Indiscernible pairs of countable sets of reals at a given projective level

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Abstract

Using an invariant modification of Jensen's "minimal Π^1_2 singleton" forcing, we define a model of **ZFC**, in which, for a given $n \ge 2$, there exists an Π^1_n unordered pair of non-OD (hence, OD-indiscernible) countable sets of reals, but there is no Σ^1_n unordered pairs of this kind.

Any two reals $x_1 \neq x_2$ are discernible by a simple formula $\varphi(x) := x < r$ for a suitable rational r. Therefore, the lowest (type-theoretic) level of sets where one may hope to find indiscernible elements, is the level of *sets of reals*. And indeed, identifying the informal notion of definability with the ordinal definability (OD), one finds indiscernible sets of reals in appropriate generic models.

Example 1. If reals $a \neq b$ in 2^ω form a Cohen-generic pair over $\mathbf{L}$, then the constructibility degrees $[a]_{\mathbf{L}} = \{x \in 2^\omega : \mathbf{L}[x] = \mathbf{L}[a]\}$ and $[b]_{\mathbf{L}}$ are OD-indiscernible disjoint sets of reals in $\mathbf{L}[a, b]$, by rather straightforward forcing arguments, see [2, Theorem 3.1] and a similar argument in [3, Theorem 2.5]. □

Example 2. As observed in [5], if reals $a \neq b$ in 2^ω form a Sacks-generic pair over $\mathbf{L}$, then the constructibility degrees $[a]_{\mathbf{L}}$ and $[b]_{\mathbf{L}}$ still are OD-indiscernible disjoint sets in $\mathbf{L}[a, b]$, with the additional advantage that the unordered pair $\{[a]_{\mathbf{L}}, [b]_{\mathbf{L}}\}$ is an OD set in $\mathbf{L}[a, b]$ because $[a]_{\mathbf{L}}$, $[b]_{\mathbf{L}}$ are the only two minimal degrees in $\mathbf{L}[a, b]$. (This argument is also presented in [3, Theorem 4.6].) In other words, it is true in such a generic model $\mathbf{L}[a, b]$ that $P = \{[a]_{\mathbf{L}}, [b]_{\mathbf{L}}\}$ is *an* OD *pair of non*-OD (hence OD-indiscernible in this case) *sets of reals*. □

Unordered OD pairs of non-OD sets of reals were called *Groszek – Laver pairs* in [4], while in the notation of [3, 6] the sets $[a]_{\mathbf{L}}$, $[b]_{\mathbf{L}}$ are *ordinal-algebraic* (meaning that they belong to a finite OD set) in $\mathbf{L}[a, b]$, but neither of the two

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sets is straightforwardly OD in $\mathbf{L}[a,b]$. From the other angle of view, any (OD or not) pair of OD-indiscernible sets $x \neq y$ is a special violation of the *Leibniz – Mycielski axiom LM* of Enayat [2] (see also [1]). [1]

Given an *unordered* pair of disjoint sets $A, B \subseteq 2^\omega$, to measure its descriptive complexity, define the equivalence relation E_{AB} on the set $A \cup B$ by $x \mathrel{\mathsf{E}_{AB}} y$ iff $x, y \in A$ or $x, y \in B$. It holds in the Sacks×Sacks generic model $\mathbf{L}[a,b]$ that $\mathsf{E}_{[a]_\mathbf{L}[b]_\mathbf{L}}$ is the restriction of the Σ^1_2 relation $\mathbf{L}[x] = \mathbf{L}[y]$ to the Δ^1_3 set

$$\begin{aligned}[a]_\mathbf{L} \cup [b]_\mathbf{L} &= \{x \in 2^\omega : x \notin \mathbf{L} \wedge \exists\, z \in 2^\omega (z \notin \mathbf{L}[x])\} \\ &= \{x \in 2^\omega : x \notin \mathbf{L} \wedge \forall\, y \in 2^\omega \cap \mathbf{L}[x]\,(y \in \mathbf{L} \vee x \in \mathbf{L}[y])\}\,.\end{aligned}$$ [2]

Thus the Groszek – Laver (unordered) pair $\{[a]_\mathbf{L}, [b]_\mathbf{L}\}$ of Example 2 can be said to be **a Δ^1_3 pair** in $\mathbf{L}[a,b]$ because so is the equivalence relation $\mathsf{E}_{[a]_\mathbf{L}[b]_\mathbf{L}}$.

Example 3. A somewhat better result was obtained in [4]: a generic model $\mathbf{L}[a,b]$ in which the E_0-equivalence classes [3] $[a]_{\mathsf{E}_0}$, $[b]_{\mathsf{E}_0}$ form a Π^1_2 Groszek – Laver pair of *countable* sets. □

Thus Δ^1_3, and even Π^1_2 Groszek – Laver pairs of countable sets in 2^ω exist in suitable extensions of $\mathbf{L}$. This is the best possible existence result since Σ^1_2 Groszek – Laver pairs do not exist by the Shoenfield absoluteness.

The main result of this paper is the following theorem. It extends the research line of our recent papers [12, 13, 14], based on some key methods and approaches outlined in Harrington's handwritten notes [7] and aimed at the construction of generic models in which this or another property of reals or pointsets holds at a given projective level.

Theorem 4. *Let $\mathfrak{n} \geq 3$. There is a generic extension $\mathbf{L}[a]$ of $\mathbf{L}$, the constructible universe, by a real $a \in 2^\omega$, such that the following is true in $\mathbf{L}[a]$:*

(i) *there exists a $\Pi^1_\mathfrak{n}$ Groszek – Laver pair of countable sets in 2^ω ;*

(ii) *every countable $\Sigma^1_\mathfrak{n}$ set consists of* OD *elements, and hence there is no $\Sigma^1_\mathfrak{n}$ Groszek – Laver pairs of countable sets.*

The proof of Theorem 4 makes use of a forcing notion $\mathbb{P} = \mathbb{P}_\mathfrak{n} \in \mathbf{L}$, defined in [12] for a given number $\mathfrak{n} \geq 2$, which satisfies the following key requirements.

1°. $\mathbb{P} \in \mathbf{L}$ and $\mathbb{P}$ consists of Silver trees in $2^{<\omega}$. A perfect tree $T \subseteq 2^{<\omega}$ is a *Silver tree*, in symbol $T \in \mathbf{ST}$, whenever there exists an infinite sequence

[1] LM claims that if $x \neq y$ then there exists an ordinal α and a (parameter-free) $\in$-formula $\varphi(\cdot)$ such that $x, y \in \mathbf{V}_\alpha$ and $\varphi(x)$ holds in $\mathbf{V}_\alpha$ but $\varphi(x)$ fails in $\mathbf{V}_\alpha$ — in this case x, y are OD-discernible (with $\alpha \in \mathbf{Ord}$ as a parameter), of course.

[2] The first line says that x is nonconstructible and not $\leqslant_\mathbf{L}$-maximal, the second line says that x is nonconstructible and $\leqslant_\mathbf{L}$-minimal; this happens to be equivalent in that model.

[3] E_0 is defined on the Cantor space 2^ω so that $x \mathrel{\mathsf{E}_0} y$ iff the set $\{n : x(n) \neq y(n)\}$ is finite.

of strings $u_k = u_k(T) \in 2^{<\omega}$ such that T consists of all strings of the form $s = u_0{}^\frown i_0{}^\frown u_1{}^\frown i_1{}^\frown u_2{}^\frown i_2{}^\frown \dots {}^\frown u_m{}^\frown i_m$, and their substrings (including Λ, the empty string), where $m < \omega$ and $i_k = 0, 1$.

2°. If $s \in T \in \mathbb{P}$ then the subtree $T\restriction_s = \{t \in T : s \subset t \vee t \subseteq s\}$ belongs to $\mathbb{P}$ as well — then clearly *the forcing* $\mathbb{P}$ *adjoins a new generic real* $a \in 2^\omega$.

3°. $\mathbb{P}$ is E_0-*invariant*, in the sense that if $T \in \mathbb{P}$ and $s \in 2^{<\omega}$ then the tree $s{\cdot}T = \{s{\cdot}t : t \in T\}$ belongs to $\mathbb{P}$ as well.[4] It follows that *if* $a \in 2^\omega$ *is* $\mathbb{P}$*-generic over* $\mathbf{L}$ *then any real* $b \in [a]_{\mathsf{E}_0}$ *is* $\mathbb{P}$*-generic over* $\mathbf{L}$ *too.*

In other words, $\mathbb{P}$ adjoins *a whole* E_0*-class* $[a]_{\mathsf{E}_0}$ *of* $\mathbb{P}$*-generic reals.*

4°. Conversely, *if* $a \in 2^\omega$ *is* $\mathbb{P}$*-generic over* $\mathbf{L}$ *and a real* $b \in 2^\omega \cap \mathbf{L}[a]$ *is* $\mathbb{P}$*-generic over* $\mathbf{L}$*, then* $b \in [a]_{\mathsf{E}_0}$.

5°. The property of "being a $\mathbb{P}$-generic real in 2^ω over $\mathbf{L}$" is (lightface) $\Pi^1_{\mathtt{n}}$ in any generic extension of $\mathbf{L}$.

6°. If $a \in 2^\omega$ is $\mathbb{P}$-generic over $\mathbf{L}$, then it is true in $\mathbf{L}[a]$ that

(1) (by 3°, 4°, 5°) $[a]_{\mathsf{E}_0}$ *is a* $\Pi^1_{\mathtt{n}}$ *set containing no* OD *elements*, but

(2) *every countable* $\Sigma^1_{\mathtt{n}}$ *set consists of* OD *elements.*[5]

Proof (Theorem 4)**.** Let $\mathbb{P} \in \mathbf{L}$ be a forcing satisfying conditions 1° – 6°. Let $a_0 \in 2^\omega$ be a real $\mathbb{P}$-generic over $\mathbf{L}$. Then, in $\mathbf{L}[a_0]$, the E_0-class $[a_0]_{\mathsf{E}_0}$ is a $\Pi^1_{\mathtt{n}}$ set containing no OD elements, by 6°(1).

Let us split the E_0-class $[a_0]_{\mathsf{E}_0}$ into two equivalence classes of the subrelation $\mathsf{E}_0^{\mathsf{even}}$ defined on 2^ω so that $x \;\mathsf{E}_0^{\mathsf{even}}\; y$ iff the set $x \triangle y = \{k : x(k) \neq y(k)\}$ contains a finite even number of elements. Thus $[a_0]_{\mathsf{E}_0} = [a_0]_{\mathsf{E}_0^{\mathsf{even}}} \cup [b]_{\mathsf{E}_0^{\mathsf{even}}}$ is this partition, where $[x]_{\mathsf{E}_0^{\mathsf{even}}}$ is the $\mathsf{E}_0^{\mathsf{even}}$-class of any $x \in 2^\omega$, and $b \in [a_0]_{\mathsf{E}_0} \smallsetminus [a_0]_{\mathsf{E}_0^{\mathsf{even}}}$ is any real E_0-equivalent but not $\mathsf{E}_0^{\mathsf{even}}$-equivalent to a_0. We claim that, in $\mathbf{L}[a_0]$, these two $\mathsf{E}_0^{\mathsf{even}}$-subclasses of $[a_0]_{\mathsf{E}_0}$ form a $\Pi^1_{\mathtt{n}}$ Groszek – Laver pair required.

Basically, we have to prove that $[a_0]_{\mathsf{E}_0^{\mathsf{even}}}$ is not OD in $\mathbf{L}[a_0]$. Suppose to the contrary that $[a]_{\mathsf{E}_0^{\mathsf{even}}}$ is OD in $\mathbf{L}[a]$, say $[a_0]_{\mathsf{E}_0^{\mathsf{even}}} = \{x \in 2^\omega : \varphi(x)\}$, where $\varphi(x)$

[4] Here $s{\cdot}t \in 2^{<\omega}$, $\mathtt{dom}(a{\cdot}t) = \mathtt{dom}\,t$, if $k < \min\{\mathtt{dom}\,s, \mathtt{dom}\,t\}$ then $(a{\cdot}t)(k) = t(k) +_2 s(k)$ (and $+_2$ is the addition mod 2), while if $\mathtt{dom}\,s \le k < \mathtt{dom}\,t$ then $(a{\cdot}t)(k) = t(k)$.

[5] Earlier results in this direction include a model in [11] with a Π^1_2 E_0-class in 2^ω, containing no OD elements — which is equivalent to case $\mathtt{n} = 2$ in 6°. The forcing employed in [11] is an invariant, as in 3°, "Silver tree" version $\mathbb{P} = \mathbb{P}_2$, of a forcing notion, call it $\mathbb{J}$, introduced by Jensen [9] to define a model with a nonconstructible minimal Π^1_2 singleton. See also 28A in [8] on Jensen's original forcing. The invariance implies that instead of a single generic real, as in [9], $\mathbb{P}_2$ adjoins a whole E_0-equivalence class $[a]_{\mathsf{E}_0}$ of $\mathbb{P}_2$-generic reals in [11]. Another version of a countable lightface Π^1_2 non-empty set of non-OD reals was obtained in [10, 15] by means of the finite-support product $\mathbb{J}^\omega$ of Jensen's forcing $\mathbb{J}$, following the idea of Ali Enayat [2]. See [12, Introduction] on a more detailed account of the problem of the existence of countable OD sets of non-OD elements.

is a $\in$-formula with ordinals as parameters. This is forced by a condition $T \in \mathbb{P}$, so that if $a \in [T]$ is $\mathbb{P}$-generic over $\mathbf{L}$ then $[a]_{\mathsf{E}_0^{\mathtt{even}}} = \{x \in 2^\omega : \varphi(x)\}$ in $\mathbf{L}[a]$.

Representing T in the form of $1°$, let $m = \mathtt{dom}(u_0)$ and let $s = 0^m{}^\frown 1$, so that $s \in 2^{<\omega}$ is the string of m 0s, followed by 1 as the rightmost term; $\mathtt{dom}\, s = m+1$. Then $s \cdot T = T$, so that the real $b = s \cdot a$ still belongs to $[T]$, and hence we have $[b]_{\mathsf{E}_0^{\mathtt{even}}} = \{x \in 2^\omega : \varphi(x)\}$ in $\mathbf{L}[b] = \mathbf{L}[a]$ by the choice of T. We conclude that $[a]_{\mathsf{E}_0^{\mathtt{even}}} = [b]_{\mathsf{E}_0^{\mathtt{even}}}$. However, on the other hand, $a \mathrel{\mathsf{E}_0^{\mathtt{even}}} b$ fails by construction since the set $a \mathbin{\triangle} b = \{m\}$ contains one (an odd number) element. The contradiction ends the proof of (i) of Theorem 4.

To prove (ii) apply $6°(2)$. □

A problem. Can (ii) of Theorem 4 be improved to the nonexistence of $\Sigma^1_{\mathbb{n}}$ Groszek – Laver pairs of not-necessarily-countable sets in the model considered?

References

[1] Ali Enayat. Leibnizian models of set theory. *J. Symb. Log.*, 69(3):775–789, 2004.

[2] Ali Enayat. On the Leibniz – Mycielski axiom in set theory. *Fundam. Math.*, 181(3):215–231, 2004.

[3] Gunter Fuchs, Victoria Gitman, and Joel David Hamkins. Ehrenfeucht's lemma in set theory. *Notre Dame J. Formal Logic*, 59(3):355–370, 2018.

[4] M. Golshani, V. Kanovei, and V. Lyubetsky. A Groszek – Laver pair of undistinguishable E_0 classes. *Mathematical Logic Quarterly*, 63(1–2):19–31, 2017.

[5] M. Groszek and R. Laver. Finite groups of OD-conjugates. *Period. Math. Hung.*, 18:87–97, 1987.

[6] Joel David Hamkins and Cole Leahy. Algebraicity and implicit definability in set theory. *Notre Dame J. Formal Logic*, 57(3):431–439, 2016.

[7] Leo Harrington. The constructible reals can be anything. Preprint dated May 1974 with several addenda dated up to October 1975:

(A) Models where Separation principles fail, May 74;

(B) Separation without Reduction, April 75;

(C) The constructible reals can be (almost) anything, Part II, May 75.

[8] Thomas Jech. *Set theory.* Springer-Verlag, Berlin-Heidelberg-New York, The third millennium revised and expanded edition, 2003.

[9] Ronald Jensen. Definable sets of minimal degree. In Yehoshua Bar-Hillel, editor, *Math. Logic Found. Set Theory, Proc. Int. Colloqu., Jerusalem 1968*, pages 122–128. North-Holland, Amsterdam-London, 1970.

[10] V. Kanovei and V. Lyubetsky. A countable definable set of reals containing no definable elements. *ArXiv e-prints*, August 2014.

[11] V. Kanovei and V. Lyubetsky. A definable E_0-class containing no definable elements. *Archive for Mathematical Logic*, 54(5):711–723, 2015.

[12] V. Kanovei and V. Lyubetsky. Definable E_0 classes at arbitrary projective levels. *Ann. Pure Appl. Logic*, 169(9):851–871, 2018.

[13] V. Kanovei and V. Lyubetsky. Definable minimal collapse functions at arbitrary projective levels. *J. Symb. Log.*, 84(1):266–289, 2019.

[14] V. Kanovei and V. Lyubetsky. Non-uniformizable sets with countable cross-sections on a given level of the projective hierarchy. *Fundam. Math.*, 245(2):175–215, 2019.

[15] V.G. Kanovei and V.A. Lyubetsky. A definable countable set not containing definable elements. *Math. Notes*, 102(3-4):338–349, 2017.