, 2^ω, and sets like **PT** exist as well.)

Definition 7.1. Let $\mathfrak{M}$ be a countable transitive model of $\mathbf{ZFC}'$. Suppose that $\mathtt{P} \in \mathfrak{M}$, $\mathtt{P} \subseteq \mathbf{LT}$ is a $\mathbf{LTF}$. Then $\mathbf{MT}(\mathtt{P}) \in \mathfrak{M}$. A set $D \subseteq \mathbf{MT}(\mathtt{P})$ is *dense in* $\mathbf{MT}(\mathtt{P})$ iff for any $\psi \in \mathbf{MT}(\mathtt{P})$ there is a multitree $\varphi \in D$ such that $\psi \preccurlyeq \varphi$.

Consider any $\preccurlyeq$-increasing sequence $\quad = \{\varphi(j)\}_{j<\omega}$ of multitrees

$$\varphi(j) = \{\langle \tau_k^{\varphi(j)}, h_k^{\varphi(j)} \rangle\}_{k<\omega} \in \mathbf{MT}(\mathtt{P}) \,,$$

generic over $\mathfrak{M}$ in the sense that it intersects every set D, $D \subseteq \mathbf{MT}(\mathtt{P})$, dense in $\mathbf{MT}(\mathtt{P})$, which belongs to $\mathfrak{M}$. Then in particular   intersects every set

$$D_{kp} = \{\varphi \in \mathbf{MT}(\mathtt{P}) : k \in |\varphi| \wedge h_k^{\varphi} \geq p\} \,, \quad k, p < \omega \,.$$

Therefore if $k < \omega$ then by definition there is an infinite sequence

$$\ldots \subseteq_5 \boldsymbol{T}_k(4) \subseteq_4 \boldsymbol{T}_k(3) \subseteq_3 \boldsymbol{T}_k(2) \subseteq_2 \boldsymbol{T}_k(1) \subseteq_1 \boldsymbol{T}_k(0)$$

of trees $\boldsymbol{T}_k(n) \in \mathbf{LC}_n(\mathtt{P})$, such that, for any j, if $k \in |\varphi(j)|$ and $n \leq h_k^{\varphi(j)}$ then $T_k^{\varphi(j)}(n) = \boldsymbol{T}_k(n)$. If $n < \omega$ and $s \in 2^n$ then we let $\boldsymbol{T}_k(s) = \boldsymbol{T}_k(n)(\to s)$; then $\boldsymbol{T}_k(s) \in \mathtt{P}$ since $\boldsymbol{T}_k(n) \in \mathbf{LC}_n(\mathtt{P})$. Then it follows from Lemma 3.4 that

$$\boldsymbol{U}_k = \textstyle\bigcap_n \boldsymbol{T}_k(n) = \bigcap_n \bigcup_{s \in 2^n} \boldsymbol{T}_k(s) \tag{1}$$

is a tree in $\mathbf{LT}$ (not necessarily in $\mathtt{P}$), as well as the trees $\boldsymbol{U}_k(\to s)$, and still by Lemma 3.4,

$$\boldsymbol{U}_k(\to s) = \boldsymbol{U}_k \cap \boldsymbol{T}_k(s) = \textstyle\bigcap_{n \geq \mathtt{lh}(s)} \boldsymbol{T}_k(n)(\to s) = \bigcap_{n \geq \mathtt{lh}(s)} \bigcup_{t \in 2^n,\, s \subseteq t} \boldsymbol{T}_k(t) \,, \tag{2}$$

and obviously $\boldsymbol{U}_k = \boldsymbol{U}_k(\to \Lambda)$.

Define a set of trees $\mathtt{U} = \{\sigma \boldsymbol{\cdot} \boldsymbol{U}_k(\to s) : k < \omega \wedge s \in 2^{<\omega} \wedge \sigma \in 2^{<\omega}\} \subseteq \mathbf{LT}$. □

The next few simple lemmas show useful effects of the genericity of ; their common motto is that the extension from $\mathtt{P}$ to $\mathtt{P} \cup \mathtt{U}$ is rather innocuous.

Lemma 7.2. *Both* $\mathtt{U}$ *and the union* $\mathtt{P} \cup \mathtt{U}$ *are* $\mathbf{LTF}$*s;* $\mathtt{P} \cap \mathtt{U} = \varnothing$.

Proof. To prove the last claim, let $T \in \mathtt{P}$ and $U = \boldsymbol{U}_k(\to s) \in \mathtt{U}$. (If $U = \sigma \boldsymbol{\cdot} \boldsymbol{U}_k(\to s)$, $\sigma \in 2^{<\omega}$, then replace T by $\sigma \boldsymbol{\cdot} T$.) The set $D(T, k)$ of all multitrees $\varphi \in \mathbf{MT}(\mathtt{P})$, such that $k \in |\varphi|$ and $T \smallsetminus T_k^{\varphi}(n)(\to s) \neq \varnothing$, where $n = h_k^{\varphi}$, belongs to $\mathfrak{M}$ and obviously is dense in $\mathbf{MT}(\mathtt{P})$. Now any multitree $\varphi(j) \in D(T, k)$ witnesses that $T \smallsetminus \boldsymbol{U}_k(\to s) \neq \varnothing$. □

Lemma 7.3. *The set* $\mathtt{U}$ *is dense in* $\mathtt{U} \cup \mathtt{P}$. *The set* $\mathtt{U} \times_{\mathsf{E}_0} \mathtt{U}$ *is dense in* $(\mathtt{P} \cup \mathtt{U}) \times_{\mathsf{E}_0} (\mathtt{P} \cup \mathtt{U})$.

Proof. Suppose that $T \in \mathtt{P}$. The set $D(T)$ of all multitrees $\varphi \in \mathbf{MT}(\mathtt{P})$, such that $T_k^{\varphi}(0) = T$ for some k, belongs to $\mathfrak{M}$ and obviously is dense in $\mathbf{MT}(\mathtt{P})$. It follows that $\varphi(j) \in D(T)$ for some j, by the choice of . Then $\boldsymbol{T}_k(\Lambda) = T$ for some k. However by construction $\boldsymbol{U}_k(\to \Lambda) = \boldsymbol{U}_k \subseteq \boldsymbol{T}_k(\Lambda)$.

Now suppose that $\langle T,T'\rangle \in \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$, so that $T'=\sigma\cdot T$, $\sigma\in 2^{<\omega}$. By Lemma 7.2 ($\mathtt{P}\cap\mathtt{U}=\varnothing$) it is impossible that one of the trees T,T' belongs to $\mathtt{P}$ and the other one to $\mathtt{U}$. Therefore we can assume that $T,T'\in\mathtt{P}$. By the first claim of the lemma, there is a tree $U\in\mathtt{U}$, $U\subseteq T$. Then $U'=\sigma\cdot U\in\mathtt{U}$ and still $U'=\sigma\cdot U$, hence $\langle U,U'\rangle\in\mathtt{U}\times_{\mathsf{E}_0}\mathtt{U}$, and it extends $\langle T,T'\rangle$. □

Lemma 7.4. *If $k,l<\omega$, $k\neq l$, and $\sigma\in 2^{<\omega}$ then $\boldsymbol{U}_k\cap(\sigma\cdot\boldsymbol{U}_l)=\varnothing$.*

Proof. The set $D'(k,l)$ of all multitrees $\varphi\in\mathbf{MT}(\mathtt{P})$, such that $k,l\in|\varphi|$ and $T^\varphi_k(n)\cap(\sigma\cdot T^\varphi_l(m))=\varnothing$ for some $n\le h^\varphi_k$, $m\le h^\varphi_l$, belongs to $\mathfrak{M}$ and is dense in $\mathbf{MT}(\mathtt{P})$. So $\varphi(j)\in D'(k,l)$ for some $j<\omega$. But then for some n,m we have $\boldsymbol{U}_k\cap(\sigma\cdot\boldsymbol{U}_l)\subseteq T^{\varphi(j)}_k(n)\cap(\sigma\cdot T^{\varphi(j)}_l(m))=\varnothing$. □

Corollary 7.5. *If $\langle U,U'\rangle\in\mathtt{U}\times_{\mathsf{E}_0}\mathtt{U}$ then there exist: $k<\omega$, strings $s,s'\in 2^{<\omega}$ with $\mathtt{lh}(s)=\mathtt{lh}(s')$, and strings $\sigma,\sigma'\in 2^{<\omega}$, such that $U=\sigma\cdot\boldsymbol{U}_k(\to s)$ and $U'=\sigma'\cdot\boldsymbol{U}_k(\to s')$.*

Proof. By definition, we have $U=\sigma\cdot\boldsymbol{U}_k(\to s)$ and $U'=\sigma'\cdot\boldsymbol{U}_{k'}(\to s')$, for suitable $k,k'<\omega$ and $s,s',\sigma,\sigma'\in 2^{<\omega}$. As $\langle U,U'\rangle\in\mathtt{U}\times_{\mathsf{E}_0}\mathtt{U}$, it follows from Lemma 7.4 that $k'=k$, hence $U'=\sigma\cdot\boldsymbol{U}_k(\to s')$. Therefore $\sigma\cdot\boldsymbol{U}_k(\to s)=\tau\cdot\sigma'\cdot\boldsymbol{U}_k(\to s')$ for some $\tau\in 2^{<\omega}$. In other words, $\boldsymbol{U}_k(\to s)=\tau'\cdot\boldsymbol{U}_k(\to s')$, where $\tau'=\sigma\cdot\sigma'\cdot\tau\in 2^{<\omega}$. It easily follows that $\mathtt{lh}(s)=\mathtt{lh}(s')$. □

The two following lemmas show that, due to the generic character of extension, those pre-dense sets which belong to $\mathfrak{M}$, remain pre-dense in the extended forcing.

Let $X\subseteq^{\mathtt{fin}}\bigcup D$ mean that there is a finite set $D'\subseteq D$ with $X\subseteq\bigcup D'$.

Lemma 7.6. *If a set $D\in\mathfrak{M}$, $D\subseteq\mathtt{P}$ is pre-dense in $\mathtt{P}$, and $U\in\mathtt{U}$, then $U\subseteq^{\mathtt{fin}}\bigcup D$. Moreover D is pre-dense in $\mathtt{U}\cup\mathtt{P}$.*

Proof. We can assume that D is in fact open dense in $\mathtt{P}$. (Otherwise replace it with the set $D'=\{T\in\mathtt{P}:\exists\, S\in D\,(T\subseteq S)\}$ which also belongs to $\mathfrak{M}$.)

We can also assume that $U=\boldsymbol{U}_k(\to s)\in\mathtt{U}$, where $k<\omega$ and $s\in 2^{<\omega}$. (The general case, when $U=\sigma\cdot\boldsymbol{U}_k(\to s)$ for some $\sigma\in 2^{<\omega}$, is reducible to the case $U=\boldsymbol{U}_k(\to s)$ by substituting the set $\sigma\cdot D$ for D.)

The set $\Delta\in\mathfrak{M}$ of all multitrees $\varphi\in\mathbf{MT}(\mathtt{P})$ such that $k\in|\varphi|$, $\mathtt{lh}(s)<h=h^\varphi_k$, and $T^\varphi_k(h)(\to t)\in D$ for all $t\in 2^h$, is dense in $\mathbf{MT}(\mathtt{P})$ by Lemma 4.1(iii) and the open density of D. Therefore there is an index j such that $\varphi(j)\in\Delta$. Let $h(j)=h^{\varphi(j)}_k$. Then the tree $S_t=T^{\varphi(j)}_k(h(j))(\to t)=\boldsymbol{T}_k(h(j))(\to t)=\boldsymbol{T}_k(t)$ belongs to D for all $t\in 2^{h(j)}$. We conclude that

$$U=\boldsymbol{U}_k(\to s)\subseteq\boldsymbol{U}_k\subseteq\textstyle\bigcup_{t\in 2^{h(j)}}\boldsymbol{T}_k(t)\subseteq\bigcup_{t\in 2^{h(j)}}S_t=\bigcup D',$$

where $D'=\{S_t:t\in 2^{h(j)}\}\subseteq D$ is finite.

To prove the pre-density claim, pick a string $t\in 2^{h(j)}$ with $s\subset t$. Then $V=\boldsymbol{U}_k(\to t)\in\mathtt{U}$ and $V\subseteq U$. However $V\subseteq\boldsymbol{T}_k(t)=S_t\in D$. Thus V witnesses that U is compatible with $S_t\in D$ in $\mathtt{U}\cup\mathtt{P}$, as required. □

Lemma 7.7. *If a set $D \in \mathfrak{M}$, $D \subseteq \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ is pre-dense in $\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ then D is pre-dense in $(\mathtt{P} \cup \mathtt{U}) \times_{\mathsf{E}_0} (\mathtt{P} \cup \mathtt{U})$.*

Proof. Let $\langle U, U' \rangle \in \mathtt{U} \times_{\mathsf{E}_0} \mathtt{U}$; the goal is to prove that $\langle U, U' \rangle$ is compatible in $(\mathtt{P} \cup \mathtt{U}) \times_{\mathsf{E}_0} (\mathtt{P} \cup \mathtt{U})$ with a condition $\langle T, T' \rangle \in D$. By Corollary 7.5, there exist: $k < \omega$ and strings $s, s', \sigma, \sigma' \in 2^{<\omega}$ such that $\mathtt{lh}(s) = \mathtt{lh}(s')$ and $U = \sigma \cdot \boldsymbol{U}_k(\to s)$, $U' = \sigma' \cdot \boldsymbol{U}_k(\to s')$. As in the proof of the previous lemma, we can assume that $\sigma = \sigma' = \Lambda$, so that $U = \boldsymbol{U}_k(\to s)$, $U' = \boldsymbol{U}_k(\to s')$. (The general case is reducible to this case by substituting the set $\{\langle \sigma \cdot T, \sigma' \cdot T' \rangle : \langle T, T' \rangle \in D\}$ for D.)

Assume that D is in fact open dense.

Consider the set $\Delta \in \mathfrak{M}$ of all multitrees $\varphi \in \mathbf{MT}(\mathtt{P})$ such that $k \in |\varphi|$, $\mathtt{lh}(s) = \mathtt{lh}(s') = n < h = h_k^\varphi$, and $\langle T_k^\varphi(h)(\to u), T_k^\varphi(h)(\to u') \rangle \in D$ whenever $u, u' \in 2^h$ and $u(h-1) \neq u'(h-1)$. The set Δ is dense in $\mathbf{MT}(\mathtt{P})$ by Lemma 5.2. Therefore $\varphi(j) \in \Delta$ for some j, so that if $u, u' \in 2^{h(j)}$, where $h(j) = h_k^{\varphi(j)} > n$, and $u(h(j)-1) \neq u'(h(j)-1)$, then

$$\langle T_k^{\varphi(j)}(h(j))(\to u), T_k^{\varphi(j)}(h(j))(\to u') \rangle = \langle \boldsymbol{T}_k(u), \boldsymbol{T}_k(u') \rangle \in D\,.$$

Now, as $h(j) > n$, let us pick $u, u' \in 2^{h(j)}$ such that $u(h(j)-1) \neq u'(h(j)-1)$ and $s \subset u$, $s' \subset u'$. Then $\langle \boldsymbol{T}_k(u), \boldsymbol{T}_k(u') \rangle \in D$. On the other hand, the pair $\langle \boldsymbol{U}_k(\to u), \boldsymbol{U}_k(\to u') \rangle$ belongs to $\mathtt{U} \times_{\mathsf{E}_0} \mathtt{U}$ by Lemma 5.3,

$$\langle \boldsymbol{U}_k(\to u), \boldsymbol{U}_k(\to u') \rangle \leq \langle \boldsymbol{U}_k(\to s), \boldsymbol{U}_k(\to s') \rangle\,,$$

and finally we have $\langle \boldsymbol{U}_k(\to u), \boldsymbol{U}_k(\to u') \rangle \leq \langle \boldsymbol{T}_k(u), \boldsymbol{T}_k(u') \rangle$. We conclude that the given condition $\langle \boldsymbol{U}_k(\to s), \boldsymbol{U}_k(\to s') \rangle$ is compatible with the condition $\langle \boldsymbol{T}_k(u), \boldsymbol{T}_k(u') \rangle \in D$, as required. □

8. Real names

In this Section, we assume that $\mathtt{P}$ is a **LTF** and $2^{<\omega} \in \mathtt{P}$. It follows by (4.1) that all trees $T[s] = (2^{<\omega})(\to s)$ (see Example 2.2) also belong to $\mathtt{P}$.

Recall that $\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ adds a pair of reals $\langle x_{\mathtt{left}}, x_{\mathtt{right}} \rangle \in 2^\omega \times 2^\omega$.

Arguing in the conditions of Definition 7.1, the goal of the following Theorem 9.3 will be to prove that, for any $(\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P})$-name c of a real in 2^ω, it is forced by the extended forcing $(\mathtt{P} \cup \mathtt{U}) \times_{\mathsf{E}_0} (\mathtt{P} \cup \mathtt{U})$ that c does not belong to sets of the form $[U]$, where U is a tree in $\mathtt{U}$, **unless** c is a name of one of reals in the E_0-class of one of the generic reals $x_{\mathtt{left}}, x_{\mathtt{right}}$ themselves.

We begin with a suitable notation.

Definition 8.1. A $(\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P})$-*real name* is a system $\mathbf{c} = \{C_n^i\}_{n<\omega, i<2}$ of sets $C_n^i \subseteq \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ such that each set $C_n = C_n^0 \cup C_n^1$ is pre-dense in $\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ and any conditions $\langle S, S' \rangle \in C_n^0$ and $\langle T, T' \rangle \in C_n^1$ are incompatible in $\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$.

If a set $G \subseteq \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ is $(\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P})$-generic at least over the collection of all sets C_n then we define $\mathbf{c}[G] \in 2^\omega$ so that $\mathbf{c}[G](n) = i$ iff $G \cap C_n^i \neq \varnothing$. □

Any $(\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P})$-real name $\mathbf{c} = \{C_n^i\}$ induces (can be understood as) a $(\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P})$-name (in the ordinary forcing notation) for a real in 2^ω.

Definition 8.2 (actions)**.** Strings in $2^{<\omega}$ can act on names $\mathbf{c} = \{C_n^i\}_{n<\omega, i<2}$ in two ways, related either to conditions or to the output.

If $\sigma, \sigma' \in 2^{<\omega}$ then define a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\langle\sigma, \sigma'\rangle \circ \mathbf{c} = \{\langle\sigma, \sigma'\rangle \cdot C_n^i\}$, where $\langle\sigma, \sigma'\rangle \cdot C_n^i = \{\langle\sigma \cdot T, \sigma' \cdot T'\rangle : \langle T, T'\rangle \in C_n^i\}$ for all n, i.

If $\rho \in 2^{<\omega}$ then define a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\rho \cdot \mathbf{c} = \{C\rho_n^i\}$, where $C\rho_n^i = C_n^{1-i}$ whenever $n < \mathtt{lh}(\rho)$ and $\rho(n) = 1$, but $C\rho_n^i = C_n^i$ otherwise. □

Both actions are idempotent. The difference between them is as follows. If $G \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic set then $(\langle\sigma, \sigma'\rangle \circ \mathbf{c})[G] = \mathbf{c}[\langle\sigma, \sigma'\rangle \circ G]$, where $\langle\sigma, \sigma'\rangle \circ G = \{\langle\sigma \cdot T, \sigma' \cdot T'\rangle : \langle T, T'\rangle \in G\}$, while $(\rho \cdot \mathbf{c})[G] = \rho \cdot (\mathbf{c}[G])$.

Example 8.3. Define a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\dot{\boldsymbol{x}}_{\mathtt{left}} = \{C_n^i\}_{n<\omega, i<2}$ such that each set $C_n^i \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ contains all pairs of the form $\langle T[s], T[t]\rangle$, where $s, t \in 2^{n+1}$ and $s(n) = i$, and a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\dot{\boldsymbol{x}}_{\mathtt{right}} = \{C_n^i\}_{n<\omega, i<2}$ such that accordingly each set $C_n^i \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ contains all pairs $\langle T[s], T[t]\rangle$, where $s, t \in 2^{n+1}$ and now $t(n) = i$. □

Then $\dot{\boldsymbol{x}}_{\mathtt{left}}$, $\dot{\boldsymbol{x}}_{\mathtt{right}}$ are names of the $\mathbb{P}$-generic reals $x_{\mathtt{left}}$, resp., $x_{\mathtt{right}}$, and each name $\sigma \cdot \dot{\boldsymbol{x}}_{\mathtt{left}}$ $(\sigma \in 2^{<\omega})$ induces a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-name of the real $\sigma \cdot (x_{\mathtt{left}}[G])$; the same for $_{\mathtt{right}}$.

9. Direct forcing a real to avoid a tree

Let $\mathbf{c} = \{C_n^i\}$, $\mathbf{d} = \{D_n^i\}$ be $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real names. Say that a condition $\langle T, T'\rangle \in \mathbf{LT} \times_{\mathsf{E}_0} \mathbf{LT}$:

- *directly forces* $\mathbf{c}(n) = i$, where $n < \omega$, $i = 0, 1$, if $\langle T, T'\rangle \le \langle S, S'\rangle$ for some $\langle S, S'\rangle \in C_n^i$;
- *directly forces* $s \subset \mathbf{c}$, where $s \in 2^{<\omega}$, iff for all $n < \mathtt{lh}(s)$, $\langle T, T'\rangle$ directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;
- *directly forces* $\mathbf{d} \ne \mathbf{c}$, iff there are strings $s, t \in 2^{<\omega}$, incomparable in $2^{<\omega}$ and such that $\langle T, T'\rangle$ directly forces $s \subset \mathbf{c}$ and $t \subset \mathbf{d}$;
- *directly forces* $\mathbf{c} \notin [U]$, where $U \in \mathbf{PT}$, iff there is a string $s \in 2^{<\omega} \smallsetminus U$ such that $\langle T, T'\rangle$ directly forces $s \subset \mathbf{c}$.

Lemma 9.1. *If $S \in \mathbb{P}$, $\langle R, R'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, and $\mathbf{c}$ is a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name, then there exists a tree $S' \in \mathbb{P}$ and a condition $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T'\rangle \le \langle R, R'\rangle$, such that $S' \subseteq S$ and $\langle T, T'\rangle$ directly forces $\mathbf{c} \notin [S']$.*

Proof. Clearly there is a condition $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T'\rangle \le \langle R, R'\rangle$, which directly forces $u \subset \mathbf{c}$ for some $u \in 2^{<\omega}$ satisfying $\mathtt{lh}(u) > \mathtt{lh}((\mathtt{stem}(S)))$. There is a string $v \in S$, $\mathtt{lh}(v) = \mathtt{lh}(u)$, incomparable with u. The tree $S' = S\restriction_v$ belongs to $\mathbb{P}$, $S' \subseteq S$ by construction, and obviously $\langle T, T'\rangle$ directly forces $\mathbf{c} \notin [S']$. □

Lemma 9.2. *If $\mathbf{c}$ is a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name, $\sigma \in 2^{<\omega}$, and a condition $\langle R, R'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ directly forces $\sigma \cdot \mathbf{c} \ne \dot{\boldsymbol{x}}_{\mathtt{left}}$, resp., $\sigma \cdot \mathbf{c} \ne \dot{\boldsymbol{x}}_{\mathtt{right}}$, then there is a stronger condition $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T'\rangle \le \langle R, R'\rangle$, which directly forces resp. $\mathbf{c} \notin [\sigma \cdot T]$, $\mathbf{c} \notin [\sigma \cdot T']$.*

Proof. We just prove the "left" version, as the "right" version can be proved similarly. So let's assume that $\langle R, R' \rangle$ directly forces $\mathbf{c} \neq \dot{\boldsymbol{x}}_{\mathtt{left}}$. There are incomparable strings $u, v \in 2^{<\omega}$ such that $\langle R, R' \rangle$ directly forces $u \subset \sigma \cdot \mathbf{c}$, hence, $\sigma \cdot u \subset \mathbf{c}$ as well, and also directly forces $v \subset \dot{\boldsymbol{x}}_{\mathtt{left}}$. Then by necessity $v \in R$, hence $T = R{\restriction}_v \in \mathtt{P}$, but $u \notin T$. Let $T' = \rho \cdot T$, where $\rho \in 2^{<\omega}$ satisfies $R' = \rho \cdot R$. By definition, the condition $\langle T, T' \rangle \in \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ directly forces $\mathbf{c} \notin [\sigma \cdot T]$ (witnessed by $s = \sigma \cdot u$), as required. $\square$

Theorem 9.3. *With the assumptions of Definition 7.1, suppose that* $\mathbf{c} = \{C^i_m\}_{m<\omega, i<2} \in \mathfrak{M}$ *is a* $(\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P})$*-real name, and for every* $\sigma \in 2^{<\omega}$ *the set*

$$D_\sigma = \{\langle T, T' \rangle \in \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P} : \langle T, T' \rangle \text{ directly forces } \mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\mathtt{left}} \text{ and } \mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\mathtt{right}}\}$$

is dense in $\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$. *Let* $\langle W, W' \rangle \in (\mathtt{P} \cup \mathtt{U}) \times_{\mathsf{E}_0} (\mathtt{P} \cup \mathtt{U})$ *and* $U \in \mathtt{U}$.

Then there is a stronger condition $\langle V, V' \rangle \in \mathtt{U} \times_{\mathsf{E}_0} \mathtt{U}$, $\langle V, V' \rangle \leq \langle W, W' \rangle$, *which directly forces* $\mathbf{c} \notin [U]$.

Proof. By construction, $U = \rho \cdot \boldsymbol{U}_K(\to s_0)$, where $K < \omega$ and $\rho, s_0 \in 2^{<\omega}$; we can assume that simply $s_0 = \Lambda$, so that $U = \rho \cdot \boldsymbol{U}_K$. Moreover we can assume that $\rho = \Lambda$ as well, so that $U = \boldsymbol{U}_K$ (for if not then replace $\mathbf{c}$ with $\rho \cdot \mathbf{c}$).

Further, by Corollary 7.5, we can assume that $W = \sigma \cdot \boldsymbol{U}_L(\to t_0) \in \mathtt{U}$ and $W' = \sigma' \cdot \boldsymbol{U}_L(\to t'_0) \in \mathtt{U}$, where $L < \omega$, $t_0, t'_0 \in 2^{<\omega}$, $\mathtt{lh}(t_0) = \mathtt{lh}(t'_0)$, and $\sigma, \sigma' \in 2^{<\omega}$. And moreover we can assume that $\sigma = \sigma' = \Lambda$, so that $W = \boldsymbol{U}_L(\to t_0)$ and $W' = \boldsymbol{U}_L(\to t'_0)$ (for if not then replace $\mathbf{c}$ with $\langle \sigma, \sigma' \rangle \circ \mathbf{c}$).

The indices K, L involved can be either equal or different.

There is an index J such that the multitree $\varphi(J)$ satisfies $K, L \in |\varphi(J)|$ and $h^{\varphi(J)}_L \geq h_0 = \mathtt{lh}(t_0) = \mathtt{lh}(t'_0)$, so that the trees $S_0 = T^{\varphi(J)}_K(0) = \boldsymbol{T}_K(0)$,

$$T_0 = T^{\varphi(J)}_L(h_0)(\to t_0) = \boldsymbol{T}_L(t_0)\,, \quad T'_0 = T^{\varphi(J)}_L(h_0)(\to t'_0) = \boldsymbol{T}_L(t'_0)$$

in $\mathtt{P}$ are defined. Note that $U \subseteq S_0$ and $W \subseteq T_0$, $W' \subseteq T'_0$ under the above assumptions.

Let $\mathscr{D}$ be the set of all multitrees $\varphi \in \mathbf{MT}(\mathtt{P})$ such that $\varphi(J) \preccurlyeq \varphi$ and for every pair $t, t' \in 2^n$, where $n = h^\varphi_L$, such that $t(n-1) \neq t'(n-1)$, the condition $\langle T^\varphi_L(t), T^\varphi_L(t') \rangle$ directly forces $\mathbf{c} \notin [T^\varphi_K(m)]$, where $m = h^\varphi_K$.

Claim 9.4. $\mathscr{D}$ *is dense in* $\mathbf{MT}(\mathtt{P})$ *above* $\varphi(J)$.

Proof. Let a multitree $\psi \in \mathbf{MT}(\mathtt{P})$ satisfy $\varphi(J) \preccurlyeq \psi$; the goal is to define a multitree $\varphi \in \mathscr{D}$, $\psi \preccurlyeq \varphi$. Let $m = h^\psi_K$, $n = h^\psi_L$, $Q = T^\psi_K(m)$, $P = T^\psi_L(n)$.

Case 1: $K \neq L$. Consider any $s \in 2^{m+1}$ and $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$. By Lemma 9.1, there is a tree $S \in \mathtt{P}$ and a condition $\langle R, R' \rangle \in \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ such that $S \subseteq Q(\to s)$, $\langle R, R' \rangle \leq \langle P(\to t), P(\to t') \rangle$, and $\langle R, R' \rangle$ directly forces $\mathbf{c} \notin [S]$. By Lemma 4.1(ii),(iv) there are trees $Q_1 \in \mathbf{LC}_{m+1}(\mathtt{P})$ and $P_1 \in \mathbf{LC}_{n+1}(\mathtt{P})$ such that $Q_1 \subseteq_{m+1} Q$, $P_1 \subseteq_{n+1} P$, $Q_1(\to s) = S$ and $\langle P_1(\to t), P_1(\to t') \rangle \leq \langle R, R' \rangle$.

Repeat this procedure so that all strings $s \in 2^{m+1}$ and all pairs of strings $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$ are considered. We obtain trees $Q' \in \mathbf{LC}_{m+1}(\mathtt{P})$

and $P' \in \mathbf{LC}_{n+1}(\mathtt{P})$ such that $Q' \subseteq_{m+1} Q$, $P' \subseteq_{n+1} P$, and if $s \in 2^{m+1}$ and $t, t' \in 2^{n+1}$, $t(n) \neq t'(n)$, the condition $\langle P'(\to t), P'(\to t')\rangle$ directly forces $\mathbf{c} \notin [Q'(\to s)]$ — hence directly forces $\mathbf{c} \notin [Q']$.

Now define a multitree $\varphi \in \mathbf{MT}(\mathtt{P})$ so that $|\varphi| = |\psi|$, $h^\varphi_k = h^\psi_k$ and $\tau^\varphi_k = \tau^\psi_k$ for all $k \notin \{K, L\}$, $h^\varphi_K = m+1$, $h^\varphi_L = n+1$, and $T^\varphi_K(m+1) = P'$, $T^\varphi_L(n+1) = Q'$ as the new elements of the Kth and Lth components. We have $\varphi \in \mathscr{D}$ and $\psi \preccurlyeq \varphi$ by construction. (Use the fact that $P' \subseteq_{n+1} P$ and $Q' \subseteq_{m+1} Q$.)

Case 2: $L = K$, and hence $m = n$ and $P = Q$. Let $h = \mathbf{spl}_n(P)$. Consider any pair $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$. In our assumptions there is a condition $\langle U, U'\rangle \in \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$, $\langle U, U'\rangle \leq \langle T(\to t), T(\to t')\rangle$, which directly forces both $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\mathtt{left}}$ and $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\mathtt{right}}$ for any $\sigma \in 2^h$. By Lemma 9.2, there is a stronger condition $\langle T, T'\rangle \in \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$, $\langle T, T'\rangle \leq \langle U, U'\rangle$, which directly forces both $\mathbf{c} \notin [\sigma \cdot T]$ and $\mathbf{c} \notin [\sigma \cdot T']$ still for all $\sigma \in 2^h$. Then as in Case 1, there is a tree $P_1 \in \mathbf{LC}_{n+1}(\mathtt{P})$, $P_1 \subseteq_{n+1} P$, such that $P_1(\to t) \subseteq T$, $P_1(\to t') \subseteq T'$.

We claim that $\langle T, T'\rangle$ directly forces $\mathbf{c} \notin [P_1]$, or equivalently, directly forces $\mathbf{c} \notin [P_1(\to s^\frown i)]$ for any $s^\frown i \in 2^{n+1}$ (then $s \in 2^n$). Indeed if $s^\frown i \in 2^{n+1}$ then $P_1(\to s^\frown i) = \sigma \cdot P_1(\to t)$ or $= \sigma \cdot P_1(\to t')$ for some $\sigma \in 2^h$ by the choice of h. Therefore $P_1(\to s^\frown i)$ is a subtree of one of the two trees $\sigma \cdot T$ and $\sigma \cdot T'$. The claim now follows from the choice of $\langle T, T'\rangle$. We conclude that the stronger condition $\langle P_1(\to t), P_1(\to t')\rangle \leq \langle T, T'\rangle$ also directly forces $\mathbf{c} \notin [P_1]$.

Repeat this procedure so that all pairs of strings $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$ are considered. We obtain a tree $P' \in \mathbf{LC}_{n+1}(\mathtt{P})$ such that $P' \subseteq_{n+1} P$, and if $t, t' \in 2^{n+1}$, $t(n) \neq t'(n)$, then $\langle P'(\to t), P'(\to t')\rangle$ directly forces $\mathbf{c} \notin [P']$.

Similar to Case 1, define a multitree $\varphi \in \mathbf{MT}(\mathtt{P})$ so that $|\varphi| = |\psi|$, $h^\varphi_k = h^\psi_k$ and $\tau^\varphi_k = \tau^\psi_k$ for all $k \neq K$, $h^\varphi_K = n+1$, and $T^\varphi_K(n+1) = P'$ as the new element of the $(K = L)$th component. Then $\varphi \in \mathscr{D}$, $\psi \preccurlyeq \varphi$. □ (Claim)

We come back to the proof of Theorem 9.3. The lemma implies that there is an index $j \geq J$ such that the multitree $\varphi(j)$ belongs to $\mathscr{D}$. Let $n = h^{\varphi(j)}_L$, $m = h^{\varphi(j)}_K$. Pick strings $t, t' \in 2^n$ such that $t_0 \subset t$, $t'_0 \subset t'$, $t(n) \neq t'(n)$. Let

$$T = T^{\varphi(j)}_L(t) = \boldsymbol{T}_L(t),\ T' = T^{\varphi(j)}_L(t') = \boldsymbol{T}_L(t'),\ S = T^{\varphi(j)}_K(m) = \boldsymbol{T}_K(m).$$

Then $\langle T, T'\rangle \in \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$, $\langle T, T'\rangle \leq \langle T_0, T'_0\rangle$, and $\langle T, T'\rangle$ directly forces $\mathbf{c} \notin [S]$.

Consider the condition $\langle V, V'\rangle \in \mathtt{U} \times_{\mathsf{E}_0} \mathtt{U}$, where $V = \boldsymbol{U}_L(\to t)$ and $V' = \boldsymbol{U}_L(\to t')$ belong to $\mathtt{U}$. (Recall that $V = \boldsymbol{U}_L(\to t)$ and $V' = \boldsymbol{U}_L(\to t')$, and hence $V' = \sigma \cdot V$ for a suitable $\sigma \in 2^{<\omega}$.) By construction we have both $\langle V, V'\rangle \leq \langle W, W'\rangle$ (as $t_0 \subseteq t, t'$) and $\langle V, V'\rangle \leq \langle T, T'\rangle \leq \langle T_0, T'_0\rangle$. Therefore $\langle V, V'\rangle$ directly forces $\mathbf{c} \notin [S]$. And finally, we have $U \subseteq T^{\varphi(j)}_K(m) = S$, so that $\langle V, V'\rangle$ directly forces $\mathbf{c} \notin [U]$, as required. □ (Theorem 9.3)

10. Jensen's forcing

In this section, **we argue in $\mathbf{L}$, the constructible universe.** Let $\le_{\mathbf{L}}$ be the canonical wellordering of $\mathbf{L}$.

Definition 10.1 (in $\mathbf{L}$)**.** Following the construction in [7, Section 3] *mutatis mutandis*, define, by induction on $\xi < \omega_1$, a countable **LTF** $\mathtt{U}_\xi \subseteq \mathbf{LT}$ as follows.

Let $\mathtt{U}_0$ consist of all trees of the form $T[s]$, see Example 2.2.

Suppose that $0 < \lambda < \omega_1$, and countable **LTF**s $\mathtt{U}_\xi \subseteq \mathbf{LT}$ are defined for $\xi < \lambda$. Let $\mathfrak{M}_\lambda$ be the least model $\mathfrak{M}$ of $\mathbf{ZFC}'$ of the form $\mathbf{L}_\kappa$, $\kappa < \omega_1$, containing $\{\mathtt{U}_\xi\}_{\xi<\lambda}$ and such that $\lambda < \omega_1^{\mathfrak{M}}$ and all sets $\mathtt{U}_\xi$, $\xi < \lambda$, are countable in $\mathfrak{M}$. Then $\mathtt{P}_\lambda = \bigcup_{\xi<\lambda} \mathtt{U}_\xi$ is countable in $\mathfrak{M}$, too. Let $\{\varphi(j)\}_{j<\omega}$ be the $\le_{\mathbf{L}}$-least sequence of multitrees $\varphi(j) \in \mathbf{MT}(\mathtt{P}_\lambda)$, $\preccurlyeq$-increasing and generic over $\mathfrak{M}_\lambda$. Define $\mathtt{U}_\lambda = \mathtt{U}$ as in Definition 7.1. This completes the inductive step.

Let $\mathtt{P} = \bigcup_{\xi<\omega_1} \mathtt{U}_\xi$. □

Proposition 10.2 (in $\mathbf{L}$)**.** *The sequence $\{\mathtt{U}_\xi\}_{\xi<\omega_1}$ belongs to $\varDelta_1^{\mathrm{HC}}$.* □

Lemma 10.3 (in $\mathbf{L}$)**.** *If a set $D \in \mathfrak{M}_\xi$, $D \subseteq \mathtt{P}_\xi$ is pre-dense in $\mathtt{P}_\xi$ then it remains pre-dense in $\mathtt{P}$. Therefore if $\xi < \omega_1$ then $\mathtt{U}_\xi$ is pre-dense in $\mathtt{P}$.*

If a set $D \in \mathfrak{M}_\xi$, $D \subseteq \mathtt{P}_\xi \times_{\mathsf{E}_0} \mathtt{P}_\xi$ is pre-dense in $\mathtt{P}_\xi \times_{\mathsf{E}_0} \mathtt{P}_\xi$ then it is pre-dense in $\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$.

Proof. By induction on $\lambda \ge \xi$, if D is pre-dense in $\mathtt{P}_\lambda$ then it remains pre-dense in $\mathtt{P}_{\lambda+1} = \mathtt{P}_\lambda \cup \mathtt{U}_\lambda$ by Lemma 7.6. Limit steps are obvious. To prove the second claim note that $\mathtt{U}_\xi$ is dense in $\mathtt{P}_{\xi+1}$ by Lemma 7.3, and $\mathtt{U}_\xi \in \mathfrak{M}_{\xi+1}$.

To prove the last claim use Lemma 7.7. □

Lemma 10.4 (in $\mathbf{L}$)**.** *If $X \subseteq \mathrm{HC} = \mathbf{L}_{\omega_1}$ then the set W_X of all ordinals $\xi < \omega_1$ such that $\langle \mathbf{L}_\xi ; X \cap \mathbf{L}_\xi \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1} ; X \rangle$ and $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ is unbounded in ω_1. More generally, if $X_n \subseteq \mathrm{HC}$ for all n then the set W of all ordinals $\xi < \omega_1$, such that $\langle \mathbf{L}_\xi ; \{X_n \cap \mathbf{L}_\xi\}_{n<\omega} \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1} ; \{X_n\}_{n<\omega} \rangle$ and $\{X_n \cap \mathbf{L}_\xi\}_{n<\omega} \in \mathfrak{M}_\xi$, is unbounded in ω_1.*

Proof. Let $\xi_0 < \omega_1$. Let M be a countable elementary submodel of $\mathbf{L}_{\omega_2}$ containing ξ_0, ω_1, X, and such that $M \cap \mathrm{HC}$ is transitive. Let $\phi : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ be the Mostowski collapse, and let $\xi = \phi(\omega_1)$. Then $\xi_0 < \xi < \lambda < \omega_1$ and $\phi(X) = X \cap \mathbf{L}_\xi$ by the choice of M. It follows that $\langle \mathbf{L}_\xi ; X \cap \mathbf{L}_\xi \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1} ; X \rangle$. Moreover, ξ is uncountable in $\mathbf{L}_\lambda$, hence $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\xi$. We conclude that $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ since $X \cap \mathbf{L}_\xi \in \mathbf{L}_\lambda$ by construction.

The second claim does not differ much: we start with a model M containing both the whole sequence $\{X_n\}_{n<\omega}$ and each particular X_n, and so on. □

Corollary 10.5 (compare to [7], Lemma 6). *The forcing notions* $\mathtt{P}$ *and* $\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ *satisfy CCC in* $\mathbf{L}$.

Proof. Suppose that $A \subseteq \mathtt{P}$ is a maximal antichain. By Lemma 10.4, there is an ordinal ξ such that $A' = A \cap \mathtt{P}_\xi$ is a maximal antichain in $\mathtt{P}_\xi$ and $A' \in \mathfrak{M}_\xi$. But then A' remains pre-dense, therefore, still a maximal antichain, in the whole set $\mathtt{P}$ by Lemma 10.3. It follows that $A = A'$ is countable. $\square$

11. The model

We view the sets $\mathtt{P}$ and $\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ (Definition 10.1) as forcing notions over $\mathbf{L}$.

Lemma 11.1 (compare to Lemma 7 in [7]). *A real* $x \in 2^\omega$ *is* $\mathtt{P}$*-generic over* $\mathbf{L}$ *iff* $x \in Z = \bigcap_{\xi<\omega_1^{\mathbf{L}}} \bigcup_{U \in \mathtt{U}_\xi} [U]$.

Proof. If $\xi < \omega_1^{\mathbf{L}}$ then $\mathtt{U}_\xi$ is pre-dense in $\mathtt{P}$ by Lemma 10.3, therefore any real $x \in 2^\omega$ $\mathtt{P}$-generic over $\mathbf{L}$ belongs to $\bigcup_{U \in \mathtt{U}_\xi} [U]$.

To prove the converse, suppose that $x \in Z$ and prove that x is $\mathtt{P}$-generic over $\mathbf{L}$. Consider a maximal antichain $A \subseteq \mathtt{P}$ in $\mathbf{L}$; we have to prove that $x \in \bigcup_{T \in A} [T]$. Note that $A \subseteq \mathtt{P}_\xi$ for some $\xi < \omega_1^{\mathbf{L}}$ by Corollary 10.5. But then every tree $U \in \mathtt{U}_\xi$ satisfies $U \subseteq^{\mathtt{fin}} \bigcup A$ by Lemma 7.6, so that $\bigcup_{U \in \mathtt{U}_\xi} [U] \subseteq \bigcup_{T \in A} [T]$, and hence $x \in \bigcup_{T \in A} [T]$, as required. $\square$

Corollary 11.2 (compare to Corollary 9 in [7]). *In any generic extension of* $\mathbf{L}$*, the set of all reals in* 2^ω $\mathtt{P}$*-generic over* $\mathbf{L}$ *is* Π_1^{HC} *and* Π_2^1.

Proof. Use Lemma 11.1 and Proposition 10.2. $\square$

Definition 11.3. From now on, we assume that $G \subseteq \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ is a set $(\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P})$-generic over $\mathbf{L}$, so that the intersection $X = \bigcap_{\langle T, T' \rangle \in G} [T] \times [T']$ is a singleton $X_G = \{\langle x_{\mathtt{left}}[G], x_{\mathtt{right}}[G] \rangle\}$. $\square$

Compare the next lemma to Lemma 10 in [7]. While Jensen's forcing notion in [7] guarantees that there is a single generic real in the extension, the forcing notion $\mathtt{P}$ we use adds a whole E_0-class (a countable set) of generic reals!

Lemma 11.4 (under the assumptions of Definition 11.3). *If* $y \in \mathbf{L}[G] \cap 2^\omega$ *then* y *is a* $\mathtt{P}$*-generic real over* $\mathbf{L}$ *iff* $y \in [x_{\mathtt{left}}[G]]_{\mathsf{E}_0} \cup [x_{\mathtt{right}}[G]]_{\mathsf{E}_0}$.

Recall that $[x]_{\mathsf{E}_0} = \{\sigma \cdot x : \sigma \in 2^{<\omega}\}$.

Proof. The reals $x_{\mathtt{left}}[G]$, $x_{\mathtt{right}}[G]$ are separately $\mathtt{P}$-generic (see Remark 5.1). It follows that any real $y = \sigma \cdot x_{\mathtt{left}}[G] \in [x_{\mathtt{left}}[G]]_{\mathsf{E}_0}$ or $y = \sigma \cdot x_{\mathtt{right}}[G] \in [x_{\mathtt{right}}[G]]_{\mathsf{E}_0}$ is $\mathtt{P}$-generic as well since the forcing $\mathtt{P}$ is by definition invariant under the action of any $\sigma \in 2^{<\omega}$.

To prove the converse, suppose towards the contrary that there is a condition $\langle T, T' \rangle \in \mathtt{P} \times_{\mathsf{E}_0} \mathtt{P}$ and a $(\mathtt{P} \times_{\mathsf{E}_0} \mathtt{P})$-real name $\mathbf{c} = \{C_n^i\}_{n<\omega,\, i=0,1} \in \mathbf{L}$

such that $\langle T,T'\rangle$ $(\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P})$-forces that $\mathbf{c}$ is $\mathbb{P}$-generic while $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ forces both formulas $\mathbf{c}\ne\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{left}}$ and $\mathbf{c}\ne\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{left}}$ for all $\sigma\in 2^{<\omega}$.

Let $C_n=C_n^0\cup C_n^1$, this is a pre-dense set in $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$. It follows from Lemma 10.4 that there exists an ordinal $\lambda<\omega_1$ such that each set $C'_n=C_n\cap(\mathbb{P}_\lambda\times_{\mathsf{E}_0}\mathbb{P}_\lambda)$ is pre-dense in $\mathbb{P}_\lambda\times_{\mathsf{E}_0}\mathbb{P}_\lambda$, and the sequence $\{C'_{ni}\}_{n<\omega,\,i=0,1}$ belongs to $\mathfrak{M}_\lambda$, where $C'_{ni}=C'_n\cap C_n^i$ — then C'_n is pre-dense in $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ too, by Lemma 10.3. Therefore we can assume that in fact $C_n=C'_n$, that is, $\mathbf{c}\in\mathfrak{M}_\lambda$ and $\mathbf{c}$ is a $(\mathbb{P}_\lambda\times_{\mathsf{E}_0}\mathbb{P}_\lambda)$-real name.

Further, as $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ forces that $\mathbf{c}\ne\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{left}}$ and $\mathbf{c}\ne\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{right}}$, the set $D(\sigma)$ of all conditions $\langle S,S'\rangle\in\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ which directly force $\mathbf{c}\ne\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{left}}$ and $\mathbf{c}\ne\sigma\cdot\dot{\boldsymbol{x}}_{\mathtt{right}}$, is dense in $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ — for every $\sigma\in 2^{<\omega}$. Therefore, still by Lemma 10.4, we may assume that the same ordinal λ as above satisfies the following: each set $D'(\sigma)=D(\sigma)\cap(\mathbb{P}_\lambda\times_{\mathsf{E}_0}\mathbb{P}_\lambda)$ is dense in $\mathbb{P}_\lambda\times_{\mathsf{E}_0}\mathbb{P}_\lambda$.

Applying Theorem 9.3 with $\mathbb{P}=\mathbb{P}_\lambda$, $\mathbb{U}=\mathbb{U}_\lambda$, and $\mathbb{P}\cup\mathbb{U}=\mathbb{P}_{\lambda+1}$, we conclude that for each tree $U\in\mathbb{U}_\lambda$ the set Q_U of all conditions $\langle V,V'\rangle\in\mathbb{P}_{\lambda+1}\times_{\mathsf{E}_0}\mathbb{P}_{\lambda+1}$ which directly force $\mathbf{c}\notin[U]$, is dense in $\mathbb{P}_{\lambda+1}\times_{\mathsf{E}_0}\mathbb{P}_{\lambda+1}$. As obviously $Q_U\in\mathfrak{M}_{\lambda+1}$, we further conclude that Q_U is pre-dense in the whole forcing $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ by Lemma 10.3. This implies that $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ forces $\mathbf{c}\notin\bigcup_{U\in\mathbb{U}_\lambda}[U]$, hence, forces that $\mathbf{c}$ is not $\mathbb{P}$-generic, by Lemma 11.1. But this contradicts to the choice of $\langle T,T'\rangle$. □

Corollary 11.5. *The set $[x_{\mathtt{left}}[G]]_{\mathsf{E}_0}\cup[x_{\mathtt{right}}[G]]_{\mathsf{E}_0}$ is Π^1_2 set in $\mathbf{L}[G]$. Therefore the 2-element set $\{[x_{\mathtt{left}}[G]]_{\mathsf{E}_0},[x_{\mathtt{right}}[G]]_{\mathsf{E}_0}\}$ is OD in $\mathbf{L}[G]$.* □

Corollary 11.6. *The E_0-classes $[x_{\mathtt{left}}[G]]_{\mathsf{E}_0}$, $[x_{\mathtt{right}}[G]]_{\mathsf{E}_0}$ are disjoint.*

Proof. Corollary 5.4 implies $x_{\mathtt{left}}[G]\ \not\mathsf{E}_0\ x_{\mathtt{right}}[G]$. □

Lemma 11.7 (still under the assumptions of Definition 11.3)**.** *Neither of the two E_0-classes $[x_{\mathtt{left}}[G]]_{\mathsf{E}_0}$, $[x_{\mathtt{right}}[G]]_{\mathsf{E}_0}$ is OD in $\mathbf{L}[G]$.*

Proof. Suppose towards the contrary that there is a condition $\langle T,T'\rangle\in G$ and a formula $\vartheta(x)$ with ordinal parameters such that $\langle T,T'\rangle$ $(\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P})$-forces that $\vartheta([\dot{\boldsymbol{x}}_{\mathtt{left}}]_{\mathsf{E}_0})$ but $\neg\vartheta([\dot{\boldsymbol{x}}_{\mathtt{right}}]_{\mathsf{E}_0})$. However both the formula and the forcing are invariant under actions of strings in $2^{<\omega}$. In particular if $\sigma\in 2^{<\omega}$ then $\langle\sigma\cdot T,\sigma\cdot T'\rangle$ still $(\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P})$-forces $\vartheta([\dot{\boldsymbol{x}}_{\mathtt{left}}]_{\mathsf{E}_0})$ and $\neg\,\vartheta([\dot{\boldsymbol{x}}_{\mathtt{right}}]_{\mathsf{E}_0})$. We can take σ which satisfies $T'=\sigma\cdot T$; thus $\langle T',T\rangle$ still $(\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P})$-forces $\vartheta([\dot{\boldsymbol{x}}_{\mathtt{left}}]_{\mathsf{E}_0})$ and $\neg\vartheta([\dot{\boldsymbol{x}}_{\mathtt{right}}]_{\mathsf{E}_0})$.[5] However $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ is symmetric with respect to the left-right exchange, which implies that conversely $\langle T',T\rangle$ has to force $\vartheta([\dot{\boldsymbol{x}}_{\mathtt{right}}]_{\mathsf{E}_0})$ and $\neg\,\vartheta([\dot{\boldsymbol{x}}_{\mathtt{left}}]_{\mathsf{E}_0})$. The contradiction proves the lemma. □

□ (Theorem 1.1)

12. Conclusive remarks

(I) One may ask whether other Borel equivalence relations E admit results similar to Theorem 1.1. Fortunately this question can be easily solved on the base of the Glimm – Effros dichotomy theorem [4].

[5] This is the argument which does not go through for the full product $\mathbb{P}\times\mathbb{P}$.

Corollary 12.1. *The following is true in the model of Theorem 1.1. Let* E *be a Borel equivalence relation on* ω^ω *coded in* $\mathbf{L}$. *Then there exists an OD pair of* E*-equivalence classes* $\{[x]_\mathsf{E},[y]_\mathsf{E}\}$ *such that neither of the classes* $[x]_\mathsf{E}$, $[y]_\mathsf{E}$ *is separately OD, iff* E *is not smooth.*

Proof. Suppose first that E is smooth. By the Shoenfield absoluteness theorem, the smoothness can be witnessed by a Borel map $\vartheta:\omega^\omega\to\omega^\omega$ coded in $\mathbf{L}$, hence, ϑ is OD itself. If $p=\{[x]_\mathsf{E},[y]_\mathsf{E}\}$ is OD in the extension then so is the 2-element set $R=\{\vartheta(z):z\in[x]_\mathsf{E}\cup[y]_\mathsf{E}\}\subseteq\omega^\omega$, whose both elements (reals), say p_x and p_y, are OD by obvious reasons. Then finally $[x]_\mathsf{E}=\vartheta^{(-1)}(p_x)$ and $[y]_\mathsf{E}=\vartheta^{(-1)}(p_y)$ are OD as required.

Now let E be non-smooth. Then by Shoenfield and the Glimm – Effros dichotomy theorem in [4], there is a continuous, coded by some $r\in\omega^\omega\cap\mathbf{L}$, hence, OD, reduction $\vartheta:2^\omega\to\omega^\omega$ of E_0 to E, so that we have $a\mathrel{\mathsf{E}_0}b$ iff $\vartheta(a)\mathrel{\mathsf{E}}\vartheta(b)$ for all $a,b\in2^\omega$. Let, by Theorem 1.1, $\{[a]_{\mathsf{E}_0},[b]_{\mathsf{E}_0}\}$ be a $\varPi^1_2$ pair of non-OD E_0-equivalence classes. By the choice of ϑ, one easily proves that $\{[\vartheta(a)]_\mathsf{E},[\vartheta(b)]_\mathsf{E}\}$ is a $\varPi^1_2(r)$ pair of non-OD E-equivalence classes. □

(II) One may ask what happens with the Groszek – Laver pairs of sets of reals in better known models. For some of them the answer tends to be in the negative. Consider *e.g.* the Solovay model of **ZFC** in which all projective sets of reals are Lebesgue measurable [14]. Arguing in the Solovay model, let $\{X,Y\}$ be an OD set, where $X,Y\subseteq2^\omega$. Then the set of *four* sets $X\smallsetminus Y$, $Y\smallsetminus X$, $X\cap Y$, $2^\omega\smallsetminus(X\cup Y)$ is still OD, and hence we have an OD equivalence relation E on 2^ω with four (or fewer if say $X\subseteq Y$) equivalence classes. By a theorem of [8] [6], either E admits an OD reduction $\vartheta:2^\omega\to2^{<\omega_1}$ to equality on $2^{<\omega_1}$ or E_0 admits a continuous reduction to E. The "or" option fails since E has finitely many classes.

The "either" option leads to a finite (not more than 4 elements) OD set $R=\mathtt{ran}\,\vartheta\subseteq2^{<\omega_1}$. An easy argument shows that then every $r\in R$ is OD, and hence so is the corresponding E-class $\vartheta^{-1}(r)$. It follows that X,Y themselves are OD.

Question 12.2. Is it true in the Solovay model that every *countable* OD set $W\subseteq\mathscr{P}(\omega^\omega)$ *of sets of reals* contains an OD element $X\in W$ (a set of reals)? □

An uncountable counterexample readily exists, for take the set of all non-OD sets of reals. As for sets $W\subseteq\omega^\omega$, any countable OD set of reals in the Solovay model consists of OD elements, *e.g.* by the result mentioned in Footnote 6.

(III) One may ask whether a forcing similar to $\mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$ with respect to the results in Section 11, exists in ground models other than $\mathbf{L}$ or $\mathbf{L}[x]$,

[6] To replace the following brief argument, one can also refer to a result by Stern implicit in [15]: in the Solovay model, if an OD equivalence relation E has at least one non-OD equivalence class then there is a pairwise E-inequivalent perfect set.

$x \in 2^\omega$. Some coding forcing constructions with perfect trees do exist in such a general frameworks, see [1, 10].

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