On Harrington's model in which Separation holds but Reduction fails at the 3rd projective level, and on some related models of Sami

Vladimir Kanovei[*]   Vassily Lyubetsky[†]

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Abstract

In a handwtitten note of 1975, Leo Harrington sketched a construction of a model of **ZFC** (no large cardinals or anything beyond **ZFC**!) in which $\mathbf{\Pi}^1_3$-Separation holds but $\mathbf{\Sigma}^1_3$-Reduction fails. The result has never appeared in a journal or book publication except for a few of old references.

MSC 03E15, 03E35

1   Introduction

The separation property for a pointclass K, or simply K-*Separation*, is the assertion that any two disjoint sets X, Y in K (in the same Polish space) can be separated by a set in $K \cap K^{\complement}$, where $K^{\complement}$ is the pointclass of complements of sets in K. The reduction property for a pointclass K, or simply K-*Reduction*, is the assertion that for any two sets X, Y in K (in the same Polish space) there exist *disjoint* sets $X' \subseteq X$, $Y' \subseteq Y$ in the same class K, such that $X' \cup Y' = X \cup Y$.

It is known classically from studies of Luzin [13, 12], Novikov [17, 18], Kuratowski [11] that Separation holds for projective classes $\mathbf{\Sigma}^1_1$ (analytic sets) and $\mathbf{\Pi}^1_2$, but fails for $\mathbf{\Pi}^1_1$ (coanalytic sets) and $\mathbf{\Sigma}^1_2$, while Reduction holds for $\mathbf{\Pi}^1_1$ and $\mathbf{\Sigma}^1_2$, but fails for $\mathbf{\Sigma}^1_1$ and $\mathbf{\Pi}^1_2$, and generally K-Reduction implies $K^{\complement}$-Separation by a simple argument.

As for the higher projective classes, Addison [1, 2] proved that the axiom of constructibility $\mathbf{V} = \mathbf{L}$ implies that Separation holds for projective classes $\mathbf{\Pi}^1_n$, $n \geq 3$, but fails for $\mathbf{\Sigma}^1_n$, $n \geq 3$ while Reduction holds for $\mathbf{\Sigma}^1_n$, $n \geq 3$, but fails for $\mathbf{\Pi}^1_n$, $n \geq 3$. On the other hand, by Martin [14], the axiom of projective

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determinacy **PD** implies that, similarly to projective level 1, $\mathbf{\Sigma}_n^1$-Separation and $\mathbf{\Pi}_n^1$-Reduction hold for all odd numbers $n \geq 3$, and, similarly to projective level 2, $\mathbf{\Pi}_n^1$-Separation and $\mathbf{\Sigma}_n^1$-Reduction hold for all even numbers $n \geq 4$.

Apparently not much is known on Separation and Reduction for higher projective classes in generic models. In a handwtitten note [4, Part C] (1975), Leo Harrington sketched a construction of a model of **ZFC** in which $\mathbf{\Pi}_3^1$-Separation holds but $\mathbf{\Sigma}_3^1$-Reduction fails. The model was a generic extension of **L** by means of the almost-disjoint coding of [7], with no reference to determinacy, large cardinals or anything beyond **ZFC**. The result has never appeared in a journal or book publication except for a few rather old references. [1]

Here we present a proof of Harrington's theorem.

Theorem 1.1 (Harrington [4], Part C)**.** *There exists a set-generic extension of* **L***, in which* $\mathbf{\Pi}_3^1$*-Separation holds but* $\mathbf{\Sigma}_3^1$*-Reduction fails, and moreover, there is a pair of lightfsce* Σ_3^1 *sets of reals, not reducible by a pair of* $\mathbf{\Sigma}_3^1$ *subsets.*

In the proof, we'll follow, more or less, the flow of Harrington's arguments, filling in details and claims wherever (we find it) necessary. We'll try to preserve even Harrington wording wherever possible. Of most notable deviations, we change Harrington's Boolean-valued approach to the poset forcing approach, as we observed that the non-absoluteness of the RO operation causes problems in understanding of the behaviour of certain BAs in different models. Of notable

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[1] Hinman [5, p. 230, end of Section V.3] communicates a much more general consistency result related to the principles of Separation and Reduction, absent even in [4], citing a paper of Harrington entitled "*Consistency and independence results in descriptive set theory*, to appear in *Ann. of Math.*", which has apparently never materialized. Moschovakis [16, Theorem 5B.3 on p. 214] mentions another Harrington's model, present in [4], in which Separation fails for both $\mathbf{\Pi}_3^1$ and $\mathbf{\Sigma}_3^1$. Another similar reference, to Harrington's models in which $\mathbf{\Sigma}_n^1$-Separation and $\mathbf{\Pi}_n^1$-Separation both fail for a given n, see Mathias [15, p. 166], a comment on P 3110. Sami [19, Thm 1.21] presents the following result with reference to Harrington:

If $n \geq 3$, then there exist generic extensions $\mathfrak{N}_1, \mathfrak{N}_2, \mathfrak{N}_3, \mathfrak{N}_4$ of **L** such that

(a) $\mathbf{\Pi}_n^1$-*Separation and* $\mathbf{\Sigma}_n^1$-*Separation fail in* $\mathfrak{N}_1$;

(b) $\mathbf{\Pi}_n^1$-*Separation holds but* $\mathbf{\Sigma}_n^1$-*Separation fails in* $\mathfrak{N}_2$;

(c) $\mathbf{\Pi}_n^1$-*Separation fails but* $\mathbf{\Sigma}_n^1$-*Separation holds in* $\mathfrak{N}_3$;

(d) $\mathbf{\Pi}_n^1$-*Separation and* $\mathbf{\Sigma}_n^1$-*Separation hold in* $\mathfrak{N}_4$.

In addition, there exists a generic extension $\mathfrak{N}$ of **L** such that

(e) $\mathbf{\Pi}_n^1$-*Separation and* $\mathbf{\Sigma}_n^1$-*Separation fail in* $\mathfrak{N}$ for all $n \geq 3$.

Here, $\mathfrak{N}_1$ is defined in [4, Part B] for $n = 3$, and a hint is given regarding the general case. A different model, in which both $\mathbf{\Sigma}_3^1$-Separation and $\mathbf{\Pi}_3^1$-Separation fail, has recently been defined in [8]. As for $\mathfrak{N}_2$, the constructible universe itself works by Addison. Models $\mathfrak{N}_3$ and $\mathfrak{N}_4$ are absent in [4], generally no generic extensions of **L** are known in which $\mathbf{\Sigma}_n^1$-Separation holds for at least one $n \geq 3$. However a generic model in which both (lightface) $\mathit{\Pi}_n^1$-Separation and $\mathit{\Sigma}_n^1$-Separation hold *for sets of integers* is given in [4, Part D]. Finally, the existence of a model $\mathfrak{N}$ for (e) is characterized in [4, Part B] as an "expression of belief".

additional details, we adjoined some amount of definitions and results related to intermediate sumbodels of generic extensions, necessary to fully understand the arguments but near completely avoided (or just hinted) in [4].

The following is Harrington's comment to Theorem 1.1 in [4, Part C].

> *The above proof was directly inspired by a result of Sami, namely: there is a model of* **ZFC** *in which* $\mathrm{Sep}(\mathbf{\Pi}_3^1, \mathbf{\Delta}_3^1)$ *holds but* $\mathrm{Red}(\Sigma_3^1, \Sigma_3^1)$ *fails for sets of reals.* (Note the lightface Σ_3^1 twice in the second part, so $\mathrm{Red}(\Sigma_3^1, \Sigma_3^1)$ is Σ_3^1-Reduction in the above sense.)

Thus result indeed can be found in Ramez Sami's PhD Thesis [19, Theorem 1.7], but it has never been published.

The next theorem presents some related results in [19, Thms 1.7,1.18,1.20].

Theorem 1.2. (I) *It is true in any extension of* **L** *by a single Cohen-generic real that* Σ_3^1*-Reduction fails,* Π_3^1*-Separation holds, and if* $n \geq 4$ *then* Σ_n^1*-Reduction holds, and hence* $\mathbf{\Sigma}_n^1$*-Reduction holds as well.* [2]

(II) *It is true in any extension of* **L** *by* $\aleph_1$ *Cohen-generic reals that if* $n \geq 3$ *then* Σ_n^1*-Reduction holds, and hence* $\mathbf{\Sigma}_n^1$*-Reduction holds as well.*

(III) *The same is true in the the Solovay model, i.e., the Levy-collapse extension of* **L** *via an inaccessible cardinal.*

We sketch the proof of claim (II) in the end of the paper. Note that (II) also holds in models obtained by adding any uncountable (not necessarily $\aleph_1$) number of Cohen-generic reals. (Because they are elementarily equivalent to the extension by $\aleph_1$ Cohen reals.) And (II) also holds in extensions by $\aleph_1$ or more Solovay-random reals.

2 Almost disjoint preliminaries

Some definitions related to the almost disjoint forcing of [7].

- $\mathbb{C} = 2^{<\omega}$ (the Cohen forcing).

- Λ (the empty string) is the weakest condition in $\mathbb{C}$.

- $\omega^{<\omega} = \{s_j : j < \omega\}$ is a fixed recursive enumeration.

- if $f \in \omega^\omega$ then $S(f) = \{j < \omega : s_j \subset f\}$.

- **ZFC$^-$** is **ZFC** without the Power Sets axiom,

  T is **ZFC$^-$** plus **V = L** and "all sets are countable".

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[2] To prove that Σ_n^1-Reduction implies the boldface $\mathbf{\Sigma}_n^1$-Reduction, it suffices to use a double-universal pair of Σ_n^1 sets, as those used in a typical proof that $\mathbf{\Sigma}_n^1$-Reduction and $\mathbf{\Sigma}_n^1$-Separation contadict each other. This argument does not work for Separation though.

- HC = all hereditarily-countable sets.

- $\langle \xi_\alpha, n_\alpha \rangle$ is the αth element of the set $\omega_1 \times \omega$, ordered lexicographically.

Definition 2.1. Reals $f_\alpha \in \omega^\omega$ are defined in $\mathbf{L}$ by induction on $\alpha < \omega_1$. Suppose that f_γ are defined for all $\gamma < \alpha$. Let $\mathbf{L}_{\mu(\alpha)} \models \mathbf{T}$ be the smallest model containing the sequence $\gamma \mapsto f_\gamma$ of already defined reals. Let f_α be the Goedel-least real $f \in \omega^\omega$, Cohen-generic over $\mathbf{L}_{\mu(\alpha)}$ and satisfying $s_{n_\alpha} \subset f$.

If $\xi = \xi_\alpha$ and $n = n_\alpha$ then let $f_{\xi n} := f_\alpha$, hence $s_n \subset f_{\xi n}$ always holds. $\qquad \square$

If $F \subseteq \omega^\omega$ then $\mathbf{JS}(F)$ is the corresponding almost-disjoint forcing, which consists of all pairs $\langle u, S \rangle$, where $u \subseteq \omega$ and $S \subseteq S(F) = \{S(f) : f \in F\}$ are finite sets, ordered so that $\langle t, S \rangle \leqslant \langle t', S' \rangle$ (the smaller condition is stronger) iff $t' \subseteq t$, $S' \subseteq S$, and $u \cap A = u' \cap A$ for all $A \in S'$.

- If $g \in 2^\omega$ then let $F_g = \{f_{\xi i} : \xi < \omega_1, i < \omega, g(i) = 0\}$.

- If $e \in 2^{<\omega}$ then let $F_e = \{f_{\xi i} : \xi < \omega_1, i < \mathtt{lh}(e), e(i) = 0\}$.

- Define $T(g, a)$ iff $\forall \xi < \omega_1 \, \forall i \in \omega \, (S(f_{\xi i}) \cap a$ is finite $\iff g(i) = 0)$.

Lemma 2.2 (see [7]). *If $g \in 2^\omega$ in a set universe $\mathbf{V}$ then the forcing $\mathbf{JS}(F_g)$ adjoins a real $a \subseteq \omega$ satisfying $T(g, a)$.* $\qquad \square$

Definition 2.3. Let $Q \in \mathbf{L}$ be the forcing notion responsible for the following two-step generic extension: 1st, we extend a ground set universe $\mathbf{V}$ by a real $g \in 2^\omega$ Cohen-generic over $\mathbf{V}$, and 2nd, we extend $\mathbf{V}[g]$ by $\mathbf{JS}(F_g)$.

Thus Q consists of all triples $p = \langle e, u, S \rangle$, where $e \in 2^{<\omega} = \mathbb{C}$ (a Cohen condition) while $\langle u, S \rangle \in \mathbf{JS}(F_e)$. The order is defined so that $p = \langle e, u, S \rangle \leqslant p' = \langle e', u', S' \rangle$ (p is stronger) iff $e' \subseteq e$ and $\langle u, S \rangle \leqslant \langle u', S' \rangle$ in $\mathbf{JS}(F_e)$. Note that $1 = \langle \Lambda, \varnothing, \varnothing \rangle \in Q$ is the largest (and weakest) element in Q.

Let $\boldsymbol{\mathcal{Q}} = Q^\omega$ (a finite-support product), with the product order $\leqslant := \leqslant_{\boldsymbol{\mathcal{Q}}}$; $p \leqslant q$ still means that p is stronger. Thus $\boldsymbol{\mathcal{Q}} = \langle \boldsymbol{\mathcal{Q}}; \leqslant \rangle$ is a forcing in $\mathbf{L}$. Its largest (= weakest) element $\mathbf{1} \in \boldsymbol{\mathcal{Q}}$ is defined by $\mathbf{1}(k) = 1$, $\forall k$. $\qquad \square$

Lemma 2.4 (definability). *The sequences $\langle f_\alpha \rangle_{\alpha < \omega_1}$ and $\langle f_{\xi n} \rangle_{\xi < \omega_1, n < \omega}$ are $\Delta_1^{\mathbf{L}_{\omega_1}}$, hence Σ_1^{HC}. The sets Q and $\boldsymbol{\mathcal{Q}}$ are $\Delta_1^{\mathbf{L}_{\omega_1}}$, hence Σ_1^{HC}. The relations of compatibility and incompatibility in Q and $\boldsymbol{\mathcal{Q}}$ are $\Delta_1^{\mathbf{L}_{\omega_1}}$, hence Σ_1^{HC}.* $\qquad \square$

Proof. To circumvent the naturally reqired $\forall$ over the given po set in the definition of incompatibility, define the binary operation $\wedge$ on Q as follows. If $p = \langle e, u, S \rangle$ and $p' = \langle e', u', S' \rangle$ belong to Q then put $p \wedge q = \langle e \wedge e', u \cup u', S \cup S' \rangle$, where $e \wedge e' = e$, or $= e'$, or $= \Lambda$ (the empty string) in cases pesp. $e' \subseteq e$, $e \subseteq e'$, or e, e' are incomparable in $\mathbb{C} = 2^{<\omega}$. Extend $\wedge$ to $\boldsymbol{\mathcal{Q}}$ componentwise. Then, both in Q and in $\boldsymbol{\mathcal{Q}}$, conditions p, q are incompatible, in symbol $p \perp q$, iff $p \wedge q \not\leqslant p$ or $p \wedge q \not\leqslant q$. This yields the result required. $\qquad \square$

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Remark 2.5. (A) The forcing $\mathcal{Q}$ adjoins sequences of the form $A = \langle g_n, a_n \rangle_{n < \omega}$, where each pair $\langle g_n, a_n \rangle$ is Q-generic, hence $g_n \in 2^\omega$ is Cohen-generic, $a_n \subseteq \omega$, and $T(g_n, a_n)$ holds. If $G \subseteq \mathcal{Q}$ is generic over a set universe $\mathbf{V}$, and $A = A[G] = \langle g_n[G], a_n[G] \rangle_{n < \omega}$ is the corresponding sequence as above, then $\mathbf{V}[G] = \mathbf{V}[A[G]]$.

(B) Any A of such a form can be converted to a real $r(A) \subseteq \omega$ by means of any recursive bijection between $(2^\omega \times \mathscr{P}(\omega))^\omega$ and $\mathscr{P}(\omega)$, thus essentially $\mathcal{Q}$ adds a real, so that if $G \subseteq \mathcal{Q}$ is generic then $r(A[G]) \subseteq \omega$ and $\mathbf{V}[G] = \mathbf{V}[r(A[G])]$.

(C) The forcing notions $\mathbf{JS}(F_g)$, Q, and any finite-support product of Q, in particular $\mathcal{Q} = Q^\omega$ and any $\mathcal{Q}^\lambda$, satisfy CCC, see *e.g.* [7, Lemma 1 in 4.6]. $\square$

The next lemma is established in [7] in a somewhat different but pretty similar case, as a theorem in Section 4.8, pp. 95–97, so we skip the proof.

Lemma 2.6. *If* $g \in 2^\omega$ *is Cohen-generic over* $\mathbf{L}$ *and* $B = \langle\langle g_n, a_n \rangle\rangle_{n < \omega}$ *is* $\mathcal{Q}$-*generic over* $\mathbf{L}[g]$ *then it holds in* $\mathbf{L}[g, B]$ *that* $\neg \exists a\, T(g, a)$. $\square$

Lemma 2.7. $\mathcal{Q}$ *is homogeneous in the sense that if* $p, q \in \mathcal{Q}$ *then there is an order automorphism* h *of* $\mathcal{Q}$ *such that* p *and* $h(q)$ *are compatible. Therefore*

(i) *if* φ *is a formula with names of elements of the ground universe* $\mathbf{V}$ *as parameters, and some* $p \in \mathcal{Q}$ *forces* φ *then* $\mathcal{Q}$ *(i.e., every* $q \in \mathcal{Q}$*) forces* φ;

(ii) *if* $\varphi(\cdot)$ *is a formula with names of elements* $\mathbf{V}$ *as parameters,* t *is a* $\mathcal{Q}$-*name, and some* $p \in \mathcal{Q}$ *forces* $\varphi(t)$*, then there is another* $\mathcal{Q}$-*name* t' *such that* $\mathcal{Q}$ *forces* $\varphi(t')$ *and* p *forces* $t = t'$.

Proof. As the supports $|p|, |q| \subseteq \omega$ are finite, there is a permutation π of ω such that the π-image of $|p|$ does not intersect $|q|$. Such a π induces h as required. Claims (i) and (ii) are well-known consequences of the homogeneity. $\square$

3 On intermediate models

Given a forcing notion $P = \langle P; \leq \rangle$ in a ground set universe $\mathbf{V}$, if a set $X \subseteq \mathbf{V}$ belongs to a P-generic extension $\mathbf{V}[G]$ of $\mathbf{V}$, then the submodel $\mathbf{V}[X] \subseteq \mathbf{V}[G]$ is a generic extension of $\mathbf{V}$. (But it is not asserted that the set X itself is generic over $\mathbf{V}$!) This issue has been exhaustively studied in terms of boolean-valued forcing (see *e.g.* Lemma 69 in [6]), which we avoid here. Instead we make use of the classical Σ-construction by Solovay [20], rendered here only for the case $P = \mathcal{Q}$ and $X \subseteq \mathbf{V}$. (See [21, Section 1] or [10] for the treatment in the case when X is not a subset of the ground set universe $\mathbf{V}$.) Basically, the results below hold for any $P \in \mathbf{L}$, and if $P \notin \mathbf{L}$ then the results also hold with P as a uniform parameter.

Definition 3.1 (Solovay [20])**.** Assume that $t \in \mathbf{V}$, $t \subseteq \mathcal{Q} \times \mathbf{V}$. ($\mathbf{V}$ being a ground set universe.) Let $X \subseteq \mathbf{V}$ be a set in a generic extension of $\mathbf{V}$. We

define $\Sigma(X, t) = \mathcal{Q} \smallsetminus \bigcup_{\alpha < \vartheta} W_\alpha$, where the increasing sequence of sets $W_\alpha \subseteq \mathcal{Q}$ is defined in $\mathbf{V}[X]$ by induction, and an ordinal ϑ is determined in the course of construction.

(1) W_0 consists of all conditions $p \in \mathcal{Q}$ such that **either** there is a set $x \in X$ such that p is incompatible in $\mathcal{Q}$ with any condition q satisfying $\langle q, x \rangle \in t$, **or** there is $x \notin X$ and a weaker condition $q \geqslant p$ such that $\langle q, x \rangle \in t$.

(2) $W_{\alpha+1}$ consists of all conditions $p \in \mathcal{Q}$ such that there is a dense set $D \in \mathbf{V}$, $D \subseteq \mathcal{Q}$ satisfying: if $q \in D$ and $q \leqslant p$ then $q \in W_\alpha$.

(3) If λ is limit then $W_\lambda = \bigcup_{\alpha < \lambda} W_\alpha$.

Note that $\alpha < \beta \implies W_\alpha \subseteq W_\beta$, hence there is an ordinal ϑ satisfying $W_\vartheta = W_{\vartheta+1}$, and then $W_\xi = W_\vartheta$ for all $\xi > \vartheta$. Finally, let $W = \bigcup_{\alpha < \vartheta} W_\alpha$. The set $\Sigma(X, t) = \mathcal{Q} \smallsetminus W$ contains all conditions p which, roughly speaking, are compatible with the assumption that $X = t[G]$ for a $\mathcal{Q}$-generic set G containing p. $\qquad\square$

A set $D \subseteq \mathcal{Q}$ is *dense* iff $\forall p \in \mathcal{Q} \, \exists q \in D \, (q \leqslant p)$, and *open* dense if in addition $(q \in D \wedge q \leqslant p) \implies p \in D$. The set $t[G] = \{x : \exists p \in G \, (\langle p, x \rangle \in t)\}$ is the *G-valuation* of t. The next lemma evaluates the length ϑ of the construction of 3.1.

Lemma 3.2. *Under the assumptions of 3.1, $\vartheta \leq \omega_1$.*

Proof. To prove $W_{\omega_1+1} = W_{\omega_1}$, let $p \in \mathcal{Q}$, hence there is a dense set $D \in \mathbf{V}$, $D \subseteq \mathcal{Q}$, satisfying $(q \in D \wedge q \leqslant p) \implies q \in W_{\omega_1}$. The set

$$D' = \{q' \in D : q' \leqslant p \vee q', p \text{ are incompatible}\}$$

is still dense and satisfies $(*)$ $(q' \in D' \wedge q' \leqslant p) \implies q' \in W_{\omega_1}$.

Let $A \subseteq D'$ be a maximal antichain; A is countable by 2.5(C), hence there is $\alpha < \omega_1$ such that $(\dagger)$ $A \bigcap W = A \cap W_\alpha$. By the maximality of A, the set

$$D'' = \{q'' \in \mathcal{Q} : \exists r \in A \, (q'' \leqslant r)\}$$

is dense, and even *open* dense. We claim that $(\ddagger)$ $(q'' \in D'' \wedge q'' \leqslant p) \implies q'' \in W_\alpha$; this implies $p \in W_\alpha \subseteq W_{\omega_1}$, ending the proof of the lemma. Thus prove $(\ddagger)$.

By definition, $q'' \leqslant r$ for some $r \in A$. Thus r, p are compatible. Therefore, as $A \subseteq D'$, we have $r \leqslant p$. To conclude, $r \leqslant p$ and $r \in D'$. It follows that $r \in W_{\omega_1}$ by $(*)$, hence $r \in W_\alpha$ by $(\dagger)$. It follows that $q'' \in W_\alpha$. (Indeed, by induction, each set W_α satisfies $q'' \leqslant r \in W_\alpha \implies q'' \in W_\alpha$.) As required. $\qquad\square$

Theorem 3.3 (Solovay [20]). *Under the assumptions of 3.1, suppose that a set $G \subseteq \mathcal{Q}$ is $\mathcal{Q}$-generic over $\mathbf{V}$, and $X = t[G]$. Then*

(i) $G \subseteq \Sigma = \Sigma(X, t)$ *and* $X = t[\Sigma]$ — *hence* $\mathbf{V}[\Sigma] = \mathbf{V}[X]$,

(ii) G *is* Σ-*generic over the intermediate model* $\mathbf{V}[\Sigma] = \mathbf{V}[X] \subseteq \mathbf{V}[G]$ — *hence* $\mathbf{V}[G]$ *is a set generic extension of* $\mathbf{V}[X]$;

(iii) *if* $G' \subseteq \Sigma$ *is* Σ-*generic over* $\mathbf{V}[\Sigma] = \mathbf{V}[X]$ *then* G' *is* $\mathbf{Q}$-*generic over* $\mathbf{V}$ *and* $t[G'] = X$. $\qquad\square$

Corollary 3.4. *Under the assumptions of Theorem 3.3, let* $\varphi(\cdot)$ *be a formula with* $\mathbf{Q}$-*names for sets in* $\mathbf{V}$ *allowed as parameters. Then* $\varphi(X)$ *is true in* $\mathbf{V}[X]$ *iff there is a condition* $p \in \Sigma$ *that* $\mathbf{Q}$-*forces* $\mathbf{V}[t[\underline{G}]] \models \varphi(t[\underline{G}])$ *over* $\mathbf{V}$.

Proof. To prove $\Longleftarrow$, the less trivial direction, assume that a condition $p \in \Sigma$ $\mathbf{Q}$-forces $\mathbf{V}[t[\underline{G}]] \models \varphi(t[\underline{G}])$ over $\mathbf{V}$. Consider a set $G' \subseteq \Sigma$, Σ-generic over $\mathbf{V}[X]$ and containing p. Then G' is $\mathbf{Q}$-generic over $\mathbf{V}$ and $t[G'] = X$. It follows that $\varphi(X)$ is true in $\mathbf{V}[X]$ by the choice of p. $\qquad\square$

Definition 3.5 (see [9]). Let $t \in \mathbf{V}$, $t \subseteq \mathbf{Q} \times \mathbf{V}$. Define, in $\mathbf{V}$, the order relation $\leqslant_t$ on $\mathbf{Q}$ as follows: $p \leqslant_t q$, iff p $\mathbf{Q}$-forces over $\mathbf{V}$ that $\dot{q} \in \Sigma(\dot{t}[\underline{G}], \dot{t})$.
  Let $\mathbf{Q}_t = \langle \mathbf{Q} ; \leqslant_t \rangle$. $\qquad\square$

Under the hypothesis that for any $p \in P$ there is a P-generic set $G \subseteq P$ containing p, the relation $p \leqslant_t q$ is equivalent to the following: if G is a set $\mathbf{Q}$-generic over $\mathbf{V}$ and containing p then $q \in \Sigma(X, t)$.

The next theorem contains the main application of the orders $\leqslant_t$.

Theorem 3.6. *Suppose that* $t \in \mathbf{V}$, $t \subseteq \mathbf{Q} \times \mathbf{V}$. *Then*

(i) $\leqslant_t$ *is a partial order relation on* $\mathbf{Q}$ *satisfying* $p \leq q \Longrightarrow p \leqslant_t q$,

(ii) *if a set* $G \subseteq \mathbf{Q}$ *is* $\mathbf{Q}$-*generic over* $\mathbf{V}$ *and* $X = t[G]$, *then the set* $\Sigma(X, t)$ *is* $\mathbf{Q}_t$-*generic over* $\mathbf{V}$ *and* $\mathbf{V}[X] = \mathbf{V}[\Sigma(X, t)]$,

(iii) *if a set* $\Sigma \subseteq \mathbf{Q}$ *is* $\mathbf{Q}_t$-*generic over* $\mathbf{V}$, *and a set* $G' \subseteq \Sigma$ *is* Σ-*generic over* $\mathbf{V}[\Sigma]$, *then* G' *is* $\mathbf{Q}$-*generic over* $\mathbf{V}$ *and* $\Sigma \in \mathbf{V}[G']$.

Proof. Claims (i), (ii) are established in [9]. To prove (iii), suppose towards the contrary that $p \in \mathbf{Q}$ forces the negation, that is, if $\Sigma \subseteq \mathbf{Q}$ is $\mathbf{Q}_t$-generic over $\mathbf{V}$, $G' \subseteq \Sigma$ is Σ-generic over $\mathbf{V}[\Sigma]$, and $p \in G'$, then either G' is **not** $\mathbf{Q}$-generic over $\mathbf{V}$ or $\Sigma \in \mathbf{V}[G']$. Consider a set $G' \subseteq \mathbf{Q}$, containing p and $\mathbf{Q}$-generic over $\mathbf{V}$. Let $X' = t[G']$. The set $\Sigma' = \Sigma(X', t)$ is $\mathbf{Q}_t$-generic over $\mathbf{V}$ by (ii), G' is Σ-generic over $\mathbf{V}[\Sigma']$ by Theorem 3.3(ii), and $X' = t[G'] \in \mathbf{V}[G']$, hence $\Sigma' = \Sigma(X', t) \in \mathbf{V}[G']$ as well. Finally, $p \in G' \subseteq \Sigma'$, which contradicts the choice of p. $\qquad\square$

Lemma 3.7. *Let* $G \subseteq \mathbf{Q}$ *be* $\mathbf{Q}$-*generic over* $\mathbf{V}$, *and* $x \subseteq \omega$ *be a real in* $\mathbf{V}[G]$. *Then there is a countable set* $t \in \mathbf{V}$, $t \subseteq \mathbf{Q} \times \omega$ *such that* $x = t[G]$ *and* $\mathbf{V}[x]$ *is a* $\mathbf{Q}_t$-*generic extension of* $\mathbf{V}$. *(But* x *itself is not asserted to be a generic.)*

Proof. By basic forcing theory, there is a set $t \in \mathbf{V}$, $t \subseteq \mathcal{Q} \times \omega$, satisfying $x = t[G]$, and by CCC (see 2.5(C)) there is a countable such t. Now use Theorem 3.6(ii). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

Remark 3.8. Under the assumptions of the theorem, $\mathcal{Q}_t$ satisfies CCC (in $\mathbf{V}$). Indeed, as $(\leqslant) \subseteq (\leqslant_t)$, any $\leqslant_t$-antichain is a $\leqslant$-antichain as well. $\qquad$ $\square$

Remark 3.9. (A) Let the *Harrington fan* $\mathrm{HF}(\mathcal{Q})$ consist of all forcing notions of the form $\mathcal{Q}_t = \langle \mathcal{Q}\,;\leqslant_t\rangle$, where $t \subseteq \mathcal{Q} \times \omega$ is at most countable.

(B) Coming back to Remark 2.5, let $\boldsymbol{\tau} \in \mathbf{L}$ be a canonical $\mathcal{Q}$-name of the real $r(A[G])$ as in 2.5(B), so that if G is $\mathcal{Q}$-generic then $\boldsymbol{\tau}[G] = r(A[G])$ and hence $\mathbf{V}[G] = \mathbf{V}[\boldsymbol{\tau}[G]]$, and moreover if $G \neq G'$ then $\boldsymbol{\tau}[G] \neq \boldsymbol{\tau}[G']$. This implies that the order $\leqslant_{\boldsymbol{\tau}}$ on $\mathcal{Q}$ coincides with $\leqslant$, but by means of a rather legthy argument, which includes the verification of the separativity of the forcing notion $\mathcal{Q}$. In order to circumwent these complications, **it will be outright assumed that the partial order $\leqslant_{\boldsymbol{\tau}}$ on $\mathcal{Q}$ coincides with $\leqslant$**, for this particular $\mathcal{Q}$-name $\boldsymbol{\tau}$, and accordingly if a set $G \subseteq \mathcal{Q}$ is generic and $\boldsymbol{\tau}[G] = r(A[G]) = \boldsymbol{r} \subseteq \omega$ then $\Sigma(\boldsymbol{r}, \boldsymbol{\tau}) = G$, so 3.3(i) still holds. With this amendment, we have $\mathcal{Q} \in \mathrm{HF}(\mathcal{Q})$.

(C) In the notation of Remark 2.5(A), let $\mathbf{c} \in \mathbf{L}$ be the canonical $\mathcal{Q}$-name for the set $\{k < \omega : g_0(k) = 0\}$, so that $\mathcal{Q}$ forces that $\mathbf{c}[G] \subseteq \omega$ is Cohen-generic over the ground universe. Then the forcing $\mathcal{Q}_{\mathbf{c}} \in \mathrm{HF}(\mathcal{Q})$ adds a Cohen real. $\quad$ $\square$

4 Absoluteness of the Σ construction

We have to consider a subtle issue related to the construction of $\Sigma(X, t)$, namely, its formal dependence of the choice of $\mathbf{V}$ in (2) of Definition 3.1. The next lemma shows that the dependence can be eliminated in a really important case.

Lemma 4.1. *Under the assumptions of Definition 3.1, suppose that, in addition, $\mathbf{V}$ is a set-generic extension of $\mathbf{L}[t]$. Then $\Sigma^{\mathbf{V}}(X, t) = \Sigma^{\mathbf{L}[t]}(X, t)$.*

Proof. Assume that $\Pi \in \mathbf{L}[t]$ is a forcing notion, and $\mathbf{V} = \mathbf{L}[t][H]$, where H is Π-generic over $\mathbf{L}[t]$. Then H is Π-generic over $\mathbf{L}[t][X]$ by the product forcing theorem. Prove by induction that $W_\alpha^{\mathbf{V}} = W_\alpha^{\mathbf{L}[t]}$.

It suffices to handle the inductive step $\alpha \to \alpha + 1$ in 3.1(2). Thus suppose that $W_\alpha^{\mathbf{V}} = W_\alpha^{\mathbf{L}[t]} = W_\alpha$ and prove $W_{\alpha+1}^{\mathbf{V}} = W_{\alpha+1}^{\mathbf{L}[t]}$. As $\mathbf{L}[t] \subseteq \mathbf{V}$, we have $W_{\alpha+1}^{\mathbf{L}[t]} \subseteq W_{\alpha+1}^{\mathbf{V}}$. To prove the opposite inclusion, suppose that $p_0 \in W_{\alpha+1}^{\mathbf{V}}$, and this is witnessed by a dense set $D \in \mathbf{V}$, $D \subseteq \mathcal{Q}$, as in 3.1(2). The goal is to prove $p_0 \in W_{\alpha+1}^{\mathbf{L}[t]}$. We have $D = \tau[H]$, where $\tau \in \mathbf{L}[t]$, $\tau \subseteq \Pi \times \mathcal{Q}$ (a Π-name of a subset of $\mathcal{Q}$). There is a condition $\pi_0 \in H$, which Π-forces, over $\mathbf{L}[t][X]$, that

$$\text{``}\tau[\underline{H}] \text{ is dense and } \forall\, p \in \tau[\underline{H}]\,(p \leqslant p_0 \implies p \in W_\alpha^{\smile})\text{ ''}, \qquad\qquad (\dagger)$$

where $W_\alpha^{\smile} = \Pi \times W_\alpha$ is the canonical Π-name for the set $W_\alpha \in \mathbf{L}[t][X]$.

We can wlog assume that (‡) $\langle \pi, p \rangle \in \tau \wedge \pi' \in \Pi \wedge \pi' \leq_\Pi \pi \implies \langle \pi', p \rangle \in \tau$.

We notice that the sets $D_1' = \{p \in \mathcal{Q} : p \leqslant p_0 \wedge \exists \pi \in \Pi \, (\pi \leq_\Pi \pi_0 \wedge \langle \pi, p \rangle \in \tau)\}$, $D_2' = \{p \in \mathcal{Q} : p_0, p$ are incompatible$\}$, and $D' = D_1' \cup D_2'$ belong to $\mathbf{L}[t]$.

We claim that D' is dense in $\mathcal{Q}$. Indeed let $p \in \mathcal{Q}$. If p is incompatible with p_0 then immediately $p \in D'$. If otherwise, then we can assume that $p \leqslant p_0$. As π_0 forces (†), there is a condition $\pi \in H$, $\pi \leq_\Pi \pi_0$, and some $p' \in \mathcal{Q}$, $p' \leqslant p$, such that π forces $\pi' \in \tau[\underline{H}]$ — that is, $\langle \pi, p' \rangle \in \tau$ by (‡). Then $p' \in D'$, as required.

We finally claim that if $p \in D'$ and $p \leqslant p_0$ then $p \in W_\alpha$. Indeed, $p \notin D_2'$, hence, $p \in D_1'$. Let this be witnessed by $\pi \in \Pi$, $\pi \leq_\Pi \pi_0$. Then π obviously forces $p \in \tau[\underline{H}]$, and hence, as π also forces (†), we conclude that $p \in W_\alpha$, as required.

Thus D' witnesses that $p_0 \in W_{\alpha+1}^{\mathbf{L}[t]}$, as required. $\qquad\qquad\square$

Corollary 4.2. *Under the assumptions of Lemma 4.1, if* $\mathbf{V}_1 = \mathbf{L}[t, Y_1]$ *and* $\mathbf{V}_2 = \mathbf{L}[t, Y_2]$, *where* $Y_1, Y_2 \in \mathbf{V}$ *and* $Y_1 \cup Y_2 \subseteq \mathbf{L}[t]$, *then*

(i) $\Sigma^{\mathbf{V}_1}(X, t) = \Sigma^{\mathbf{V}_2}(X, t) = \Sigma^{\mathbf{V}}(X, t) = \Sigma^{\mathbf{L}[t]}(X, t)$.

(ii) $\leqslant_t^{\mathbf{V}_1} = \leqslant_t^{\mathbf{V}_2} = \leqslant_t^{\mathbf{V}} = \leqslant_t^{\mathbf{L}[t]}$.

Proof. If $\mathbf{V}$ is a set-generic extension of $\mathbf{L}[t]$ then any subextension $\mathbf{L}[t, Y]$, where $Y \in \mathbf{V}, Y \subseteq \mathbf{L}[t]$, is a set-generic extension of $\mathbf{L}[t]$ as well by [3] or Theorem 3.6. This implies (i) by Lemma 4.1, and then (ii) also follows by a routine argument. $\qquad\qquad\square$

Blanket agreement 4.3. We'll freely use the notation $\Sigma(X, t)$ and $\leqslant_t$ without reference to the ground universe, due to Corollary 4.2. Indeed, the universes considered will always be subuniverses of a fixed generic extension of $\mathbf{L}$.

Corollary 4.4. *Under the assumptions of Lemma 4.1, the relation* $\leqslant_t$ *belongs to* $\mathbf{L}[t]$, *and we have* $\mathbf{L}[t, X] = \mathbf{L}[t, \Sigma(X, t)]$. $\qquad\qquad\square$

5 Definability of the Σ construction

Consider the sets $\mathbb{E} = \{A \in \mathbf{L} : A$ is a maximal antichain in $\mathcal{Q}\}$ and

$$\Sigma = \{\langle X, t, p \rangle : X \subseteq \omega \wedge t \in \mathbf{L}_{\omega_1}, t \subseteq \mathcal{Q} \times \omega \wedge p \in \Sigma(X, t)\}.$$

We have $\Sigma \subseteq \mathbf{L}_{\omega_1}$, and also $\mathbb{E} \subseteq \mathbf{L}_{\omega_1}$ since $\mathcal{Q}$ is CCC.

Lemma 5.1. $\mathbb{E}$ *is* $\Sigma_1^{\mathbf{L}_{\omega_1}}$.

Proof. See Section 2 on $\mathbf{ZFC}^-$ and $\mathbf{T}$. Let $\mathbf{T}^+$ be the theory $\mathbf{ZFC}^-$ plus the axiom saying that every set belongs to a countable model $\mathbf{L}_\alpha \models \mathbf{T}$. We claim that the following are equivalent: (1) $A \in \mathbb{E}$,

(2) $\exists\,\lambda < \omega_1 (\mathbf{L}_\lambda \models \mathbf{T}^+ \wedge A \in \mathbf{L}_\lambda \wedge A$ is a maximal antichain in $\mathcal{Q} \cap \mathbf{L}_\lambda)$,

(3) $\forall\,\lambda < \omega_1 (\mathbf{L}_\lambda \models \mathbf{T}^+ \wedge A \in \mathbf{L}_\lambda \implies A$ is a maximal antichain in $\mathcal{Q} \cap \mathbf{L}_\lambda)$.

If this is established then Lemma 2.4 leads to the definability result required.

It remains to prove the claim. (1) $\implies$ (3) is obvious. To prove (3) $\implies$ (2), consider any countable elementary submodel $\mathbf{L}_\lambda$ of $\mathbf{L}_{\omega_1}$ containing A. Then $\mathbf{L}_\lambda$ is a model of $\mathbf{T}^+$ (as so is $\mathbf{L}_{\omega_1}$). Thus A is a maximal antichain in $\mathcal{Q} \cap \mathbf{L}_\lambda$.

Now prove (2) $\implies$ (1), the nontrivial implication. Suppose that $\lambda < \omega_1$, $A \in \mathbf{L}_\lambda \models \mathbf{T}^+$, and A is a maximal antichain in $\mathcal{Q} \cap \mathbf{L}_\lambda$. Then A is an antichain in $\mathcal{Q}$, since being antichain means that $p \wedge q \not\leqslant p$ or $\not\leqslant q$ for any $p \neq q$ in A, and $\wedge$ is an absolute operation (see the proof of Lemma 2.4). It remains to prove that A is a maximal antichain.

Suppose to the contrary that some $p \in \mathcal{Q} \smallsetminus \mathbf{L}_\lambda$ is incompatible with every $q \in A$. By definition there are finitely many reals of the form $f_{\xi n}$ with $\xi \geq \lambda$, occurring in p, so we may write $p = p(f_{\xi_0, n_0}, \dots, f_{\xi_k, n_k})$, $k < \omega$, $\xi_i \geq \lambda$ for all i. (The substitution form $p(f_{\xi_0, n_0}, \dots, f_{\xi_k, n_k})$ is naturally chosen so that $p(f_{\eta_0, n_0}, \dots, f_{\eta_k, n_k}) \in \mathcal{Q}$ for any other string of ordinals $\eta_0, \dots, \eta_k < \omega_1$.) The reals $f_{\xi n}$ with $\xi < \lambda$ may occur as well, but they belong to $\mathbf{L}_\lambda$ and are not to be explicitly mentioned. But anyway there is an ordinal $\nu < \lambda$ such that all $f_{\xi n}$ with $\xi < \lambda$, occuring in p, actually satisfy $\xi < \nu$, and in addition $A \in \mathbf{L}_\nu \models \mathbf{T}$.

Note that by construction the string of reals $\langle f_{\xi_0, n_0}, \dots, f_{\xi_k, n_k} \rangle \in (\omega^\omega)^{n+1}$ is Cohen generic over $\mathbf{L}_\lambda$, hence over $\mathbf{L}_\nu$ as well. Therefore the property

$$\text{``}p(f_{\xi_0, n_0}, \dots, f_{\xi_k, n_k}) \text{ is incompatible with every } q \in A\text{''}$$

is forced over $\mathbf{L}_\nu$, in the sense that there exist strings $e_0, \dots, e_k \in \omega^{<\omega}$, such that $e_i \subset f_{\xi_i, n_i}$, $\forall\, i$, and if $\langle y_0, \dots, y_n \rangle$ is Cohen generic over $\mathbf{L}_\nu$ with $e_i \subset y_i$, $\forall\, i$, then still $p(y_0, \dots, y_k)$ is incompatible with every $q \in A$.

It remains to note that, since $\nu < \lambda$, there exist intermediate ordinals $\eta_0, \dots, \eta_k \in \lambda \smallsetminus \nu$ such that the reals $y_i = f_{\eta_i, n_i}$ satisfy $e_i \subset y_i$, $\forall\, i$. Then $\langle y_0, \dots, y_n \rangle$ is Cohen generic over $\mathbf{L}_\nu$, hence $p' = p(y_0, \dots, y_k)$ is incompatible with each $q \in A$ by the above. And on the other hand $p' \in \mathcal{Q} \cap \mathbf{L}_\lambda$, a contradiction. $\qquad\square$

Lemma 5.2. Σ *is definable in* HC $=$ hereditarily-countable sets *by a conjunction of the Σ_1 formula "$t \in \mathbf{L}_{\omega_1}$" and a Π_1 formula $\boldsymbol{\sigma}(X, t, p)$.*

Proof. Assume that $X \subseteq \omega$, $t \in \mathbf{L}$, $t \subseteq \mathcal{Q} \times \omega$, and $p \in \mathcal{Q}$. Let a *maximal p-antichain* be any maximal antichain $A \subseteq \mathcal{Q}$ such that if $q \in A$ then either q, p are incompatible or $q \leqslant p$. Come back to the sets W_α in 3.1. As $t \in \mathbf{L}$, Lemma 4.1 allows us to consider only dense sets $D \in \mathbf{L}$ in 3.1(2).

If $\alpha < \omega_1$ then let an α-*ladder* be any sequence $\langle W'_\xi, \mathscr{A}_\xi \rangle_{\xi \leq \alpha}$ such that each $W'_\xi \subseteq \mathcal{Q}$ is at most countable, each $\mathscr{A}_\xi \subseteq \mathbb{E}$ is at most countable, and

(1') $W'_0 \subseteq W_0$ (the latter defined as in 3.1(1));

($2'$) if $\xi + 1 \leq \alpha$ and $p \in W'_{\xi+1}$ then there is a maximal p-antichain $A \in \mathscr{A}_\xi$ such that $r \in W'_\xi$ holds for all $r \in A, r \leqslant p$;

($3'$) if ξ is limit then $W'_\xi = \bigcup_{\eta < \xi} W'_\eta$.

We assert that if $p \in \mathcal{Q}$ and $\alpha < \omega_1$ then:

($*$) $\quad p \in W_\alpha$ *iff there is an α-ladder $\langle W'_\xi, \mathscr{A}_\xi \rangle_{\xi \leq \alpha}$ such that $p \in W'_\alpha$.*

If this is established then $\bigcup_{\alpha < \omega_1} W_\alpha$ becomes a Σ_1^{HC} set. (The incompatibility in 3.1(1), to which ($1'$) refers, is handled by Lemma 2.4.) Then $\Sigma(X,t) = \mathcal{Q} \smallsetminus \bigcup_{\alpha < \omega_1} W_\alpha$ becomes a Π_1^{HC} set, and the lemma easily follows. Note that the union needn't exceed ω_1 by Lemma 3.2.

In the direction $\Longleftarrow$ of ($*$), we prove by induction that $W'_\xi \subseteq W_\xi$. The nontrivial step is ($2'$). Let $p \in W'_{\xi+1}$, and let this be witnessed by $A \in \mathscr{A}_\xi$ in the sense of ($2'$). As $A \in \mathbb{E}$, the set $D = \{q \in \mathcal{Q} : \exists\, r \in A\, (q \leqslant r)\}$ is dense and $D \in \mathbf{L}$. It remains to prove that if $q \in D, q \leqslant p$, then $q \in W_\xi$, see 3.1(2). Indeed, by construction there is $r \in A$ with $q \leqslant r$. But A is a p-antichain, hence either r, p are incompatible or $r \leqslant p$. However $q \leqslant p$, $q \leqslant r$, excluding the 'either' case. Thus $r \leqslant p$. It follows by the choice of A that $r \in W'_\xi$. Thus $r \in W_\xi$ by the inductive hypothesis. We conclude that $q \in W_\xi$ as well, since $q \leqslant p$.

We prove $\Longrightarrow$ in ($*$) by induction on α. The nontrivial step is still ($2'$). Thus suppose that $p \in W_{\alpha+1}$, and let this be witnessed by a dense set $D \in \mathbf{L}$, $D \subseteq \mathcal{Q}$ in the sense of 3.1(2). Let D' consist of all $q \in D$ such that $q \leqslant p$ or q is incompatible with p; then $D' \in \mathbf{L}$ is still dense and witnesses $p \in W_{\alpha+1}$. Consider a maximal antichain $A \subseteq D'$ in $\mathbf{L}$, so that $A \in \mathbb{E}$. Then A is a maximal p-antichain by the definition of D', and if $q \in A$, $q \leqslant p$ then $q \in W_\alpha$ by the choice of D, hence, by the inductive hypothesis, there is an α-ladder $\langle W_\xi(q), \mathscr{A}_\xi(q) \rangle_{\xi \leq \alpha}$ satisfying $q \in W_\xi(q)$. To accomplish the proof of $\Longrightarrow$ in ($*$), define an $(\alpha+1)$-ladder by $W'_\xi = \bigcup_{q \in A, q \leqslant p} W_\xi(q)$ and $\mathscr{A}_\xi = \bigcup_{q \in A, q \leqslant p} \mathscr{A}_\xi(q)$ for $\xi \leq \alpha$, and separately $W'_{\alpha+1} = \{p\}$ and $\mathscr{A}_{\alpha+1} = \{A\}$. $\qquad \square$

6 The model

Here we start the proof of Theorem 1.1. The key idea of [4, Part C] consists in making use of the ω_1-long iterated extension of $\mathbf{L}$, where the forcing at each step is the finite-support product of all elements of the fan $\mathrm{HF}(\mathcal{Q})$ defined within the extension obtained at the previous step of the iteration. We are going to define such an extension as a submodel of a more elementary background set universe $\mathfrak{M}$.

To define the latter, we consider the forcing notion $\mathcal{Q}^{\omega_1 \times \omega_1} \in \mathbf{L}$ (finite support). As $\mathcal{Q} = Q^\omega$, the forcing $\mathcal{Q}^{\omega_1 \times \omega_1}$ is order-isomorphic to Q^{ω_1}, of course. The forcing $\mathcal{Q}^{\omega_1 \times \omega_1} \in \mathbf{L}$ naturally adjoins an array of mutually $\mathcal{Q}$-

generic sets $G^{\nu\gamma} \subseteq \boldsymbol{\mathcal{Q}}$, $\nu, \gamma < \omega_1$ to $\mathbf{L}$. We let $\mathfrak{M}$, *the background model*, be the extension $\mathbf{L}[\langle G^{\nu\gamma} \rangle_{\nu,\gamma<\omega_1}]$.

If $u \in \mathbf{L}$, $u \subseteq \omega_1 \times \omega_1$, then let $\mathfrak{M}{\upharpoonright}_u = \mathbf{L}[\langle G^{\mu\delta} \rangle_{\langle\mu,\delta\rangle\in u}]$. In particular, if $\nu, \gamma < \omega_1$ then put $\mathfrak{M}_\nu = \mathfrak{M}{\upharpoonright}_{\nu\times\omega_1} = \mathbf{L}[\langle G^{\mu\delta} \rangle_{\mu<\nu,\delta<\omega_1}]$ and $\mathfrak{M}_{\nu\gamma} = \mathfrak{M}{\upharpoonright}_{\nu\times\gamma}$.

Lemma 6.1 (by 2.5(C))**.** $\mathfrak{M}$ *preserves all* $\mathbf{L}$-*cardinals. If* $x \in \mathfrak{M}$ *is a real then* x *belongs to some* $\mathfrak{M}_{\nu\gamma}$, $\nu, \gamma < \omega_1$. *Every* $\mathfrak{M}_{\nu\gamma}$ *is a* $\boldsymbol{\mathcal{Q}}$-*generic extension of* $\mathbf{L}$. $\qquad\qquad\square$

The actual model for Theorem 1.1 will be a certain subuniverse $\mathfrak{N} \subseteq \mathfrak{M}$.

Definition 6.2. Arguing in $\mathfrak{M}$, we define, by transfinite induction on ν, an array of countable $\boldsymbol{\mathcal{Q}}$-names $\boldsymbol{t}_{\nu\gamma} \subseteq \boldsymbol{\mathcal{Q}} \times \omega$, such that

(1) if $\nu, \gamma < \omega_1$ then $\langle \boldsymbol{t}_{\mu\delta} \rangle_{\mu\leq\nu,\delta<\omega_1} \in \mathfrak{M}_\nu$, and $\langle \boldsymbol{t}_{\mu\delta} \rangle_{\mu\leq\nu,\delta<\gamma} \in \mathfrak{M}_{\nu\gamma}$, so that each particular $\boldsymbol{t}_{\nu\gamma}$ belongs to $\mathfrak{M}_{\nu,\gamma+1}$.

We also define *derived objects*, namely

(2) reals $\boldsymbol{r}_{\nu\gamma} = \boldsymbol{t}_{\nu\gamma}[G^{\nu\gamma}] \subseteq \omega$, sets $\boldsymbol{\Sigma}_{\nu\gamma} = \Sigma(\boldsymbol{r}_{\nu\gamma}, \boldsymbol{t}_{\nu\gamma}) \subseteq \boldsymbol{\mathcal{Q}}$, forcing notions $\boldsymbol{\mathcal{Q}}_{\nu\gamma} = \langle \boldsymbol{\mathcal{Q}} ; \leqslant_{\boldsymbol{t}_{\nu\gamma}} \rangle$, and

(3) models $\mathfrak{N}_\nu = \mathbf{L}[\langle \boldsymbol{t}_{\mu\delta}, \boldsymbol{r}_{\mu\delta} \rangle_{\mu<\nu,\delta<\omega_1}]$, $\mathfrak{N}_{\nu\gamma} = \mathbf{L}[\langle \boldsymbol{t}_{\mu\delta}, \boldsymbol{r}_{\mu\delta} \rangle_{\mu<\nu,\delta<\gamma}]$ $(\gamma < \omega_1)$;

which, by construction and the results of Section 3, satisfy the following:

(4) $\boldsymbol{r}_{\nu\gamma} \in \mathfrak{M}_{\nu+1,\gamma+1}[G^{\nu\gamma}]$, $\boldsymbol{\mathcal{Q}}_{\nu\gamma} = \langle \boldsymbol{\mathcal{Q}} ; \leqslant_{\boldsymbol{t}_{\nu\gamma}} \rangle$ is a forcing notion in $\mathfrak{M}_{\nu,\gamma+1}$, and $\boldsymbol{\Sigma}_{\nu\gamma} \subseteq \boldsymbol{\mathcal{Q}}$ is a set $\boldsymbol{\mathcal{Q}}_{\nu\gamma}$-generic over $\mathfrak{M}_{\nu,\gamma+1}$ and over $\mathfrak{M}_\nu$, and satisfying $\mathbf{L}[\boldsymbol{t}_{\nu\gamma}, \boldsymbol{\Sigma}_{\nu\gamma}] = \mathbf{L}[\boldsymbol{t}_{\nu\gamma}, \boldsymbol{r}_{\nu\gamma}]$, by Corollary 4.4 ;

(5) if $\nu, \gamma < \omega_1$ then the arrays $\langle \leqslant_{\boldsymbol{t}_{\mu\delta}}, \boldsymbol{\mathcal{Q}}_{\mu\delta} \rangle_{\mu\leq\nu,\delta<\omega_1}$, $\langle \boldsymbol{\Sigma}_{\mu\delta}, \boldsymbol{r}_{\mu\delta} \rangle_{\mu<\nu,\delta<\omega_1}$ belong to $\mathfrak{M}_\nu$, and the arrays $\langle \leqslant_{\boldsymbol{t}_{\mu\delta}}, \boldsymbol{\mathcal{Q}}_{\mu\delta} \rangle_{\mu\leq\nu,\delta<\gamma}$, $\langle \boldsymbol{\Sigma}_{\mu\delta}, \boldsymbol{r}_{\mu\delta} \rangle_{\mu<\nu,\delta<\gamma}$ belong to $\mathfrak{M}_{\nu\gamma}$;

(6) therefore $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$ and $\mathfrak{N}_{\nu\gamma} \subseteq \mathfrak{M}_{\nu\gamma}$, $\forall\, \gamma$.

Now the step. Suppose that $\nu < \omega_1$ and all sets $\boldsymbol{t}_{\mu\delta} \subseteq \boldsymbol{\mathcal{Q}} \times \omega$ $(\mu < \nu, \delta < \omega_1)$ are defined, so that $\langle \boldsymbol{t}_{\mu\delta} \rangle_{\mu<\nu,\delta<\omega_1} \in \mathfrak{M}_\nu$, and if $\gamma < \omega_1$ then $\langle \boldsymbol{t}_{\mu\delta} \rangle_{\mu<\nu,\delta<\gamma} \in \mathfrak{M}_{\nu\gamma}$; this is slightly weaker than (1) since does not include ν itself. Then $\boldsymbol{\mathcal{Q}}_{\mu\gamma}$, $\boldsymbol{\Sigma}_{\mu\gamma}$, $\boldsymbol{r}_{\mu\gamma}$, $\mathfrak{N}_\nu$, $\mathfrak{N}_{\nu\gamma}$ as in (2), (3) are defined as well. The goal is to define $\boldsymbol{t}_{\nu\gamma}$, $\gamma < \omega_1$.

Note that $\mathfrak{M}_\nu = \mathbf{L}[\langle G^{\mu\gamma} \rangle_{\mu<\nu,\gamma<\omega_1}]$ is a $\boldsymbol{\mathcal{Q}}^{\omega_1}$-generic extension of $\mathbf{L}$, hence GCH is true in $\mathfrak{M}_\nu$, and hence in $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$ as well. Therefore it holds in $\mathfrak{N}_{\nu\xi}$ that there exist only $\aleph_1$-many countable sets $t \subseteq \boldsymbol{\mathcal{Q}} \times \omega$; let $\langle t^\xi_{\nu\eta} \rangle_{\eta<\omega_1}$ be the Gödel-least (relative to $\langle \boldsymbol{t}_{\mu\delta}, \boldsymbol{r}_{\mu\delta} \rangle_{\mu<\nu,\delta<\xi}$ as the parameter) enumeration of all such t in $\mathfrak{N}_{\nu\xi}$.

Let $\Omega = \{\gamma + 1 : \gamma < \omega_1\}$ (successor ordinals). Fix a bijection $\mathfrak{b} : \Omega$ onto $\omega_1 \times \omega_1$, $\mathfrak{b} \in \mathbf{L}$, satisfying $\mathfrak{b}^{-1}(\xi, \eta) > \max\{\xi, \eta\}$ for all $\xi, \eta < \omega_1$. If $\gamma = \mathfrak{b}^{-1}(\xi, \eta) \in$

Ω then let $\boldsymbol{t}_{\nu\gamma} = t_{\nu\eta}^{\xi}$. Put $\boldsymbol{t}_{\nu\gamma} = \varnothing$ for all limit $\gamma < \omega_1$. The enumeration $\langle \boldsymbol{t}_{\nu\gamma} \rangle_{\gamma < \omega_1}$ involves all at most countable sets $t \in \mathfrak{N}_\nu$, $t \subseteq \boldsymbol{\mathcal{Q}} \times \omega$, the whole sequence $\langle \boldsymbol{t}_{\nu\gamma} \rangle_{\gamma < \omega_1}$ belongs to $\mathfrak{N}_\nu$, and if $\gamma < \omega_1$ then the subsequence $\langle \boldsymbol{t}_{\nu\delta} \rangle_{\delta < \gamma}$ belongs to $\mathfrak{N}_{\nu\gamma}$.

This ends the inductive step.

After the inductive construction is accomplished, we let

$$\mathfrak{N}{\restriction}_u = \mathbf{L}[\langle \boldsymbol{t}_{\nu\gamma}, \boldsymbol{r}_{\nu\gamma} \rangle_{\langle \nu, \gamma \rangle \in u}] = \mathbf{L}[\langle \boldsymbol{t}_{\nu\gamma}, \boldsymbol{\Sigma}_{\nu\gamma} \rangle_{\langle \nu, \gamma \rangle \in u}], \quad \text{for all } u \subseteq \omega_1 \times \omega_1,$$

and then $\mathfrak{N} = \mathfrak{N}{\restriction}_{\omega_1 \times \omega_1}$, and, equivalently to (3), $\mathfrak{N}_\nu = \mathfrak{N}{\restriction}_{\nu \times \omega_1}$ and $\mathfrak{N}_{\nu\gamma} = \mathfrak{N}{\restriction}_{\nu \times \gamma}$ for $\nu, \gamma < \omega_1$. We have by construction:

(7) the whole sequence $\langle \boldsymbol{t}_{\nu\gamma} \rangle_{\gamma < \omega_1}$ belongs to $\mathfrak{N}_\nu$, and if $\gamma < \omega_1$ then the subsequence $\langle \boldsymbol{t}_{\nu\delta} \rangle_{\delta < \gamma}$ belongs to $\mathfrak{N}_{\nu\gamma}$. $\qquad\square$

The next lemma explains further details.

Lemma 6.3. *Assume that $\nu < \omega_1$, $t \in \mathfrak{N}_\nu$, $t \subseteq \boldsymbol{\mathcal{Q}} \times \omega$ is at most countable.*

Then there is an ordinal $\gamma < \omega_1$ such that $t = \boldsymbol{t}_{\nu\gamma}$.

In this case, $\boldsymbol{\mathcal{Q}}_t = \boldsymbol{\mathcal{Q}}_{\nu\gamma}$, the set $\boldsymbol{\Sigma}_{\nu\gamma} \in \mathfrak{N}_{\nu+1}$ is $\boldsymbol{\mathcal{Q}}_t$-generic over $\mathfrak{M}_\nu$, hence over the model $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$ as well, and the real $\boldsymbol{r}_{\nu\gamma} = t[G^{\nu\gamma}]$ satisfies $\mathbf{L}[\boldsymbol{t}_{\nu\gamma}, \boldsymbol{r}_{\nu\gamma}] = \mathbf{L}[\boldsymbol{t}_{\nu\gamma}, \boldsymbol{\Sigma}_{\nu\gamma}]$ and belongs to $\mathfrak{N}_{\nu+1}$.

Proof. Recall that $G^{\nu\gamma}$ is $\boldsymbol{\mathcal{Q}}$-generic over $\mathfrak{M}_\nu$, hence over $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$ as well. It remains to use Theorem 3.6. $\qquad\square$

Remark 6.4. If $\nu < \omega_1$ then by construction the collection of all forcing notions $\boldsymbol{\mathcal{Q}}_{\nu\gamma}$, $\gamma < \omega_1$, is equal to the Harrington fan $\mathrm{HF}(\boldsymbol{\mathcal{Q}})$ computed in $\mathfrak{N}_\nu$. Thus $\mathfrak{N}$ can be viewed as the ω_1-long iterated $\mathrm{HF}(\boldsymbol{\mathcal{Q}})$-generic extension of $\mathbf{L}$.

In particular, by 3.9(B), there is an index $\gamma < \omega_1$ such that $\boldsymbol{\mathcal{Q}}_{\nu\gamma} = \boldsymbol{\mathcal{Q}}$, hence, by Lemma 6.3, $\mathfrak{N}_{\nu+1}$ contains a set $\boldsymbol{\mathcal{Q}}$-generic over $\mathfrak{M}_\nu$ and over $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$. Similarly by 3.9(C) there is an index $\gamma < \omega_1$ such that $\boldsymbol{t}_{\nu\gamma} = \mathbf{c} \in \mathbf{L}$ (see 3.9(C) on $\mathbf{c}$), and hence $\boldsymbol{\mathcal{Q}}_{\nu\gamma} = \boldsymbol{\mathcal{Q}}_{\mathbf{c}}$ adds a Cohen real $\boldsymbol{r}_{\nu\gamma} \subseteq \omega$ over $\mathfrak{N}_\nu$. $\qquad\square$

7 Key lemmas

Lemma 7.1. *If $x \in 2^\omega \cap \mathfrak{N}$, then there is a set $G \in \mathfrak{N}$, $\boldsymbol{\mathcal{Q}}$-generic over $\mathbf{L}[x]$, hence there are reals $g, a \in \mathfrak{N}$ such that the pair $\langle g, a \rangle$ is Q-generic over $\mathbf{L}[x]$.*

Proof. By Lemma 6.1, x belongs to some $\mathfrak{N}_\nu \subseteq \mathfrak{M}_\nu$, $\nu < \omega_1$. By 6.4, the submodel $\mathfrak{N}_{\nu+1}$ contains a set $\boldsymbol{\mathcal{Q}}$-generic set over $\mathfrak{N}_\nu$, hence over $\mathbf{L}[x]$ as well. $\qquad\square$

Lemma 7.2. *Assume that $\nu, \gamma < \omega_1$, $u \in \mathbf{L}$, $u \subsetneqq \nu \times \gamma$, and $W = (\nu \times \gamma) \smallsetminus u$. Then $\mathfrak{K} = (\mathfrak{N}{\restriction}_u)[\langle G^{\mu\delta} \rangle_{\langle \mu, \delta \rangle \in W}]$ is a $\boldsymbol{\mathcal{Q}}$-generic extension of $\mathfrak{N}{\restriction}_u$, and $\mathfrak{N}_{\nu\gamma} \subseteq \mathfrak{K}$.*

Proof. Let $\boldsymbol{\mathcal{Q}}(\mu,\delta) = \boldsymbol{\mathcal{Q}}$, for all $\langle \mu, \delta \rangle \in W$. Note that $\mathfrak{K}$ is a $\prod_{\langle \mu, \delta \rangle \in W} \boldsymbol{\mathcal{Q}}(\mu, \delta)$-generic extension of $\mathfrak{N}{\restriction}_u$ by construction, hence essentially a $\boldsymbol{\mathcal{Q}}^\omega$-generic extension, yet $\boldsymbol{\mathcal{Q}}^\omega$ is isomorphic to $\boldsymbol{\mathcal{Q}}$ as a forcing. To prove $\mathfrak{N}_{\nu\gamma} \subseteq \mathfrak{K}$, we check, by induction, that $\langle \boldsymbol{t}_{\kappa\delta}, \boldsymbol{r}_{\kappa\delta} \rangle_{\delta<\gamma} \in \mathfrak{K}$ for all $\kappa < \nu$. The induction hypothesis is $\kappa < \nu$ and $\mathfrak{N}_{\kappa\gamma} \subseteq \mathfrak{K}$, and the goal is to "effectively" prove that then $\langle \boldsymbol{t}_{\kappa\delta}, \boldsymbol{r}_{\kappa\delta} \rangle_{\delta<\gamma} \in \mathfrak{K}$. We first remind that $\langle \boldsymbol{t}_{\kappa\delta} \rangle_{\delta<\gamma} \in \mathfrak{N}_{\kappa\gamma}$ by (7) of Definition 6.2. Now, for any particular $\delta < \gamma$, if $\langle \kappa, \delta \rangle \in u$ then $\boldsymbol{r}_{\kappa\delta}$ belongs to $\mathfrak{N}{\restriction}_u$, hence, to $\mathfrak{K}$ as well, while if $\langle \kappa, \delta \rangle \in W$ then $G^{\kappa\delta}$ belongs to $\mathfrak{K}$, hence $\boldsymbol{r}_{\kappa\delta} = \boldsymbol{t}_{\kappa\delta}[G^{\kappa\delta}] \in \mathfrak{K}$, as required. □

Definition 7.3 (autonomous sets). A set $x \in \mathfrak{N}$, $x \subseteq \mathbf{L}$, is *autonomous* if there is a countable set $u \in \mathbf{L}$, $u \subseteq \omega_1 \times \omega_1$ such that $\mathbf{L}[x] = \mathfrak{N}{\restriction}_u$. □

Lemma 7.4 (in $\mathfrak{N}$). *If z is an autonomous real and $t \in \mathbf{L}[z]$, $t \subseteq \boldsymbol{\mathcal{Q}} \times \mathbf{L}[z]$, then there is a real b such that $\langle z, b \rangle$ is autonomous and $\mathbf{L}[z,b] = \mathbf{L}[z][\Sigma]$, where $b = t[\Sigma]$ and $\Sigma \subseteq \boldsymbol{\mathcal{Q}}$, Σ is $\boldsymbol{\mathcal{Q}}_t$-generic over $\mathbf{L}[z]$.*

Proof. Let a countable $u \in \mathbf{L}$, $u \subseteq \omega_1 \times \omega_1$ witness that z is autonomous. Then $u \subseteq \nu \times \omega_1$ for some $\nu < \omega_1$, and $t \in \mathfrak{N}_\nu$. By Lemma 6.3, $t = \boldsymbol{t}_{\nu\gamma}$ for some γ, and then $b = \boldsymbol{r}_{\nu\gamma}$ is as required. To see that $\langle z, b \rangle$ is autonomous note that $\mathbf{L}[z,b] = \mathfrak{N}{\restriction}_v$, where $v = u \cup \{\langle \nu, \gamma \rangle\}$. □

Lemma 7.5 (in $\mathfrak{N}$). *Let $x \in 2^\omega$ and $\varphi(\cdot)$ be a Σ_3^1 formula. Then*

 (i) *if $\boldsymbol{\mathcal{Q}}$ forces $\varphi(\dot{x})$ over $\mathbf{L}[x]$ then $\varphi(x)$ is true (in $\mathfrak{N}$);*[3]

 (ii) *if x is autonomous and $\varphi(x)$ is true (in $\mathfrak{N}$) then $\boldsymbol{\mathcal{Q}}$ forces $\varphi(\dot{x})$ over $\mathbf{L}[x]$.*

Proof. (i) holds by Lemma 7.1, since the truth of Σ_3^1 formulas passes to bigger models by Shoenfield. To prove (ii), let $\mathbf{L}[x] = \mathfrak{N}{\restriction}_u = \mathbf{L}[\langle \boldsymbol{t}_{\nu\gamma}, \boldsymbol{r}_{\nu\gamma} \rangle_{\langle \nu, \gamma \rangle \in u}]$, where $u \in \mathbf{L}$, $u \subseteq \omega_1 \times \omega_1$ is countable. Let $\varphi(\cdot)$ be $\exists z\, \psi(z, \cdot)$, ψ being Π_2^1. Assume that $\varphi(x)$ is true in $\mathfrak{N}$. There is a real $z \in \mathfrak{N}$ such that $\psi(z,x)$ is true in $\mathfrak{N}$. There is an ordinal $\mu < \omega_1$, such that $u \subsetneqq \mu \times \mu$ and $z \in \mathfrak{N}_{\mu\mu} = \mathbf{L}[\langle \boldsymbol{t}_{\nu\gamma}, \boldsymbol{r}_{\nu\gamma} \rangle_{\nu,\gamma<\mu}]$. Then $z \in \mathfrak{K} = (\mathfrak{N}{\restriction}_u)[\langle G^{\mu\delta} \rangle_{\langle \mu, \delta \rangle \in W}]$ by Lemma 7.2, where $W = (\mu \times \mu) \setminus u$. And $\mathfrak{K}$ is a $\boldsymbol{\mathcal{Q}}$-generic extension of $\mathfrak{N}{\restriction}_u = \mathbf{L}[x]$ still by Lemma 7.2.

On the other hand, $\psi(z,x)$ is true in $\mathfrak{K}$ by Shoenfield, hence $\varphi(x)$ is true in $\mathfrak{K}$ as well. It follows that a condition in $\boldsymbol{\mathcal{Q}}$ forces $\varphi(\dot{x})$ over $\mathbf{L}[x]$. We conclude by Lemma 2.7 that $\boldsymbol{\mathcal{Q}}$ forces $\varphi(\dot{x})$ over $\mathbf{L}[x]$. □

Recall that HC = hereditarily countable sets (in $\mathfrak{N}$).

Lemma 7.6. *If φ is a Σ_n formula containing $\boldsymbol{\mathcal{Q}}$-names in HC (names of sets in $\mathrm{HC}[\underline{G}]$), and if $p \in \boldsymbol{\mathcal{Q}}$, then $p \|{\vdash}_{\boldsymbol{\mathcal{Q}}}^{\mathrm{HC}} \varphi$ is a Σ_n^{HC} assertion about p, φ.*

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[3] In this lemma, $\dot{x} = \{\mathbf{1}\} \times x \in \mathbf{L}[x]$ is a $\boldsymbol{\mathcal{Q}}$-name of x itself.

Proof (sketch). If φ is a Σ_2^1 formula then, by the Mostowski absoluteness, $p \Vdash_{\boldsymbol{\mathcal{Q}}}^{\mathrm{HC}} \varphi$ iff $p \Vdash_{\boldsymbol{\mathcal{Q}}}^{M} \varphi$ over some countable transitive model M of a sufficient fragment of **ZFC**, which is a Σ_1^{HC} relation. But, Σ_1^{HC} relations are the same as Σ_2^1. This covers the case $n = 1$ of the lemma.

Step $\Sigma_n \to \Pi_n$. Suppose that φ is a Σ_n formula. Then $p \Vdash_{\boldsymbol{\mathcal{Q}}}^{\mathrm{HC}} \neg \varphi$ iff $\forall q \left(q \leqslant p \implies \neg q \Vdash_{\boldsymbol{\mathcal{Q}}}^{\mathrm{HC}} \varphi \right)$. This leads to a Π_n formula.

Step $\Pi_n \to \Sigma_{n+1}$. Let $\varphi(x)$ be a Π_n formula. Then $p \Vdash_{\boldsymbol{\mathcal{Q}}}^{\mathrm{HC}} \exists x \, \varphi(x)$ iff there is a $\boldsymbol{\mathcal{Q}}$-name $t \in \mathrm{HC}$ such that $p \Vdash_{\boldsymbol{\mathcal{Q}}}^{\mathrm{HC}} \varphi(t)$. This leads to a Σ_{n+1} formula. $\square$

Lemma 7.7. *There is a recursive correspondence $\varphi \mapsto \varphi^*$ between Σ_3^1 formulas (and hence between Π_3^1 formulas as well) such that for all reals $b \in \mathfrak{N}$, $\boldsymbol{\mathcal{Q}}$ forces $\varphi(\dot{b})$ over $\mathbf{L}[b]$ if and only if $\mathbf{L}[b] \models \varphi^*(b)$.*

Proof. Let $\boldsymbol{\sigma}(z, t, p)$ be a Π_1 formula provided by Lemma 5.2.

Given φ a Σ_3^1 formula, define $\varphi^*(b)$ iff:

$$\exists t \in \mathbf{L} \, \exists p \left(\mathbf{L}[b] \models \boldsymbol{\sigma}(b, t, p) \text{ and } p \Vdash_{\boldsymbol{\mathcal{Q}}} \varphi(\dot{b}) \text{ over } \mathbf{L}[b] \right). \text{[4]}$$

Prove that φ^* is as required. Since Σ_n^1 formulas correspond to the Σ_{n-1} definability in HC, φ^* is a Σ_2^{HC} formula by Lemma 7.6, hence essentially a Σ_3^1 formula.

Now suppose that b is a real in $\mathfrak{N}$ and $\boldsymbol{\mathcal{Q}}$ forces $\varphi(\dot{b})$ over $\mathbf{L}[b]$. We have to prove that $\varphi^*(b)$ holds in $\mathbf{L}[b]$. Note that $b \in \mathfrak{N} \subseteq \mathfrak{M}$. Hence $b \in \mathfrak{M}_{\nu\nu}$ for some $\nu < \omega_1$. But $\mathfrak{M}_{\nu\nu}$ is a $\boldsymbol{\mathcal{Q}}^{\nu \times \nu}$-generic extension, hence, a $\boldsymbol{\mathcal{Q}}$-generic extension of $\mathbf{L}$. Let say $\mathfrak{M}_{\nu\nu} = \mathbf{L}[H]$, where $H \subseteq \boldsymbol{\mathcal{Q}}$, $H \in \mathfrak{M}$ is $\boldsymbol{\mathcal{Q}}$-generic over $\mathbf{L}$; H need not be in $\mathfrak{N}$. Thus $b = t[H]$, where $t \in \mathbf{L}, t \subseteq \boldsymbol{\mathcal{Q}} \times \omega$. Let $\Sigma = \Sigma(b, t)$; then $\Sigma \subseteq \boldsymbol{\mathcal{Q}}$ and $\mathbf{L}[\Sigma] = \mathbf{L}[b]$. Consider any $p \in \Sigma$; thus $\boldsymbol{\sigma}(b, t, p)$ holds in $\mathbf{L}$. Under our assumptions, $p \Vdash_{\boldsymbol{\mathcal{Q}}} \varphi(\dot{b})$ over $\mathbf{L}[b]$, hence we have $\varphi^*(b)$ in $\mathbf{L}[b]$.

To prove the converse, assume that $\varphi^*(b)$ holds in $\mathbf{L}[b]$, and this is witnessed by $t \in \mathbf{L}$ and p. In particular $\boldsymbol{\sigma}(b, t, p)$ holds, thus $p \in \Sigma = \Sigma(b, t) \subseteq \boldsymbol{\mathcal{Q}}$. Moreover, $p \Vdash_{\boldsymbol{\mathcal{Q}}} \varphi(\dot{b})$ over $\mathbf{L}[b]$. It follows that if $G \subseteq \boldsymbol{\mathcal{Q}}$ is generic over $\mathbf{L}[b]$ and $p \in G$ then $\varphi(b)$ is true in $\mathbf{L}[b][G]$. Thus $\boldsymbol{\mathcal{Q}}$ forces $\varphi(\dot{b})$ over $\mathbf{L}[b]$ by Lemma 2.7. $\square$

8 Reduction fails

In the remainder, we are going to prove that $\mathfrak{N}$ is a model for Theorem 1.1. The following is the first part of the proof.

Theorem 8.1. *In $\mathfrak{N}$, there is a pair of Σ_3^1 sets of reals, not reducible to a pair of boldface $\boldsymbol{\Sigma}_3^1$ sets.*

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[4] $\dot{b} = \{\mathbf{1}\} \times b \in \mathbf{L}[b]$ is the canonical $\boldsymbol{\mathcal{Q}}$-name of a real $b \subseteq \omega$, where $\mathbf{1}$ is the largest condition in $\boldsymbol{\mathcal{Q}}$.

Proof. [5] **Arguing in** $\mathfrak{N}$, consider the Σ_3^1 set $A = \{g : \exists\, a\, T(g,a)\}$, and let $U \subseteq 2^\omega$ be a Σ_3^1 set, universal in the sense that $\{e < \omega : e^\smallfrown z \in U\} \notin \Pi_3^1(z)$ for every $z \in \omega^\omega$, where $e^\smallfrown z$ adds $e \in \omega$ as the leftmost term to $z \in \omega^\omega$.

Consider the Σ_3^1 sets $\omega^\omega \times A$, $U \times \omega^\omega$.

Suppose to the contrary that, in $\mathfrak{N}$, there are $\boldsymbol{\Sigma}_3^1$ sets $A', U' \subseteq 2^\omega$, such that

$$A' \subseteq \omega^\omega \times A,\ U' \subseteq U \times \omega^\omega,\ A' \cap U' = \varnothing,\ A' \cup U' = (\omega^\omega \times A) \cup (U \times \omega^\omega).$$

Let z be an autonomous real such that A', U' are $\Sigma_3^1(z)$.

Lemma 8.2 (in $\mathfrak{N}$). *Assume that $d \in 2^\omega \smallsetminus U$. Then $\{y : \langle d,y\rangle \in A'\} = A$ and there is $y \in 2^\omega$ such that $\langle d,y\rangle \in A'$ and y is Cohen generic over $\mathbf{L}[z,d]$.*

Proof. If $\langle d,y\rangle \in A'$ then $y \in A$ by construction. Conversely assume that $y \in A$. Then $\langle d,y\rangle \in \omega^\omega \times A$, but $\langle d,y\rangle \notin U \times \omega^\omega$ (as $d \notin U$). Therefore $\langle d,y\rangle \in A'$ as required. Prove the second claim. By Lemma 7.1, there is a pair $\langle g,a\rangle$ in $\mathfrak{N}$, Q-generic over $\mathbf{L}[z,d]$. Thus $g \in 2^\omega$ is Cohen-generic over $\mathbf{L}[z,d]$ while $a \subseteq \omega$ satisfies $T(g,a)$, hence $g \in A$. Thus we are done with $y = g$. $\square$ (Lemma 8.2)

Consider the sets $K = \{e^\smallfrown z : e < \omega \wedge e^\smallfrown z \notin U\} \notin \Sigma_3^1(z)$ and

$$K' = \{e^\smallfrown z : \exists\, y\, (e < \omega \wedge \langle\langle z,e\rangle,y\rangle \in A' \wedge y \text{ is } C\text{-generic over } \mathbf{L}[z])\}.$$

Clearly $K \notin \Sigma_3^1(z)$ by the choice of U, while K' is $\Sigma_3^1(z)$, and we have $K \subseteq K'$ by Lemma 8.2. Thus $K \subsetneqq K'$. We conclude that there is an integer e such that $e^\smallfrown z \in U$ and $e^\smallfrown z \in K'$, so that

$$\exists\, y\, (\langle e^\smallfrown z, y\rangle \in A' \wedge y \text{ is Cohen-generic over } \mathbf{L}[z]).$$

Fix such a number $e < \omega$.

Let $A'' = \{y : \langle e^\smallfrown z, y\rangle \in A'\}$; $A'' \subseteq A$. We claim that A'' is $\Delta_3^1(z)$. Indeed $y \notin A'' \iff \langle e^\smallfrown z, y\rangle \notin A'$. But $e^\smallfrown z \in U$, hence $\langle e^\smallfrown z, y\rangle \in U \times \omega^\omega$. Therefore $\langle e^\smallfrown z, y\rangle \notin A' \iff \langle e^\smallfrown z, y\rangle \in U'$. This yields the Π_3^1 definition for A''.

Let φ be a Π_3^1 formula such that $y \in A'' \iff \varphi(z,y)$ in $\mathfrak{N}$. By the choice of e, there is a real $g \in A''$, Cohen-generic over $\mathbf{L}[z]$. So $\varphi(z,g)$ is true in $\mathfrak{N}$. It follows by Lemma 7.5 that $\mathcal{Q}$ forces $\varphi(\dot{z},\dot{g})$ over $\mathbf{L}[z,g]$.

So by Lemma 7.7 we have $\mathbf{L}[z,g] \models \varphi^*(z,g)$. By the genericity of g, there is a Cohen condition $\bar{g} \in \mathbb{C}$, $\bar{g} \subseteq g$ such that $\bar{g} \Vdash$ " $\mathbf{L}[z,\dot{g}] \models \varphi^*(z,\dot{g})$ ".

Recall that z is autonomous. Let this be witnessed by a countable $u \in \mathbf{L}$, $u \subseteq \nu \times \vartheta$, where $\nu, \vartheta < \omega_1$; thus $\mathbf{L}[z] = \mathfrak{N}{\restriction}_u = \mathbf{L}[\langle \boldsymbol{t}_{\mu\delta}, \boldsymbol{r}_{\mu\delta}\rangle_{\langle\mu,\delta\rangle\in u}] \subseteq \mathfrak{M}_{\nu\vartheta} \subseteq \mathfrak{M}_\nu$. By 6.4, there is an ordinal $\gamma < \omega_1$ such that $\boldsymbol{t}_{\nu\gamma} = \mathbf{c}$ and $\mathcal{Q}_{\nu\gamma}$ adds a Cohen real over $\mathfrak{M}_\nu$, so $\boldsymbol{r}_{\nu\gamma} \in 2^\omega$ is a Cohen real over $\mathfrak{M}_\nu$. Changing appropriately a

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[5] Harrington relates the idea of the proof to Sami.

finite number of values $r_{\nu\gamma}(k)$, we get another real $g' \in 2^\omega$, Cohen-generic over $\mathfrak{M}_\nu$, and satisfying $\bar{g} \subseteq g'$ and still $(\mathfrak{N}\restriction_u)[g'] = (\mathfrak{N}\restriction_u)[r_{\nu\gamma}]$, or in other words, $\mathbf{L}[z, g'] = (\mathfrak{N}\restriction_u)[g'] = \mathfrak{N}\restriction_v$, where $v = u \cup \{\langle \nu, \gamma \rangle\}$. Thus $\langle z, g' \rangle$ is autonomous.

To conclude, $\langle z, g' \rangle$ is autonomous and $\mathbf{L}[z, g'] \models \varphi^*(z, g')$ by the choice of g' and of $\bar{g} \in \mathbb{C}$. It follows by Lemma 7.7 that $\varphi(z, g')$ is true in some/every $\mathcal{Q}$-generic extension of $\mathbf{L}[z, g']$. Therefore $\varphi(z, g')$ is true in $\mathfrak{N}$ by Shoenfield.

Thus $g' \in A''$ by the choice of φ, hence $g' \in A$ and $\exists a\, T(g', a)$ holds in $\mathfrak{N}$. Thus $\mathcal{Q}$ forces $\exists a\, T(g', a)$ over $\mathbf{L}[z, g']$ by Lemma 7.5(ii), as $\langle z, g' \rangle$ is autonomous.

Lemma 8.3. $\mathcal{Q}$ forces $\exists a\, T(g', a)$ over $\mathbf{L}[g']$.

Proof. It suffices to get a $\mathcal{Q}$-generic extension of $\mathbf{L}[g']$ in which $\exists a\, T(g', a)$ holds. Consider a set G', $\mathcal{Q}$-generic over $\mathbf{L}[z, g']$. Then $\exists a\, T(g', a)$ holds in $\mathbf{L}[z, g', G']$ by the above. Recall that z belongs to a $\mathcal{Q}$-generic extension $\mathfrak{M}_{\nu\vartheta} \subseteq \mathfrak{M}_\nu$ of $\mathbf{L}$. Therefore $\mathbf{L}[z]$ is a $\mathcal{Q}_t$-generic extension of $\mathbf{L}$ by Lemma 3.7, where $t \subseteq \mathcal{Q} \times \omega$ is countable. In other words, $\mathbf{L}[z] = \mathbf{L}[\Sigma]$, where $\Sigma \subseteq \mathcal{Q}$ is $\mathcal{Q}_t$-generic over $\mathbf{L}$.

On the other hand, g' is Cohen-generic over $\mathbf{L}[z]$ while G' is $\mathcal{Q}$-generic over $\mathbf{L}[z, g']$. It follows, by the product forcing theorem, that G' is $\mathcal{Q}$-generic over $\mathbf{L}[g']$ and $\mathbf{L}[z, g', G'] = \mathbf{L}[g', G'][\Sigma]$ is a $\mathcal{Q}_t$-generic extension of $\mathbf{L}[g', G']$.

Let $G \subseteq \Sigma$ be Σ-generic over $\mathbf{L}[g', G'][\Sigma]$. Then $\mathbf{L}[g', G', \Sigma, G]$ is a $\mathcal{Q}$-generic extension of $\mathbf{L}[g', G']$ by Theorem 3.6(iii), hence a $\mathcal{Q}$-generic extension of $\mathbf{L}[g']$ as well because $\mathcal{Q} \times \mathcal{Q}$ is order-isomorphic to $\mathcal{Q}$. Finally $\exists a\, T(g', a)$ holds in $\mathbf{L}[g', G', \Sigma, G]$ by Shoenfield as it holds in $\mathbf{L}[z, g', G']$, a smaller model. $\square$ (Lemma)

But this contradicts Lemma 2.6. $\hspace{5cm} \square$ (Theorem 8.1)

9 Separation holds

The next theorem is the final part of the proof of Theorem 1.1.

Theorem 9.1. In $\mathfrak{N}$, any pair of disjoint $\mathbf{\Pi}_3^1$ sets is separable by a $\mathbf{\Delta}_3^1$ set.

Proof. We argue in $\mathfrak{N}$. Let $\varphi_0(\cdot, \cdot)$, $\varphi_1(\cdot, \cdot)$ be $\mathit{\Pi}_3^1$ formulas and z be an autonomous real parameter, such that the $\mathbf{\Pi}_3^1$ sets $A_i = \{x : \varphi_i(z, x)\}$, $i = 0, 1$, are disjoint. Let $\varphi_i^*(z, x)$ be the $\mathit{\Pi}_3^1$-formulas provided by Lemma 7.7. Then the sets $B_i = \{x : \mathbf{L}[z, x] \models \varphi_i^*(z, x)\}$, $i = 0, 1$, are $\mathbf{\Pi}_3^1$, and $A_i \subseteq B_i$ by Lemma 7.5.

We claim that $B_0 \cap B_1 = \varnothing$. Assume to the contrary that $b \in B_0 \cap B_1$. The goal of the following argument is to get another real $b' \in B_0 \cap B_1$, with the extra property that $\langle z, b' \rangle$ is autonomous.

As z is autonomous, Lemma 7.2 implies that b belongs to a $\mathcal{Q}$-generic extension $\mathbf{L}[z][G]$ of $\mathbf{L}[z]$. Therefore, by Lemma 3.7, there is a countable set $t \subseteq \mathcal{Q} \times \omega$, $t \in \mathbf{L}[z]$, such that $b = t[G]$ and $\mathbf{L}[z,b]$ is a $\mathcal{Q}_t$-generic extension of $\mathbf{L}[z]$. By Lemma 2.7(ii), $\mathcal{Q}$ forces "$\varphi_0^*(\dot{z}, t[\underline{G}])$ and $\varphi_1^*(\dot{z}, t[\underline{G}])$ hold in $\mathbf{L}[z, t[\underline{G}]]$" over $\mathbf{L}[z]$, therefore $\mathcal{Q}_t$ forces "$\varphi_0^*(\dot{z}, t[\underline{G}]) \wedge \varphi_0^*(\dot{z}, t[\underline{G}])$" over $\mathbf{L}[z]$. On the other hand, by Lemma 7.4, there is a real b' such that $\langle z, b' \rangle$ is autonomous and $\mathbf{L}[z, b'] = \mathbf{L}[z][\Sigma]$, where $b' = t[\Sigma]$ and $\Sigma \subseteq \mathcal{Q}$ is $\mathcal{Q}_t$-generic over $\mathbf{L}[z]$. Thus $\varphi_0^*(z, b')$ and $\varphi_1^*(z, b')$ hold in $\mathbf{L}[z, b']$, hence $b' \in B_0 \cap B_1$, and $\langle z, b' \rangle$ is autonomous.

But then $b' \in A_0 \cap A_1$ by Lemmas 7.7 and 7.5, contradiction.

Thus we have $B_0 \cap B_1 = \varnothing$. Then we separate B_0 from B_1 by a $\Delta_3^1(p)$ set of reals by a standard argument. Indeed let $\leq_{zx}^{\mathrm{G}}$ be the canonical "good" Gödel wellordering of the reals in $\mathbf{L}[z, x]$. Let $\exists y\, \vartheta_0(z, x, y)$ be the canonical transformation of $\neg\, \varphi_1^*(z, x)$ to Σ_3^1-form, and $\exists y\, \vartheta_1(z, x, y)$ be the canonical transformation of $\neg\, \varphi_0^*(z, x)$ to Σ_3^1-form, so that ϑ_i are Π_2^1-formulas and

$$B_i \subseteq C_i = \{x : \mathbf{L}[x, z] \models \exists y\, \vartheta_i(z, x, y)\}, \quad i = 0, 1.$$

Here C_i is the complement to B_{1-i}, thus $C_0 \cup C_1 =$ all reals. If $x \in C_i$ then let $y_i(x)$ be the $\leq_{zx}^{\mathrm{G}}$-least real y satisfying $\mathbf{L}[x, z] \models \vartheta_i(z, x, y)$. The sets

$$
\begin{aligned}
D_0 &= \{x \in C_0 : x \notin C_1 \vee y_0(x) \leq_{zx}^{\mathrm{G}} y_1(x)\} \\
D_1 &= \{x \in C_1 : x \notin C_0 \vee y_1(x) <_{zx}^{\mathrm{G}} y_0(x)\}
\end{aligned}
$$

then satisfy $A_i \subseteq B_i \subseteq D_i \subseteq C_i$ and $D_0 \cup D_1 =$ all reals. On the other hand, a standard argument (as in the proof of Σ_n^1-reduction in $\mathbf{L}$) shows that both D_i are Σ_3^1 sets. It follows that D_0 is a Δ_3^1 set separating A_0 from A_1. $\square$

10 Comments and questions

We may note the following substantial inventions in Harrington's proof.

- The *localization property* in $\mathfrak{N}$, that is, the reduction of the truth of a formula $\varphi(x)$ in the final model $\mathfrak{N}$, first, to the truth in $\mathcal{Q}$-generic extensions of $\mathbf{L}[x]$ by Lemma 7.5, and second, to the truth in $\mathbf{L}[x]$ itself by Lemma 7.7. This is quite similar to the "important lemma" of Solovay [20, page 18], but achieved in a much less friendly generic model.

- The *Harrington fan* construction of 3.9 (see also Remark 6.4) which allows to inhibit (by Lemma 7.4) the fact that Lemma 7.5(ii) holds only for autonomous reals.

- Unlike the Separation counterexamples in specific models in [4, Part B] or say [8], the Reduction counterexample as in Theorem 8.1 (which Harrington grants to Sami) is not something explicitly designed by the intended definability structure of generic reals in the model considered.

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Harrington ends [4, Part C] with the following remark:

> We believe that this result [= Theorem 1.1] can be generalized by replacing 3 by any integer $n \geq 3$. We also believe that that this result can be improved so as to obtain a model of **ZFC** in which both $\mathrm{Sep}(\mathbf{\Pi}_3^1, \mathbf{\Delta}_3^1)$ and $\mathrm{Sep}(\mathbf{\Sigma}_3^1, \mathbf{\Delta}_3^1)$ hold. At the moment though these beliefs are just expressions of faith (or is it hope?).

The second part of this "expressions of faith or hope" was partially materialized in [4, Part C], where, for an arbitrary $n \geq 3$, a model of **ZFC** is presented, in which $\mathrm{Sep}(\mathit{\Pi}_n^1, \mathit{\Delta}_n^1)$ and $\mathrm{Sep}(\mathit{\Sigma}_n^1, \mathit{\Delta}_n^1)$ (note the lightface classes!) both hold for sets of integers. (The proof is given for $n = 3$ only.) The rest presumably remains as open as it was in 1970s.

11 Reduction holds in extensions by Cohen reals

Here we sketch **the proof of Claim** (II) **of Theorem 1.2.**

Let the set universe $\mathbf{V}$ be an extension of $\mathbf{L}$ by a transfinite sequence of Cohen-generic reals. The following is a known property of the Cohen forcing $\mathbb{C} = 2^{<\omega}$ and Cohen extensions.

Lemma 11.1. *If $x, y \in \mathbf{V}$ are reals then either $y \in \mathbf{L}[x]$ or $\mathbf{L}[x, y]$ is a $\mathbb{C}$-generic extension of $\mathbf{L}[x]$. In particular, y belongs to a $\mathbb{C}$-generic extension of $\mathbf{L}[x]$.* $\qquad\square$

Case $n = 3$. We claim that if $\varphi(x)$ is a $\mathit{\Sigma}_3^1$ or $\mathit{\Pi}_3^1$ formula with x as the only real parameter, then

$$\varphi(x) \text{ holds in } \mathbf{V} \quad \text{iff} \quad \mathbf{L}[x] \models \underbrace{\Lambda \ \mathbb{C}\text{-forces } \varphi(\dot{x}) \text{ over the universe.}}_{\varphi^*(x)} \quad (1)$$

Here $\Lambda \in \mathbb{C}$ (the empty sequence) is the weakest Cohen condition, and $\dot{x} = \{\Lambda\} \times x$ is the canonical Cohen name for a set x in the ground model.

To prove (1) for a $\mathit{\Sigma}_3^1$ formula $\varphi(x) := \exists x_1 \psi(x, y)$, ψ being $\mathit{\Pi}_2^1$, assume that $\mathbf{V} \models \varphi(x)$, hence there is a real $y \in \mathbf{V}$ satisfying $\psi(x, y)$. But y belongs to a $\mathbb{C}$-generic extension of $\mathbf{L}[x][g]$ of $\mathbf{L}[x]$ by Lemma 11.1. Then $\exists y \, \psi(x, y)$ is true in $\mathbf{L}[x, g]$ by Shoenfield, and hence $\exists y \, \psi(\dot{x}, y)$ is $\mathbb{C}$-forced by Λ over $\mathbf{L}[x]$ (by the homogeneity of $\mathbb{C}$), that is, $\mathbf{L}[x] \models \varphi^*(x)$. Conversely let $\mathbf{L}[x] \models \varphi^*(x)$. Let $g \in \mathbf{V}$ be a $\mathbb{C}$-generic real over $\mathbf{L}[x]$. Then $\mathbf{L}[x, g] \models \varphi(x)$, hence $\mathbf{V} \models \varphi(x)$ by Shoenfield.

To check (1) for a $\mathit{\Pi}_3^1$ formula $\Phi(x) := \neg \varphi(x)$, φ being $\mathit{\Sigma}_3^1$, assume first that $\mathbf{V} \models \Phi(x)$. Then $\mathbf{V} \not\models \varphi(x)$, hence by (1) $\varphi(\dot{x})$ is not $\mathbb{C}$-forced by Λ over $\mathbf{L}[x]$, thus by the homogeneity $\Phi(\dot{x})$ is forced, that is, $\mathbf{L}[x] \models \Phi^*(\dot{x})$. Conversely if $\mathbf{L}[x] \models \Phi^*(\dot{x})$, then definitely $\mathbf{L}[x] \not\models \varphi^*(\dot{x})$, thus $\mathbf{V} \not\models \varphi(x)$, and hence $\mathbf{V} \models \Phi(x)$.

Pretty similar to the proof of Lemma 7.6, $\varphi^*(\cdot)$ is a formula of type Σ_3^1, resp., Π_3^1 formula provided φ itself is of this type. The next lemma will be used below.

Lemma 11.2. *Let $\varphi(x)$ be a Σ_3^1 or Π_3^1 formula. Let g be a real $\mathbb{C}$-generic over $\mathbf{L}[x]$, and $\mathbf{L}[x, g] \models \varphi^*(x)$. Then $\mathbf{L}[x] \models \varphi^*(x)$.*

Proof. By the homogeneity, it suffices to get a $\mathbb{C}$-generic real h over $\mathbf{L}[x]$, such that $\mathbf{L}[x, h] \models \varphi(x)$. Let g' be $\mathbb{C}$-generic over $\mathbf{L}[x, g]$. Then $\mathbf{L}[x, g, g'] \models \varphi(x)$, so it remains to make use of $h = \langle g, g' \rangle$. $\qquad\qquad\square$

Now, consider Σ_3^1 sets $A_0 = \{x : \varphi_0(x, z)\}$ and $A_1 = \{x : \varphi_1(x, z)\}$ in $\mathbf{V}$, where φ_i are Σ_3^1 formulas. Then $A_i = \{x : \mathbf{L}[x, z] \models \varphi_i^*(x, z)\}$ by the above, where $\varphi_i^*(x, z) := \exists y \Phi_i(x, z, y)$ are Σ_3^1 formulas by the claim, so Φ_i are Π_2^1 formulas.

If $x \in A_i$ then let $y_i(x, z)$ be the $\leq_{xz}^{\mathrm{G}}$-least real y satisfying $\mathbf{L}[x, z] \models \Phi_i(x, z, y)$. The sets

$$
\begin{aligned}
B_0 &= \{x \in A_0 : x \notin A_1 \vee y_0(x, z) \leq_{xz}^{\mathrm{G}} y_1(x, z)\} \\
B_1 &= \{x \in A_1 : x \notin A_0 \vee y_1(x, z) <_{xz}^{\mathrm{G}} y_0(x, z)\}
\end{aligned}
$$

then satisfy $B_i \subseteq A_i$ and $B_0 \cap B_1 = \varnothing$, and belong to Σ_3^1.

Case $n \geq 4$. Let say $n = 4$ exactly; it will be clear how to treat the general case. Suppose that $\varphi(x) := \exists y \, \psi(x, y)$ is a Σ_4^1 formula, ψ being Π_3^1. Then a Π_3^1 formula $\psi^*(x, y)$ has been defined as above, such that $\mathbf{V} \models \psi(x, y)$ iff $\mathbf{L}[x, y] \models \psi^*(x, y)$, for all reals $x, y \in \mathbf{V}$. We claim that then

$$
\mathbf{V} \models \varphi(x) \quad \text{iff} \quad \mathbf{L}[x] \models \underbrace{\Lambda \ \mathbb{C}\text{-forces "}\exists y \, \psi^*(\dot{x}, y)\text{" over the universe.}}_{\varphi^*(x)} \quad (2)
$$

Indeed assume that $\mathbf{V} \models \varphi(x)$, hence there is a real $y \in \mathbf{V}$ satisfying $\psi(x, y)$. It follows that $\mathbf{L}[x, y] \models \psi^*(x, y)$. But y belongs to a $\mathbb{C}$-generic extension of $\mathbf{L}[x]$ by Lemma 11.1. Therefore, $\exists y \, \psi^*(\dot{x}, y)$ is $\mathbb{C}$-forced by Λ over $\mathbf{L}[x]$ (by the homogeneity of $\mathbb{C}$), that is, $\mathbf{L}[x] \models \varphi^*(x)$.

To prove the converse, let $\mathbf{L}[x] \models \varphi^*(x)$. Let $g \in \mathbf{V}$ be a Cohen-generic real over $\mathbf{L}[x]$. Then we have $\mathbf{L}[x, g] \models \exists y \, \psi^*(x, y)$. Let this be witnessed by a real $y \in \mathbf{L}[x, g]$, thus $\mathbf{L}[x, g] \models \psi^*(x, y)$. However, by Lemma 11.1, either $\mathbf{L}[x, g] = \mathbf{L}[x, y]$ — and then $\mathbf{L}[x, y] \models \psi^*(x, y)$ and hence $\mathbf{V} \models \psi(x, y)$ and $\mathbf{V} \models \varphi(x)$, or $\mathbf{L}[x, g]$ is a $\mathbb{C}$-generic extension of $\mathbf{L}[x, y]$ — and then we still have $\mathbf{L}[x, y] \models \psi^*(x, y)$ by Lemma 11.2, and then $\mathbf{V} \models \varphi(x)$ as just above.

The proof of Reduction for a pair of Σ_4^1-sets A_0, A_1 in $\mathbf{V}$ goes on, on the base of (2), exactly as in the case $n = 3$ above.

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