Countable OD sets of reals belong to the ground model

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Abstract

It is true in the Cohen, random, dominaning, and Sacks generic extensions, that every countable ordinal-definable set of reals belongs to the ground universe. Stronger results hold in the Solovay model.

1 Introduction

It is known from descriptive set theory that countable definable sets of reals have properties inavailable for arbitrary sets of reals of the same level of definability. Thus all elements of a countable Δ^1_1 set of reals are Δ^1_1 themselves while an uncountable Δ^1_1 set does not necessarily contan a Δ^1_1 real. This difference vanishes to some extent at higher levels of projective hierarchy, as it is demonstrated that some non-homogeneous forcing notions lead to models of **ZFC** with countable Π^1_2 non-empty sets of reals with no OD (ordinal-definable) elements [11] [1], and such a set can even have the form of a Π^1_2 E_0-equivalence class [12].

On the other hand, one may expect that homogeneous forcing notions generally yield opposite results. We prove the following theorems.

Theorem 1.1. *Let a be one of the following generic reals over the universe $\mathbf{V}$:*

(I) *a Cohen-generic real over $\mathbf{V}$;*

(II) *a Solovay-random real over $\mathbf{V}$;*

(III) *a dominating-forcing real over $\mathbf{V}$;*

(IV) *a Sacks (perfect-set generic) real over $\mathbf{V}$.*

Then it is true in $\mathbf{V}[a]$ that if $X \subseteq 2^\omega$ is a countable OD set then $X \in \mathbf{V}$.

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[1] The model presented in [11] was obtained via the countable product of Jensen's minimal Δ^1_3 real forcing [6]. Such a product-forcing model was earlier considered by Enayat [4].

Theorem 1.2. (i) *It is true in the first Solovay model*[2] *that every non-empty OD countable or finite set* $\mathscr{X}$ **of sets of reals** *contains an OD element, and hence consists of OD elements as the notion of being OD is OD itself.*

(ii) *It is true in the second Solovay model*[2] *that every non-empty OD countable or finite set* $\mathscr{X}$ *of any kind, contains an OD element, and hence consists of OD elements, by the same reason.*

Regarding (ii), Theorem 4.8 in Caicedo and Ketchersid [3] contains a similar result under a different **AC**-incompatible hypothesis on the top of $\mathbf{ZF} + \mathbf{DC}$.

One may expect such theorems to be true in any suitably homogeneous generic models. However it does not seem to be an easy task to manufacture a proof of sufficient degree of generality, because of various *ad hoc* arguments lacking a common denominator, which we have to make use of, specifically for the Cohen, random, and dominating cases of Theorem 1.1, and a totally different argument used for Theorem 1.2.

To explain the method of the proof of Theorem 1.1 in parts I, II, III (the Sacks case is quite elementary), let T be a name of a counterexample. We pick a pair of reals a, b, each being generic over the ground set universe $\mathbf{V}$, and satisfying $\mathbf{V}[a] = \mathbf{V}[b]$. Then the interpretations $T[\![a]\!]$, $T[\![b]\!]$ of T resp. via a and via b coincide as each of them is defined by the same formula (with ordinals) in the same universe: $T[\![a]\!] = T[\![b]\!]$. In the same time, the pair $\langle a, b\rangle$ is a product generic pair over a suitable *countable* model $\mathfrak{M}$, or close to be such in the sense that at least $\mathfrak{M}[a] \cap \mathfrak{M}[b] \cap 2^\omega \subseteq \mathfrak{M}$. However $T[\![a]\!] \subseteq \mathfrak{M}[a]$ and $T[\![b]\!] \subseteq \mathfrak{M}[b]$, so in fact $T[\![a]\!] = T[\![b]\!] \subseteq \mathfrak{M}$, as required.

This scheme works rather transparently in the Cohen (Section 2) and Solovay-random (Section 3) cases, but contains a couple of nontrivial lemmas (5.5 and especially 5.6 with a lengthy proof) in the dominating case (Section 5).

We add an alternative and rather elementary proof for the Cohen and Solovay-random cases (Section 4), which makes use of some old folklore results related to degrees of reals in those extensions over the ground model. We finish in Section 7 with a proof of Theorem 1.2.

2 Cohen-generic case

Here we prove Case I of Theorem 1.1. We begin with some notation and a couple of preliminary lemmas.

Assume that $u, v \in 2^\omega \cup 2^{<\omega}$ are dyadic sequences, possibly of different (finite or infinite) length. We let $u \bullet v$ (the termwise action of u on v) be a dyadic

[2] See Definition 7.1 below on the Solovay models. See [7, 10, 13] and Stern [17] on different aspects of definability in the Solovay models.

sequence defined so that $\mathtt{dom}\, u^\frown v = \mathtt{dom}\, v$ (independently of the length $\mathtt{dom}\, u$ of u) and if $j < \mathtt{dom}\, v$ then

$$(u^\frown v)(j) = \begin{cases} 1 - v(j) & , \text{ whenever } \quad j < \mathtt{dom}\, u \wedge u(j) = 1\,, \\ v(j) & , \text{ otherwise } \quad . \end{cases}$$

In particular, if $z \in 2^\omega \cup 2^{<\omega}$ then $x \mapsto z^\frown x$ $(x \in 2^\omega)$ is a homeomorphism of 2^ω while $p \mapsto z^\frown p$ $(p \in 2^{<\omega})$ is an order automorphism of $2^{<\omega}$.

Let $\mathbf{Coh} = 2^{<\omega}$ be the Cohen forcing.

Lemma 2.1. *Let $\mathfrak{M}$ be a transitive model of a large fragment of* **ZFC**. *Then*

(i) *if a pair $\langle a, b\rangle \in 2^\omega \times 2^\omega$ is $(\mathbf{Coh} \times \mathbf{Coh})$-generic over $\mathfrak{M}$ then $\mathfrak{M}[a] \cap \mathfrak{M}[b] = \mathfrak{M}$* — this is a well-known theorem on product forcing;

(ii) *if a pair $\langle a, b\rangle \in 2^\omega \times 2^\omega$ is $(\mathbf{Coh} \times \mathbf{Coh})$-generic over $\mathfrak{M}$ then so is the pair $\langle a, a^\frown b\rangle$* ;

(iii) *if $\mathfrak{M}$ is countable and $p, q \in \mathbf{Coh}$ then there are reals $a, b \in 2^\omega$, $\mathbf{Coh}$-generic over $\mathbf{V}$ and such that $p \subset a$, $q \subset b$, $\mathbf{V}[a] = \mathbf{V}[b]$, and the pair $\langle a, b\rangle$ is $(\mathbf{Coh} \times \mathbf{Coh})$-generic over $\mathfrak{M}$.*

Proof. (ii) Otherwise there is a condition $\langle p, q\rangle \in \mathbf{Coh}\times\mathbf{Coh}$ with $\mathtt{dom}\, p = \mathtt{dom}\, q$, which forces the opposite over $\mathfrak{M}$. By the countability, there is a real $a \in 2^\omega$ in $\mathbf{V}$ $\mathbf{Coh}$-generic over $\mathfrak{M}$, with $p \subset a$; $\mathfrak{M}[a]$ is a set in $\mathbf{V}$. Let $r = p^\frown q$ and let $c \in \mathfrak{M}$ be $\mathbf{Coh}$-generic over $\mathfrak{M}[a]$, with $r \subset c$. Then $b = a^\frown c$ is $\mathbf{Coh}$-generic over $\mathfrak{M}[a]$ by obvious reasons, $c = a^\frown b$, and $q = p^\frown r \subset b = a^\frown c$. Finally $\langle a, b\rangle$ is $(\mathbf{Coh} \times \mathbf{Coh})$-generic over $\mathfrak{M}$ by the product forcing theorem, a contradiction.

(iii) Assuming wlog that $\mathtt{dom}\, p = \mathtt{dom}\, q$, we let $r = p^\frown q$. Once again, there is a real $c \in 2^\omega$ in $\mathbf{V}$, $\mathbf{Coh}$-generic over $\mathfrak{M}$, with $r \subset c$. Let $a \in 2^\omega$ be $\mathbf{Coh}$-generic over $\mathbf{V}$, hence over $\mathfrak{M}[c]$, too, and satisfying $p \subset a$. Then the real $b = c^\frown a$ is $\mathbf{Coh}$-generic over $\mathbf{V}$ (since $c \in \mathbf{V}$), $\mathbf{V}[b] = \mathbf{V}[a]$, and $q = r^\frown p \subset b$.

Finally the pair $\langle a, c\rangle$ is $(\mathbf{Coh} \times \mathbf{Coh})$-generic over $\mathfrak{M}$ by the product forcing theorem, therefore $\langle a, b\rangle = \langle a, a^\frown c\rangle$ is $(\mathbf{Coh} \times \mathbf{Coh})$-generic over $\mathfrak{M}$ by (ii). □

Proof (Theorem 1.1, case I)**.** Let $a_0 \in 2^\omega$ be a real $\mathbf{Coh}$-generic over the universe $\mathbf{V}$. First of all, note this: it suffices to prove that (it is true in $\mathbf{V}[a_0]$ that) if $Z \subseteq 2^\omega$ is a countable OD set then $Z \subseteq \mathbf{V}$. Indeed, as the Cohen forcing is homogeneous, any statement about sets in $\mathbf{V}$, the ground model, is decided by the weakest condition.

Thus let $Z \subseteq 2^\omega$ be a countable OD set in $\mathbf{V}[a_0]$.

Suppose to the contrary that $Z \not\subseteq \mathbf{V}$.

There is a formula $\varphi(z)$ with an unspecified ordinal γ_0 as a parameter, such that $Z = \{z \in 2^\omega : \varphi(z)\}$ in $\mathbf{V}[a_0]$, and then there is a condition $p_0 \in \mathbf{Coh}$ such that $p_0 \subset a_0$ and p_0 **Coh**-forces that $\{z \in 2^\omega : \varphi(z)\}$ is a countable set and (by the contrary assumption) also forces $\exists\, z\, (z \notin \mathbf{V} \wedge \varphi(z))$.

There is a sequence $\{t_n\}_{n<\omega} \in \mathbf{V}$ of **Coh**-names, such that if $x \in 2^\omega$ is Cohen generic and $p_0 \subset x$ then it is true in $\mathbf{V}[x]$ that $\{z \in 2^\omega : \varphi(z)\} = \{t_n[\![x]\!] : n < \omega\}$, where $t[\![x]\!]$ is the interpretation of a **Coh**-name t by a real $x \in 2^\omega$. Let $T \in \mathbf{V}$ be the canonical **Coh**-name for $\{t_n[\![\dot a]\!] : n < \omega\}$. Thus we assume that

(1) p_0 **Coh**-forces, over $\mathbf{V}$, that $T[\![\dot a]\!] = \{x \in 2^\omega : \varphi(x)\} \not\subseteq \check{\mathbf{V}}$,

where $\dot a$ is the canonical **Coh**-name for the **Coh**-generic real, and $\check{\mathbf{V}}$ is a name for the ground model (of "old" sets).

We continue towards **getting a contradiction from** (1). Pick a regular cardinal $\kappa > \alpha_0$, sufficiently large for the set $\mathbf{H}_\kappa$ to contain γ_0 and all names t_n and T. Consider a countable elementary submodel $\mathfrak{M}$ of $\mathbf{H}_\kappa$ containing γ_0, all t_n, T. Let $\pi : \mathfrak{M} \to \mathfrak{M}'$ be the Mostowski collapse onto a transitive set $\mathfrak{M}'$. As **Coh** is countable, we have $\pi(\mathbf{Coh}) = \mathbf{Coh}$, $\pi(t_n) = t_n$, $\pi(T) = T$, so $T \in \mathfrak{M}'$.

Now pick reals $a, b \in 2^\omega$ **Coh**-generic over $\mathbf{V}$ by Lemma 2.1(iii), such that $p_0 \subset a$, $p_0 \subset b$, $\mathbf{V}[a] = \mathbf{V}[b]$, and the pair $\langle a, b\rangle$ is $(\mathbf{Coh} \times \mathbf{Coh})$-generic over $\mathfrak{M}'$. In particular, as $\mathbf{V}[a] = \mathbf{V}[b]$, we have $T[\![a]\!] = T[\![b]\!] \not\subseteq \mathbf{V}$ by (1). On the other hand, $\mathfrak{M}'[a] \cap \mathfrak{M}'[b] \subseteq \mathfrak{M}'$ by Lemma 2.1(i), therefore $T[\![a]\!] \cap T[\![b]\!] \subseteq \mathfrak{M}'[a] \cap \mathfrak{M}'[b] \subseteq \mathfrak{M}' \subseteq \mathbf{V}$, contrary to the above.

□ (Theorem 1.1, case I)

3 Solovay-random case

Here we prove Case II of Theorem 1.1.

Let $\boldsymbol\lambda$ be the standard probability Lebesgue measure on 2^ω. The Solovay-random forcing **Rand** consists of all trees $\tau \subseteq 2^{<\omega}$ with no endpoints and no isolated branches, and such that the set $[\tau] = \{x \in 2^\omega : \forall\, n\, (x\restriction n \in \tau)\}$ has positive measure $\boldsymbol\lambda([\tau]) > 0$. The forcing **Rand** depends on the ground model, so that "random over a model $\mathfrak{M}$" will mean "$(\mathbf{Rand} \cap \mathfrak{M})$-generic over $\mathfrak{M}$".

Lemma 3.1 (trivial in the Cohen case)**.** *If $\mathfrak{M} \subseteq \mathfrak{N}$ are TM of a large fragment of* **ZFC***, and $a \in 2^\omega$ is random over $\mathfrak{N}$ then a is random over $\mathfrak{M}$, too.*

Proof. It suffices to prove that if $A \in \mathfrak{M}$ is a maximal antichain in $\mathbf{Rand} \cap \mathfrak{M}$ then A remains such in $\mathbf{Rand} \cap \mathfrak{N}$, which is rather clear since being a maximal antichain in **Rand** amounts to 1) countability, 2) pairwise intersections being null sets (those of $\boldsymbol\lambda$-measure 0), and 3) the union being a co-null set. □

Unlike the Cohen-generic case, a random pair of reals is **not** a (**Rand** $\times$ **Rand**)-generic pair. The notion of a random pair is rather related to forcing by closed sets in $2^\omega \times 2^\omega$ (or trees which generate them, or equivalently Borel sets) of positive product measure (non-null). This will lead to certain changes of arguments, with respect to the Cohen-generic case of Section 2.

We'll make use of the following known characterisation of random pairs.

Proposition 3.2. *Let $\mathfrak{M}$ be a transitive model of a large fragment of* **ZFC**, *and $a, b \in 2^\omega$. Then the following three assertions are equivalent*:

1) *the pair $\langle a, b\rangle$ is a random pair over $\mathfrak{M}$;*
2) *a is random over $\mathfrak{M}$ and b is random over $\mathfrak{M}[a]$;*
3) *b is random over $\mathfrak{M}$ and a is random over $\mathfrak{M}[b]$.* □

Lemma 3.3. *Let $\mathfrak{M}$ be a transitive model of a large fragment of* **ZFC**. *Then*

(i) *if a pair $\langle a, b\rangle \in 2^\omega \times 2^\omega$ is random over $\mathfrak{M}$ then $\mathfrak{M}[a] \cap \mathfrak{M}[b] \cap 2^\omega \subseteq \mathfrak{M}$;*

(ii) *if a pair $\langle a, b\rangle \in 2^\omega \times 2^\omega$ is random over $\mathfrak{M}$ then so is the pair $\langle a, a \dot{+} b\rangle$;*

(iii) *if $\mathfrak{M}$ is countable and $\tau \in$ **Rand** then there are reals $a, b \in [\tau]$, random over $\mathbf{V}$, such that $\mathbf{V}[a] = \mathbf{V}[b]$, and the pair $\langle a, b\rangle$ is random over $\mathfrak{M}$.*

Proof. (i) This is somewhat more difficult than in the Cohen-generic case of Lemma 2.1(i). Assume towards the contrary that $x \in \mathfrak{M}[a] \cap \mathfrak{M}[b] \cap 2^\omega$ but $x \notin M$. The random forcing admits continuous reading of real names, meaning that there are continuous maps $f, g : 2^\omega \to 2^\omega$, coded in $\mathfrak{M}$ and such that $x = f(a) = g(b)$. Let the contrary assumption be forced by a Borel set $P \subseteq 2^\omega \times 2^\omega$ of positive product measure, coded in $\mathfrak{M}$ and containing $\langle a, b\rangle$; in particular, P (random pair)-forces that $f(\dot a_{\mathtt{lef}}) = g(\dot a_{\mathtt{rig}})$.[3] By the Lebesgue density theorem, we can wlog assume that every point $\langle x, y\rangle \in P$ has density 1.

We claim that $f(x) = g(y)$ for all $\langle x, y\rangle \in P$. Indeed if $\langle x_0, y_0\rangle \in P$ and $f(x_0) \neq g(y_0)$ then say $f(x_0)(n) = 0 \neq g(y_0)(n) = 1$ for some n. As f, g are continuous, there is a nbhd Q of $\langle x_0, y_0\rangle$ in P such that $f(x)(n) = 0 \neq g(y)(n) = 1$ for all $\langle x, y\rangle \in Q$. But Q' is a non-null set by the density 1 assumption. It follows that Q forces that $f(\dot a_{\mathtt{lef}}) \neq g(\dot a_{\mathtt{rig}})$, a contradiction.

Let a *cell* be any Borel set $Q \subseteq P$ such that f, g are constant on Q, that is, there is a real r such that $f(x) = g(y) = r$ for all $\langle x, y\rangle \in Q$. Note that in this case, if Q is non-null then Y forces $f(\dot a_{\mathtt{lef}}) = g(\dot a_{\mathtt{rig}}) = r \in \mathfrak{M}$, therefore to prove (i) it suffices to show the existence of a non-null cell $Q \subseteq P$.

Let $P_x = \{y : \langle x, y\rangle \in P\}$ and $P^y = \{x : \langle x, y\rangle \in P\}$, cross-sections. By Fubini, the sets $X = \{x : \boldsymbol{\lambda}(P_x) > 0\}$ and $Y = \{y : \boldsymbol{\lambda}(P^y \cap X) > 0\}$ are non-null. Let $y_0 \in Y$ and let $X' = P^{y_0} \cap X$, a non-null set. By construction,

[3] $\dot a_{\mathtt{lef}}, \dot a_{\mathtt{rig}}$ are canonical names for the left, resp., right of the terms of a random pair.

if $x \in X'$ then the cross-section P_x is non-null, and hence $Q = \{\langle x, y\rangle \in P : x \in X'\}$ is non-null by Fubini. We claim that Q is a cell. Indeed suppose that $\langle x, y\rangle \in Q$. Then $x \in X'$, therefore $\langle x, y_0\rangle \in P$, and we have $f(x) = g(y_0)$ by the above claim. However $\langle x, y\rangle \in P$, hence similarly $g(y) = f(x)$. Thus $g(y) = f(x) = g(y_0) = \text{Const}$ on Q, as required.

(ii) The contrary assumption implies the existence (in $\mathfrak{M}$) of a *non-null* Borel set $P \subseteq 2^\omega \times 2^\omega$ and a *null* Borel set $Q \subseteq 2^\omega \times 2^\omega$ such that the map $\langle x, y\rangle \mapsto \langle x, x \dot{} y\rangle$ maps P into Q. However this map is obviously measure-preserving, a contradiction.

(iii) The set $P = \{\langle x, x \dot{} y\rangle : x, y \in [\tau]\}$ is non-null, hence, by Fubini, so is the projection $Y = \{y : \boldsymbol{\lambda}(P^y) > 0\}$, where $P^y = \{x : \langle x, y\rangle \in P\}$, as above. Let, in $\mathbf{V}$, $y \in Y$ be random over $\mathfrak{M}$. Then P^y is non-null, so we can pick a real $a \in P^y$ random over $\mathbf{V}$ hence, over $\mathfrak{M}[y]$, too. Then the pair $\langle a, y\rangle$ belongs to P and is random over $\mathfrak{M}$ by Proposition 3.2. Let $b = a \dot{} y$. It follows by (ii) that the pair $\langle a, b\rangle$ is random over $\mathfrak{M}$ as well. And $a, b \in [\tau]$ by construction. Finally b is random over $\mathbf{V}$ since so is a while $y \in \mathbf{V}$. □

Proof (Theorem 1.1, case II)**.** As above (the Cohen case), the **contrary assumption** leads to a formula $\varphi(z)$ with $\gamma_0 \in \text{Ord}$ as a parameter, a condition $\tau_0 \in \mathbf{Rand}$ in $\mathbf{V}$ which **Rand**-forces, over $\mathbf{V}$, that the set $\{z \in 2^\omega : \varphi(z)\}$ is countable and $\exists\, z\, (z \notin \check{\mathbf{V}} \wedge \varphi(z))$, a sequence $\{t_n\}_{n<\omega} \in \mathbf{V}$ of **Rand**-names for reals in $\mathbb{Z}^\omega$, and a canonical **Rand**-name $T \in \mathbf{V}$ for $\{t_n[\![\dot a]\!] : n < \omega\}$, such that

(2) if $x \in [\tau_0]$ is a random real over $\mathbf{V}$, then it is true in $\mathbf{V}[x]$ that

$$\{z \in 2^\omega : \varphi(z)\} = \{t_n[\![x]\!] : n < \omega\} = T[\![x]\!] \not\subseteq \mathbf{V}\,.$$

Pick a regular cardinal $\kappa > \alpha_0$, sufficiently large for the set $\mathbf{H}_\kappa$ to contain γ_0 and all names t_n and T. Consider a countable elementary submodel $\mathfrak{M}$ of $\mathbf{H}_\kappa$ containing γ_0, all names t_n and T, and **Rand**. Let $\pi : \mathfrak{M} \to \mathfrak{M}'$ be the Mostowski collapse onto a transitive set $\mathfrak{M}'$. Unlike the Cohen case, the set $\mathbf{Rand}' = \pi(\mathbf{Rand})$ is equal to $\mathbf{Rand} \cap \mathfrak{M}'$, just the random forcing in $\mathfrak{M}'$, but still $\pi(t_n) = t_n$ for all n, since by the ccc property of **Rand** we can assume that t_n is a hereditarily countable set, and accordingly $\pi(T) = T$.

Pick reals $a, b \in [\tau_0]$ random over $\mathbf{V}$ by Lemma 3.3(iii), such that $\mathbf{V}[a] = \mathbf{V}[b]$, and the pair $\langle a, b\rangle$ is random over $\mathfrak{M}'$. As $\mathbf{V}[a] = \mathbf{V}[b]$, we have $T[\![a]\!] = T[\![b]\!] \not\subseteq \mathbf{V}$ by (2). But $\mathfrak{M}'[a] \cap \mathfrak{M}'[b] \subseteq \mathfrak{M}'$ by Lemma 2.1(i), therefore $T[\![a]\!] \cap T[\![b]\!] \subseteq \mathfrak{M}'[a] \cap \mathfrak{M}'[b] \subseteq \mathfrak{M}' \subseteq \mathbf{V}$, and we get a contratiction required.

□ (Theorem 1.1, case II)

4 Cohen and random cases: a different proof

Here we present a shorter proof of Cases I and II of Theorem 1.1, based on the following lemmas.

Lemma 4.1. *Let $a \in 2^\omega$ be Cohen-generic over a transitive model $\mathfrak{M}$, and $b \in 2^\omega \cap \mathfrak{M}[a]$, a real in the extension. Then*

(i) *either $b \in \mathfrak{M}$ or there is a real $b' \in 2^\omega$, Cohen-generic over $\mathfrak{M}$ and satisfying $\mathfrak{M}[b] = \mathfrak{M}[b']$;*

(ii) *either $\mathfrak{M}[b] = \mathfrak{M}[a]$ or $\mathfrak{M}[a]$ is a Cohen-generic extension of $\mathfrak{M}[b]$.* □

Lemma 4.2. *Let $a \in 2^\omega$ be random over a transitive model $\mathfrak{M}$, and $b \in 2^\omega \cap \mathfrak{M}[a]$, a real in the extension. Then*

(i) *either $b \in \mathfrak{M}$ or there is a real $b' \in 2^\omega$, random over $\mathfrak{M}$ and satisfying $\mathfrak{M}[b] = \mathfrak{M}[b']$;*

(ii) *either $\mathfrak{M}[b] = \mathfrak{M}[a]$ or $\mathfrak{M}[a]$ is a random extension of $\mathfrak{M}[b]$.* □

The lemmas are known in set theoretic folklore, yet we are not able to suggest any reference. In particular Lemma 4.1(ii) is rather simple on the base on general results on intermediate models by Grigorieff [5] since any subforcing of the Cohen forcing either is trivial or is equivalent to Cohen forcing.

Proof (Theorem 1.1, case I, from Lemma 4.1)**.** In $\mathfrak{M}[a]$, let b belong to a countable OD set $X = \{x \in 2^\omega : \varphi(x)\}$, where φ is a formula containing ordinals. As $b \in \mathfrak{M}[a]$, there is a Borel function f, coded in $\mathfrak{M}$, such that $b = f(a)$. We have to prove that $b \in \mathfrak{M}$. Let $\dot a$ be a canonical **Coh**-name for the generic real.

We have two cases, by Lemma 4.1(ii).

Case 1: $\mathfrak{M}[b] = \mathfrak{M}[a]$. Then there is a Borel function g, coded in $\mathfrak{M}$, such that $a = g(b)$. There is a Cohen condition $u \in \mathbf{Coh}$ which satisfies $u \subset a$ and forces $\dot a = g(f(\dot a))$, $\varphi(f(\dot a))$, and the sentence "$\{x \in 2^\omega : \varphi(x)\}$ is countable".

Now, the set A of all reals $a' \in 2^\omega$, Cohen-generic over $\mathfrak{M}$ and satisfying $u \subset a'$ and $\mathfrak{M}[a'] = \mathfrak{M}[a]$, belongs to $\mathfrak{M}$ and definitely is uncountable in $\mathfrak{M}$. If $a' \in A$ then $f(a')$ satisfies $\varphi(f(a'))$ in $\mathfrak{M}[a'] = \mathfrak{M}[a]$ and hence belongs to X. Furthermore if $a' \ne a'' \in A$ then $f(a') \ne f(a'')$ since $a' = g(f(a')$ and $a'' = g(f(a'')$. We conclude that X is uncountable, a contradiction.

Case 2: $\mathfrak{M}[a]$ is a Cohen-generic extension of $\mathfrak{M}[b]$. Let $\psi(x)$ be the formula saying: "$x \in 2^\omega$ and **Coh** forces $\varphi(\dot x)$, where $\dot x$ is a canonical **Coh**-name for x in any transitive graund model containing x. As **Coh** is a homogeneous forcing notion, the set $Y = X \cap \mathfrak{M}[b]$ coincides with the set $\{x \in 2^\omega : \psi(x)\}$ defined in $\mathfrak{M}[b]$, and $b \in Y$. Finally $\mathfrak{M}[b]$ is a Cohen extension of $\mathfrak{M}$ by Lemma 4.1(i) (or else just $b \in \mathfrak{M}$), and it remails to apply the result in Case 1 to Y. □

Proof (Theorem 1.1, case II, from Lemma 4.2)**.** Similar. □

It is really temptating to prove the dominating case of the theorem by this same rather simple method. However we cannot establish any result similar to lemmas 4.1, 4.2 for dominating forcing. Some relevant results by Palumbo [15, 14] fall short of what would be useful here. Generally, a remark in [14, Section 4] casts doubts that even claims (i) of the lemmas hold for dominating-generic extensions in any useful form. This is why we have to process the dominating case of Theorem 1.1 the hard way in the next section.

5 Dominating case

Here we prove Case III of Theorem 1.1.

Let $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$, integers of both signs.

We let the dominating forcing **DF** consist of all pairs $\langle n, f\rangle$ such that $f \in \mathbb{Z}^\omega$ (that is, f is an infinite sequence of integers) and $n < \omega$. We order **DF** so that $\langle n, f\rangle \leq \langle n', f'\rangle$ (the bigger is stronger) iff $n \leq n'$, $f{\restriction}n = f'{\restriction}n$, and $f \leq f'$ componentwise, that is, $f(k) \leq f'(k)$ holds for all $k < \omega$.[4]

A modified version $\mathbf{DF}'$ consists of all pairs $\langle u, h\rangle$, where $u \in \mathbb{Z}^{<\omega}$, $h \in \mathbb{Z}^\omega$. Each such pair is identified with the pair $\langle \mathtt{dom}\, u, u^\frown h\rangle \in \mathbf{DF}$, where $^\frown$ denotes the concatenation, and the order on $\mathbf{DF}'$ is induced by this identification.

Definition 5.1. If $G \subseteq \mathbf{DF}$ is a generic filter then $a_G = \bigcup_{\langle n,f\rangle\in G} f{\restriction}n$ belongs to $\mathbb{Z}^\omega$; we call a_G *a dominating-generic real*. More exactly, if $\mathfrak{M}$ is a transitive model and a set $G \subseteq \mathbf{DF} \cap \mathfrak{M}$ is $(\mathbf{DF} \cap \mathfrak{M})$-generic over $\mathfrak{M}$ then say that a_G is a *dominating-generic* (DG, in brief) real over $\mathfrak{M}$. □

Remark 5.2. Unfortunately there is no result similar to Proposition 3.2 for the dominating forcing, since if a is a DG real over $\mathfrak{M}$ and b is a DG real over $\mathfrak{M}[a]$ then a is definitely not DG over $\mathfrak{M}[b]$. This will make our arguments here somewhat more complex than in the Solovay-random section. □

If u, v are finite or infinite sequences of integers in $\mathbb{Z}$ then let $u \oplus v$ be a sequence defined by componentwise sum, so that $\mathtt{dom}\,(u \oplus v) = \mathtt{dom}\, v$ (independently of the length $\mathtt{dom}\, u$) and if $j < \mathtt{dom}\, v$ then $(u \oplus v)(j) = u(j) + v(j)$. If in addition $\mathtt{dom}\, u = \mathtt{dom}\, v$ then $u \ominus v$ is defined similarly.

For instance $f \oplus g$ and $f \ominus g$ are defined for all $f, g \in \mathbb{Z}^\omega$.

Lemma 5.3. *If $\mathfrak{M} \subseteq \mathfrak{N}$ are TM of a large fragment of* **ZFC***, and $a \in \mathbb{Z}^\omega$ is* DG *over $\mathfrak{N}$ then a is* DG *over $\mathfrak{M}$, too.*

[4] This slightly differs from the standard definition, as *e.g.* in Bartoszyński – Judah [2, 3.1] where $f \in \omega^\omega$. The difference does not change any forcing properties, but leads to a more friendly setup since **DF** as defined here is a group under componentwise addition.

Proof. It suffices to prove that if $A \in \mathfrak{M}$ is a maximal antichain in $\mathbf{DF} \cap \mathfrak{M}$ then A remains such in $\mathbf{DF} \cap \mathfrak{N}$. Note that A is countable in $\mathfrak{M}$ since $\mathbf{DF}$ is a ccc forcing, therefore A is effectively coded by a real $r \in \mathfrak{M}$ so that being a maximal antichain is a Π^1_1 property of r. It remains to refer to the Mostowski absoluteness theorem. □

Lemma 5.4. *If $\mathfrak{M}$ is a TM of a large fragment of* **ZFC***, $h \in \mathfrak{M} \cap \mathbb{Z}^\omega$, and $a \in \mathbb{Z}^\omega$ is a* DG *real over $\mathfrak{M}$ then $a \oplus h$, $a \ominus h$ are* DG *over $\mathfrak{M}$, too.*

Proof. The maps $\langle n, f\rangle \mapsto \langle n, f \oplus h\rangle$ and $\langle n, f\rangle \mapsto \langle n, f \ominus h\rangle$ are order-automorphisms of $\mathbf{DF} \cap \mathfrak{M}$ in $\mathfrak{M}$. □

Lemma 5.5. *If $\mathfrak{M}$ is a TM of a large fragment of* **ZFC***, $a \in \mathbb{Z}^\omega$ is a* DG *real over $\mathfrak{M}$, and $b \in \mathbb{Z}^\omega$ is a* DG *real over $\mathfrak{M}[a]$, then $\mathfrak{M}[a] \cap \mathfrak{M}[b] \cap 2^\omega \subseteq \mathfrak{M}$.*

Proof. Otherwise the opposite is forced over $\mathfrak{M}[a]$ by a condition $\langle n, f\rangle \in \mathbf{DF} \cap \mathfrak{M}[a]$; thus $f \in \mathbb{Z}^\omega \cap \mathfrak{M}[a]$. To be more precise, $\langle n, f\rangle$ ($\mathbf{DF} \cap \mathfrak{M}$)-forces $\underline{\mathfrak{M}}[\dot b] \cap \underline{\mathfrak{M}}[a] \cap 2^\omega \not\subseteq \underline{\mathfrak{M}}$ over $\mathfrak{M}[a]$, where $\underline{\mathfrak{M}}$ is a suitable name for $\mathfrak{M}$ as a class in $\mathfrak{M}[a]$, and $\dot b$ is a canonical name for the DG real over $\mathfrak{M}[a]$.

We claim that any other condition $\langle n', f'\rangle \in \mathbf{DF} \cap \mathfrak{M}[a]$ forces the same. Suppose to the contrary that in fact some $\langle n', f'\rangle \in \mathbf{DF} \cap \mathfrak{M}[a]$ forces $\underline{\mathfrak{M}}[\dot b] \cap \underline{\mathfrak{M}}[a] \cap 2^\omega \subseteq \underline{\mathfrak{M}}$ over $\mathfrak{M}[a]$. We can wlog assume that $n' = n$ and the n-tails of f and f' coincide: $f(j) = f'(j)$ for all $j \geq n$. Now let $b \in \mathbb{Z}^\omega$ be a DG real over $\mathfrak{M}[a]$ compatible with $\langle n, f\rangle$, that is, $b{\restriction}n = f{\restriction}n$ and $f \leq b$ componentwise. Let $b' \in \mathbb{Z}^\omega$ be defined so that $b'(j) = b(j)$ for all $j \geq n$, but $b{\restriction}n = f'{\restriction}n$; then b' is a DG real over $\mathfrak{M}[a]$ compatible with $\langle n, f'\rangle$. Then by construction we have $\mathfrak{M}[b] \cap \mathfrak{M}[a] \cap 2^\omega \not\subseteq \mathfrak{M}$ but $\mathfrak{M}[b'] \cap \mathfrak{M}[a] \cap 2^\omega \subseteq \mathfrak{M}$. However obviously $\mathfrak{M}[b] = \mathfrak{M}[b']$, a contradiction which completes the claim.

We conclude that if $b \in \mathbb{Z}^\omega$ is any DG real over $\mathfrak{M}[a]$ then $\mathfrak{M}[b] \cap \mathfrak{M}[a] \cap 2^\omega \not\subseteq \mathfrak{M}$. As a itself is generic over $\mathfrak{M}$, there is a condition $\langle m, h\rangle \in \mathbf{DF} \cap \mathfrak{M}$ such that $\mathfrak{M}[b] \cap \mathfrak{M}[a] \cap 2^\omega \not\subseteq \mathfrak{M}$ holds whenever $a \in \mathbb{Z}^\omega$ is DG over $\mathfrak{M}$ compatible with $\langle m, h\rangle$ and $b \in \mathbb{Z}^\omega$ is DG over $\mathfrak{M}[a]$.

Now let $\kappa = 2^{\aleph_0}$ in $\mathfrak{M}$, and let $\lambda = \kappa^+$ be the next cardinal in $\mathfrak{M}$. Let

$$\mathbb{Q} = \{\langle m', h'\rangle \in \mathbf{DF} \cap \mathfrak{M} : \langle m, h\rangle \leq \langle m', h'\rangle\}.$$

Consider the finite-support forcing product $\mathbb{Q}^\lambda$ in $\mathfrak{M}$. A $\mathbb{Q}^\lambda$-generic extension of $\mathfrak{M}$ has the form $\mathfrak{N} = \mathfrak{M}[\{a_\xi\}_{\xi<\lambda}]$, where $a_\xi \in 2^\omega$ are pairwise DG reals over $\mathfrak{M}$, compatible with $\langle m, h\rangle$, in particular $\mathfrak{M}[a_\xi] \cap \mathfrak{M}[a_\eta] = \mathfrak{M}$ whenever $\xi \neq \eta$.

Consider a ($\mathbf{DF} \cap \mathfrak{N}$)-generic extension $\mathfrak{N}[b]$ of $\mathfrak{N}$, so that $b \in \mathbb{Z}^\omega$ is a DG real over $\mathfrak{N}$. Then b is DG over each $\mathfrak{M}[a_\xi]$ by Lemma 5.3. It follows by the above that $\mathfrak{M}[b] \cap \mathfrak{M}[a_\xi] \cap 2^\omega \not\subseteq \mathfrak{M}$. Let $z_\xi \in \mathfrak{M}[b] \cap \mathfrak{M}[a_\xi] \cap 2^\omega \setminus \mathfrak{M}$, for all $\xi < \lambda$.

Note that if $\xi \neq \eta$ then $z_\xi \neq z_\eta$ since $\mathfrak{M}[a_\xi] \cap \mathfrak{M}[a_\eta] = \mathfrak{M}$, see above. Thus we have λ-many different reals in $\mathfrak{M}[b]$. However $\mathfrak{M}[b]$ is a CCC extension of $\mathfrak{M}$ by Lemma 5.3, and hence there cannot be more (in the sense of cardinality) reals in $\mathfrak{M}[b]$ than in $\mathfrak{M}$. The contradiction ends the proof. □

Lemma 5.6. *If $\mathfrak{M}$ is a TM of a large fragment of* **ZFC**, *$a \in \mathbb{Z}^\omega$ is a* DG *real over $\mathfrak{M}$, and $b \in \mathbb{Z}^\omega$ is a* DG *real over $\mathfrak{M}[a]$, then $\mathfrak{M}[b] \cap \mathfrak{M}[a \oplus b] \cap 2^\omega \subseteq \mathfrak{M}$.*

One may want to prove the lemma by proving that $\langle b, a \oplus b\rangle$ is dominating product-generic over $\mathfrak{M}$ due to the genericity of a. But in fact this is not the case. Indeed if $\langle b, a \oplus b\rangle$ is dominating product-generic over $\mathfrak{M}$ then a transparent forcing argument shows that $a = (a \oplus b) \ominus b$ is simply Cohen-generic over $\mathfrak{M}$, contrary to a being DG.

Proof. By Lemma 5.4, $a \oplus b$ is DG over $\mathfrak{M}[a]$, and hence over $\mathfrak{M}$ by Lemma 5.4 Therefore the **contrary assumption** implies a pair of $(\mathbf{DF} \cap \mathfrak{M})$-real names $\sigma, \tau \in \mathfrak{M}$ such that $\sigma[\![b]\!] = \tau[\![a \oplus b]\!] \in 2^\omega \smallsetminus \mathfrak{M}$, where $t[\![b]\!]$ is the b-interpretation of σ.

Let us present the two-step iterated forcing $\mathbb{P} \in \mathfrak{M}$ which produces $\mathfrak{M}[a][b]$ as $\mathbf{DF} * \mathbf{DF}'$, with $\mathbf{DF}'$, not $\mathbf{DF}$, as the second stage. Then $\mathbb{P}$ consists of all quadruples, or double-pairs, of the form $p = \langle\langle m_p, f_p\rangle, \langle u_p, t_p\rangle\rangle = \langle m_p, f_p, u_p, t_p\rangle$, where $\langle m_p, f_p\rangle \in \mathbf{DF} \cap \mathfrak{M}$, $u_p \in \mathbb{Z}^{<\omega}$, and $t_p \in \mathfrak{M}$ is a $\mathbf{DF}$-name for an element of $\mathbb{Z}^\omega$, with a suitable order. We shall use $\dot a$, $\dot b$ as canonical $\mathbb{P}$-names of the DG real over $\mathfrak{M}$ and DG real over $\mathfrak{M}[a]$, respectively.

By the contrary assumption, there is a condition $p_0 = \langle m_0, f_0, u_0, t_0\rangle \in \mathbb{P}$ which $\mathbb{P}$-forces, over $\mathfrak{M}$, the formula $\sigma[\![\dot b]\!] = \tau[\![\dot a \oplus \dot b]\!] \in 2^\omega \smallsetminus \underline{\mathfrak{M}}$, so that

(3) if $\langle a, b\rangle \in \mathbb{Z}^\omega$ is a pair $\mathbb{P}$-generic over $\mathfrak{M}$ (so a is DG over $\mathfrak{M}$ and b DG over $\mathfrak{M}[a]$) and compatible with p_0, then $\sigma[\![b]\!] = \tau[\![a \oplus b]\!] \in 2^\omega \smallsetminus \mathfrak{M}$.

Let $n_0 = \mathtt{dom}\, u_0$. We can assume that $n_0 \leq m_0$; otherwise change m_0 to n_0.

By simple strengthening, we find a stronger condition $p_1 = \langle m_1, f_1, u_1, t_1\rangle$ in $\mathbb{P}$, $p_1 \geq p_0$, such that $m_0 \leq n_1 = \mathtt{dom}\, u_1 \leq m_1$.

Claim 5.7. *If conditions $p_2 = \langle m, f, u_2, t_2\rangle$ and $p_3 = \langle m, f, u_3, t_3\rangle$ (same m, f!) in $\mathbb{P}$ satisfy $p_1 \leq p_2$, $p_1 \leq p_3$, and in addition $k < \omega$, $z \in \{0, 1\}$, and p_2 $\mathbb{P}$-forces $\sigma[\![\dot b]\!](k) = z$ then so does p_3.*

Proof (Claim)**.** Otherwise there are conditions p_2 and p_3 as in the claim, such that p_2 $\mathbb{P}$-forces $\sigma[\![\dot b]\!](k) = 0$ while p_3 $\mathbb{P}$-forces $\sigma[\![\dot b]\!](k) = 1$. We can wlog assume that $\mathtt{dom}\, u_3 = \mathtt{dom}\, u_2 =$ some n and $m_1 \leq n \leq m$, so overall

$$n_0 = \mathtt{dom}\, u_0 \leq m_0 \leq n_1 = \mathtt{dom}\, u_1 \leq m_1 \leq n = \mathtt{dom}\, u_2 = \mathtt{dom}\, u_3 \leq m\,. \quad (4)$$

And we can wlog assume that

(5) $t_2 = t_3 =$ some $t \in \mathbb{Z}^{<\omega}$, thus $p_2 = \langle m, f, u_2, t\rangle$ $\mathbb{P}$-forces $\sigma[\![\dot b]\!](k) = 0$ while $p_3 = \langle m, f, u_3, t\rangle$ (same $m, f, t!$) $\mathbb{P}$-forces $\sigma[\![\dot b]\!](k) = 1$.

Indeed just let $t = \sup\{t_2, t_3\}$ termwise, thus $t \in \mathfrak{M}$ is a $(\mathbf{DF} \cap \mathfrak{M})$-name saying: I am a real in $\mathbb{Z}^\omega$ and each value $t(j)$ is equal to $\sup\{t_2(j), t_3(j)\}$.

It is clear that the difference between the conditions p_2 and p_3 of (5) is located in the set $U = \{j : u_2(j) \neq u_3(j)\} \subseteq [n_1, n) = \{j : n_1 \leq j < n\}$, which we divide into subsets $U_2 = \{j : u_3(j) < u_2(j)\}$ and $U_3 = \{j : u_2(j) < u_3(j)\}$. Now define $f_2, f_3 \in \mathbb{Z}^\omega$ as follows:

$$\left.\begin{aligned} f_3(j) &= \begin{cases} f(j) + u_2(j) - u_3(j), & \text{whenever} \quad j \in U_2 \\ f(j), & \text{otherwise} \end{cases} ; \\ f_2(j) &= \begin{cases} f(j) + u_3(j) - u_2(j), & \text{whenever} \quad j \in U_3 \\ f(j), & \text{otherwise} \end{cases} ; \end{aligned}\right\} \tag{6}$$

so that $f \leq f_2$ and $f \leq f_3$ termwise, the difference between f, f_2, f_3 is still located in $U \subseteq [n_1, n)$, and the termwise sums $(f_2 \restriction n) \oplus u_2$, $(f_3 \restriction n) \oplus u_3$ coincide.

Note that $q_2 = \langle m, f_2\rangle$ and $\langle m, f_3\rangle$ are conditions in $\mathbf{DF} \cap \mathfrak{M}$, and $f_2 \restriction n_1 = f_3 \restriction n_1 = f \restriction n_1$ by construction. Let $a_0 \in \mathbb{Z}^\omega$ be a DG real over $\mathfrak{M}$, compatible with the condition $\langle m, f\rangle$, so that

(a) $f \restriction m \subset a_0$ and $f \leq a_0$ termwise,

Accordingly define $a_2, a_3 \in \mathbb{Z}^\omega$ so that

(b) $a_2 \restriction n = f_2 \restriction n$, $a_3 \restriction n = f_3 \restriction n$, and $a_3(j) = a_2(j) = a_0(j)$ for all $j \geq n$, so that $f_2 \leq a_2$ and $f_3 \leq a_3$ termwise.

Then a_2, a_3 are DG reals over $\mathfrak{M}$, compatible with resp. $\langle m, f_2\rangle$, $\langle m, f_3\rangle$.

Now come back to the name t which occurs in conditions p_2, p_3 in (5). As t is a $(\mathbf{DF} \cap \mathfrak{M})$-name for a real in $\mathbb{Z}^\omega$, in fact the interpretations $t[\![a_0]\!]$, $t[\![a_2]\!]$, $t[\![a_3]\!]$ belong to $\mathbb{Z}^\omega \cap \mathfrak{M}[a_0]$. Moreover, as soon as the finite strings $f \restriction n$, u_2, u_3 (of length n) are given, the reals $a_2 = H_2(a_0)$ and $a_3 = H_3(a_0)$ are defined by simple functions H_2 and H_3 whose definitions are contained in (b) and (6). Let $t' \in \mathfrak{M}$ be a $(\mathbf{DF} \cap \mathfrak{M})$-name for a real in $\mathbb{Z}^\omega$, explicitly defined as the termwise supremum of $t[\![\dot a]\!]$, $t[\![H_2(\dot a)]\!]$, $t[\![H_3(\dot a)]\!]$, so that in particular

(c) $t'[\![a_0]\!](j) = \sup\{t[\![a_0]\!](j), t[\![a_2]\!](j), t[\![a_3]\!](j)\}$ for all $j < \omega$.

Note that $q_2 = \langle m, f_2, u_2, t\rangle$ and $q_3 = \langle m, f_3, u_3, t\rangle$ are still conditions in $\mathbb{P}$, and $f_2 \restriction n_1 = f_3 \restriction n_1 = f \restriction n_1$ by construction. As $n_0 \leq m_0 \leq n_1$ by (4), it follows that $p_0 \leq q_2$ and $p_0 \leq q_3$. (We do not claim that $p_1 \leq q_{2,3}$ or $p_{2,3} \leq q_{2,3}$!) By the choice of a_0 there is a real $b_2 \in \mathbb{Z}^\omega$ such that $\langle a_0, b_2\rangle$ is a $\mathbb{P}$-generic pair in $\mathbb{Z}^\omega \times \mathbb{Z}^\omega$, compatible with the condition $p_2' = \langle m, f, u_2, t'\rangle$, so that

(d) $u_2 \subset b_2$, and $u_2{}^\frown t'[\![a_0]\!] \le b_2$ termwise.

We further define $b_3 \in \mathbb{Z}^\omega$ so that

(e) $u_3 \subset b_3$, and $b_3(j) = b_2(j)$ for all $j \ge n = \mathtt{dom}\, u_2 = \mathtt{dom}\, u_3$, hence $u_3{}^\frown t'[\![a_0]\!] \le b_3$ termwise by (d).

It follows that $\langle a_0, b_3\rangle$ is a $\mathbb{P}$-generic pair, compatible with $p_3 = \langle m, f, u_3, t\rangle$. We conclude by (5) that

(7) $\sigma[\![b_2]\!](k) = 0$ while $\sigma[\![b_3]\!](k) = 1$, thus $\sigma[\![b_2]\!] \ne \sigma[\![b_3]\!]$.

Then the pairs $\langle a_2, b_2\rangle$ and $\langle a_3, b_3\rangle$ are $\mathbb{P}$-generic over $\mathfrak{M}$, and we have

(8) $a_2 \oplus b_2 = a_3 \oplus b_3$ — therefore $\tau[\![a_2 \oplus b_2]\!] = \tau[\![a_3 \oplus b_3]\!]$,

since $(a_2 \restriction n) \oplus (b_2 \restriction n) = (f_2 \restriction n) \oplus u_2 = (f_3 \restriction n) \oplus u_3 = (a_3 \restriction n) \oplus (b_3 \restriction n)$ by construction, and if $n \le j$ then $a_3(j) = a_2(j) = a_0(j)$ and $b_3(j) = b_2(j)$.

Assume for a moment that

(9) the pairs $\langle a_2, b_2\rangle$, $\langle a_3, b_3\rangle$ are compatible with the conditions resp. q_2, q_3.

Then, as $p_0 \le q_2, q_3$, we have $\sigma[\![b_2]\!] = \tau[\![a_2 \oplus b_2]\!]$ and $\sigma[\![b_3]\!] = \tau[\![a_3 \oplus b_3]\!]$, by (3). It follows that $\sigma[\![b_2]\!] = \sigma[\![b_3]\!]$ by (8), which is a contradiction with (7), and this proves the claim. Thus it remains to establish (9), which amounts to

(9)$*$: $f_2 \restriction m \subset a_2$, $f_3 \restriction m \subset a_3$, and $f_2 \le a_2$, $f_3 \le a_3$ termwise,

(9)$\dagger$: $u_2 \subset b_2$, $u_3 \subset b_3$, and

(9)$\ddagger$: $u_2{}^\frown t[\![a_2]\!] \le b_2$ and $u_3{}^\frown t[\![a_3]\!] \le b_3$ termwise.

Beginning with (9)$*$, note that $f_2 \restriction n \subset a_2$ by (b), while if $n \le j < m$ then $a_2(j) = a_0(j) = f(j)$ by (b) and (a), and $f_2(j) = f(j)$ by construction, hence $a_2(j) = f_2(j)$, and $f_2 \restriction m \subset a_2$ is verified. Similarly, if $j \ge m$ then $f_2(j) = f(j)$ and $a_2(j) = a_0(j)$, but $f(j) \le a_0(j)$ by (a), hence $f_2(j) \le a_2(j)$.

Claim (9)$\dagger$ immediately follows from (d), (e).

As regards for (9)$\ddagger$, we have $t[\![a_2]\!] \le t'[\![a_0]\!]$ and $t[\![a_3]\!] \le t'[\![a_0]\!]$ componentwise by (c). It remains to refer to (d) and (e). □ (Claim 5.7)

A standard consequence of the claim is that p_1 $\mathbb{P}$-forces that $\sigma[\![\dot b]\!] \in \underline{\mathfrak{M}}[\dot a]$. However $p_0 \le p_1$ and p_0 forces the opposite, a contradiction. □ (Lemma 5.6)

Proof (Theorem 1.1, case III)**.** As above, the **contrary assumption** leads to a formula $\varphi(z)$ with $\gamma_0 \in \mathrm{Ord}$ as a parameter, a condition $p_0 = \langle m_0, f_0\rangle \in \mathbf{DF}$ in $\mathbf{V}$ which $\mathbf{DF}$-forces, over $\mathbf{V}$, that the set $\{z \in 2^\omega : \varphi(z)\}$ is countable and $\exists\, z\, (z \notin \check{\mathbf{V}} \wedge \varphi(z))$, a sequence $\{t_n\}_{n<\omega} \in \mathbf{V}$ of $\mathbf{DF}$-names for reals in $\mathbb{Z}^\omega$, and a canonical $\mathbf{DF}$-name $T \in \mathbf{V}$ for $\{t_n[\![\dot a]\!] : n < \omega\}$, such that

(10) if $x \in \mathbb{Z}^\omega$ is a DG real, over $\mathbf{V}$, compatible with p_0 then it is true in $\mathbf{V}[x]$ that $\{z \in 2^\omega : \varphi(z)\} = \{t_n[\![x]\!] : n < \omega\} = T[\![x]\!] \not\subseteq \mathbf{V}$.

Pick a regular cardinal $\kappa > \alpha_0$, sufficiently large for $\mathbf{H}_\kappa$ to contain γ_0 and all names t_n and T. Consider a countable elementary submodel $\mathfrak{M}$ of $\mathbf{H}_\kappa$ containing γ_0, all t_n, T, and $\mathbf{DF}$. Let $\pi : \mathfrak{M} \to \mathfrak{M}'$ be the Mostowski collapse onto a transitive set $\mathfrak{M}'$. We have $\pi(t_n) = t_n$ for all n (as by the ccc property of $\mathbf{DF}$ we can assume that t_n is a hereditarily countable set), and $\pi(T) = T$.

By the countability, there is a real $a \in \mathbb{Z}^\omega$ in $\mathbf{V}$, DG over $\mathfrak{M}'$. We can wlog assume that $a(j) = 0$ for all $j < m_0$ and $a(j) \geq 0$ for all $j \geq m_0$.

Let $b \in \mathbb{Z}^\omega$ be a real DG over $\mathbf{V}$, compatible with p_0. In our assumptions, the real $b' = a \oplus b \in \mathbb{Z}^\omega$ also is DG over $\mathbf{V}$ and compatible with p_0, and $\mathbf{V}[b'] = \mathbf{V}[b]$ (since $a \in \mathbf{V}$). Then $T[\![b]\!] = T[\![b']\!]$ by (10).

On the other hand, b is DG over $\mathfrak{M}'[a]$ as well by Lemma 5.3. It follows by Lemma 5.6 that $\mathfrak{M}[b] \cap \mathfrak{M}[b'] \cap 2^\omega \subseteq \mathfrak{M}$, therefore

$$T[\![b]\!] \cap T[\![b']\!] \subseteq \mathfrak{M}'[b] \cap \mathfrak{M}'[b'] \subseteq \mathfrak{M}' \subseteq \mathbf{V}\,,$$

so that $T[\![b]\!] = T[\![b']\!] \subseteq \mathbf{V}$, and we get a contratiction required with (10).

□ (Theorem 1.1, case III)

6 Sacks case

It is a known property of Sacks-generic extensions $\mathbf{V}[a]$ that if $b \in 2^\omega$ is a real in $\mathbf{V}[a]$ then either $b \in \mathbf{V}$ or b itself is Sacks-generic over $\mathbf{V}$ and $\mathbf{V}[b] = \mathbf{V}[a]$. Thus if $X \in \mathbf{V}[a]$ is an OD set of reals in $\mathbf{V}[a]$ and $X \not\subseteq \mathbf{V}$ then there is a perfect set $Y \subseteq 2^\omega$ coded in $\mathbf{V}$, such that every Sacks-generic real $b \in Y$ in $\mathbf{V}[a]$ belongs to X. However it is true in $\mathbf{V}[a]$ that every (non-empty) perfect set coded in $\mathbf{V}$ contains uncountably many reals Sacks-generic over $\mathbf{V}$.

This is a rather transparent argument, so we can skip details.

□ (Theorem 1.1, case IV)

7 The Solovay model

Definition 7.1. The *first Solovay model* is a model of **ZFC** defined as a generic extension $\mathbf{L}[G]$ of $\mathbf{L}$ by the Levy collapse below an inaccessible cardinal in $\mathbf{L}$. The *second Solovay model* is a model of $\mathbf{ZF} + \mathbf{DC}$ equal to the collection of all hereditarily real-ordinal definable (HROD) sets in the first model, $\mathbf{L}[G]$. □

Thus we explicitly consider the case when the ground **ZFC** model of the Solovay models considered is the constructible model. Theorem 1.2 is true for an arbitrary ground model (with a strongly inaccessible cardinal), but we stick

to the particular case to avoid some minor unrelated complications. We'll make use of the following result, implicit in Stern [17, proof of 3.2] and [7].

Proposition 7.2. *It holds in either of the Solovay models, that if an OD equivalence relation on ω^ω has at most countably many equivalence classes then all of them are OD sets.* □

Our first proof of Theorem 1.2(i) was presented in [8]. Further research demonstrated though that the proof was a largely unnecessary roundabout, and the result can be obtained by a rather brief reduction to 7.2. We also note that the case, when $\mathscr{X}$ is a (non-empty OD countable) **set of reals** in Theorem 1.2(i), is well known and is implicitly contained in the proof of the perfect set property for ROD sets of reals by Solovay [16]. Hovever the proofs known for this particular case (as, *e.g.*, in [7] or Stern [17]) do not work for sets $\mathscr{X} \subseteq \mathscr{P}(2^\omega)$.

Proof (Theorem 1.2(i))**.** *Arguing in the first Solovay model*, let $\mathscr{X}$ be a non-empty OD countable **set of sets of reals**; we have to prove that $\mathscr{X}$ contains an OD element (an OD set of reals). Consider a particular case first.

Case 1: $\mathscr{X}$ consists of *pairwise disjoint* sets of reals. If x, y are reals then define $x \mathsf{E} y$ iff either both x, y do not belong to $\bigcup \mathscr{X}$ or x, y belong to the same set $X \in \mathscr{X}$. This is an OD equivalence relation with countably many equivalence classes, and hence each E-class is an OD set by 7.2, as required.

Case 2: general. Let $\mathscr{C}$ be the set of all countable sets C of reals, such that if $X \neq Y$ belong to $\mathscr{X}$ then already $X \cap C \neq Y \cap C$. Note that $\mathscr{C} \neq \varnothing$ as $\mathscr{X}$ is countable. If $X \in \mathscr{X}$ then let P_X be the set of all pairs of the form $\langle C, X \cap C \rangle$, where $C \in \mathscr{C}$. Then $P_X \cap P_Y = \varnothing$ whenever $X \neq Y$ belong to $\mathscr{X}$. We conclude that $\mathscr{P} = \{ P_X : X \in \mathscr{X} \}$ is a countable collection of pairwise disjoint non-empty sets P_X of pairs of the form $\langle C, C' \rangle$, where $C' \subseteq C$ are countable sets of reals.

There exists an OD coding of such pairs by reals, that is, an OD map $x \mapsto \langle C_x, C'_x \rangle$, where $x \in \omega^\omega$ is a real, $C'_x \subseteq C_x$ are countable sets of reals for any x, and for any such pair $\langle C, C' \rangle$ there is at least one $x \in \omega^\omega$ such that $C = C_x$ and $C' = C'_x$. It follows from the above that the derived sets

$$Q_X = \{ x \in \omega^\omega : \langle C_x, C'_x \rangle \in P_X \}, \quad X \in \mathscr{X},$$

form a countable OD family $\mathscr{Q} = \{ Q_X : X \in \mathscr{X} \}$ of pairwise disjoint non-empty sets of reals. By the result in Case 1, all sets $Q_X \in \mathscr{Q}$ are OD. But if any Q_X is OD then so is both $P_X = \{ \langle C_x, C'_x \rangle : x \in Q_X \}$ and X itself.

□ (Theorem 1.2(i))

Proof (Theorem 1.2(ii))**.** *Arguing in the second Solovay model*, let $X \neq \varnothing$ be an OD set. Let $x_0 \in X$. We make use of the fact that, in this model, every

set is real-ordinal definable (ROD). Thus x_0 is ROD; there is an $\in$-formula $\varphi(\cdot,\cdot,\cdot)$, an ordinal α_0, and a real $r_0 \in 2^\omega$ such that $x_0 = F(\alpha_0, r_0)$, where

$$F(\alpha, r) = \begin{cases} \text{the only } x \text{ satisfying } \varphi(\alpha, r, x), & \text{whenever} \quad \exists\,!\, x\, \varphi(\alpha, r, x) \\ \varnothing, & \text{otherwise} \quad . \end{cases}$$

Let $R_0 = \{r \in 2^\omega : F(\alpha_0, r) \in X\}$, and if $r, q \in R_0$ then define $r \mathrel{\mathsf{E}} q$ iff $F(\alpha_0, r) = F(\alpha_0, q)$. Then E is an OD equivalence relation on an OD set R_0. Moreover E has countably many classes (since X is countable). It remains to refer to Proposition 7.2.

$\square$ (Theorem 1.2(ii))

8 Problems

Problem 8.1. Is the stronger result as in Theorem 1.2(i) (for a set of sets of reals) still true in the generic extensions mentioned in Theorem 1.1? $\square$

Problem 8.2. Is it still true in the first Solovay model that every nonempty countable OD set (of any kind) contains an OD element? $\square$

Problem 8.3. Do some other simple generic extensions by a real (other than Cohen-generic, Solovay-random, dominating, ans Sacks) admit results similar to Theorem 1.1 and also those similar to the old folklore lemmas 4.1 and 4.2 above? It would also be interesting to investigate the state of affairs in different 'coding by a real' models as those defined in [1, 9]. $\square$

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