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Definable selector for $\mathbf{\Delta}^0_2$ sets modulo countable

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Abstract

A set is effectively chosen in every class of $\mathbf{\Delta}^0_2$ sets modulo countable.

Let $\mathsf{E}_{\aleph_0}$ be the equivalence relation of equality modulo countable, that is, $X \mathrel{\mathsf{E}_{\aleph_0}} Y$ iff the symmetric difference $X \,\Delta\, Y$ is (at most) countable. Does there exist an *effective selector*, *i.e.*, an effective choice of an element in each $\mathsf{E}_{\aleph_0}$-equivalence class of sets of certain type? The answer depends on the type of sets considered. For instance, the question answers in the positive for the class of closed sets in Polish spaces by picking the only perfect set in each $\mathsf{E}_{\aleph_0}$-equivalence class of closed sets. On the other hand, effective selectors for $\mathsf{E}_{\aleph_0}$ do not exist in the domain of $\mathbf{F}_\sigma$ sets, *e.g.*, in the Solovay model (in which the axiom of choice AC holds and all ROD [1] sets are LM and have the Baire property) by [3, Theorem 5.5].

Our goal here is to prove that $\mathbf{F}_\sigma$ is the best possible for such a negative result.

Theorem 1. *There exists a definable selector for* $\mathsf{E}_{\aleph_0}$ *in the domain of* $\mathbf{\Delta}^0_2$ *sets in Polish spaces.* ($\mathbf{\Delta}^0_2$ = all sets simultaneously $\mathbf{F}_\sigma$ and $\mathbf{G}_\delta$.)

Proof (Theorem)**.** We'll make use of the following lemma.

Lemma 2. *If* X *is a countable* $\mathbf{G}_\delta$ *set in a Polish space then the* **closure** $\overline{X}$ *is countable. Therefore if* $X \mathrel{\mathsf{E}_{\aleph_0}} Y$ *are* $\mathbf{\Delta}^0_2$ *sets then* $\overline{X} \mathrel{\mathsf{E}_{\aleph_0}} \overline{Y}$.

Proof (Lemma)**.** Otherwise X is a countable dense $\mathbf{G}_\delta$ set in an uncountable Polish space $\overline{X}$, which is not possible. □ (Lemma)

Difference hierarchy. It is known (see *e.g.* [2, 22.E]) that every $\mathbf{\Delta}^0_2$ set A in a Polish space $\mathbb{X}$ admits a representation in the form $A = \bigcup_{\eta<\vartheta}(F_\eta \smallsetminus H_\eta)$, where $\vartheta < \omega_1$ and $F_0 \supseteq H_0 \supseteq F_1 \supseteq H_1 \supseteq \dots F_\eta \supseteq H_\eta \supseteq \dots$ is a decreasing sequence of closed sets in $\mathbb{X}$, defined by induction so that $F_0 = \mathbb{X}$, $H_\eta = \overline{F_\eta \smallsetminus A}$, $F_{\eta+1} = H_\eta \cap \overline{F_\eta \cap A}$, and the intersection on limit steps. The induction stops as soon as $F_\vartheta = \varnothing$.

The key idea of the proof of Theorem 1 is to show that if $A \mathrel{\mathsf{E}_{\aleph_0}} B$ are $\mathbf{\Delta}^0_2$ sets then the corresponding sequences of closed sets

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[1] ROD = real-ordinal definable, OD = ordinal-definable.

$$\left.\begin{array}{l} F_0^A \supseteq H_0^A \supseteq F_1^A \supseteq H_1^A \supseteq \dots F_\eta^A \supseteq H_\eta^A \supseteq \dots \\ F_0^B \supseteq H_0^B \supseteq F_1^B \supseteq H_1^B \supseteq \dots F_\eta^B \supseteq H_\eta^B \supseteq \dots \end{array}\right\} \quad (\eta < \vartheta = \vartheta^A = \vartheta^B), \ ^2$$

satisfying $A = \bigcup_{\eta<\vartheta}(F_\eta^A \smallsetminus H_\eta^A)$ and $B = \bigcup_{\eta<\vartheta}(F_\eta^B \smallsetminus H_\eta^B)$ as above, also satisfy

$(*)$ $F_\eta^A \mathsf{E}_{\aleph_0} F_\eta^B$ and $H_\eta^A \mathsf{E}_{\aleph_0} H_\eta^B$ — for all $\eta < \vartheta$.

It follows that the perfect kernels[3] $\mathbf{PK}(F_\eta^A)$, $\mathbf{PK}(F_\eta^B)$ coincide: $\mathbf{PK}(F_\eta^A) = \mathbf{PK}(F_\eta^B)$, and $\mathbf{PK}(H_\eta^A) = \mathbf{PK}(H_\eta^B)$ as well. Therefore the sets $\Phi(A) = \bigcup_{\eta<\vartheta}(\mathbf{PK}(F_\eta^A) \smallsetminus \mathbf{PK}(H_\eta^A))$ and $\Phi(B)$ coincide (whenever $A \mathsf{E}_{\aleph_0} B$ are $\mathbf{\Delta}_2^0$ sets), and $A \mathsf{E}_{\aleph_0} \Phi(A)$ holds for each $\mathbf{\Delta}_2^0$ set A, so Φ is a selector required, ending the proof of the theorem.

Thus it remains to prove $(*)$. We argue by induction.

We have $F_0^A = F_0^B = \mathbb{X}$ (the underlying Polish space).

Suppose that $F_\eta^A \mathsf{E}_{\aleph_0} F_\eta^B$; prove that $H_\eta^A \mathsf{E}_{\aleph_0} H_\eta^B$. By definition, we have $H_\eta^A = \overline{F_\eta^A \smallsetminus A}$ and $H_\eta^B = \overline{F_\eta^B \smallsetminus B}$, where $(F_\eta^A \smallsetminus A) \mathsf{E}_{\aleph_0} (F_\eta^B \smallsetminus B)$ (recall that $A \mathsf{E}_{\aleph_0} B$ is assumed), hence $H_\eta^A \mathsf{E}_{\aleph_0} H_\eta^B$ holds by Lemma 2.

It's pretty similar to show that if $F_\eta^A \mathsf{E}_{\aleph_0} F_\eta^B$ (and then $H_\eta^A \mathsf{E}_{\aleph_0} H_\eta^B$ by the above) then $F_{\eta+1}^A \mathsf{E}_{\aleph_0} F_{\eta+1}^B$ holds. This accomplishes the step $\eta \to \eta + 1$.

Finally the limit step is rather obvious. □ (Theorem 1)

Problem 3. Coming back to the mentioned result of [3, Theorem 5.5], it is a challenging problem to prove that the equivalence relation $\mathsf{E}_{\aleph_0}$ on $\mathbf{F}_\sigma$ sets is not ROD-reducible to the equality of Bodel sets in the Solovay model.

Remark 4. As established in [1], it is true in some models (including *e.g.* Cohen and random extensions of $\mathbf{L}$) that every OD and Borel set is OD-Borel (*i.e.*, has an OD Borel code). In such a model, there is an effective choice of a set and its Borel code, by an OD function, in every $\mathsf{E}_{\aleph_0}$-class of Borel sets containing an OD set. □

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[2] A shorter sequence is extended to the longer one by empty sets if necessary.

[3] $\mathbf{PK}(X)$, the *perfect kernel*, is the largest perfect subset of a closed set X.