Definable Hamel bases and $\mathsf{AC}_\omega(\mathbb{R})$

Vladimir Kanovei
IITP RAS
Bolshoy Karetny per. 19, build.1
Moscow 127051 Russia

Ralf Schindler
Institut für Mathematische Logik und Grundlagenforschung, Universität Münster
Einsteinstr. 62, 48149 Münster, Germany

Abstract

There is a model of ZF with a Δ^1_3 definable Hamel bases in which $\mathsf{AC}_\omega(\mathbb{R})$ fails.

Answering a question from [9, p. 433] it was shown in [1] that there is a Hamel basis in the Cohen–Halpern–Lévy model. In this paper we show that in a variant of this model, there is a projective, in fact Δ^1_3, Hamel basis.

Throughout this paper, by a *Hamel basis* we always mean a basis for $\mathbb{R}$, construed as a vector space over $\mathbb{Q}$. We denote by E the *Vitali equivalence relation*, xEy iff $x-y\in\mathbb{Q}$ for x, $y\in\mathbb{R}$. We also write $[x]_E=\{y\colon yEx\}$ for the E–equivalence class of x. A transversal for the set of all E–equivalence classes picks exactly one member from each $[x]_E$. The range of any such transversal is also called a *Vitali set*.

A set $\Lambda\subset\mathbb{R}$ is a *Luzin set* iff Λ is uncountable but $\Lambda\cap M$ is at most countable for every meager set $M\subset\mathbb{R}$. A set $S\subset\mathbb{R}$ is a *Sierpiński set* iff S is uncountable but $S\cap N$ is at most countable for every null set $N\subset\mathbb{R}$ ("null" in the sense of Lebesgue measure). A set $B\subset\mathbb{R}$ is a *Bernstein set* iff $B\cap P\neq\emptyset\neq P\setminus B$ for every perfect set $P\subset\mathbb{R}$. A *Burstin basis* is a Hamel basis which is also a Bernstein set. It is easy to see that $B\subset\mathbb{R}$ is a Burstin basis iff B is a Hamel basis and $B\cap P\neq\emptyset$ for every perfect $P\subset\mathbb{R}$.

By $\mathsf{AC}_\omega(\mathbb{R})$ we mean the statement that for all sequences $(A_n\colon n<\omega)$ such that $\emptyset\neq A_n\subset\mathbb{R}$ for all $n<\omega$ there is some choice function $f\colon\omega\to\mathbb{R}$, i.e., $f(n)\in A_n$ for all $n<\omega$.

D. Pincus and K. Prikry study the Cohen-Halpern-Lévy model H in [9]. The model H is obtained by adding a countable set of Cohen reals (say over L) without adding their enumeration; H does not satisfy $\mathsf{AC}_\omega(\mathbb{R})$. It is shown in [9] that there is a Luzin set in H, so that in ZF, the existence of a Luzin set does not even imply $\mathsf{AC}_\omega(\mathbb{R})$. [1, Theorems 1.7 and 2.1] show that in H there is a Bernstein set as well as a Hamel basis. As in ZF the existence of a Hamel basis implies the existence of a Vitali set, the latter also reproves Feferman's result (see [9]) according to which there is a Vitali set in H.

Therefore, in ZF the conjunction of the following statements (1), (3), and (5) (which in ZF implies (4)) does not yield $\mathsf{AC}_\omega(\mathbb{R})$.

(1) There is a Luzin set.

(2) There is a Sierpiński set.

(3) There is a Bernstein set.

(4) There is a Vitali set.

(5) There is a Hamel basis.

(6) There is a Burstin basis.

(2) is false in H, see [1, Lemma 1.6]. We aim to prove that in ZF, the conjunction of *all* of these statements does not imply $\mathsf{AC}_\omega(\mathbb{R})$, even if the respective sets are required to be projective. What we have at this point is:

Theorem 0.1 *There is a model of* ZF *plus* $\neg\mathsf{AC}_\omega(\mathbb{R})$ *in which the following hold true.*

(a) There is a Δ^1_2 *Luzin set.*

(b) There is a Δ^1_2 *Sierpiński set.*

(c) There is a Δ^1_3 *Bernstein set.*

(d) There is a Δ^1_3 *Hamel basis.*

1 Jensen's perfect set forcing, revisited.

In what follows, we shall mostly think of reals as elements of the Cantor space ${}^\omega 2$. We shall need a variant of the Cohen-Halpern-Lévy model. In order to construct our model, we need to introduce a variant of Jensen's variant of Sacks forcing, see [6] (see also [7, Definition 6.1]), which we shall call $\mathbb{P}$. The reason why we can't work with Jensen's forcing directly is that it does not seem to have the Sacks property (see e.g. [2, Definition 2.15]).

By way of notation, if $\mathbb{Q}$ is a forcing and $N > 0$ is any ordinal, then $\mathbb{Q}(N)$ denotes the finite support product of N copies of $\mathbb{Q}$, ordered component-wise. In this paper, we shall only consider $\mathbb{Q}(N)$ for $N \leq \omega$. If α is a limit ordinal, then $<_{J_\alpha}$ denotes the canonical well-ordering of J_α, see [10, Definition 5.14 and p. 79],[1] and $<_L = \bigcup\{<_{J_\alpha} : \alpha \text{ is a limit ordinal }\}$.

Let us work in L until further notice. Let us first define $(\alpha_\xi, \beta_\xi : \xi < \omega_1)$ as follows: α_ξ = the least $\alpha > \sup(\{\beta_{\bar\xi} : \bar\xi < \xi\})$ such that $J_\alpha \models \mathsf{ZFC}^-$,[2] and β_ξ = the least $\beta > \alpha_\xi$ such that $\rho_\omega(J_\beta) = \omega$ (see [10, Definition 11.22]; $\rho_\omega(J_\beta) = \omega$ is equivalent with $\mathcal{P}(\omega) \cap J_{\beta+\omega} \not\subset J_\beta$).

We shall also make use of a sequence $(f_\xi : \xi < \omega)$ which is defined as follows. Let $(\bar f_\xi : \xi < \omega)$ be defined by the following trivial recursion: $\bar f_\xi$ be the $<_L$-least f such that $f \in ({}^\omega J_{\omega_1} \cap J_{\omega_1}) \setminus \{\bar f_{\bar\xi} : \bar\xi < \xi\}$. Then if π denotes the Gödel pairing function, see [10, p. 35], we let $f_{\pi((\xi_1,\xi_2))} = \bar f_{\xi_1}$. We will then have that $f_\xi \in J_{\alpha_\xi}$ for all ξ, and for each $f \in ({}^\omega J_{\omega_1} \cap J_{\omega_1})$ the set of ξ such that $f = f_\xi$ is cofinal in ω_1.

Let us then define $(\mathbb{P}_\xi, \mathbb{Q}_\xi : \xi \leq \omega_1)$. Each $\mathbb{P}_\xi$ will consist of perfect trees $T \subset {}^{<\omega}2$ such that if $T \in \mathbb{P}_\xi$ and $s \in T$, then $T_s = \{t \in T : t \subset s \vee s \subset t\} \in \mathbb{P}_\xi$ as well.[3] Each $\mathbb{P}_\xi$ will be construed as a p.o. by stipulating $T \leq T'$ (T "is stronger than" T') iff $T \subset T'$. We will have that $\mathbb{P}_\xi \in J_{\alpha_\xi}$ and $\mathbb{P}_{\bar\xi} \subset \mathbb{P}_\xi$ whenever $\bar\xi \leq \xi \leq \omega_1$.

[1] The reader unfamiliar with the J-hierarchy may read L_α instead of J_α.

[2] Here, ZFC^- denotes ZFC without the power set axiom. Every J_α satisfies the strong form of AC according to which every set is the surjective image of some ordinal.

[3] We denote by $x \subset y$ the fact that x is a (not necessarily proper) subset of y.

To start with, let $\mathbb{P}_0$ be the set of all basic clopen sets $U_s = \{t \in {}^{<\omega}2 \colon t \subset s \vee s \subset t\}$, where $s \in {}^{\omega}2$. If $\lambda \leq \omega_1$ is a limit ordinal, then $\mathbb{P}_\lambda = \bigcup\{\mathbb{P}_\xi \colon \xi < \lambda\}$.

Now fix $\xi < \omega_1$, and suppose that $\mathbb{P}_\xi$ has already been defined. We shall define $\mathbb{Q}_\xi$ and $\mathbb{P}_{\xi+1}$.

Let $g_\xi \in {}^{\omega}J_{\alpha_\xi}$ be the following ω-sequence. If there is some $N < \omega$ such that f_ξ is an ω-sequence of subsets of $\mathbb{P}_\xi(N)$, each of which is predense in $\mathbb{P}_\xi(N)$, then for each $n < \omega$ let $g_\xi(n)$ be the open dense set

$$\{(T_1, \ldots, T_N) \in \mathbb{P}_\xi(N) \colon \exists (T_1', \ldots, T_N') \in f_\xi(n)\, (T_1, \ldots, T_N) \leq (T_1', \ldots, T_N')\},$$

and write $N_\xi = N$. Otherwise we just set $g_\xi(n) = \mathbb{P}_\xi(1)$ for each $n < \omega$, and write $N_\xi = 1$. Let d_ξ be the $<_{J_{\beta_\xi+\omega}}$-least $d \in {}^{\omega\times\omega}(\mathcal{P}(\mathbb{P}_\xi) \cap J_{\alpha_\xi}) \cap J_{\beta_\xi+\omega}$ such that

(i) for each $(n, N) \in \omega \times \omega$, $d(n, N)$ is an open dense subset of $\mathbb{P}_\xi(N)$ which exists in J_{β_ξ},

(ii) for each $N < \omega$ and each open dense subset D of $\mathbb{P}_\xi(N)$ which exists in J_{β_ξ} there is some $n < \omega$ with $d(n, N) \subset D$,

(iii) $d(n, N_\xi) \subset g_\xi(n)$ for each $n < \omega$, and

(iv) $d(n+1, N) \subset d(n, N)$ for each $(n, N) \in \omega \times \omega$.

Let us now look at the collection of all systems $(T_s^m \colon m < \omega, s \in {}^{<\omega}2)$ with the following properties.

(a) $T_s^m \in \mathbb{P}_\xi$ for all m, s,

(b) for each $T \in \mathbb{P}_\xi$ there are infinitely many $m < \omega$ with $T_\emptyset^m = T$,

(c) $T_t^m \leq T_s^m$ for all m, $t \supset s$,

(d) $\mathrm{stem}(T^m_{s^\frown 0})$ and $\mathrm{stem}(T^m_{s^\frown 1})$ are incompatible elements of T_s^m for all m, s,

(e) if $(m, s) \neq (m', s')$, where $m, m' < n$ and $\mathrm{lh}(s) = \mathrm{lh}(s') = n + 1$ for some n, then $\mathrm{stem}(T_s^m)$ and $\mathrm{stem}(T_{s'}^{m'})$ are incompatible, and

(f) for all $N \leq n < \omega$ and all pairwise different (m_1, s_1), …, (m_N, s_N) with m_1, …, $m_N < n$ and s_1, …, $s_N \in {}^{n+1}2$,

$$(T_{s_1}^{m_1}, \ldots, T_{s_N}^{m_N}) \in d_\xi(n, N).$$

It is easy to work in $J_{\beta_\xi+\omega}$ and construct initial segments $(T_s^m \colon m < \omega, s \in {}^{<\omega}2, \mathrm{lh}(s) \leq n)$ of such a system by induction on $n < \omega$. Notice that (f) formulates a constraint only for m_1, …, $m_N < \mathrm{lh}(s_1) - 1 = \ldots = \mathrm{lh}(s_N) - 1$, and writing $n = \mathrm{lh}(s_1) - 1$, there are $\sum_{N=1}^{n} \frac{(n\cdot 2^{n+1})!}{(n\cdot 2^{n+1}-N)!}$ (i.e., finitely many) such constraints.

We let $(T_{s,\xi}^m \colon m < \omega, s \in {}^{<\omega}2)$ be the $<_{\beta_\xi+\omega}$-least such system $(T_s^m \colon m < \omega, s \in {}^{<\omega}2)$. For every $m < \omega$, $s \in {}^{<\omega}2$, we let

$$A_{s,\xi}^m = \bigcap_{n \geq \mathrm{lh}(s)} \Big(\bigcup_{\substack{t \supset s \\ \mathrm{lh}(t) = n}} T_t^m \Big) = \{\mathrm{stem}(T_{t,\xi}^m) \restriction k \colon t \supset s, k < \omega\}.$$

Notice that (e) implies that

(1) $\quad A^m_{s,\xi} \cap A^{m'}_{s',\xi}$ is finite, unless $m = m'$ and $s \subset s'$ or $s' \subset s$.

(1) will imply that $A^m_{s,\xi}$ and $A^{m'}_{s',\xi}$ will be incompatible in every $\mathbb{P}_\eta$, $\eta > \xi$, unless $m = m'$ and $s \subset s'$ or $s' \subset s$.

We set $\mathbb{Q}_\xi = \{A^m_{s,\xi} \colon m < \omega, s \in {}^{<\omega}2\}$. Finally, we set $\mathbb{P}_{\xi+1} = \mathbb{P}_\xi \cup \mathbb{Q}_\xi$.

Lemma 1.1 *Let $N < \omega$, $\xi < \omega_1$.*

$$D = \{(T_1, \dots, T_N) \in \mathbb{Q}_\xi(N) \colon \mathit{stem}(T_i) \perp \mathit{stem}(T_j) \text{ for } i \neq j\}$$

is dense in $\mathbb{P}_{\xi+1}(N)$.[4]

Proof. Let $(T_1, \dots, T_N) \in \mathbb{P}_{\xi+1}(N)$. For $i \in \{1, \dots, N\}$ such that $T_i \in \mathbb{P}_\xi$ pick some $m_i < \omega$ such that $T_i = T^{m_i}_{\emptyset,\xi}$, and write $s_i = \emptyset$. This is possible by (b). If $i \in \{1, \dots, N\}$ is such that $T_i \in \mathbb{Q}_\xi$, then say $T_i = A^{m_i}_{s_i,\xi}$. Now pick $n > \max(\{m_1, \dots, m_N\})$ and $t_1 \supset s_1$, …, $t_N \supset s_N$ such that $\mathrm{lh}(t_1) = \ldots = \mathrm{lh}(t_N) = n + 1$ and the (m_i, t_i) are pairwise different.

Then by (e) the finite sequences $\mathrm{stem}(T^{m_i}_{t_i,\xi})$ are pairwise incompatible, so that by $A^{m_i}_{t_i,\xi} \leq T^{m_i}_{t_i,\xi}$, the $A^{m_i}_{t_i,\xi}$ are pairwise incompatible. But then $(A^{m_1}_{t_i,\xi}, \dots, A^{m_N}_{t_N,\xi}) \in D$ and $(A^{m_1}_{t_i,\xi}, \dots, A^{m_N}_{t_N,\xi}) \leq (T_1, \dots, T_N)$. $\square$

Lemma 1.2 (Sealing) *Let $N < \omega$, $\xi < \omega_1$. If $D \in J_{\beta_\xi}$ is predense in $\mathbb{P}_\xi(N)$, then D is predense in all $\mathbb{P}_\eta(N)$, $\eta \geq \xi$, $\eta \leq \omega_1$.*

Proof by induction on η. The cases $\eta = \xi$ and η being a limit ordinal are trivial. Suppose $\eta \geq \xi$, $\eta < \omega_1$, and D is predense in $\mathbb{P}_\eta(N)$. Write $D' = \{(T_1, \dots, T_N) \in \mathbb{P}_\eta(N) \colon \exists (T'_1, \dots, T'_N) \in D\, (T_1, \dots, T_N) \leq (T'_1, \dots, T'_N)\}$. As $\beta_\xi \leq \beta_\eta$, $D' \in J_{\beta_\xi}$ and by (ii) and (iv) there is some $n_0 < \omega$ with $d_\eta(n, N) \subset D'$ for every $n > n_0$.

To show that D' (and hence D) is predense in $\mathbb{P}_{\eta+1}(N)$, by Lemma 1.1 it suffices to show that for all $(T_1, \dots, T_N) \in \mathbb{Q}_\eta(N)$ there is some $(T'_1, \dots, T'_N) \in \mathbb{Q}_\eta(N)$, $(T'_1, \dots, T'_N) \leq (T_1, \dots, T_N)$, and $(T'_1, \dots, T'_N)$ is below some element of D'.

So let $(A^{m_1}_{s_1,\eta}, \dots, A^{m_N}_{s_N,\eta}) \in \mathbb{Q}_\eta(N)$ be arbitrary. Let

$$n > \max(\{n_0, N-1, m_1, \dots, m_N, \mathrm{lh}(s_1), \dots, \mathrm{lh}(s_N)\}),$$

and let $t_1 \supset s_1$, …, $t_N \supset s_N$ be such that $\mathrm{lh}(t_1) = \ldots = \mathrm{lh}(t_N) = n + 1$. By increasing n further if necessary, we may certainly assume that t_1, …, t_N are picked in such a way that (m_1, t_1), …, (m_N, t_N) are pairwise different. Then

$$(T^{m_1}_{t_1,\eta}, \dots, T^{m_N}_{t_N,\eta}) \in d_\eta(n, N) \subset D'$$

by (f). But

$$(A^{m_1}_{t_1,\eta}, \dots, A^{m_N}_{t_N,\eta}) \leq (T^{m_1}_{t_1,\eta}, \dots, T^{m_N}_{t_N,\eta}),$$

and also

$$(A^{m_1}_{t_1,\eta}, \dots, A^{m_N}_{t_N,\eta}) \leq (A^{m_1}_{s_1,\eta}, \dots, A^{m_N}_{s_N,\eta}),$$

which means that $(A^{m_1}_{s_1,\eta}, \dots, A^{m_N}_{s_N,\eta})$ is compatible with an element of D'. $\square$

[4] Here, $\mathrm{stem}(T_i) \perp \mathrm{stem}(T_j)$ means that the stem of T_i is incompatible with the stem of T_j.

Corollary 1.3 *Let $N < \omega$, $\xi < \omega_1$.*

$$\{(T_1, \ldots, T_N) \in \mathbb{Q}_\xi(N) \colon \textit{stem}(T_i) \perp \textit{stem}(T_j) \textit{ for } i \neq j\}$$

is predense in $\mathbb{P}(N)$.

Lemma 1.4 *Let $N < \omega$. $\mathbb{P}(N)$ has the c.c.c.*

Proof. Let $A \subset \mathbb{P}(N)$ be a maximal antichain, $A \in L$. Let $j \colon J_\beta \to J_{\omega_2}$ be elementary and such that $\beta < \omega_1$ and $\{\mathbb{P}, A\} \subset \mathrm{ran}(j)$. Write $\xi = \mathrm{crit}(j)$. We have that $j^{-1}(\mathbb{P}(N)) = \mathbb{P}(N) \cap J_\xi = \mathbb{P}_\xi(N)$ and $j^{-1}(A) = A \cap J_\xi = A \cap \mathbb{P}_\xi(N) \in J_\beta$ is a maximal antichain in $\mathbb{P}_\xi(N)$. Moreover, $\beta_\xi > \beta$, so that by Lemma 1.3 $A \cap \mathbb{P}_\xi(N)$ is predense in $\mathbb{P}(N)$. This means that $A = A \cap \mathbb{P}_\xi$ is countable. $\square$

Lemma 1.5 *Let $N < \omega$. $(c_1, \ldots c_N) \in {}^N({}^\omega 2)$ is $\mathbb{P}(N)$-generic over L iff for all $\xi < \omega_1$ there is an injection $t \colon \{1, \ldots, N\} \to \mathbb{Q}_\xi$ such that for all $i \in \{1, \ldots, N\}$, $c_i \in [t(i)]$.*

Proof. "$\Longrightarrow$": This readily follows from Corollary 1.3.

"$\Longleftarrow$": Let $A \subset \mathbb{P}(N)$ be a maximal antichain, $A \in L$. By Lemma 1.4, we may certainly pick some $\xi < \omega_1$ with $A \subset \mathbb{P}_\xi(N)$ and $A \in J_{\alpha_\xi}$. Say n_0 is such that $d_\xi(n, N) \subset \{(T_1, \ldots, T_N) \in \mathbb{P}_\xi \colon \exists (T'_1, \ldots, T'_N) \in A \, (T_1, \ldots, T_N) \leq (T'_1, \ldots, T'_N)\}$ for all $n \geq n_0$. By our hypothesis, we may pick pairwise different (m_1, s_1), $\ldots$, (m_N, s_N) with $\mathrm{lh}(s_1) = \ldots = \mathrm{lh}(s_N) = n + 1$ for some $n \geq n_0$ and $c_i \in [T^{m_i}_{s_i}, \xi]$ for all $i \in \{1, \ldots, N\}$. But then $(T^{m_1}_{s_i,\xi}, \ldots, T^{m_N}_{s_N})$ is below an element of A, which means that the generic filter given by $(c_1, \ldots, c_N)$ meets A. $\square$

Corollary 1.6 *Let $N < \omega$, and let $(c_1, \ldots c_N) \in {}^N({}^\omega 2)$ be $\mathbb{P}(N)$-generic over L. If $x \in L[(c_1, \ldots c_N)]$ is $\mathbb{P}$-generic over L, then $x \in \{c_1, \ldots c_N\}$.*

Proof. If $x \in L[(c_1, \ldots c_N)]$ is $\mathbb{P}$-generic over L, then $(c_1, \ldots c_N, x) \in {}^{N+1}({}^\omega 2)$ is $\mathbb{P}(N+1)$-generic over L, hence $x \notin L[(c_1, \ldots c_N)]$. Contradiction! $\square$

Corollary 1.7 *Let $N < \omega$, and let $(c_1, \ldots c_N) \in {}^N({}^\omega 2)$ be $\mathbb{P}(N)$-generic over L. Then inside $L[(c_1, \ldots c_N)]$, $\{c_1, \ldots c_N\}$ is a (lightface) Π^1_2 set.*

Proof. Let $\varphi(x)$ express that for all $\xi < \omega_1$ there is some $T \in \mathbb{Q}_\xi$ such that $x \in [T]$. The formula $\varphi(x)$ may be written in a Π^1_2 fashion, and it defines $\{c_1, \ldots c_N\}$ inside $L[(c_1, \ldots c_N)]$. $\square$

Lemma 1.8 (Sacks property) *Let $N < \omega$, and let g be $\mathbb{P}(N)$-generic over L. For each $f \colon \omega \to \omega$, $f \in L[a]$, there is some $g \in L$ with domain ω such that for each $n < \omega$, $f(n) \in g(n)$ and*[5] $\mathrm{Card}(g(n)) \leq (n+1) \cdot 2^{n+1}$.

Proof. Let $\tau \in L^{\mathbb{P}(N)}$, $\tau^g = f$. Let $(A_n \colon n < \omega) \in L$ be such that for each n, A_n is a maximal antichain of $\vec{T} \in \mathbb{P}(N)$ such that $\exists m < \omega \, \vec{T} \Vdash \tau(\check{n}) = \check{m}$. We may pick some $\xi < \omega_1$ such that $\bigcup\{A_n \colon n < \omega\} \subset \mathbb{P}_\xi(N)$ and $(A_n \colon n < \omega) = f_\xi$.

[5] In what follows, the only thing that will matter is that the bound on $\mathrm{Card}(g(n))$ only depends on n and not on the particular g.

By Lemma 1.5, there are pairwise different (m_1, s_1), ..., (m_N, s_N) such that

$$(A^{m_1}_{s_1,\xi}, \dots, A^{m_N}_{s_N,\xi}) \in g.$$

Let

$$n > \max(\{N-1, m_1, \dots, m_N, \mathrm{lh}(s_1), \dots, \mathrm{lh}(s_N)\}).$$

If $t_1 \supset s_1$, ..., $t_N \supset t_N$ are such that $\mathrm{lh}(t_1) = \dots = \mathrm{lh}(t_N) = n+1$, then $(T^{m_1}_{t_1,\xi}, \dots, T^{m_N}_{t_N,\xi}) \in d_\xi(n, N) \subset A_n$, so that also

$$\exists m < \omega\, (T^{m_1}_{t_1,\xi}, \dots, T^{m_N}_{t_N,\xi}) \Vdash \tau(\check{n}) = \check{m}.$$

Therefore, if we let

$$g(n) = \{m < \omega \colon \exists t_1 \supset s_1, \dots \exists t_N \supset t_N\, (\mathrm{lh}(t_1) = \dots = \mathrm{lh}(t_N) = n+1 \wedge (T^{m_1}_{t_1,\xi}, \dots, T^{m_N}_{t_N,\xi}) \Vdash \tau(\check{n}) = \check{m})\},$$

then $(A^{m_1}_{s_1,\xi}, \dots, A^{m_N}_{s_N,\xi}) \Vdash \tau(\check{n}) \in (g(n))\check{}$, hence $f(n) \in g(n)$, and $\mathrm{Card}(g(n)) = N \cdot 2^{n+1} \leq (n+1) \cdot 2^{n+1}$ for all but finitely many n. $\square$

2 The variant of the Cohen-Helpern-Lévy model.

Let us force with $\mathbb{P}(\omega)$ over L, and let g be a generic filter. Let c_n, $n < \omega$, denote the Jensen reals which g adds. Let us write $A = \{c_n \colon n < \omega\}$ for the set of those Jensen reals. The model

$$H = H(L) = \mathsf{HOD}^{L[g]}_{A \cup \{A\}}$$

of all sets which inside $L[g]$ are hereditarily definable from parameters in $\mathrm{OR} \cup A \cup \{A\}$ is the variant of the Cohen–Halpern–Lévy model (over L) which we shall work with. For the case of Jensen's original forcing this model was first considered in [4].

For any finite $a \subset A$, we write $L[a]$ for the model constructed from the finitely many reals in a.

Lemma 2.1 *Inside H, A is a (lightface) Π^1_2 set.*

Proof. Let $\varphi(-)$ be the Π^1_2 formula from the proof of Lemma 1.7. If $H \models \varphi(x)$, $x \in L[a]$, $a \in [A]^{<\omega}$, then $L[a] \models \varphi(x)$ by Shoenfield, so $x \in a \subset A$. On the other hand, if $c \in A$, then $L[c] \models \varphi(c)$ and hence $H \models \varphi(c)$ again by Shoenfield. $\square$

Fixing some Gödelization of formulae (or some enumeration of all the rud functions, resp.) at the outset, each $L[a]$, $a \in [A]^{<\omega}$, comes with a unique canonical global well–ordering $<_a$ of $L[a]$ by which we mean the one which is induced by the *natural* order of the elements of a and the fixed Gödelization device in the usual fashion. The assignment $a \mapsto <_a$, $a \in [A]^{<\omega}$, is hence in H.[6] This is a crucial fact.

Let us fix a bijection

(2) $$e \colon \omega \to \omega \times \omega,$$

and let us write $((n)_0, (n)_1) = e(n)$.

We shall also make use the following. Cf. [1, Lemma 1.2].

[6] More precisely, the ternary relation consisting of all (a, x, y) such that $x <_a y$ is definable over H.

Lemma 2.2 *(1) Let $a \in [A]^{<\omega}$ and $X \subset L[a]$, $X \in H$, say $X \in \mathsf{HOD}^{L[g]}_{b \cup \{A\}}$, where $b \supseteq a$, $b \in [A]^{<\omega}$. Then $X \in L[b]$.*

(2) There is no well–ordering of the reals in H.

(3) A has no countable subset in H.

(4) $[A]^{<\omega}$ has no countable subset in H.

Proof sketch. (1) Every permutation $\pi \colon \omega \to \omega$ induces an automorphism e_π of $\mathbb{P}(\omega)$ by sending p to q, where $q(\pi(n)) = p(n)$ for all $n < \omega$. It is clear that no e_π moves the canonical name for A, call it $\dot{A}$. Let us also write $\dot{c}_n$ for the canonical name for c_n, $n < \omega$. Now if a, and b are as in the statement of (1), say $b = \{c_{n_1}, \ldots, c_{n_k}\}$, if p, $q \in \mathbb{P}(\omega)$, if $\pi \restriction \{n_1, \ldots, n_k\} = \mathrm{id}$, $p \restriction \{n_1, \ldots, n_k\}$ is compatible with $q \restriction \{n_1, \ldots, n_k\}$, and $\mathrm{supp}(\pi(p)) \cap \mathrm{supp}(q) \subseteq \{n_1, \ldots, n_k\}$, if $x \in L$, if α_1, …, α_m are ordinals, and if φ is a formula, then

$$\begin{aligned} p \Vdash^{\mathbb{P}(\omega)}_L \ \varphi(\check{x}, \check{\alpha}_1, \ldots \check{\alpha}_m, \dot{c}_{n_1}, \ldots \dot{c}_{n_k}, \dot{A}) \quad &\Longleftrightarrow \\ \pi(p) \Vdash^{\mathbb{P}(\omega)}_L \ \varphi(\check{x}, \check{\alpha}_1, \ldots \check{\alpha}_m, \dot{c}_{n_1}, \ldots \dot{c}_{n_k}, \dot{A}) & \end{aligned}$$

and $\pi(p)$ is compatible with q, so that the statement $\varphi(\check{x}, \check{\alpha}_1, \ldots \check{\alpha}_m, \dot{c}_{n_1}, \ldots \dot{c}_{n_k}, \dot{A})$ will be decided by conditions $p \in \mathbb{P}(\omega)$ with $\mathrm{supp}(p) \subseteq \{n_1, \ldots, n_k\}$. But every set in $L[b]$ is coded by a set of ordinals, so if X is as in (1), this shows that $X \in L[b]$.

(2) Every real is a subset of L. Hence by (1), if $L[g]$ had a well–ordering of the reals in $\mathsf{HOD}^{L[g]}_{a \cup \{A\}}$, some $a \in [A]^{<\omega}$, then every real of H would be in $L[a]$, which is nonsense.

(3) Assume that $f \colon \omega \to A$ is injective, $f \in H$. Let $x \in {}^\omega\omega$ be defined by $x(n) = f((n)_0)((n)_1)$, so that $x \in H$. By (1), $x \in L[a]$ for some $a \in [A]^{<\omega}$. But then $\mathrm{ran}(f) \subset L[a]$, which is nonsense, as there is some $n < \omega$ such that $c_n \in \mathrm{ran}(f) \setminus a$.

(4) This readily follows from (3). □ (Lemma 2.2)

Let us recall another standard fact.

(3) $$\text{If } a,\, b \in [A]^{<\omega}, \text{ then } L[a] \cap L[b] = L[a \cap b].$$

To see this, let us assume without loss of generality that $a \setminus b \neq \emptyset \neq b \setminus a$, and say $a \setminus b = \{c_n \colon n \in I\}$ and $b \setminus a = \{c_n \colon n \in J\}$, where I and J are non–empty disjoint finite subsets of ω. Then $a \setminus b$ and $b \setminus a$ are mutually $\mathbb{P}(I)$- and $\mathbb{P}(J)$-generic over $L[a \cap b]$. But then $L[a] \cap L[b] = L[a \cap b][a \setminus b] \cap L[a \cap b][b \setminus a] = L[a \cap b]$, cf. [10, Problem 6.12].

For any $a \in [A]^{<\omega}$, we write $\mathbb{R}_a = \mathbb{R} \cap L[a]$ and $\mathbb{R}^+_a = \mathbb{R}_a \setminus \bigcup\{\mathbb{R}_b \colon b \subsetneq a\}$. $(\mathbb{R}^+_a \colon a \in [A]^{<\omega})$ is a partition of $\mathbb{R}$: By Lemma 2.2 (1),

(4) $$\mathbb{R} \cap H = \bigcup\{\mathbb{R}^+_a \colon a \in [A]^{<\omega}\},$$

and $\mathbb{R}_a \cap \mathbb{R}_b = \mathbb{R}_{a \cap b}$ by (3), so that

(5) $$\mathbb{R}^+_a \cap \mathbb{R}^+_b = \emptyset \text{ for } a, b \in [A]^{<\omega}, a \neq b.$$

For $x \in \mathbb{R}$, we shall also write $a(x)$ for the unique $a \in [A]^{<\omega}$ such that $x \in \mathbb{R}^+_a$, and we shall write $\#(x) = \mathrm{Card}(a(x))$.

Adrian Mathias showed that in the original Cohen–Halpern–Lévy model there is an definable function which assigns to each x an ordering $<_x$ such that $<_x$ is a well–ordering iff x can be well–ordered, cf. [8, p. 182]. The following is a special simple case of this, adapted to the current model H.

Lemma 2.3 *(A. Mathias) In H, the union of countably many countable sets of reals is countable.*

Proof. Let us work inside H. Let $(A_n : n < \omega)$ be such that for each $n < \omega$, $A_n \subset \mathbb{R}$ and there exists some surjection $f : \omega \to A_n$. For each such pair n, f let $y_{n,f} \in {}^\omega\omega$ be such that $y_{n,f}(m) = f((m)_0)((m)_1)$. If $a \in [A]^{<\omega}$ and $y_{n,f} \in \mathbb{R}_a$, then $A_n \in L[a]$. By (3), for each n there is a unique $a_n \in [A]^{<\omega}$ such that $A_n \in L[a_n]$ and $b \supset a_n$ for each $b \in [A]^{<\omega}$ such that $A_n \in L[b]$. Notice that A_n is also countable in $L[a_n]$.

Using the function $n \mapsto a_n$, an easy recursion yields a surjection $g : \omega \to \bigcup\{a_n : n < \omega\}$: first enumerate the finitely many elements of a_0 according to their natural order, then enumerate the finitely many elements of a_1 according to their natural order, etc. As A has no countable subset, $\bigcup\{a_n : n < \omega\}$ must be finite, say $a = \bigcup\{a_n : n < \omega\} \in [A]^{<\omega}$. But then $\{A_n : n < \omega\} \subset L[a]$. (We don't claim $(A_n : n < \omega) \in L[a]$.)

For each $n < \omega$, we may now let f_n the $<_a$–least surjection $f : \omega \to A_n$. Then $f(n) = f_{(n)_0}((n)_1)$ for $n < \omega$ defines a surjection from ω onto $\bigcup\{A_n : n < \omega\}$, as desired. □ (Lemma 2.3)

The following is not true in the original Cohen–Halpern–Lévy model. Its proof exploits the Sacks property, Lemma 1.8.

Lemma 2.4 *(1) Let $M \in H$ be a null set in H. There is then a G_δ null set M' with $M' \supset M$ whose code is in L.*

(2) Let $M \in H$ be a meager set in H. There is then an F_σ meager set M' with $M' \supset M$ whose code is in L.

Proof. (1) Let $M \in H$ be a null set in H.

Let us work in H. Let $(\epsilon_n : n < \omega)$ be any sequence of positive reals. Let $\bigcup_{s \in X} U_s \supset H$, where $X \subset {}^{<\omega}2$ and $\mu(\bigcup\{U_s : s \in X\}) \leq \epsilon_0$.[7] Let $e : \omega \to X$ be onto. Let $(k_n : n < \omega)$ be defined by: $k_n =$ the smallest k (strictly bigger than k_{n-1} if $n > 0$) such that $\mu(\bigcup\{U_s : s \in e"\omega \setminus k\}) \leq \epsilon_n$. Write $k_{-1} = 0$. We then have that $\mu(\bigcup\{U_s : s \in e"[k_{n-1}, k_n)\}) \leq \epsilon_n$ for every $n < \omega$.

Now fix $\epsilon > 0$. Let

$$\epsilon_n = \frac{\epsilon}{n \cdot 2^{2n+2}},$$

and let $(k_n : n < \omega)$ and $e : \omega \to {}^{<\omega}2$ be such that $\bigcup_{s \in X} U_s \supset H$ and $\mu(\bigcup\{U_s : s \in e"[k_{n-1}, k_n)\}) \leq \epsilon_n$ for every $n < \omega$. We may now apply Lemma 1.8 inside $L[a]$ for some $a \in [A]^{<\omega}$ such that $\{e, (k_n : n < \omega)\} \subset L[a]$ and find a function $g \in L$ with domain ω such that for each $n < \omega$, $g(n)$ is a finite union U_n of basic open sets such that $\{U_s : s \in e"[k_{n-1}, k_n)\} \subset U_n$ and $\mu(U_n) \leq \frac{1}{2^{n+1}}$. But then $\mathcal{O} = \bigcup\{O_n : n < \omega\} \supset M$ is open, $\mathcal{O}$ is coded in L (i.e., there is $Y \in L$, $Y \subset {}^{<\omega}2$, with $\mathcal{O} = \bigcup\{U_s : s \in Y\}$), and $\mu(\mathcal{O}) \leq \epsilon$.

[7] Here, μ denotes Lebesge measure.

We may hence for every $n < \omega$ let $\mathcal{O}_n$ be an open set with $\mathcal{O}_n \supset M$, $\mu(\mathcal{O}_n) \leq \frac{1}{n+1}$, and whose code in L is $<_L$-least among all the codes giving such a set. Then $\bigcap\{\mathcal{O}_n : n < \omega\}$ is a G_δ null set with code in L and which covers M.

(2) Let $M \in H$ be a meager set in H, say $M = \bigcup\{N_n : n < \omega\}$, where each N_n is nowhere dense.

Let us again work in H. It is easy to verify that a set $P \subset {}^\omega 2$ is nowhere dense iff there is some $z \in {}^\omega 2$ and some strictly increasing $(k_n : n < \omega)$ such that for all $n < \omega$,

$$\{x \in {}^\omega 2 : x \restriction [k_n, k_{n+1}) = z \restriction [k_n, k_{n+1})\} \cap P = \emptyset. \tag{6}$$

Look at $f : \omega \to \omega$, where $f(m) = k_{n+1}$ for the least n with $m \leq k_n$. We may first apply Lemma 1.8 inside $L[a]$ for some $a \in [A]^{<\omega}$ such that $f \in L[a]$ and get a function $g : \omega \to \omega$, $g \in L$, such that $g(m) \geq f(m)$ for all $m < \omega$. Write $\ell_0 = 0$ and $\ell_{n+1} = g(\ell_n)$, so that for each n there is some n' with

$$\ell_n \leq k_{n'} < k_{n'+1} \leq \ell_{n+1}. \tag{7}$$

Define $e : \omega \to \omega$ by $e(n) = \sum_{q=0}^{n} (q+1) \cdot 2^{q+1}$. We may now apply Lemma 1.8 inside $L[a]$ for some $a \in [A]^{<\omega}$ such that $f \in L[a]$ and get some $n \mapsto (z_i^n : i \leq (n+1) \cdot 2^{n+1})$ inside L such that for all n, i, $z_i^n : e(n) \to 2$, and for all n there is some i with $z \restriction e(n) = z_i^n$. From this we get some $z' : \omega \to \omega$, $z' \in L$, such that for all n there is some n' with $z' \restriction [\ell_{n'}, \ell_{n'+1}) = z \restriction [\ell_{n'}, \ell_{n'+1})$. But then, writing

$$D = \{x \in {}^\omega 2 : \exists n \, x \restriction [\ell_{e(n)}, \ell_{e(n+1)}) = z' \restriction [\ell_{e(n)}, \ell_{e(n+1)})\}, \tag{8}$$

$D \in L$, and D is open and dense.

We may hence for every $n < \omega$ let $\mathcal{O}_n$ be an open dense set with $\mathcal{O}_n \cap N_n = \emptyset$, whose code in L is $<_L$-least among all the codes giving such a set. Then $\bigcup\{{}^\omega 2 \setminus \mathcal{O}_n : n < \omega\}$ is an F_σ meager set with code in L and which covers M. □

Corollary 2.5 *In H, there is a Δ^1_2 Sierpiński set as well as a Δ^1_2 Luzin set.*

Proof. There is a Δ^1_2 Luzin set in L. By Lemma 2.4 (2), any such set is still a Luzin set in H. The same is true with "Luzin" replaced by "Sierpiński" and Lemma 2.4 (2) replaced by Lemma 2.4 (1). □

Lemma 2.6 *In H, there is a Δ^1_3 Bernstein set.*

Proof. In this proof, let us think of reals as elements of the Cantor space ${}^\omega 2$. Let us work in H.

We let

$$B = \{x \in \mathbb{R} : \exists \text{ even } n \, (2^n < \#(x) \leq 2^{n+1})\} \quad \text{and}$$
$$B' = \{x \in \mathbb{R} : \exists \text{ odd } n \, (2^n < \#(x) \leq 2^{n+1})\}.$$

Obviously, $B \cap B' = \emptyset$.

Let $P \subset \mathbb{R}$ be perfect. We aim to see that $P \cap B \neq \emptyset \neq P \cap B'$.

Say $P = [T] = \{x \in {}^\omega 2 \colon \forall n\, x \restriction n \in T\}$, where $T \subseteq {}^{<\omega}2$ is a perfect tree. Modulo some fixed natural bijection ${}^{<\omega}2 \leftrightarrow \omega$, we may identify T with a real. By (4), we may pick some $a \in [A]^{<\omega}$ such that $T \in L[a]$. Say $\mathrm{Card}(a) < 2^n$, where n is even.

Let $b \in [A]^{2^{n+1}}$, $b \supset a$, and let $x \in \mathbb{R}_b^+$. In particular, $\#(x) = 2^{n+1}$. It is easy to work in $L[b]$ and construct some $z \in [T]$ such that $x \leq_T z \oplus T$,[8] e.g., arrange that if $z \restriction m$ is the k^{th} splitting node of T along z, where $k \leq m < \omega$, then $z(m) = 0$ if $x(k) = 0$ and $z(m) = 1$ if $x(k) = 1$.

If we had $\#(z) \leq 2^n$, then $\#(z \oplus T) \leq \#(z) + \#(T) < 2^n + 2^n = 2^{n+1}$, so that $\#(x) < 2^{n+1}$ by $x \leq_T z \oplus T$. Contradiction! Hence $\#(z) > 2^n$. By $z \in L[b]$, $\#(z) \leq 2^{n+1}$. Therefore, $z \in P \cap B$.

The same argument shows that $P \cap B' \neq \emptyset$. B (and also B') is thus a Bernstein set.

We have that $x \in B$ iff

$$\exists a \in [A]^{<\omega}\, \exists \text{ even } n\, \exists J_\alpha[a]$$
$$(x \in J_\alpha[a] \wedge 2^n < \mathrm{Card}(a) \leq 2^{n+1} \wedge \forall b \subsetneq a\, \forall J_\beta[b] x \notin J_\beta[b]),$$

which is true iff

$$\forall a \in [A]^{<\omega}\, \forall J_\alpha[a]\, (x \in J_\alpha[a] \rightarrow \exists a' \subset a\, \exists \text{ even } n\, \exists J_{\alpha'}[a']$$
$$(x \in J_{\alpha'}[a'] \wedge 2^n < \mathrm{Card}(a) \leq 2^{n+1} \wedge \forall b \subsetneq a'\, \forall J_\beta[b] x \notin J_\beta[b])).$$

By Lemma 2.1, this shows that B is Δ^1_3. $\square$

Recall that for any $a \in [A]^{<\omega}$, we write $\mathbb{R}_a = \mathbb{R} \cap L[a]$. Let us now also write $\mathbb{R}_{<a} = \mathrm{span}(\bigcup\{\mathbb{R}_b \colon b \subsetneq a\})$, and $\mathbb{R}^*_a = \mathbb{R}_a \setminus \mathbb{R}_{<a}$. In particular, $\mathbb{R}_{<\emptyset} = \{0\}$ by our above convention that $\mathrm{span}(\emptyset) = \{0\}$, and $\mathbb{R}^*_\emptyset = (\mathbb{R} \cap L) \setminus \{0\}$.

The proof of Claim 2.8 below will show that

$$\mathbb{R} \cap H = \mathrm{span}(\bigcup\{\mathbb{R}^*_a \colon a \in [A]^{<\omega}\}). \tag{9}$$

Also, we have that $\mathbb{R}^*_a \subset \mathbb{R}^+_a$, so that by (5),

$$\mathbb{R}^*_a \cap \mathbb{R}^*_b = \emptyset \text{ for } a, b \in [A]^{<\omega}, a \neq b. \tag{10}$$

Lemma 2.7 *In H, there is a Δ^1_3 Hamel basis.*

Proof. We call $X \subset \mathbb{R}^*_a$ *linearly independent over* $\mathbb{R}_{<a}$ iff whenever

$$\sum_{n=1}^{m} q_n \cdot x_n \in \mathbb{R}_{<a},$$

where $m \in \mathbb{N}$, $m \geq 1$, and $q_n \in \mathbb{Q}$ and $x_n \in X$ for all n, $1 \leq n \leq m$, then $q_1 = \ldots = q_m = 0$. In other words, $X \subset \mathbb{R}^*_a$ is linearly independent over $\mathbb{R}_{<a}$ iff

$$\mathrm{span}(X) \cap \mathbb{R}_{<a} = \{0\}.$$

[8] Here, $(x \oplus y)(2n) = x(n)$ and $(x \oplus y)(2n+1) = y(n)$, $n < \omega$.

We call $X \subset \mathbb{R}^*_a$ *maximal linearly independent over* $\mathbb{R}_{<a}$ iff X is linearly independent over $\mathbb{R}_{<a}$ and no $Y \supsetneq X$, $Y \subset \mathbb{R}^*_a$ is still linearly independent over $\mathbb{R}_{<a}$. In particular, $X \subset \mathbb{R}^*_\emptyset = (\mathbb{R} \cap L) \setminus \{0\}$ is linearly independent over $\mathbb{R}_{<\emptyset} = \{0\}$ iff X is a Hamel basis for $\mathbb{R} \cap L$.

For any $a \in [A]^{<\omega}$, we let $b_a = \{x^a_i \colon i < \theta^a\}$, some $\theta^a \leq \omega_1$, be the unique set such that

(i) for each $i < \theta^a$, x^a_i is the $<_a$-least $x \in \mathbb{R}^*_a$ such that $\{x^a_j \colon j < i\} \cup \{x\}$ is linearly independent over $\mathbb{R}_{<a}$, and

(ii) b_a is maximal linearly independent over $\mathbb{R}_{<a}$.

By the above crucial fact, the function $a \mapsto b_a$ is well–defined and *exists inside* H. In particular,

$$B = \bigcup \{b_a \colon a \in [A]^{<\omega}\}$$

is an element of H.

We claim that B is a Hamel basis for the reals of H, which will be established by Claims 2.8 and 2.9.

Claim 2.8 $\mathbb{R} \cap H \subset \text{span}(B)$.

Proof of Claim 2.8. Assume not, and let $n < \omega$ be the least size of some $a \in [A]^{<\omega}$ such that $\mathbb{R}^*_a \setminus \text{span}(B) \neq \emptyset$. Pick $x \in \mathbb{R}^*_a \setminus \text{span}(B) \neq \emptyset$, where $\text{Card}(a) = n$.

We must have $n > 0$, as $b_\emptyset$ is a Hamel basis for the reals of L. Then, by the maximality of b_a, while b_a is linearly independent over $\mathbb{R}_{<a}$, $b_a \cup \{x\}$ cannot be linearly independent over $\mathbb{R}_{<a}$. This means that there are $q \in \mathbb{Q}$, $q \neq 0$, $m \in \mathbb{N}$, $m \geq 1$, and $q_n \in \mathbb{Q} \setminus \{0\}$ and $x_n \in b_a$ for all n, $1 \leq n \leq m$, such that

$$z = q \cdot x + \sum_{n=1}^{m} q_n \cdot x_n \in \mathbb{R}_{<a}.$$

By the definition of $\mathbb{R}_{<a}$ and the minimality of n, $z \in \text{span}(\bigcup \{b_c \colon c \subsetneq a\})$, which then clearly implies that $x \in \text{span}(\bigcup \{b_c \colon c \subseteq a\}) \subset \text{span}(B)$.

This is a contradiction! $\square$ (Claim 2.8)

Claim 2.9 *B is linearly independent.*

Proof of Claim 2.9. Assume not. This means that there are $1 \leq k < \omega$, $a_i \in [A]^{<\omega}$ pairwise different, $m_i \in \mathbb{N}$, $m_i \geq 1$ for $1 \leq i \leq k$, and $q^i_n \in \mathbb{Q} \setminus \{0\}$ and $x^i_n \in b_{a_i}$ for all i and n with $1 \leq i \leq k$ and $1 \leq n \leq m_i$ such that

$$\sum_{n=1}^{m_1} q^1_n \cdot x^1_n + \ldots + \sum_{n=1}^{m_k} q^k_n \cdot x^k_n = 0. \tag{11}$$

By the properties of b_{a_i}, $\sum_{n=1}^{m_i} q^i_n \cdot x^i_n \in \mathbb{R}^*_{a_i}$, so that (11) buys us that there are $z_i \in \mathbb{R}^*_{a_i}$, $z_i \neq 0$, $1 \leq i \leq k$, such that

$$z_1 + \ldots + z_k = 0. \tag{12}$$

There must be some i such that there is no j with $a_j \supsetneq a_i$, which implies that $a_j \cap a_i \subsetneq a_i$ for all $j \neq i$. Let us assume without loss of generality that $a_j \cap a_1 \subsetneq a_1$ for all j, $1 < j \leq k$.

Let $a_1 = \{c_\ell \colon \ell \in I\}$, where $I \in [\omega]^{<\omega}$, and let $a_j \cap a_1 = \{c_\ell \colon \ell \in I_j\}$, where $I_j \subsetneq I$, for $1 < j \leq l$.

In what follows, a *nice name* τ for a real is a name of the form

$$\tau = \bigcup_{n,m<\omega} \{(n,m)^\vee\} \times A_{n,m}, \tag{13}$$

where each $A_{n,m}$ is a maximal antichain of conditions of the forcing in question deciding that $\tau(\check{n}) = \check{m}$.

We have that z_1 is $\mathbb{P}(I)$–generic over L, so that we may pick a nice name $\tau_1 \in L^{\mathbb{P}(I)}$ for z_1 with $(\tau_1)^{g \restriction I} = z_1$. Similarly, for $1 < j \leq k$, z_j is $\mathbb{P}(I_j)$–generic over $L[g \restriction (\omega \setminus I)]$, so that we may pick a nice name $\tau_j \in L[g \restriction (\omega \setminus I)]^{\mathbb{P}(I_j)}$ for z_j with $(\tau_j)^{g \restriction I_j} = z_j$. We may construe each τ_j, $1 < j \leq k$, as a name in $L[g \restriction (\omega \setminus I)]^{\mathbb{P}(I)}$ by replacing each $p \colon I_j \to \mathbb{P}$ in an antichain as in (13) by $p' \colon I \to \mathbb{P}$, where $p'(\ell) = p(\ell)$ for $\ell \in I_j$ and $p'(\ell) = \emptyset$ otherwise. Let $p \in g \restriction I$ be such that

$$p \Vdash^{\mathbb{P}(I)}_{L[g \restriction (\omega \setminus I)]} \tau_1 + \tau_2 + \ldots + \tau_k = 0.$$

We now have that inside $L[g \restriction (\omega \setminus I)]$, there are nice $\mathbb{P}(I)$–names τ_j', $1 < j \leq k$ (namey, τ_j, $1 < j \leq k$), such that still inside $L[g \restriction (\omega \setminus I)]$

(1) $p \Vdash^{\mathbb{P}(I)} \tau_1 + \tau_2' + \ldots + \tau_k' = 0$, and

(2) for all j, $1 < j \leq k$ and for all p in one of the antichains of the nice name τ_j', $\mathrm{supp}(p) \subseteq I_j$.

By Lemma 1.4, the nice names τ_1, τ_2', …, τ_k' may be coded by reals, and both (1) and (2) are arithmetic in such real codes for τ_1, τ_2', …, τ_k', so that by $\tau_1 \in L^{\mathbb{P}(I)}$ and Σ^1_1–absoluteness between L and $L[g \restriction (\omega \setminus I)]$ there are inside L nice $\mathbb{P}(I)$–names τ_j', $1 < j \leq k$, such that in L, (1) and (2) hold true. But then, writing $z_j' = (\tau_j')^{g \restriction I}$, we have by (2) that $z_j' \in \mathbb{R}_{I_j}$ for $1 < j \leq k$, and $z_1 + z_2' + \ldots + z_k' = 0$ by (1). But then $z_1 \in \mathbb{R}^*_I \cap \mathbb{R}_{<I}$, which is absurd. □ (Claim 2.9)

We now have that $x \in B$ iff

$$\begin{gathered}\exists a \in [A]^{<\omega}\, \exists J_\alpha[a]\, \exists (x_i \colon i \leq \theta) \in J_\alpha[a]\, \exists X \subset \theta + 1\, (\text{ the } x_i \text{ enumerate the first} \\ \theta + 1 \text{ reals in } J_\alpha[a] \text{ acc. to } <_a \wedge\, \theta \in X \wedge x = x_\theta \wedge \\ \forall i \in \theta \setminus X \exists J_\beta[a]\, J_\beta[a] \models \{x_j \colon j \in X \cap i\} \cup \{x_i\} \text{ is not linearly independent over } \mathbb{R}_{<a} \wedge \\ \forall i \in X\, \forall J_\beta[a]\, J_\beta[a] \models \{x_j \colon j \in X \cap i\} \cup \{x_i\} \text{ is linearly independent over } \mathbb{R}_{<a})),\end{gathered}$$

which is true iff

$$\begin{gathered}\forall a \in [A]^{<\omega}\, \forall J_\alpha[a]\, \forall (x_i \colon i \leq \theta) \in J_\alpha[a]\, \forall X \subset \theta + 1\, ((\text{ the } x_i \text{ enumerate the first} \\ \theta + 1 \text{ reals in } J_\alpha[a] \text{ acc. to } <_a \wedge\, x = x_\theta \wedge \\ \forall i \in (\theta+1) \setminus X \exists J_\beta[a]\, J_\beta[a] \models \{x_j \colon j \in X \cap i\} \cup \{x_i\} \text{ is not linearly independent over } \mathbb{R}_{<a} \wedge \\ \forall i \in X\, \forall J_\beta[a]\, J_\beta[a] \models \{x_j \colon j \in X \cap i\} \cup \{x_i\} \text{ is linearly independent over } \mathbb{R}_{<a}) \to \\ \theta \in X).\end{gathered}$$

By Lemma 2.1, this shows that B is Δ^1_3. □

References

[1] Beriashvili, M., Schindler, R., Wu, L., and Yu, L., *Hamel bases and well-ordering the continuum*, Proc. Amer. Math. Soc. 146 (2018), pp. 3565-3573.

[2] Brendle, J., Castiblanco, F., Schindler, R., Wu, L., and Yu, L., *A model with everything except for a well-ordering of the reals*, submitted.

[3] Cohen, P., *Set theory and the continuum hypothesis*, Benjamin, New York 1966.

[4] Enayat, A., *On the Leibniz-Mycielski axiom in set theory*, Fundam. Math. **181** (2004), pp. 215-231.

[5] Halpern, J.D., and Lévy, A., *The Boolean prime ideal theorem does not imply the axiom of choice*, Proc. Sympos. Pure Math. **13** part I, Amer. Math. Soc., Providence, R.I., 1971, pp. 83-134.

[6] Jensen, R.B., *Definable Sets of Minimal Degree*, Studies in Logic and the Foundations of Mathematics Volume 59, 1970, Pages 122-128.

[7] Kanovei, V., and Lyubetsky, V., *A countable definable set of reals containing no definable elements*, to appear.

[8] Mathias, A., *The order extension principle*, in: Proceedings of Symposia in Pure Mathematics vol. 13 part II: Axiomatic Set Theory, T. Jech (ed.), American Mathematical Society, 1974.

[9] Pincus, D., and Prikry, K., *Luzin sets and well ordering the continuum*, Proc. Americ. Math. Soc. **49** (2), 1975, pp. 429–435.

[10] Schindler, R., *Set theory. Exploring independence and truth*, Springer–Verlag 2012.