19TH CENTURY REAL ANALYSIS, FORWARD AND BACKWARD

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ABSTRACT. 19th century real analysis received a major impetus from Cauchy's work. Cauchy mentions variable quantities, limits, and infinitesimals, but the meaning he attached to these terms is not identical to their modern meaning.

Some Cauchy historians work in a conceptual scheme dominated by an assumption of a teleological nature of the evolution of real analysis toward a preordained outcome. Thus, Gilain and Siegmund-Schultze assume that references to *limite* in Cauchy's work necessarily imply that Cauchy was working with an Archimedean continuum, whereas infinitesimals were merely a convenient figure of speech, for which Cauchy had in mind a complete justification in terms of Archimedean limits. However, there is another formalisation of Cauchy's procedures exploiting his *limite*, more consistent with Cauchy's ubiquitous use of infinitesimals, in terms of the *standard part principle* of modern infinitesimal analysis.

We challenge a misconception according to which Cauchy was allegedly forced to teach infinitesimals at the *Ecole Polytechnique*. We show that the debate there concerned mainly the issue of *rigor*, a separate one from *infinitesimals*. A critique of Cauchy's approach by his contemporary de Prony sheds light on the meaning of rigor to Cauchy and his contemporaries. An attentive reading of Cauchy's work challenges received views on Cauchy's role in the history of analysis, and indicates that he was a pioneer of infinitesimal techniques as much as a harbinger of the *Epsilontik*.

Keywords: butterfly model; continuity; infinitesimals; *limite*; standard part; variable quantity; Cauchy; de Prony

CONTENTS

> Since Weierstrass's time, we have held a fairly contemptuous view of the infinitesimalists which I regard as unfair. – Ivor Grattan-Guinness

1. INTRODUCTION

Cauchy exploited the concepts of variable quantity, limit, and infinitesimal in his seminal 1821 textbook *Cours d'Analyse* (CdA). However, the meaning he attached to those terms is not identical to their modern meanings. While Cauchy frequently used infinitesimals in CdA, some

scholars have argued that Cauchyan infinitesimals are merely short-hand for prototypes of ϵ, δ techniques. Moreover, one can legitimately ask whether the material found in CdA was actually taught by Cauchy in the classroom of the *Ecole Polytechnique* (EP). A valuable resource that sheds information on such issues is the archive of summaries of courses and various *Conseil* meetings at the EP, explored by Guitard ([29], 1986), Gilain ([23], 1989), and others. Among the key figures at EP at the time was Gaspard de Prony, whose critique of Cauchy's teaching will be examined in Sections 3.5 and 3.6. While de Prony was critical of Cauchy, a careful examination of the criticism indicates that de Prony's main target was what he felt was excesssive rigor, rather than an alleged absence of infinitesimals. While scholars sometimes claim that Cauchy avoided infinitesimals in the 1820s, de Prony's comments and other primary documents indicate otherwise.

1.1. **_Limites_.** Cauchy defined limits as follows in his *Cours d'Analyse* (CdA):

> On nomme quantité *variable* celle que l'on considère comme devant recevoir successivement plusieurs valeurs différentes les unes des autres. ... Lorsque les valeurs successivement attribuées à une même variable s'approchent indéfiniment d'une valeur fixe, de manière à finir par en différer aussi peu que l'on voudra, cette dernière est appelée la *limite* de toutes les autres.[1] (Cauchy [15], 1821, p. 4; emphasis in the original)

Here Cauchy defines limits in terms of a primitive notion of a *variable quantity*. As Robinson pointed out, Cauchy "assign[ed] a central role to the notion of a variable which tends to a limit, in particular to the limit zero" (Robinson [40], 1966, p. 276).

   Elsewhere in CdA, Cauchy used what appears to be a somewhat different notion of limit, as for example when the value of the derivative is extracted from the ratio of infinitesimals Δy and Δx (see Section 2.1). Two distinct approaches used by Cauchy are analyzed in Section 1.2.

1.2. **A-track and B-track for the development of analysis.** The article Katz–Sherry [33] introduced a distinction between two types of procedures in the writing of the pioneers of infinitesimal calculus:

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[1]Translation from [14, p. 6]: "We call a quantity *variable* if it can be considered as able to take on successively many different values. ... When the values successively attributed to a particular variable indefinitely approach a fixed value in such a way as to end up by differing from it by as little as we wish, this fixed value is called the *limit* of all the other values."

(A) procedures in pioneering work in analysis that can be based on an Archimedean continuum (or the A-track approach), cf. [1]; and

(B) procedures that can be based on a Bernoullian (i.e., infinitesimal-enriched) continuum (the B-track approach), as they appear in Leibniz, Bernoulli, Euler, and others.

This is not an exhaustive distinction, but one that helps broaden the lens of a historiography often handicapped by self-imposed limitations of a Weierstrassian type; see Section 1.4.

Here we use the term *procedure* in a broad sense that encompasses algorithms but is not limited to them. For instance, Euler's proof of the infinite product formula for the sine function is a rather coherent procedure though it can hardly be described as an algorithm; see [4] for an analysis of Euler's proof.

Like Leibniz, Cauchy used both A-track and B-track techniques in his work. The sample discussed in Section 3.8 below illustrates his A-track work. Elsewhere, as we document in this article and in earlier work (see e.g., [11]), Cauchy used B-track techniques, as well.

1.3. **What is Cauchy's *limite*?** Scholars who stress Cauchy's use of the limit concept rely on a traditional but flawed dichotomy of infinitesimals *vs* limits. The dichotomy is flawed because limits are present whether one works with an Archimedean or Bernoullian continuum (see Section 1.2). In fact, the definition of derivative found in Cauchy (see Section 2.1) suggests that he works with the B-track version of limits which is referred to as the *standard part function* in modern infinitesimal analysis; see Section 4, formula (4.3). Thus the real issue is whether Cauchy's continuum was Archimedean or Bernoullian, and the genuine dichotomy is between A-track ϵ, δ techniques and B-track infinitesimal techniques.

1.4. **Butterfly model.** The articles (Bair et al. [3]), (Bair et al. [4]), and (Fletcher et al. [22]) argued that some historians of mathematics operate within a conceptual scheme described in (Hacking [30], 2014) as a *butterfly model* of development.

Inspired in part by (Mancosu [38], 2009), Ian Hacking proposes a distinction between the *butterfly model* and the *Latin model*, namely the contrast between a model of a deterministic (genetically determined) biological development of animals like butterflies (the egg–larva–cocoon–butterfly development), as opposed to a model of a contingent historical evolution of languages like Latin.

Historians working within the butterfly paradigm often assume that the evolution of mathematical rigor has a natural direction, leading *forward* to the Archimedean framework as developed by Weierstrass and others (what Boyer referred to as "the great triumvirate" [13, p. 298]). Such historians also tend to interpret the qualifier *rigorous* as necessarily implying *Archimedean*, as we illustrate in Section 1.5.

1.5. **Siegmund-Schultze on *Cours d'Analyse*.** As an illustration of butterfly model thinking by modern historians, we turn to a review by historian Siegmund-Schultze of an English edition of CdA (Bradley–Sandifer [14], 2009). The review illustrates the poignancy of Grattan-Guinness' comment quoted in our epigraph. The comment appears in (Grattan-Guinness [26], 1970, p. 379) in the context of a discussion of CdA.

Siegmund-Schultze's Zentralblatt (Zlb) review ([43], 2009) of the English edition of CdA contains two items of interest:

(SS1) Siegmund-Schultze quotes part of Cauchy's definition of continuity via infinitesimals, and asserts that Cauchy's use of infinitesimals was a step backward: "There has been ... an intense historical discussion in the last four decades or so how to interpret certain apparent remnants of the past or – as compared to J. L. Lagrange's (1736–1813) rigorous 'Algebraic Analysis' – even *steps backwards* in Cauchy's book, particularly his use of infinitesimals..." ([43]; emphasis added).

(SS2) Siegmund-Schultze quotes Cauchy's comments (in translation) on rigor in geometry, and surmises that the framework for CdA was Archimedean, similarly to Euclid's geometry: "a non-Archimedean interpretation of the continuum would clash with the Euclidean theory, which was still the basis of Cauchy's book. Indeed, Cauchy writes in the 'introduction' to the Cours d'Analyse: 'As for methods, I have sought to give them all the *rigor that one demands in geometry*, ...'" (ibid.; emphasis added).

Siegmund-Schultze's Zbl review goes on to continue the quotation from Cauchy:

> "... in such a way as never to revert to reasoning drawn from the *generality of algebra*. Reasoning of this kind, although commonly admitted, particularly in the passage from convergent to divergent series and from real quantities to imaginary expressions, can, it seems to me, only occasionally be considered as inductions suitable for presenting the truth, since they accord so little with

the precision so esteemed in the mathematical sciences."
(Cauchy as quoted in [43]; emphasis added).

Cauchy's objections here have to do with the cavalier use of divergent series, based on a heuristic principle Cauchy called the *generality of algebra*, by his illustrious predecessors Euler and Lagrange, rather than with the issue of using or not using infinitesimals, contrary to Siegmund-Schultze's claim. We will evaluate Siegmund-Schultze's claims further in Section 1.6.

1.6. **Analysis of a review.** The Zbl review quoted in Section 1.5 tends to confirm the diagnosis following Hacking. Namely, the comment on infinitesimals quoted in (SS1) leading specifically *backward* will surely be read by the Zbl audience as indicative of an assumption of an organic (butterfly model) *forward* direction (culminating in the *great triumvirate*).

Similarly, the comment quoted in (SS2) appears to take it for granted that Euclid's framework, being rigorous, was necessarily Archimedean. Yet the facts are as follows:

(i) Books I through IV of *The Elements* are developed without the Archimedean axiom;
(ii) developments around 1900 showed conclusively that the completeness property of $\mathbb{R}$ is irrelevant to the development of Euclidean geometry, and in fact the latter can be developed in the context of non-Archimedean fields.

Indeed, Hilbert proved that these parts of Euclidean geometry can be developed in a non-Archimedean plane (modulo some specific assumptions such as circle–circle intersection and postulation of the congruence theorems); see further in [5, Section 5].

While Euclid relied on the Archimedean axiom to develop his *theory of proportion*, Hilbert obtained all the results of Euclidean geometry including the theory of proportion and geometric similarity without such a reliance; see Hartshorne ([31], 2000, Sections 12.3–12.5 and 20–23) or Baldwin ([2], 2017).

Furthermore, starting with Descartes' *Geometry*, mathematicians implicitly relied on ordered field properties rather than the ancient theory of proportion.

Moreover, it is difficult to understand how Siegmund-Schutze would reconcile his two claims. If Cauchy used Euclidean Archimedean mathematics exclusively, as implied by (SS2), then what exactly were the entities that constituted a step backward, as claimed in (SS1)? Siegmund-Schultze's counterfactual claims are indicative of butterfly-model thinking as outlined in Section 1.4.

Like the Zbl review by Siegmund-Schultze, the Cauchy scholarship of Gilain tends to be colored by teleological assumptions of the sort detailed above, as we argue in Sections 2 and 3.

A number of historians and mathematicians have sought to challenge the received views on Cauchy's infinitesimals, as we detail in Sections 1.7 through 1.9.

1.7. **Robinson on received views.** Abraham Robinson noted that the received view of the development of the calculus

> [would] lead us to expect that, following the rejection of Leibniz' theory by Lagrange and D'Alembert, infinitely small and infinitely large quantities would have no place among the ideas of Cauchy, who is generally regarded as the founder of the modern approach, or that they might, at most, arise as figures of speech, as in 'x tends to infinity'. However, this expectation is mistaken. [40, p. 269].

Robinson described Cauchy's approach as follows:

> Cauchy regarded his theory of infinitely small quantities as a satisfactory foundation for the theory of limits and (d'Alembert's suggestion notwithstanding) he *did not introduce the latter in order to replace the former.* His proof procedures thus involved *both* infinitely small (and infinitely large) quantities *and* limits. [40, p. 271] (emphasis added)

Note Robinson's focus on Cauchy's *procedures* (for a discussion of the procedure/ontology dichotomy, see Błaszczyk et al. [9]). After quoting Cauchy's definition of derivative, Robinson notes:

> Later generations have overlooked the fact that in this definition Δx and Δy were explicitly supposed to be infinitely small. Indeed according to our present standard ideas, we take $f'(x)$ to be the limit [of] $\Delta y/\Delta x$ as Δx tends to zero, whenever that limit exists, without any mention of infinitely small quantities. Thus, as soon as we consider limits, the assumption that Δx and Δy

> are infinitesimal is completely redundant. It is therefore
> the more interesting that the assumption is there, and,
> indeed, appears again and again also in Cauchy's later
> expositions of the same topic (Cauchy [1829, 1844]). [40,
> p. 274]

Robinson's conclusion is as follows:

> We are forced to conclude that Cauchy's mental picture
> of the situation was significantly different from the pic-
> ture adopted today, in the Weierstrass tradition. (ibid.)

It is such received views in what Robinson refers to as the *Weierstrass
tradition* that we wish to reconsider here.

1.8. **Grattan-Guinness on Cauchy's infinitesimals.** Robinson's
1966 comments on the Weierstrassian tradition cited in Section 1.7
were echoed by historians Ivor Grattan-Guinness and Detlef Laugwitz.
Thus, fourteen years later, Grattan-Guinness wrote:

> [Cauchy's definition of infinitesimal] is *in contrast to* the
> view adopted from the Weierstrassians onwards (and oc-
> casionally earlier), where an infinitesimal is a variable
> with *limit* zero... (Grattan-Guinness [27], 1980, p. 110;
> emphasis added)

Concerning the term *limit*, it is necessary to disassociate the following
two issues:

(Ca1) the issue of whether or not limits were at the base of Cauchy's
approach;

(Ca2) the issue of Cauchy's systematic use of infinitesimals as numbers
in his textbooks and research articles.

1.9. **Laugwitz on Cauchy's infinitesimals.** As far as item (Ca2)
is concerned, Laugwitz acknowledged that Cauchy started using in-
finitesimals systematically in the 1820s (whereas his attitude toward
them during the preceding decade was more ambiguous and limits may
have played a larger role):

> ... *after 1820, Cauchy developed his analysis by utilizing
> infinitesimals in a deliberate and consequent manner.*
> (Laugwitz [36], 1989, p. 196; emphasis in the original)

Laugwitz' position is consistent with Gilain's observation that infinites-
imals first appeared in Cauchy's course summary during the academic
year 1820–1821:

> Année 1820–1821 ... Notons aussi l'apparition, pour la
> première fois dans les *Matières des leçons*, des notions

> de quantités infiniment petites et infiniment grandes
> (leçon 3).[2] (Gilain [23], §52, 1989)

In 1997, Laugwitz elaborated on the subject (of Cauchy's endorsement
of infinitesimals circa 1820) in the following terms:

> Cauchy avoided the use of the infinitely small. This pro-
> voked growing criticism on the part of his colleagues,
> including the physicist Petit, who emphasized the di-
> dactical and practical advantages of the use of infin-
> itely small magnitudes. In 1819 and in 1820, the Con-
> seil d'Instruction at the Ecole exerted strong pressure
> on Cauchy, but this alone would not have made this
> rather stubborn man change his mind. *Around 1820,*
> *he must have realized that infinitesimal considerations*
> *were a powerful research method* at a time when he was
> in a state of constant rivalry, especially with Poisson.
> (Laugwitz [37], 1997, p. 657; emphasis added)

In the textbook *Cours d'Analyse* [15], *limite* is not the only central
foundational concept for Cauchy, as we argue in Section 2.

We challenge a common misconception according to which Cauchy
was forced to teach infinitesimals at the *Ecole Polytechnique* allegedly
against his will. We show that the debate there concerned mainly the
issue of *rigor*, a separate one from *infinitesimals*; see Section 3.

2. CAUCHY'S *limite* AND *infiniment petit*

In this section we will analyze the meaning of Cauchy's terms *limite*
and *infiniment petit*.

2.1. **Differentials and infinitesimals.** In his work, Cauchy care-
fully distinguishes between differentials ds, dt which to Cauchy are
noninfinitesimal variables, on the one hand, and infinitesimal incre-
ments $\Delta s, \Delta t$, on the other:

> ...soit s une variable distincte de la variable primitive t.
> En vertu des définitions adoptées, le rapport entre les
> différentielles ds, dt, sera la limite du rapport entre les

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[2]Translation: "Year 1820–1821 ... We also note the appearance, for the first time
in the *Lesson summaries*, of the notions of infinitely small and infinitely large
quantities (lesson 3)."

> *accroissements infiniment petits* $\Delta s, \Delta t$.[3] (Cauchy [17],
> 1844, p. 11; emphasis added)

Cauchy goes on to express such a relation by means of a formula in terms of the infinitesimals Δs and Δt:

> On aura donc
> $$\frac{ds}{dt} = \text{ lim.} \frac{\Delta s}{\Delta t} \qquad (2.1)$$
> (ibid., equation (1); the period after lim in "lim." in the original; equation number (2.1) added)

Cauchy's procedure involving the passage from the ratio of infinitesimals like $\frac{\Delta s}{\Delta t}$ to the value of the derivative $\frac{ds}{dt}$ as in equation (2.1) has a close parallel in Robinson's infinitesimal analysis, where it is carried out by the standard part function; see equations (4.1) and (4.2) in Section 4.

Paraphrasing this definition in Archimedean terms would necessarily involve elements that are not explicit in Cauchy's definition. Thus Cauchy's "lim." finds a closer proxy in the notion of standard part, as in formula (4.3), than in any notion of limit in the context of an Archimedean continuum; see also Bascelli et al. ([6], 2014).

2.2. **Definite integrals and infinitesimals.** Similar remarks apply to Cauchy's 1823 definition of the definite integral which exploits a partition of the domain of integration into infinitesimal subintervals. Here Cauchy writes: "D'après ce qui a été dit dans la dernière leçon, si l'on divise $X - x_0$ en élémens[4] infiniment petits $x_1 - x_0, x_2 - x_1 \ldots X - x_{n-1}$, la somme

(1) $S = (x_1 - x_0)f(x_0) + (x_1 - x_2)f(x_1) + \ldots + (X - x_{n-1})f(x_{n-1})$

convergera vers une *limite* représentée par l'intégrale définie

(2) $$\int_{x_0}^{X} f(x)dx.$$

Des principes sur lesquels nous avons fondé cette proposition il résulte, etc." (Cauchy [16], 1823, Leçon 22, p. 85; emphasis added).

Note that there is a misprint in Cauchy's formula (1): the difference $(x_1 - x_2)$ should be $(x_2 - x_1)$. In this passage, Cauchy refers to the successive differences $x_1 - x_0$, $x_2 - x_1$, $X - x_{n-1}$ as *infinitely small elements*.

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[3]Translation: "Let s be a variable different from the primitive variable t. By virtue of the definitions given, the ratio of the differentials ds, dt will be the limit of the ratio of the infinitely small increments $\Delta s, \Delta t$."

[4]We preserved the original spelling.

Analogous partitions into infinitesimal subintervals are exploited in Keisler's textbook [34] (and throughout the literature on infinitesimal analysis; see e.g., [24, p. 153]). Cauchy's use of *limite* in the passage above is another instance of limit in the context of a Bernoullian continuum, which parallels the use of the standard part function (see Section 4) enabling the transition from a sum of type (1) above to the definite integral (2), similar to the definition of the derivative analyzed in Section 2.1.

2.3. **Un infiniment petit in Cauchy.** What is the precise meaning of Cauchy's *infiniment petit* (infinitely small)? All of Cauchy's textbooks on analysis contain essentially the same definition up to slight changes in word order:

> Lorsque les valeurs numériques successives d'une même variable décroissent indéfiniment, de manière à s'abaisser au-dessous de tout nombre donné, cette variable devient ce qu'on nomme un *infiniment petit* ou une quantité *infiniment petite*. Une variable de cette espèce a zéro pour limite.[5] [15, p. 4] (emphasis in the original)

An examination of the books [15], [16] reveals that Cauchy typically *did not* define his infinitely small literally as a variable whose limit is zero. Namely, he rarely wrote "an infinitely small *is* a variable, etc." but said, rather, that a variable *becomes* (*devient*) an infinitely small.

Thus, the passage cited above is the first definition of the infinitely small in *Cours d'Analyse*. The next occurrence is on page 26 there, again using *devient*, and emphasizing *infiniment petite* by means of italics. On page 27 Cauchy summarizes the definition as follows: "Soit α une quantité infiniment petite, c'est-à-dire, une variable dont la valeur numérique décroisse indéfiniment." This is a summary of the definition already given twice, the expression "infiniment petite" is not italicized, and "is" is used in place of "becomes" as shorthand for the more detailed and precise definitions appearing earlier in Cauchy's textbook. An identical definition with *devient* appears in his 1823 textbook [16, p. 4].

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[5]Translation: "When the successive numerical values of such a variable decrease indefinitely, in such a way as to fall below any given number, this variable becomes what we call *infinitesimal*, or an *infinitely small quantity*. A variable of this kind has zero as its limit" [14, p. 7].

Cauchy's term *becomes* implies a change of nature or *type*.[6] Namely, a variable is not quite an infinitesimal yet, but only serves to *generate* or *represent* one, as emphasized by Laugwitz:

> Cauchy never says what his infinitesimals *are*; we are told only how infinitesimals can be *represented*. (Laugwitz [35], 1987, p. 271)

See also Sad et al. [41]. This indicates that Cauchy considered an infinitesimal as a separate type of mathematical entity, distinct from variable or sequence.

2.4. **Variable quantities, infinitesimals, and limits.** To comment more fully on Cauchy's passage cited in Section 2.3, note that there are three players here:

(A) variable quantity;
(B) infinitesimal;
(C) limit zero.

We observe that the notion of variable quantity is the primitive notion in terms of which both infinitesimals and limits are defined (see Section 1.1 for Cauchy's definition of limit in terms of variable quantity). This order of priorities is confirmed by the title of Cauchy's very first lesson in his 1823 book:

> 1.^{re} Leçon. Des variables, de leurs limites, et des quantités infiniment petites [16, p. ix]

Thus, Cauchy is proposing a definition and an observation:

(Co1) a variable quantity that diminishes indefinitely becomes an infinitesimal; and

(Co2) such a variable quantity has zero as limit.

Here item (Co2) is merely a restatement of the property of diminishing indefinitely in terms of the language of limits. As noted in Section 1, Robinson pointed out that Cauchy assigned a central role to the notion of a variable which tends to a limit. Cauchy's notion of limit here is close to the notion of limit of his predecessor Lacroix (see Section 3.9).

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[6]To illustrate such a change in modern terms, note that in the context of the traditional construction of the real numbers in terms of Cauchy sequences $u = (u_n) \in \mathbb{Q}^{\mathbb{N}}$ of rational numbers, one never says that a real number *is* a sequence, but rather that a sequence *represents* or *generates* the real number, or to use Cauchy's terminology, *becomes* a real number. A related construction of hyperreal numbers out of sequences of real numbers, where a sequence tending to zero generates an infinitesimal, is summarized in Section 4.

2.5. **Assigning a sign to an infinitesimal.** Cauchy often uses the notation α for a generic infinitesimal, in both his 1821 and 1823 textbooks. In his 1823 textbook Cauchy assumes that α is either positive or negative:

> Cherchons maintenant la limite vers laquelle converge l'expression $(1 + \alpha)^{\frac{1}{\alpha}}$, tandis que α s'approche indéfiniment de zéro. Si l'on suppose d'abord la quantité α positive et de la forme $\frac{1}{m}$, m désignant un nombre entier variable et susceptible d'un accroissement indéfini, on aura $(1 + \alpha)^{\frac{1}{\alpha}} = \left(1 + \frac{1}{m}\right)^m$ ... Supposons enfin que α devienne une quantité négative. Si l'on fait dans cette hypothèse $1 + \alpha = \frac{1}{1+\beta}$, β sera une quantité positive, qui convergera elle-même vers zéro, etc. [16, pp. 2–4]

It is well known that variable quantities or sequences that generate Cauchyan infinitesimals are not necessarily monotone. Indeed, Cauchy himself gives a non-monotone example at the beginning of CdA:

$$\frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \frac{1}{8}, \frac{1}{7}, \&c. \dots \ [15, \text{p. } 27]$$

This poses a problem since it is not obvious how to assign a sign plus or minus to an arbitrary null sequence (i.e., a sequence tending to zero).

When Cauchy actually uses infinitesimals in proofs and applications, he assumes that they can be manipulated freely in arithmetic operations and other calculations. While formal order theory is a few decades away and is not to be found as such in Cauchy, he does appear to assume that a definite sign can be attached to an infinitesimal. Besides assuming that they have a well-defined sign, Cauchy also routinely applies arithmetic operations to infinitesimals.

This creates a difficulty to those who consider that Cauchy merely used the term "infinitely small" as shorthand for a sequence with limit 0, since it is unclear how to assign a sign to an arbitrary null sequence, whereas Cauchy does appear to assign a sign to his infinitesimals.

Which process exactly did Cauchy envision when he spoke of a sequence *becoming* an infinitesimal? Cauchy does not explain. However, Cauchy's assumption that each infinitesimal has a sign suggests that a sequence is not identical to the infinitesimal it generates.

Even monotone sequences are not closed under arithmetic operations. Namely, such operations necessarily lead to non-monotone ones, including ones that change sign.

Cauchy routinely assumes in his work, particularly on integrals, that one can freely add infinitesimals and obtain other infinitesimals, i.e., that the numbers involved are closed under arithmetic operations.

Such an assumption is valid in modern theories of ordered fields properly extending $\mathbb{R}$, but if one is working with sequences, such an assumption leads to a dilemma:

(1) either one only works with monotone ones, in which case one gets into a problem of closedness under natural arithmetic operations;

(2) or one works with arbitrary sequences, in which case the assumption that a sequence can be declared to be either positive or negative becomes problematic.

Cauchy was probably not aware of the difficulty that that one can't *both* assign a specific sign to α, and also have the freedom of applying arithmetic operations to infinitesimals. The point however is that the way he uses infinitesimals indicates that both conditions are assumed, even though from the modern standpoint the justification provided is insufficient. In other words, Cauchy's *procedures* are those of an infinitesimal-enriched framework, though the *ontology* of such a system is not provided.

Cauchy most likely was not aware of the problem, for otherwise he may have sought to address it in one way or another. He did have some interest in asymptotic behavior of sequences. Thus, in some of his texts from the late 1820s he tried to develop a theory of the order of growth at infinity of functions. Such investigations were eventually picked up by du Bois-Reymond, Borel, and Hardy; see Borovik–Katz ([11], 2012) for details.

2.6. **Gilain on omnipresence of limits.** Gilain refers to Cauchy's course in 1817 as a

> cours très important historiquement, où les bases de la nouvelle analyse, notamment celle de l'*Analyse algébrique* de 1821, sont posées... [23, §30]

He goes on to note "l'omniprésence du concept de limite" (ibid.). How are we to evaluate Gilain's claim as to the "omnipresence" of the concept of limit?

With regard to Cauchy's pre-1820 courses such as the one in 1817 mentioned by Gilain, there appears to be a consensus among scholars already noted in Section 1.8 concerning the absence of infinitesimals. As far as Cauchy's 1821 book is concerned, the presence (perhaps even "omnipresence" as per Gilain) of limits in the definition of infinitesimals goes hand-in-hand with the fact that Cauchy defined both limits and infinitesimals in terms of the primitive notion of a *variable quantity* (see

beginning of Section 1 as well as Section 2.4). It is therefore difficult to agree with Gilain when he claims to know the following:

> On sait que Cauchy définissait le concept d'infiniment petit à l'aide du concept de limite, qui avait le premier rôle (voir Analyse algébrique, p. 19; . . .) [23, note 67]

Here Gilain claims that it is the concept of *limite* that played a primary role in the definition of infinitesimal, with reference to page 19 in the 1897 *Ouevres Complètes* edition of CdA [15]. The corresponding page in the 1821 edition is page 4. We quoted Cauchy's definition in Section 2.3 and analyzed it in Section 2.4. An attentive analysis of the definition indicates that it is more accurate to say that it is the concept of variable quantity (rather than *limite*) that "avait le premier rôle."

Cauchy exploited the notion of limit in [15, Chapter 2, §3] in the proofs of Theorem 1 and Theorem 2. Theorem 1 compares the convergence of the difference $f(x + 1) - f(x)$ and that of the ratio $\frac{f(x)}{x}$. Theorem 2 compares the convergence of $\frac{f(x+1)}{f(x)}$ and $[f(x)]^{\frac{1}{x}}$. These proofs can be viewed as prototypes of ϵ, δ arguments. On the other hand, neither of the two proofs mentions infinitesimals. Therefore neither can support Gilain's claim to the effect that Cauchy allegedly used limits as a basis for defining infinitesimals. The proof of Theorem 1 is analyzed in more detail in Section 3.8.

Cauchy's procedures exploiting infinitesimals have stood the test of time and proved their applicability in diverse areas of mathematics, physics, and engineering.

Gilain and some other historians assume that the appropriate modern proxy for Cauchy's *limite* necessarily operates in the context of an Archimedean continuum (see Section 2.4). Yet the vitality and robustness of Cauchy's infinitesimal procedures is obvious given the existence of proxies in modern theories of infinitesimals. What we argue is that modern infinitesimal proxies for Cauchy's procedures are more faithful to the original than Archimedean proxies that typically involve anachronistic paraphrases of Cauchy's briefer definitions and arguments.

This article does not address the historical *ontology* of infinitesimals (a subject that may require separate study) but rather the *procedures* of infinitesimal calculus and analysis as found in Cauchy's oeuvre (see [9] for further details on the procedure/ontology dichotomy).

2.7. **Limite and infinity.** As we noted in Section 1.3, the use of the term *limite* by Cauchy could be misleading to a modern reader. Consider for example its use in the passage cited in Section 2.3. The

fact that Cauchy is not referring here to a modern notion of limit is evident from his very next sentence:

> Lorsque les valeurs numériques successives d'une même variable croissent de plus en plus, de manière à s'élever au-dessus de tout nombre donné, on dit que cette variable a pour limite l'infini positif indiqué par le signe ∞ s'il s'agit d'une variable positive...[7] [16, p. 4]

In today's calculus courses, it is customary to give an (ϵ, δ) or (ϵ, N) definition of limit of, say, a sequence, and then introduce infinite 'limits' in a broader sense when the sequence diverges to infinity. But Cauchy does not make a distinction between convergent limits and divergent infinite limits.

Scholars ranging from Sinaceur ([44], 1973) to Nakane ([39], 2014) have pointed out that Cauchy's notion of limit is distinct from the Weierstrassian *Epsilontik* one (this is particularly clear from Cauchy's definition of the derivative analyzed in Section 2.1); nor did Cauchy ever give an ϵ, δ *definition* of limit, though prototypes of ϵ, δ *arguments* do occasionally appear in Cauchy; see Section 1.2.

3. Minutes of meetings, Poisson, and de Prony

Here we develop an analysis of the third of the misconceptions diagnozed in Borovik–Katz ([11], 2012, Section 2.5), namely the idea that Cauchy was forced to teach infinitesimals at the *Ecole Polytechnique* allegedly against his will. We show that the debate there concerned mainly the issue of *rigor*, a separate one from *infinitesimals*.

Minutes of meetings at the *Ecole* are a valuable source of information concerning the scientific and pedagogical interactions there in the 1820s.

3.1. **Cauchy pressured by Poisson and de Prony.** Gilain provides detailed evidence of the pressure exerted by Siméon Denis Poisson, Gaspard de Prony, and others on Cauchy to simplify his analysis course. Thus, in 1822

> Poisson et de Prony... insistent [sur la] nécessité... de simplifier l'enseignement de l'analyse, en multipliant les exemples numériques et en réduisant beaucoup la partie analyse algébrique placée au début du cours. [23, §61]

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[7]Translation: "When the successive numerical values [i.e., absolute values] of the same variable grow larger and larger so as to rise above each given number, one says that this variable has limit positive infinity denoted by the symbol ∞ when the variable is positive."

Similarly, in 1823, Cauchy's course was criticized for being too complicated:

> des voix se sont élevées pour trouver trop compliquées les feuilles de cours en question et il était décidé de proposer au Ministre la nomination d'une commission qui serait chargée chaque année de l'examen des feuilles d'analyse et des modifications éventuelles à y apporter. [23, §72]

The critics naturally include Poisson and de Prony:

> Cette commission, effectivement mise en place, comprendra, outre Laplace, président, les examinateurs de mathématiques (Poisson et de Prony),... (ibid.)

The complaints continue in 1825 as François Arago declares that

> ce qu'il y a de plus utile à faire pour le cours d'analyse, c'est de le simplifier. [23, §84]

At this stage Cauchy finally caves in and declares (in third person):

> il ne s'attachera plus à donner, comme il a fait jusqu'à présent, des démonstrations parfaitement *rigoureuses*. [23, §86] (emphasis added)

Note however that in these discussions, the issue is mainly that of *rigor* (i.e., too many proofs) rather than choice of a particular approach to the foundations of analysis. While Cauchy's commitment to simplify the course may have entailed skipping the proofs in the style of the *Epsilontik* of Theorems 1 and 2 in [15, Chapter 2, §3] (see end of Section 2.4), it may have also entailed skipping the proofs of as many as *eight* theorems concerning the properties of *infinitesimals* of various orders in [15, Chapter 2, §1], analyzed in [11, Section 2.3].

3.2. **Reports by de Prony.** Gilain notes that starting in 1826, there is a new source of information concerning Cauchy's course, namely the reports by de Prony:

> de Prony reproche de façon générale à Cauchy de *ne pas utiliser suffisamment* les considérations géométriques et les infiniment petits, tant en analyse qu'en mécanique. [23, §101] (emphasis added)

Thus with regard to the post-1820 period, only starting in 1826 do we have solid evidence that not merely excessive *rigor* but also insufficient use of *infinitesimals* was being contested. Even here, the complaint is not an alleged *absence* of infinitesimals, but merely *insufficient use* thereof. We will examine de Prony's views in Section 3.5.

3.3. **Course summaries.** According to course summaries reproduced in [23], Cauchy taught both continuous functions and infinitesimals (and presumably the definition of continuity in terms of infinitesimals after 1820) in the *première année* during the academic years 1825–1826, 1826–1827, 1827–1828, and 1828–1829 (the summaries for the *première année* during the 1829–1830 academic year, Cauchy's last at the *Ecole Polytechnique*, are not provided). All these summaries contain identical comments on continuity and infinitesimals for those years:

> Des fonctions en général, et des fonctions *continues* en particulier. – Représentation géométrique des fonctions *continues* d'une seule variable. – Du rapport entre l'accroissement d'une fonction et l'accroissement de la variable. – Valeur que prend ce rapport quand les accroissemens deviennent *infiniment petits*. (Cauchy as quoted by Gilain; emphasis added)

In 1827 for the first time we find a claim of an actual *absence* of infinitesimals from Cauchy's teaching. Thus, on 12 january 1827,

> le cours de Cauchy a de nouveau été mis en cause pour sa difficulté, (le gouverneur affirmant que des élèves avaient déclaré qu'ils ne le comprenaient pas), et son *non-usage* de la méthode des infiniment petits (voir document C12).[8] [23, §103] (emphasis added)

Tellingly, this comment by Gilain is accompanied by a footnote 111 where Gilain acknowledges that in the end Cauchy did use infinitesimals that year in his treatment of the theory of contact of curves; see Section 3.4 for details.

3.4. **Cauchy taken to task.** Gilain writes that during the 1826–1827 academic year, Cauchy was taken to task in the *Conseil de Perfectionnement* of the *École Polytechnique* for allegedly not teaching infinitesimals (see [23, §103]). Gilain goes on to point out in his footnote 111

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[8]To comment on Gilain's "document C12" (denoted C_{12} in [23]), it is necessary to reproduce what the document actually says: "Un membre demande si le professeur expose la méthode des infiniment petits, ainsi que le voeu en a été exprimé." What was apparently Cauchy's response to this query is reproduced in the next paragraph of document C12: "On répond que le commencement du cours ne pourra être fondé sur les notions infinitésimales que l'année prochaine, parce que le cours de cette année était commencé à l'époque où cette disposition a été arrêtée; que M. Cauchy s'occupe de la rédaction de ses feuilles, en conséquence, et qu'il a promis de les communiquer bientôt à la commission de l'enseignement mathématique."
Thus, the actual contents of document C12 indicate that Gilain's claim of "*non-usage*" is merely an extrapolation.

that Cauchy exploited infinitesimals anyway that year, in developing the theory of contact of curves:

> S'il ne fonde pas le calcul différentiel et intégral sur la 'méthode' des infiniment petits, Cauchy n'en utilise pas moins de façon importante ces objets (considérés comme des variables dont la limite est zéro),[9] en liaison notamment avec l'exposition de la théorie du contact des courbes. [23, note 111]

It emerges that Cauchy did use infinitesimals that year in his treatment of a more advanced topic (theory of contact). Thus Cauchy's actual scientific practice was not necessarily dependent on his preliminary definitions. There is conflicting evidence as to whether Cauchy used infinitesimals (as developed in [15] and [16]) in the introductory part of his course that year. As we mentioned in Section 3.2, the course summary for 1826–1827 does include both continuity and infinitesimals.

3.5. **Critique by de Prony.** Gilain describes de Prony's criticism of Cauchy as follows:

> [De Prony] critique notamment l'emploi de la méthode des limites par Cauchy au lieu de celle des infiniment petits, faisant appel ici à l'autorité posthume de Laplace, décédé depuis le 5 mars 1827 (voir document C14). [23, §105]

Here Gilain is referring to the following comments by de Prony:

> Les démonstrations des formules generales[10] du mouvement varié se sont encor trouvées melées de considerations relatives aux *limites*; ... (de Prony as quoted in Grattan-Guinness [28], 1990, p. 1339; emphasis in the original)

Having specified the target of his criticism, namely Cauchy's concept of *limite*, de Prony continues:

> ... il me semble qu'en employant, immediatement et exclusivement, la methode des infiniment petits, on abrege et on simplifie les raisonnements sans nuire à la clarté;

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[9]Gilain's parenthetical remark here is an editorial comment for which he provides no evidence. The remark reveals more about Gilain's own default expectations (see Section 1) than about Cauchy's actual foundational stance.

[10]The spelling as found in (Gilain [23, Document C_{14}]) is *générales* (i.e., the modern French spelling). Gilain similarly replaced *encor* by *encore*, *melées* by *mêlées*, *immediatement* by *immédiatement*, *methode* by *méthode*, *abrege* by *abrège*, and *collegue* by *collègue*.

> rappellons nous combien cette methode était recommandée
> par l'illustre collegue [Laplace] que la mort nous a en-
> levé. (ibid.)

What is precisely the nature of de Prony's criticism of Cauchy's approach to analysis? Does his criticism focus on excessive rigor, or on infinitesimals, as Gilain claims? The answer depends crucially on understanding de Prony's own approach, explored in Section 3.6.

3.6. **De Prony on small oscillations.** In his work *Mécanique philosophique*, de Prony considers infinitesimal oscillations of the pendulum (de Prony [19], 1799, p. 86, §125). He gives the familiar formula for the period or more precisely halfperiod, namely

$$\pi\sqrt{\tfrac{a}{g}}$$

where a is the length of the cord, and g is acceleration under gravity. Limits are not mentioned. In the table on the following page 87, he states the property of *isochronism*, meaning that the halfperiod $\pi\sqrt{\tfrac{a}{g}}$ is independent of the size of the infinitesimal amplitude. This however is not true literally but only up to a passage to limits, or taking the standard part;[11] see Section 4. Thus de Prony's own solution to the conceptual difficulties involving limits/standard parts in this case is merely to ignore the difficulties and suppress the limits.

In his article "Suite des leçons d'analyse," de Prony lets $n = Az$ ([18], 1796, p. 237). He goes on to write down the formula

$$\cos z = \frac{\left[\cos\frac{z}{n} + \sin\frac{z}{n}\sqrt{-1}\,\right]^n + \left[\cos\frac{z}{n} - \sin\frac{z}{n}\sqrt{-1}\,\right]^n}{2}$$

as well as a similar formula for the sine function. Next, de Prony makes the following remark:

> Je remarque maintenant qu'à mesure que A diminue et n
> augmente, ces équations s'approchent de devenir
>
> $$\cos z = \frac{\left[1 + \frac{z\sqrt{-1}}{n}\right]^n + \left[1 - \frac{z\sqrt{-1}}{n}\right]^n}{2} \qquad (3.1)$$

(ibid.; labeling (3.1) added)

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[11]Even if literally infinitesimal amplitudes are admitted, there is still a discrepancy disallowing one to claim that the halfperiod is literally $\pi\sqrt{\tfrac{a}{g}}$. This difficulty can be overcome in the context of modern infinitesimal analysis; see Kanovei et al. ([32], 2016).

De Prony's formula (3.1) is correct only up to taking the standard part of the right-hand side (for infinite n). Again de Prony handles the conceptual difficulty of dealing with infinite and infinitesimal numbers by *suppressing* limits or standard parts. Note that both of de Prony's formulas are taken verbatim from (Euler [21], 1748, §133 – §138).[12]

It is reasonable to assume that de Prony's criticism of Cauchy's teaching of prospective engineers had to do with what Prony saw as excessive fussiness in dealing with what came to be viewed later as conceptual difficulties of passing to the limit, i.e., taking the standard part. Note that in the comment by de Prony cited at the beginning of this section, he does *not* criticize Cauchy for not using infinitesimals, but merely for excessive emphasis on technical detail involving *limites*. Therefore Gilain's claim to the contrary cited at the beginning of Section 3.5 amounts to massaging the evidence by putting a tendentious spin on de Prony's criticism.

3.7. **Foundations, limits, and infinitesimals.** Can one claim that Cauchy established the foundations of analysis on the concept of infinitesimal?

The notions of infinitesimal, limit, and variable quantity are all fundamental for Cauchy. One understands them only by the definition which explains how they interact. If Cauchy established such foundations it was on the concept of a variable quantity, as analyzed in Section 2.4.

Can one claim that Cauchy conferred upon *limite* a central role in the architecture of analysis? The answer is affirmative if one takes note of the frequency of the occurrence of the term in Cauchy's oeuvre; similarly, Cauchy conferred upon infinitesimals a central role in the said architecture.

A more relevant issue, however, is the precise meaning of the term *limite* as used by Cauchy. As we saw in Section 2.1 he used it in the *differential* calculus in a sense closer to the *standard part function* than to any limit concept in the context of an Archimedean continuum; and as we saw in Section 2.2, he used it in the *integral* calculus in a sense closer to the *standard part* than any Archimedean counterpart.

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[12]Schubring lodges the following claim concerning de Prony: "The break with previous tradition, which was probably the most visible to his contemporaries, was the exclusion and rejection of infiniment petits by the analytic method. In de Prony the infiniment petits were excluded from the foundational concepts of his teaching by simply not being mentioned; only in a heading did they appear in a quotation, as 'so-called analysis of the infinitely small quantities'" (Schubring [42], 2005, p. 289). Schubring's assessment of de Prony's attitude toward infinitesimals seems about as apt as his assessment of Cauchy's; see (Błaszczyk et al. [10], 2017).

Did Cauchy ever seek a justification of infinitesimals in terms of limits? Hardly so, since he expressed both concepts in terms of a primitive notion of variable quantity. In applications of analysis, Cauchy makes no effort to justify infinitesimals in terms of limits.

3.8. **Cauchy's A-track arguments.** Let us examine in more detail the issue of ϵ, δ arguments in Cauchy, as found in [15, Section 2.3, Theorem 1] (already mentioned in Section 2.6). Cauchy seeks to show that if the difference $f(x+1) - f(x)$ converges towards a certain limit k, for increasing values of x, then the ratio $\frac{f(x)}{x}$ converges at the same time towards the same limit; see [14, p. 35].

Cauchy chooses $\epsilon > 0$, and notes that we can give the number h a value large enough so that, when x is equal to or greater than h, the difference $f(x+1) - f(x)$ is always contained between $k - \epsilon$ and $k + \epsilon$. Cauchy then arrives at the formula

$$\frac{f(h+n) - f(h)}{n} = k + \alpha,$$

where α is a quantity contained between the limits $-\epsilon$ and $+\epsilon$, and eventually obtains that the ratio $\frac{f(x)}{x}$ has for its limit a quantity contained between $k - \epsilon$ and $k + \epsilon$.

This is a fine sample of a prototype of an ϵ, δ proof in Cauchy. However, as pointed out by Sinkevich, Cauchy's proofs are all missing the tell-tale sign of a modern proof in the tradition of the Weierstrassian *Epsilontik*, namely exhibiting an explicit functional dependence of δ (or in this case h) on ϵ (Sinkevich [45], 2016).

One of the first occurrences of a modern definition of continuity in the style of the *Epsilontik* can be found in Schwarz's summaries of 1861 lectures by Weierstrass; see (Dugac [20], 1973, p. 64), (Yushkevich [47], 1986, pp. 74–75). This definition is a verbal form of a definition featuring a correct quantifier order (involving alternations of quantifiers).

The salient point here is that this sample of Cauchy's work has no bearing on Cauchy's infinitesimals. Nor does it imply that infinitesimals are merely variables tending to zero, since the term *infinitely small* does not occur in this proof at all. Nor does Cauchy's argument show that he thought of limits in anything resembling post-Weierstrassian terms since his recurring definition of limit routinely falls back on the primitive notion of a variable quantity, rather than on any form of an alternating quantifier string, whether verbal or not.

3.9. **Lacroix, Laplace, and Poisson.** The Bradley–Sandifer edition quotes a revealing comment of Cauchy's on the importance of infinitesimals. The comment is found in Cauchy's introduction:

> In speaking of the continuity of functions, I could not
> dispense with a treatment of the principal properties
> of infinitely small quantities, properties which serve as
> the foundation of the infinitesimal calculus. (Cauchy as
> translated in [14, p. 1])

Bradley and Sandifer then go on to note: "It is interesting that Cauchy does *not* also mention limits here" (ibid., note 6; emphasis added).

The circumstances of the publication of the 1821 *Cours d'Analyse* indicate that attaching fundamental importance to infinitesimals rather than limits (noted by Bradley and Sandifer) was Cauchy's personal choice, rather than being dictated by the constraints of his teaching at the *École Polytechnique*. Indeed, unlike Cauchy's later textbooks, his 1821 book was not commissioned by the *École* but was rather written upon the personal request of Laplace and Poisson, as acknowledged in (Gilain [23], 1989, note 139).

Sinaceur points out that Cauchy's definition of limit resembles, not that of Weierstrass, but rather that of Lacroix[13] dating from 1810 (see [44, p. 108–109]).[14] This is acknowledged in (Grabiner [25], 1981, p. 80).

Cauchy's kinematic notion of limit was expressed, like his notion of infinitesimal α, in terms of a primitive notion of *variable quantity* (see Section 2.4). Thus, Cauchy's comment that when a variable becomes an infinitesimal α, the limit of such a variable is zero, can be interpreted in two ways. It can be interpreted in the context of an Archimedean continuum. Alternatively, it could be interpreted as the statement that the assignable part of α is zero, in the context of a Bernoullian (i.e., infinitesimal-enriched) continuum, or in modern terminology, that the *standard part* of α is zero; see Section 4.

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[13]As a student at the *Polytechnique*, Cauchy attended Lacroix's course in analysis in 1805; see (Belhoste [8], 1991, p. 10, 243).

[14]Sinaceur explicitly denies Cauchy the honor of having published the first arithmetic definition of limits, by writing: "Or, 1) l'épsilonisation n'est pas l'œuvre de Cauchy, mais celle de Weierstrass ; . . . on ne peut dire qu'il en donne une définition purement arithmétique ou purement analytique. Sa définition . . . n'enveloppe pas moins d'intuition géométrique que celle contenue dans le *Traité* de Lacroix. . ."

4. Modern infinitesimals in relation to Cauchy's procedures

While set-theoretic justifications for either A-track or B-track modern framework are obviously not to be found in Cauchy, Cauchy's *procedures* exploiting infinitesimals find closer proxies in Robinson's framework for analysis with infinitesimals than in a Weierstrassian framework. In this section we outline a set-theoretic construction of a hyperreal extension $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$, and point out specific similarities between procedures using the hyperreals, on the one hand, with Cauchy's procedures, on the other.

Let $\mathbb{R}^{\mathbb{N}}$ denote the ring of sequences of real numbers, with arithmetic operations defined termwise. Then we have ${}^*\mathbb{R} = \mathbb{R}^{\mathbb{N}} / \mathrm{MAX}$ where MAX is the maximal ideal consisting of all "negligible" sequences (u_n). Here a sequence is negligible if it vanishes for a set of indices of full measure ξ, namely, $\xi\big(\{n \in \mathbb{N} : u_n = 0\}\big) = 1$. Here $\xi \colon \mathcal{P}(\mathbb{N}) \to \{0, 1\}$ is a finitely additive probability measure taking the value 1 on cofinite sets, where $\mathcal{P}(\mathbb{N})$ is the set of subsets of $\mathbb{N}$. The subset $\mathcal{F}_\xi \subseteq \mathcal{P}(\mathbb{N})$ consisting of sets of full measure ξ is called a nonprincipal ultrafilter. These originate with (Tarski [46], 1930). The set-theoretic presentation of a Bernoullian continuum (see Section 1.2) outlined here was therefore not available prior to that date.

The field $\mathbb{R}$ is embedded in ${}^*\mathbb{R}$ by means of constant sequences. The subring ${}^\flat\mathbb{R} \subseteq {}^*\mathbb{R}$ consisting of the finite elements of ${}^*\mathbb{R}$ admits a map **st** to $\mathbb{R}$, known as *standard part*

$$\mathbf{st} \colon {}^\flat\mathbb{R} \to \mathbb{R}, \tag{4.1}$$

which rounds off each finite hyperreal number to its nearest real number (the existence of such a map **st** is the content of the *standard part principle*). This enables one, for instance, to define the derivative of $s = f(t)$ as

$$f'(t) = \frac{ds}{dt} = \mathbf{st}\left(\frac{\Delta s}{\Delta t}\right) \tag{4.2}$$

(here $\Delta s \neq 0$ is infinitesimal) which parallels Cauchy's definition of derivative (see equation (2.1) in Section 2.1) more closely than any *Epsilontik* definition. Limit is similarly defined in terms of **st**, e.g., by setting

$$\lim_{t \to 0} f(t) = \mathbf{st}(f(\epsilon)) \tag{4.3}$$

where ϵ is a nonzero infinitesimal, in analogy with Cauchy's limit as analyzed in Section 1.3. For additional details on Robinson's framework see e.g., [22].

5. Conclusion

The oft-repeated claim (as documented e.g., in [3]; [7]) that "Cauchy's infinitesimal is a variable with limit 0" (see Gilain's comment cited in Section 3.4) is a reductionist view of Cauchy's foundational stance, at odds with much compelling evidence in Cauchy's writings, as we argued in Sections 2 and 3.

Gilain, Siegmund-Schultze, and some other historians tend to adopt a butterfly model for the development of analysis, to seek proxies for Cauchy's procedures in a default modern Archimedean framework, and to view his infinitesimal techniques as an evolutionary dead-end in the history of analysis. Such an attitude was criticized by Grattan-Guinness, as discussed in Section 1. The fact is that, while Cauchy did use an occasional epsilon in an Archimedean sense, his techniques relying on infinitesimals find better proxies in a modern framework exploiting a Bernoullian continuum.

Robinson first proposed an interpretation of Cauchy's *procedures* in the framework of a modern theory of infinitesimals in [40] (see Section 1.7). A *set-theoretic foundation* for infinitesimals could not have been provided by Cauchy for obvious reasons, but Cauchy's *procedures* find closer proxies in modern infinitesimal frameworks than in modern Archimedean ones.

Acknowledgments

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