On Russell typicality in Set Theory*

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Abstract

By Tzouvaras, a set is nontypical in the Russell sense, if it belongs to a countable ordinal definable set. The class **HNT** of all hereditarily nontypical sets satisfies all axioms of **ZF** and the double inclusion $\mathbf{HOD} \subseteq \mathbf{HNT} \subseteq \mathbf{V}$ holds. Several questions about the nature of such sets, recently proposed by Tzouvaras, are solved in this paper. In particular, a model of **ZFC** is presented in which $\mathbf{HOD} \subsetneqq \mathbf{HNT} \subsetneqq \mathbf{V}$, and another model of **ZFC** in which **HNT** does not satisfy the axiom of choice.

1 Introduction

One of the fundamental directions in modern set theory is the study of important classes of sets in the set theoretic universe $\mathbf{V}$, which themselves satisfy the axioms of set theory. Gödel's class $\mathbf{L}$ of all *constructive* sets traditionally belongs to such classes, as well as the class **HOD** of all *hereditarily ordinal definable* sets, see [11] or [13, § 7A]. Both $\mathbf{L}$ and **HOD** are transitive classes of sets in which all the axioms of the **ZFC** set theory, with the axiom of choice **AC**, are fulfilled (even if the universe $\mathbf{V}$ itself only satisfies **ZF** without the axiom of choice). These classes satisfy $\mathbf{L} \subseteq \mathbf{HOD} \subseteq \mathbf{V}$, and as it was established in early works on modern axiomatic set theory, the class **HOD** can be strictly between the classes $\mathbf{L} \subseteq \mathbf{V}$ in suitable generic extensions of $\mathbf{L}$.

Recent studies have shown considerable interest in other classes of sets based on the key concept of ordinal definability, which also satisfy set-theoretic axioms. In particular, the classes of *nontypical* and *hereditarily nontypical* sets are considered, whose name Tzouvaras [30, 29] connects with philosophical and mathematical studies of Bertrand Russell and the works of van Lambalgen [27] et al. on the axiomatization of the concept of randomness.

Definition 1.1. The set x is *nontypical*, for short $x \in \mathbf{NT}$, if it belongs to a countable **OD** (ordinal definable) set. The set x is *hereditarily nontypical*, for short $x \in \mathbf{HNT}$, if it itself, all its elements, elements of elements, and so on, are all nontypical, in other words the *transitive closure* $\mathrm{TC}(x)$ satisfies $\mathrm{TC}(x) \subseteq \mathbf{NT}$. □

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The classes **NT** and **HNT** in this definition correspond to $\mathbf{NT}_{\aleph_1}$ and $\mathbf{HNT}_{\aleph_1}$ in the basic definition system of [29]. Similarly defined narrower classes $\mathbf{NT}_{\aleph_0}$ (elements of finite ordinally definable sets) and $\mathbf{HNT}_{\aleph_0}$ in [29] are identical to *algebraically definable* and *hereditarily algebraically definable* sets that have been investigated in recent papers [6, 9, 10] and are not considered in this article.

The class **NT** is not necessarily transitive, but the smaller class $\mathbf{HNT} \subseteq \mathbf{NT}$ is transitive and, as shown in [29], satisfies all axioms of **ZF** (the axiom of choice **AC** not included), and also satisfies the relation $\mathbf{HOD} \subseteq \mathbf{HNT} \subseteq \mathbf{V}$. As for the axiom of choice **AC**, if $\mathbf{V} = \mathbf{L}$ (the constructibility axiom), then $\mathbf{HNT} = \mathbf{HOD} = \mathbf{L}$ obviously holds, so in this case **AC** holds in **HNT**.

Problem 1.2 (essentially Tzouvaras [29, § 2])**.** Is it compatible with **ZFC** that the axiom of choice **AC does not hold** in **HNT**?

The next problem of Tzouvaras [29, 2.15] aims to clarify the possibility of the precise *equalities* in the relation $\mathbf{HOD} \subseteq \mathbf{HNT} \subseteq \mathbf{V}$.

Problem 1.3. Are the next sentences compatible with **ZFC**?

(I) $\mathbf{HOD} = \mathbf{HNT} \subsetneqq \mathbf{V}$;

(II) $\mathbf{HOD} \subsetneqq \mathbf{HNT} = \mathbf{V}$;

(III) $\mathbf{HOD} \subsetneqq \mathbf{HNT} \subsetneqq \mathbf{V}$.

We answer all these questions in the positive. This is the main result of this article. It is contained in the theorems 2.1,4.1,5.1,10.1 below. The answer will be given through the construction of four corresponding models of **ZFC** by the method of generic extensions of the constructible universe **L**.

We begin with a model for Problem 1.3(I) in Section 2. It occurs that it is true in the extension $\mathbf{L}[a]$ of **L** by a single Cohen generic real that $\mathbf{L} = \mathbf{HOD} = \mathbf{HNT} \subsetneqq \mathbf{L}[a]$ (Theorem 2.1 below). This is based on our earlier result [19] that the Cohen real a does not belong to a countable **OD** set in $\mathbf{L}[a]$.

As for Problem 1.3(II), we make use (Section 4) of a forcing notion $\mathbb{P}$ introduced in [14] in order to define a generic real $a \in 2^\omega$ whose E_0-equivalence class $[a]_{\mathsf{E}_0}$ is a lightface Π^1_2 set with no **OD** element.

A positive answer to Problem 1.3(III) is given in Sections 5–9 by means of the forcing notion $\mathbb{P} \times$ (the Cohen forcing). This includes the study of some aspects of the behavior of Borel functions in Sections 6,7.

The article ends with a positive solution to Problem 1.2 in Section 10. We make use of the finite-support product $\mathbf{P}^{<\omega}$ of Jensen's forcing notion as in [12].

2 Model I in which not all sets are nontypical

The following theorem solves the problem 1.3(I) in the positive. We make use of a well-known forcing notion.

Theorem 2.1. *If $a \in 2^\omega$ is a Cohen generic real over* $\mathbf{L}$ *then it is true in* $\mathbf{L}[a]$ *that* $\mathbf{L} = \mathbf{HOD} = \mathbf{HNT} \subsetneqq \mathbf{L}[a]$.

Recall that Cohen generic extensions involve the forcing notion $\mathbb{C} = 2^{<\omega}$ (all finite dyadic sequences). Countable **OD** sets in Cohen extensions are investigated in our papers [23, 19, 22]. In particular, we'll use the following result here.

Lemma 2.2 (Thm 1.1 in [19]). *Let $a \in 2^\omega$ be Cohen-generic over the set universe* $\mathbf{V}$. *Then it holds in* $\mathbf{V}[a]$ *that if $Z \subseteq 2^\omega$ is a countable* **OD** *set then* $Z \in \mathbf{V}$. $\square$

This result admits the following extension for the case $\mathbf{V} = \mathbf{L}$:

Corollary 2.3. *Let $a \in 2^\omega$ be Cohen-generic over the constructive universe* $\mathbf{L}$. *Then it holds in* $\mathbf{L}[a]$ *that if* $X \in \mathbf{L}$ *and $A \subseteq 2^X$ is countable* **OD** *then* $A \subseteq \mathbf{L}$.

Proof. As $\mathbb{C}$ is countable, there is a set of $Y \subseteq X$, $Y \in \mathbf{L}$, countable in $\mathbf{L}$ and such that if $a \neq b$ belong to 2^X then $a(x) \neq b(x)$ for some $x \in Y$. Then Y is countable and **OD** in $\mathbf{L}[a]$, so the *projection* $B = \{a{\restriction}Y : a \in A\}$ of the set A will also be countable and **OD** in $\mathbf{L}[a]$. We have $B \subseteq \mathbf{L}$ by the lemma. (The set Y here can be identified with ω.) Hence, each $b \in B$ is **OD** in $\mathbf{L}[a]$. However, if $a \in A$ and $b = a{\restriction}Y$, then by the choice of Y it holds in $\mathbf{L}[a]$ that a is the only element in A satisfying $a{\restriction}Y = b$. Hence, $a \in \mathbf{OD}$. $\square$

Proof (Theorem 2.1). The fact that $\mathbf{L} = \mathbf{HOD}$ in $\mathbf{L}[a]$ is a standard consequence of the homogeneity of the Cohen forcing $\mathbb{C}$. Further, it is clear that $\mathbf{HOD} \subseteq \mathbf{HNT}$. Let's prove the inverse relation $x \in \mathbf{HNT} \implies x \in \mathbf{L}$ in $\mathbf{L}[a]$ by induction on the set-theoretic rank $\mathtt{rk}\, x$ of sets $x \in \mathbf{L}[a]$. Since each set consists only of sets of strictly lower rank, it is sufficient to check that if a set $H \in \mathbf{L}[a]$ satisfies $H \subseteq \mathbf{L}$ and $H \in \mathbf{HNT}$ in $\mathbf{L}[a]$ then $H \in \mathbf{L}$. Here we can assume that in fact $H \subseteq \mathbf{Ord}$, since $\mathbf{L}$ allows an **OD** wellordering. Thus, let $H \subseteq \lambda \in \mathbf{Ord}$. Additionally, since $H \in \mathbf{HNT}$, we have, in $\mathbf{L}[a]$, a countable **OD** set $A \subseteq \mathscr{P}(\lambda)$ containing H. However, $A \in \mathbf{L}$ by Corollary 2.3. This implies $H \in \mathbf{L}$. $\square$

3 Perfect trees and Silver trees

Our results will involve forcing notions that consist of perfect trees and Silver trees. Here we introduce the relevant terminology from our earlier works [14, 20, 21].

By $2^{<\omega}$ we denote the set of all *tuples* (finite sequences) of terms $0, 1$, including the empty tuple Λ. The length of a tuple s is denoted by $\mathtt{lh}\, s$, and $2^n = \{s \in 2^{<\omega} : \mathtt{lh}\, s = n\}$ (all tuples of length n). A tree $\varnothing \neq T \subseteq 2^{<\omega}$ is *perfect*, symbolically $T \in \mathbf{PT}$, if it has no endpoints and isolated branches. In this case, the set

$$[T] = \{a \in 2^\omega : \forall\, n\, (a{\restriction}n \in T)\}$$

of all *branches* of T is a perfect set in 2^ω. Note that $[S] \cap [T] = \varnothing$ iff $S \cap T$ is finite.

- If $u \in T \in \mathbf{PT}$, then a *portion* (or a *pruned tree*) $T\restriction_u \in \mathbf{PT}$ is defined by $T\restriction_u = \{s \in T : u \subset s \vee s \subseteq u\}$.

- A tree $S \subseteq T$ is *clopen* in T iff it is equal to the union of a finite number of portions of T. This is equivalent to $[S]$ being clopen in $[T]$.

A tree $T \subseteq 2^{<\omega}$ is a *Silver tree*, symbolically $T \in \mathbf{ST}$, if there is an infinite sequence of tuples $u_k = u_k(T) \in 2^{<\omega}$, such that T consists of all tuples of the form

$$s = u_0{}^\frown i_0{}^\frown u_1{}^\frown i_1{}^\frown u_2{}^\frown i_2{}^\frown \ldots {}^\frown u_n{}^\frown i_n$$

and their sub-tuples, where $n < \omega$ and $i_k = 0, 1$. In this case the *stem* $\mathtt{stem}(T) = u_0(T)$ is equal to the largest tuple $s \in T$ with $T = T\restriction_s$, and $[T]$ consists of all infinite sequences $a = u_0{}^\frown i_0{}^\frown u_1{}^\frown i_1{}^\frown u_2{}^\frown i_2{}^\frown \cdots \in 2^\omega$, where $i_k = 0, 1$, $\forall\, k$. Put

$$\mathtt{spl}_n(T) = \mathtt{lh}\, u_0 + 1 + \mathtt{lh}\, u_1 + 1 + \cdots + \mathtt{lh}\, u_{n-1} + 1 + \mathtt{lh}\, u_n\,.$$

In particular, $\mathtt{spl}_0(T) = \mathtt{lh}\, u_0$. Thus $\mathtt{spl}(T) = \{\mathtt{spl}_n(T) : n < \omega\} \subseteq \omega$ is the set of all *splitting levels* of the Silver tree T.

Action. Let $\sigma \in 2^{<\omega}$. If $v \in 2^{<\omega}$ is another tuple of length $\mathtt{lh}\, v \geq \mathtt{lh}\, \sigma$, then the tuple $v' = \sigma \boldsymbol{\cdot} v$ of the same length $\mathtt{lh}\, v' = \mathtt{lh}\, v$ is defined by $v'(i) = v(i) +_2 \sigma(i)$ (addition modulo 2) for all $i < \mathtt{lh}\, \sigma$, but $v'(i) = v(i)$ whenever $\mathtt{lh}\, \sigma \leq i < \mathtt{lh}\, v$. If $\mathtt{lh}\, v < \mathtt{lh}\, \sigma$, then we just define $\sigma \boldsymbol{\cdot} v = (\sigma \restriction \mathtt{lh}\, v) \boldsymbol{\cdot} v$.

If $a \in 2^\omega$, then similarly $a' = \sigma \boldsymbol{\cdot} a \in 2^\omega$, $a'(i) = a(i) +_2 \sigma(i)$ for $i < \mathtt{lh}\, \sigma$, but $a'(i) = a(i)$ for $i \geq \mathtt{lh}\, \sigma$. If $T \subseteq 2^{<\omega}$, $X \subseteq 2^\omega$, then the sets

$$\sigma \boldsymbol{\cdot} T = \{\sigma \boldsymbol{\cdot} v : v \in T\} \quad \text{and} \quad \sigma \boldsymbol{\cdot} X = \{\sigma \boldsymbol{\cdot} a : a \in X\}$$

are *shifts* of the tree T and the set X accordingly.

Lemma 3.1 ([21], 3.4)**.** *If $n < \omega$ and $u, v \in T \cap 2^n$, then $T\restriction_u = v \boldsymbol{\cdot} u \boldsymbol{\cdot} (T\restriction_v)$.*

If $t \in T \in \mathbf{ST}$ and $\sigma \in 2^{<\omega}$, then $\sigma \boldsymbol{\cdot} T \in \mathbf{ST}$ and $T\restriction_s \in \mathbf{ST}$. □

Definition 3.2 (refinements). Assume that $T, S \in \mathbf{ST}$, $S \subseteq T$, $n < \omega$. We define $S \subseteq_n T$ (the tree S *n-refines* T) if $S \subseteq T$ and $\mathtt{spl}_k(T) = \mathtt{spl}_k(S)$ for all $k < n$. This is equivalent to ($S \subseteq T$ and) $u_k(S) = u_k(T)$ for all $k < n$, of course. □

Then $S \subseteq_0 T$ is equivalent to $S \subseteq T$, and $S \subseteq_{n+1} T$ implies $S \subseteq_n T$ (and $S \subseteq T$), but if $n \geq 1$ then $S \subseteq_n T$ is equivalent to $\mathtt{spl}_{n-1}(T) = \mathtt{spl}_{n-1}(S)$.

Lemma 3.3. *Assume that $T, U \in \mathbf{ST}$, $n < \omega$, $h > \mathtt{spl}_{n-1}(T)$, $s_0 \in 2^h \cap T$, and $U \subseteq T\restriction_{s_0}$. Then there is a unique tree $S \in \mathbf{ST}$ such that $S \subseteq_n T$ and $S\restriction_{s_0} = U$.*

If in addition U is clopen in T then S is clopen in T as well.

Proof (sketch)**.** Define a tree S so that $S \cap 2^h = T \cap 2^h$, and if $t \in T \cap 2^h$ then, by Lemma 3.1, $S\restriction_t = (t \boldsymbol{\cdot} s_0) \boldsymbol{\cdot} U$; then $S\restriction_{s_0} = U$. To check that $S \in \mathbf{ST}$, we can easily compute the tuples $u_k(S)$. Namely, as $U \subseteq T\restriction_{s_0}$, we have $s_0 \subseteq u_0(U) = \mathtt{stem}(U)$, hence $\ell = \mathtt{lh}\,(u_0(U)) \geq h > m = \mathtt{spl}_{n-1}(T)$. Then $u_k(S) = u_k(T)$ for all $k < n$, $u_n(S) = u_0(U) \restriction [m, \ell)$ (thus $u_n(S) \in 2^{\ell - m}$), and $u_k(S) = u_k(U)$ for all $k > n$. □

Lemma 3.4 ([21], Lemma 4.4)**.** *Let* $\ldots \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0$ *be a sequence of trees in* **ST***. Then* $T = \bigcap_n t_n \in$ **ST***.* $\square$

Proof (sketch)**.** By definition we have $u_k(T_n) = u_k(T_{n+1})$ for all $k \le n$. Then one easily computes that $u_n(T) = u_n(T_n)$ for all n. $\square$

4 Model II in which there are more nontypical sets than HOD sets

The following theorem solves the problem 1.3(II) positively.

Theorem 4.1. *There is a generic extension of the constructible universe* **L***, in which it is true that* $\mathbf{HOD} \subsetneqq \mathbf{HNT} = \mathbf{V}$.

Recall that the equivalence relation E_0 is defined on 2^ω so that $a \mathrel{\mathsf{E}_0} b$ iff the set $a \mathbin{\Delta} b = \{k : a(k) \neq b(k)\}$ is finite.

To prove Theorem 4.1, we will use an **OD** E_0-equivalence class

$$[a]_{\mathsf{E}_0} = \{b \in 2^\omega : a \mathrel{\mathsf{E}_0} b\} = \{\sigma \bullet a : \sigma \in 2^{<\omega}\}$$

of a non-**OD** generic real $a \in 2^\omega$, introduced in [14] and also applied in [7, 21, 20]. This is done by a forcing notion $\mathbb{P}$ having the following key properties, see [14].

1*. $\mathbb{P} \in \mathbf{L}$ consists of Silver trees: $\mathbb{P} \subseteq \mathbf{ST}$.

2*. If $u \in T \in \mathbb{P}$ and $\sigma \in 2^{<\omega}$ then $T{\restriction}_u \in \mathbb{P}$ and $\sigma \bullet T \in \mathbb{P}$ — this is the property of *invariance* w.r.t. shifts and portions.

3*. $\mathbb{P}$ satisfies the countable antichain condition CCC in **L**.

4*. The forcing $\mathbb{P}$ ajoins a generic real $a \in 2^\omega$ to **L**, whose E_0-class $[a]_{\mathsf{E}_0} = \{b \in 2^\omega : b \mathrel{\mathsf{E}_0} a\}$ is a (countable) **OD**, and even $\mathit{\Pi}^1_2$ (lightface) set in $\mathbf{L}[a]$.

5*. If a real $a \in 2^\omega$ is $\mathbb{P}$-generic over **L**, then a is not **OD** in the generic extension $\mathbf{L}[a]$. (This property is an elementary consequence of the invariance property as in 2*, see Lemma 7.5 in [14].)

Proof (Theorem 4.1)**.** Let a real $a \in 2^\omega$ be $\mathbb{P}$-generic over **L**. According to 4* the real a itself belongs to **HNT** in $\mathbf{L}[a]$, hence the equality $\mathbf{HNT} = \mathbf{V}$ holds in $\mathbf{L}[a]$. On the other hand, $a \notin \mathbf{OD}$ in $\mathbf{L}[a]$ by 5*, thus $\mathbf{HOD} \subsetneqq \mathbf{HNT}$ in $\mathbf{L}[a]$, as required. (A more thorough analysis based on 2* shows that $\mathbf{HOD} = \mathbf{L}$ in $\mathbf{L}[a]$.) $\square$

5 Model III: nontypical sets in general position

The following theorem positively solves Problem 1.3(III), providing a model in which hereditarily nontypical sets are strictly between **HOD** and **V**.

Theorem 5.1. *There is a generic extension of* **L***, in which* $\mathbf{HOD} \subsetneqq \mathbf{HNT} \subsetneqq \mathbf{V}$.

The proof of this theorem (to be completed in Section 9) is based on a combination of ideas from the proof of theorems 2.1 and 4.1. In fact, the forcing notion involved will be equal to the product $\mathbb{P} \times \mathbb{C}$. However, we will have to consider in more detail the inductive construction of the set $\mathbb{P}$, as well as some questions related to continuous and Borel functions and the construction of Silver trees.

In the remainder, if $v \in \omega^{<\omega}$ (a tuple of natural numbers), then we define $\mathscr{N}_v = \{x \in \omega^\omega : v \subset x\}$, the *Baire interval* or *portion* in the Baire space ω^ω.

6 Reduction of Borel functions to continuous ones

A classical theorem claims that in Polish spaces every Borel function is continuous on a suitable dense $\mathbf{G}_\delta$ set (Theorem 8.38 in Kechris [26]). It is also known that a Borel map defined on 2^ω is continuous on a suitable Silver tree. The next lemma combines these two results. Our interest in functions defined on $2^\omega \times \omega^\omega$ is motivated by further applications to reals in generic extensions of the form $\mathbf{L}[a, x]$, where $a \in 2^\omega$ is $\mathbb{P}$-generic real for some $\mathbb{P} \subseteq \mathbf{ST}$ while $x \in \omega^\omega$ is just Cohen generic.

Lemma 6.1. *Let $T \in \mathbf{ST}$ and $f : 2^\omega \times \omega^\omega \to 2^\omega$ be a Borel map. There is a Silver tree $S \subseteq T$ and a dense $\mathbf{G}_\delta$ set $D \subseteq \omega^\omega$ such that f is continuous on $[S] \times D$.*

Proof. By the abovementioned classical theorem, f is already continuous on some dense $\mathbf{G}_\delta$ set $Z \subseteq [T] \times \omega^\omega$. It remains to define a Silver tree $S \subseteq T$ and a dense $\mathbf{G}_\delta$ set $D \subseteq \omega^\omega$ such that $[S] \times D \subseteq Z$. This will be our goal.

We have $Z = \bigcap_n Z_n$, where each $Z_n \subseteq [T] \times \omega^\omega$ is open dense.

We will define $S = \bigcup_k S_k$, where Silver trees S_k satisfy $\ldots \subseteq_4 S_3 \subseteq_3 S_2 \subseteq_2 s_1 \subseteq_1 S_0 = T$ as in the lemma 3.4. Tuples $w_k \in \omega^{<\omega}$, $k < \omega$ will also be defined.

We fix a recursive enumeration $\omega \times \omega^{<\omega} = \{\langle N_k, v_k \rangle : k < \omega\}$.

At step 0 we already have $S_0 = T$. Assume that the tree S_k has already been defined. We claim that there exist:

(A) a tuple $w_k \in \omega^{<\omega}$ and a Silver tree $S_{k+1} \subseteq_{k+1} S_k$, **clopen** in S_k (see Section 3), such that $v_k \subseteq w_k$ and $[S_{k+1}] \times \mathscr{N}_{w_k} \subseteq Z_{N_k}$.

Now let $N = N_k$, $v = v_k$. Put $h = \mathtt{spl}_{k+1}(S_k)$. Consider any tuple $t \in 2^h \cap S_k$. Since Z_N is open dense, there exist a tuple $v_1 \in \omega^{<\omega}$ and a Silver tree $A \subseteq S_k {\restriction}_t$, clopen in S_k (for example, a portion in S_k) such that $v \subseteq v_1$ and $[A] \times \mathscr{N}_{v_1} \subseteq Z_N$. According to Lemma 3.3, there exists a Silver tree $U_1 \subseteq_{k+1} S_k$, clopen in S_k along with A, such that $U_1 {\restriction}_t = A$, so $[U_1 {\restriction}_t] \times \mathscr{N}_{v_1} \subseteq Z_N$ by construction.

Now take another tuple $t' \in 2^h \cap S_k$, and similarly find $v_2 \in \omega^{<\omega}$ and a Silver tree $A \subseteq U_1 {\restriction}_{t'}$, clopen in U_1, such that $v_1 \subseteq v_2$ and $[A] \times \mathscr{N}_{v_2} \subseteq Z_N$. Once again there is a Silver tree $U_2 \subseteq_{k+1} U_1$, clopen in S_k and such that $[U_2 {\restriction}_{t'}] \times \mathscr{N}_{v_2} \subseteq Z_N$.

We iterate this construction over all tuples $t \in 2^h \cap S_k$, $\subseteq_{k+1}$-shrinking trees and extending tuples in $\omega^{<\omega}$. We get a Silver tree $U \subseteq_{k+1} S_k$, clopen in S_k, and tuple $w \in \omega^{<\omega}$, that $v \subseteq w$ and $[U] \times \mathscr{N}_w \subseteq Z_N$. Take $w_k = w$, $S_{k+1} = U$. This completes the inductive step.

As a result we get a sequence $\ldots \subseteq_4 S_3 \subseteq_3 S_2 \subseteq_2 S_1 \subseteq_1 S_0 = T$ of Silver trees S_k, and tuples $w_k \in \omega^{<\omega}$ $(k < \omega)$, which satisfy (A) for all k.

We put $S = \bigcap_k S_k$; then $S \in \mathbf{ST}$ by Lemma 3.4, and $S \subseteq T$.

If $n < \omega$ then let $W_n = \{w_k : N_k = n\}$. Then $D_n = \bigcup_{w \in W_n} \mathscr{N}_w$ is an open dense set in ω^ω. Indeed, let $v \in \omega^{<\omega}$. Consider k such that that $v_k = v$ and $N_k = n$. By construction, we have $v \subseteq w_k \in W_n$, as required. We conclude that the set $D = \bigcap_n D_n$ is dense $\mathbf{G}_\delta$.

To check $[S] \times D \subseteq Z$, let $n < \omega$; we show that $[S] \times D \subseteq Z_n$. Let $a \in [S]$ and $x \in D$, in particular $x \in D_n$, so $x \in \mathscr{N}_{w_k}$ for some k with $N_k = n$. However, $[S_{k+1}] \times \mathscr{N}_{w_k} \subseteq Z_n$ by (A), and at the same time obviously $a \in [S_{k+1}]$. We conclude that in fact $\langle a, x\rangle \in Z_n$, as required. $\square$ (Lemma 6.1)

7 Normalization of Borel maps

Definition 7.1. A map $f : 2^\omega \times \omega^\omega \to 2^\omega$ is *normalized on* $T \in \mathbf{ST}$ *for* $\mathbb{U} \subseteq \mathbf{ST}$ if there exists a dense $\mathbf{G}_\delta$ set $X \subseteq \omega^\omega$ such that f is continuous on $[T] \times X$ and:

– either (1) there are tuples $v \in \omega^{<\omega}$, $\sigma \in 2^{<\omega}$ such that $f(a, x) = \sigma \cdot a$ for all $a \in [T]$ and $x \in \mathscr{N}_v \cap X$, where $\mathscr{N}_v = \{x \in \omega^\omega : v \subset x\}$;

– or (2) $f(a, x) \notin \bigcup_{\sigma \in 2^{<\omega} \wedge S \in \mathbb{U}} \sigma \cdot [S]$ for all $a \in [T]$ and $x \in X$. $\square$

Theorem 7.2. *Let* $\mathbb{U} = \{T_0, T_1, T_2, \ldots\} \subseteq \mathbf{ST}$ *and* $f : 2^\omega \times \omega^\omega \to 2^\omega$ *be a Borel map. There is a set* $\mathbb{U}' = \{S_0, S_1, S_2, \ldots\} \subseteq \mathbf{ST}$*, such that* $S_n \subseteq T_n$ *for all* n *and* f *is normalized on* S_0 *for* $\mathbb{U}'$.

Proof. First of all, according to Lemma 6.1, there is a Silver tree $T' \subseteq T_0$ and a dense $\mathbf{G}_\delta$ set $W \subseteq \omega^\omega$ such that f is continuous on $[T'] \times W$. And since any dense $\mathbf{G}_\delta$ set $X \subseteq \omega^\omega$ is homeomorphic to ω^ω, we can w.l.o.g. assume that $W = \omega^\omega$ and $T' = T_0$. Thus, we simply suppose that f is already *continuous on* $[T_0] \times \omega^\omega$.

Assume that option (1) of the definition of 7.1 does not take place, *i.e.*

$(*)$ if $X \subseteq \omega^\omega$ is dense $\mathbf{G}_\delta$, and $v \in \omega^{<\omega}$, $\sigma \in 2^{<\omega}$, $S \in \mathbf{ST}$, $S \subseteq T_0$, then there are reals $a \in [S]$ and $x \in \mathscr{N}_v \cap X$ such that $f(a, x) \neq \sigma \cdot a$.

We'll define Silver trees $S_n \subseteq T_n$ and a dense $\mathbf{G}_\delta$ set $X \subseteq \omega^\omega$ satisfying (2) of Definition 7.1, that is, in our case, the relation $f(a, x) \notin \bigcup_{\sigma \in 2^{<\omega} \wedge n < \omega} \sigma \cdot [S_n]$ will be fulfilled for all $a \in [S_0]$ and $x \in X$.

The construction of the trees is organized in the form $S_n = \bigcup_k S^n_k$, where Silver trees S^n_k satisfy $\cdots \subseteq_4 S^n_3 \subseteq_3 S^n_2 \subseteq_2 S^n_1 \subseteq_1 S^n_0 = T_n$ as in Lemma 3.4 for each $n < \omega$. A series of tuples $w_k \in \omega^{<\omega}$ $(k < \omega)$ will also be defined, they will help us to construct a dense $\mathbf{G}_\delta$ set $X \subseteq \omega^\omega$ required.

We fix any enumeration $\omega \times 2^{<\omega} \times \omega^{<\omega} = \{\langle N_k, \sigma_k, v_k\rangle : k < \omega\}$.

At step 0 of the construction, we put $S^n_0 = T_n$ for all n.

Assume that $k < \omega$ and all Silver trees S^n_k, $n < \omega$ are already defined.

(B) We put $S^n_{k+1} = S^n_k$ for all $n > 0,\ n \neq N_k$.

As for the trees S^0_{k+1} and $S^{N_k}_{k+1}$, we claim that there exist:

(C) a tuple $w_k \in \omega^{<\omega}$ and Silver trees $S^0_{k+1} \subseteq_{k+1} S^0_k,\ S^{N_k}_{k+1} \subseteq_{k+1} S^{N_k}_k$ such that $v_k \subseteq w_k$ and $f(a,x) \notin \sigma_k \cdot [S^N_{k+1}]$ for all $a \in [S^0_{k+1}]$ and $x \in \mathscr{N}_{w_k}$.

For brevity, let $N = N_k,\ \sigma = \sigma_k,\ v = v_k$. Put $h = \mathtt{spl}_{k+1}(S^0_k)$, $m = \mathtt{spl}_{k+1}(S^N_k)$.

Case 1: $N > 0$. Take any pair of tuples $s \in 2^h \cap S^0_k$, $t \in 2^m \cap S^N_k$ and any reals $a_0 \in [S^0_k \restriction_s]$ and $x_0 \in \omega^\omega$. Consider any real $b_0 \in [S^N_k \restriction_t]$ not equal to $\sigma \cdot f(a_0, x_0)$. Let's say $b_0(\ell) = i \neq j = (\sigma \cdot f(a_0, x_0))(\ell)$, where $i, j \leq 1,\ \ell < \omega$. By the continuity of f, there is a tuple $v_1 \in \omega^{<\omega}$ and Silver tree $A \subseteq S^0_k \restriction_s$ such that $v \subseteq v_1 \subset x_0$, $a_0 \in [A]$, and $(\sigma \cdot f(a,x))(\ell) = j$ for all $x \in \mathscr{N}_{v_1}$ and $a \in [A]$. It is also clear that $B = \{\tau \in S^N_k \restriction_t : \mathtt{lh}\,\tau \leq \ell \vee \tau(\ell) = i\}$ is a Silver tree containing b_0, and $b(\ell) = i$ for all $b \in [B]$. According to Lemma 3.3, there are Silver trees $U_1 \subseteq_{k+1} S^0_k$ and $V_1 \subseteq_{k+1} S^N_k$, such that $U_1 \restriction_s = A$ and $V_1 \restriction_t = B$, hence by construction we have $\sigma \cdot f(a,x) \notin [V_1 \restriction_t]$ for all $a \in [U_1 \restriction_s]$ and $x \in \mathscr{N}_{v_1}$.

Now consider another pair of tuples $s \in 2^h \cap S^0_k$, $t \in 2^m \cap S^N_k$. We similarly get Silver trees $U_2 \subseteq_{k+1} U_1$ and $V_2 \subseteq_{k+1} V_1$, and a tuple $v_2 \in \omega^{<\omega}$, such that $v_1 \subseteq v_2$ and $\sigma \cdot f(a,x) \notin [V_2(\to t')]$ for all $a \in [U_2 \restriction_{s'}]$ and $x \in \mathscr{N}_{v_2}$. In this case, we have $V_2 \restriction_t \subseteq V_1 \restriction_t$ and $U_2 \restriction_s \subseteq U_1 \restriction_s$, so that what has already been achieved at the previous step is preserved.

We iterate through all pairs of $s \in 2^h \cap S^0_k$, $t \in 2^m \cap S^N_k$, $\subseteq_{k+1}$-shrinking trees and extending tuples in $\omega^{<\omega}$ at each step. This results in a pair of Silver trees $U \subseteq_{k+1} S^0_k$, $V \subseteq_{k+1} S^N_k$ and a tuple $w \in \omega^{<\omega}$ such that $v \subseteq w$ and $\sigma \cdot f(a,x) \notin [V]$ for all reals $a \in [U]$ and $x \in \mathscr{N}_w$. Now to fulfill (C), take $w_k = w$, $S^0_{k+1} = U$, and $S^{N_k}_{k+1} = V$. Recall that here $N_k = N > 0$.

Case 2: $N = 0$. Here the construction somewhat changes, and hypothesis $(*)$ will be used. We claim that there exist:

(D) a tuple $w_k \in \omega^{<\omega}$ and a Silver tree $S^0_{k+1} \subseteq_{k+1} S^0_k$ such that $v_k \subseteq w_k$ and $f(a,x) \notin \sigma_k \cdot [S^0_{k+1}]$ for all $a \in [S^0_{k+1}]$ and $x \in \mathscr{N}_{w_k}$.

As above, let $\sigma = \sigma_k,\ v = v_k$. Take any pair of tuples $s, t \in 2^h \cap S^0_k$, where $h = \mathtt{spl}_{k+1}(S^0_k)$ as above. Thus $S^0_k \restriction_t = t \cdot s \cdot (S^0_k \restriction_s)$, by Lemma 3.1. According to $(*)$, there are reals $x_0 \in \mathscr{N}_v$ and $a_0 \in [S^0_k \restriction_s]$ satisfying $f(a_0, x_0) \neq \sigma \cdot s \cdot t \cdot a_0$, or equivalently, $\sigma \cdot f(a_0, x_0) \neq s \cdot t \cdot a_0$.

Similarly to Case 1, we have $(\sigma \cdot f(a_0, x_0))(\ell) = i \neq j = (s \cdot t \cdot a_0)(\ell)$ for some $\ell < \omega$ and $i, j \leq 1$. By the continuity of f, there is a tuple $v_1 \in \omega^{<\omega}$ and a Silver tree $A \subseteq S^0_k \restriction_s$, clopen in S^0_k, such that $v \subseteq v_1 \subset x_0$, $a_0 \in [A]$, and $(\sigma \cdot f(a,x))(\ell) = j$ but $(s \cdot t \cdot a)(\ell) = j$ for all $x \in \mathscr{N}_{v_1}$ and $a \in [A]$. Lemma 3.3 gives us a Silver tree $U_1 \subseteq_{k+1} S^0_k$, clopen in S^0_k as well, such that $U_1 \restriction_s = A$ — and then $U_1 \restriction_t = s \cdot t \cdot A$. Therefore $\sigma \cdot f(a,x) \notin [U_1 \restriction_t]$ holds for all $a \in [U_1 \restriction_s]$ and $x \in \mathscr{N}_{v_1}$ by construction.

Having worked out all pairs of tuples $s,t \in 2^h \cap S^0_k$, we obtain a Silver tree $U \subseteq_{k+1} S^0_k$ and a tuple $w \in \omega^{<\omega}$, such that $v \subseteq w$ and $\sigma{\cdot}f(a,x) \notin [U]$ for all $a \in [U]$ and $x \in \mathscr{N}_w$. Now to fulfill (D), take $w_k = w$ and $S^0_{k+1} = U$.

To conclude, we have for each n a sequence $\ldots \subseteq_4 S^n_3 \subseteq_3 S^n_2 \subseteq_2 S^n_1 \subseteq_1 S^n_0 = T_n$ of Silver trees S^n_k, along with tuples $w_k \in \omega^{<\omega}$ $(k < \omega)$, and these sequences satisfy the requirements (B) and (C) (equivalent to (D) in case $N_k = 0$).

We put $S_n = \bigcap_k S^n_k$. Then $S_n \in \mathbf{ST}$ by Lemma 3.4, and $S_n \subseteq T_n$.

If $n < \omega$ then let $W_{n\sigma} = \{w_k : N_k = n \wedge \sigma_k = \sigma\}$. The set $X_{n\sigma} = \bigcup_{w \in W_{n\sigma}} \mathscr{N}_w$ is then open dense in ω^ω. Indeed, if $v \in \omega^\omega$ then we take k such that $v_k = v$, $N_k = n$, $\sigma_k = \sigma$; then $v \subseteq w_k \in W_{n\sigma}$ by construction. Therefore, $X = \bigcap_{n<\omega, \sigma \in 2^{<\omega}} X_{n\sigma}$ is a dense $\mathbf{G}_\delta$ set. Now to check property (2) of Definition 7.1, consider any $n < \omega$, $\sigma \in 2^{<\omega}$, $a \in [S_0]$, $x \in X$; we show that $f(a,x) \notin \sigma{\cdot}[S_n]$.

By construction, we have $x \in X_{n\sigma}$, *i.e.* $x \in \mathscr{N}_{w_k}$, where $k \in W_{n\sigma}$, so that $N_k = n$, $\sigma_k = \sigma$. Now $f(a,x) \notin \sigma{\cdot}[S_n]$ directly follows from (C) for this k, since $S_0 \subseteq S^0_{k+1}$ and $S_n \subseteq S^n_{k+1}$. □ (Theorem 7.2)

8 The forcing notion for Model III

Using the standard encoding of Borel sets, as *e.g.* in [28] or [13, § 1D], we fix a coding of Borel functions $f : 2^\omega \times \omega^\omega \to 2^\omega$. As usual, it includes a $\mathit{\Pi}^1_1$-*set*[1] *of codes* $\mathbf{BC} \subseteq \omega^\omega$, and for each code $r \in \mathbf{BC}$ a certain Borel function $F_r : 2^\omega \times \omega^\omega \to 2^\omega$ coded by r. We assume that each Borel function has some code, and there is a $\mathit{\Sigma}^1_1$ relation $\mathfrak{S}(\cdot,\cdot,\cdot,\cdot)$ and a $\mathit{\Pi}^1_1$ relation $\mathfrak{P}(\cdot,\cdot,\cdot,\cdot)$ such that for all $r \in \mathbf{BC}$, $x \in \omega^\omega$, and $a,b \in 2^\omega$ it holds $F_r(a,x) = b \iff \mathfrak{S}(r,a,x,b) \iff \mathfrak{P}(r,a,x,b)$.

If $\mathbb{U} \subseteq \mathbf{ST}$, then $\mathtt{Clos}(\mathbb{U})$ denotes the set of all trees of the form $\sigma{\cdot}(T{\restriction}_s)$, where $\sigma \in 2^{<\omega}$ and $s \in T \in \mathbb{U}$, *i.e.* the closure of $\mathbb{U}$ w.r.t. both shifts and portions.

The following construction is maintained in L. We define a sequence of countable sets $\mathbb{U}_\alpha \subseteq \mathbf{ST}$, $\alpha < \omega_1$ satisfying the following conditions $1^\dagger$–$5^\dagger$.

$1^\dagger$. Each $\mathbb{U}_\alpha \subseteq \mathbf{ST}$ is countable, $\mathbb{U}_0$ consists of a single tree $2^{<\omega}$.

We then define $\mathbb{P}_\alpha = \mathtt{Clos}(\mathbb{U}_\alpha)$, $\mathbb{P}_{<\alpha} = \bigcup_{\xi<\alpha} \mathbb{P}_\xi$. These sets are obviously closed with respect to shifts and portions, that is $\mathtt{Clos}(\mathbb{P}_\alpha) = \mathbb{P}_\alpha$ and $\mathtt{Clos}(\mathbb{P}_{<\alpha}) = \mathbb{P}_{<\alpha}$.

$2^\dagger$. For every $T \in \mathbb{P}_{<\alpha}$ there is a tree $S \in \mathbb{U}_\alpha$, $S \subseteq T$.

Let $\mathbf{ZFC}^-$ be the subtheory of the theory $\mathbf{ZFC}$, containing all axioms except the power set axiom, and additionally containing an axiom asserting the existence of the power set $\mathscr{P}(\omega)$. This implies the existence of $\mathscr{P}(X)$ for any countable X, the existence of ω_1 and 2^ω, as well as the existence of continual sets like 2^ω or $\mathbf{ST}$.

By $\mathfrak{M}_\alpha$ we denote the smallest model of $\mathbf{ZFC}^-$ of the form $\mathbf{L}_\lambda$ containing the sequence $\langle \mathbb{U}_\xi \rangle_{\xi<\alpha}$, in which α and all sets $\mathbb{U}_\xi$, $\xi < \alpha$, are countable.

[1] The letters $\mathit{\Sigma}$ and $\mathit{\Pi}$ denote effective (lightface) projective classes.

$3^\dagger$. If a set $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}_{<\alpha}$ is dense in $\mathbb{P}_{<\alpha}$, and $U \in \mathbb{U}_\alpha$, then $U \subseteq^{\mathtt{fin}} \bigcup D$, meaning that there is a finite set $D' \subseteq D$ such that $U \subseteq \bigcup D'$.

Given that $\mathtt{Clos}(\mathbb{P}_{<\alpha}) = \mathbb{P}_{<\alpha}$, this is automatically transferred to all trees $U \in \mathbb{P}_\alpha$ as well. It follows that D remains predense in $\mathbb{P}_{<\alpha} \cup \mathbb{P}_\alpha$.

To formulate the next property, we fix an enumeration $\mathbf{ST} \times \omega^\omega = \{\langle T_\xi, b_\xi \rangle : \xi < \omega_1\}$ in $\mathbf{L}$, which 1) is definable in $\mathbf{L}_{\omega_1}$, and 2) each value in $\mathbf{ST} \times \omega^\omega$ is taken uncountably many times.

$4^\dagger$. If $T_\alpha \in \mathbb{P}_{<\alpha}$ then there is a tree $S \in \mathbb{U}_\alpha$ satisfying $S \subseteq T$, on which F_{b_α} is normalized for $\mathbb{U}_\alpha$ in the sense of Definition 7.1.

$5^\dagger$. The sequence $\langle \mathbb{U}_\alpha \rangle_{\alpha<\omega_1}$ is $\in$-definable in $\mathbf{L}_{\omega_1}$.

The construction goes on as follows. **Arguing in L**, suppose that $\alpha < \omega_1$, the subsequence $\langle \mathbb{U}_\xi \rangle_{\xi<\alpha}$ has been defined, and the sets $\mathbb{P}_\xi = \mathtt{Clos}(\mathbb{U}_\xi)$ (for $\xi < \alpha$), $\mathbb{P}_{<\alpha}$, $\mathfrak{M}_\alpha$ are defined as above.

Lemma 8.1 (in $\mathbf{L}$)**.** *Under these assumptions, there is a countable set* $\mathbb{U}_\alpha \subseteq \mathbf{ST}$ *satisfying* $2^\dagger$, $3^\dagger$, $4^\dagger$.

Proof. The existence of a countable set $\mathbb{U}_\alpha \subseteq \mathbf{ST}$ satisfying $2^\dagger$, $3^\dagger$ is known from our earlier papers, see [14, § 4], [21, § 9 and 10], [20, § 10]. If now the tree T_α belongs to $\mathbb{P}_{<\alpha}$ (if not then we don't worry about $4^\dagger$), then we consider, according to $2^\dagger$, a tree $T \in \mathbb{U}_\alpha$ satisfying $T \subseteq T_\alpha$. Using Theorem 7.2, we shrink each tree $U \in \mathbb{U}_\alpha$ to a tree $U' \in \mathbf{ST}$, $U' \subseteq U$, so that the function F_{b_α} is normalized on T' for $\mathbb{U}' = \{U' : U \in \mathbb{U}_\alpha\}$. Finally take $\mathbb{U}'$ as $\mathbb{U}_\alpha$ and T' as S to fulfill $4^\dagger$. □ (Lemma)

To accomplish the construction, we take $\mathbb{U}_\alpha$ to be the smallest, in the sense of the Gödel wellordering of $\mathbf{L}$, of those sets that exist by Lemma 8.1. Since the whole construction is relativized to $\mathbf{L}_{\omega_1}$, the requirement $5^\dagger$ is also met.

We put $\mathbb{P}_\alpha = \mathtt{Clos}(\mathbb{U}_\alpha)$ for all $\alpha < \omega_1$, and $\mathbb{P} = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha$.

The following result, in part related to CCC, is a fairly standard consequence of $3^\dagger$, see for example [14, 6.5], [20, 12.4], or [12, Lemma 6]; we will skip the proof.

Lemma 8.2 (in $\mathbf{L}$)**.** *The forcing notion* $\mathbb{P}$ *belongs to* $\mathbf{L}$, *satisfies* $\mathbb{P} = \mathtt{Clos}(\mathbb{P})$ *and satisfies CCC in* $\mathbf{L}$. □

Lemma 8.3 (in $\mathbf{L}$)**.** *Let* $T \in \mathbb{P}$ *and* $f : 2^\omega \times \omega^\omega \to 2^\omega$ *be a Borel function. There is an ordinal* $\alpha < \omega_1$ *and a tree* $S \in \mathbb{U}_\alpha$, $S \subseteq T$, *on which* f *is normalized for* $\mathbb{U}_\alpha$.

Proof. By the choice of the enumeration of pairs in $\mathbf{ST} \times \omega^\omega$, there is an ordinal $\alpha < \omega_1$ such that $T \in \mathbb{P}_{<\alpha}$ and $T = T_\alpha$, $f = F_{r_\alpha}$. It remains to refer to $4^\dagger$. □

9 Model III: finalization

We use the product $\mathbb{P} \times \mathbb{C}$ of the forcing notion $\mathbb{P}$ defined in $\mathbf{L}$ and satisfying conditions $1^\dagger$–$5^\dagger$ as above, and the Cohen forcing, here in the form of $\mathbb{C} = \omega^{<\omega}$.

Theorem 9.1. *Let a pair of reals $\langle a_0, x_0\rangle$ be $\mathbb{P}\times\mathbb{C}$-generic over* $\mathbf{L}$. *Then it is true in* $\mathbf{L}[a,x]$ *that* $\mathbf{HOD} \subsetneqq \mathbf{HNT} \subsetneqq \mathbf{V}$, *and more precisely*:

(i) a_0 *is not* **OD** *in* $\mathbf{L}[a_0, x_0]$;

(ii) a_0 *belongs to* **HNT** *in* $\mathbf{L}[a_0, x_0]$;

(iii) x_0 *does not belong to* **HNT** *in* $\mathbf{L}[a_0, x_0]$.

Proof. (i) By the forcing product theorem, a_0 is a $\mathbb{P}$-generic real over $\mathbf{L}[x]$. However the forcing notion $\mathbb{P}$ is invariant w.r.t. shifts by construction, that is if $T \in \mathbb{P}$ and $\sigma \in 2^{<\omega}$ then $\sigma \cdot T \in \mathbb{P}$. Now the result required is obtained by an elementary argument, see Lemma 7.5 in [14].

(ii) It is sufficient to prove that the E_0-equivalence class $[a_0]_{\mathsf{E}_0}$ of our generic real a_0 is an **OD** set in $\mathbf{L}[a_0, x_0]$. According to $5^\dagger$, it suffices to establish the equality

$$[a_0]_{\mathsf{E}_0} = \bigcap\nolimits_{\xi<\omega_1} \bigcup\nolimits_{T\in\mathbb{P}_\xi} [T]\,. \qquad (*)$$

Note that every set $\mathbb{P}_\xi$ is pre-dense in $\mathbb{P}$; this follows from $3^\dagger$ and $4^\dagger$, see, for example, Lemma 6.3 in [14]. This immediately implies $a_0 \in \bigcup_{T\in\mathbb{P}_\xi}[T]$ for each ξ. Yet all sets $\mathbb{P}_\xi$ are invariant w.r.t. shifts by construction. Thus we have $\subseteq$ in (*).

To prove the inverse inclusion, assume that a real $b \in 2^\omega$ belongs to the right-hand side of (*) in $\mathbf{L}[a_0, x_0]$. It follows from Lemma 8.2 (and the countability of DC) that the forcing $\mathbb{P}\times\mathbb{C}$ preserves cardinals. We conclude that that $b = g(a_0, x_0)$ for some Borel function $g = F_q : 2^\omega \times \omega^\omega \to 2^\omega$ with a code $q \in \mathbf{BC} \cap \mathbf{L}$. **Assume to the contrary that** $b = g(a_0, x_0) \notin [a_0]_{\mathsf{E}_0}$. Since $x_0 \in \omega^\omega$ is a $\mathbb{C}$-generic real over $\mathbf{L}[a_0]$ by the forcing product theorem, this assumption is forced, so that there is a tuple $u \in \mathbb{C} = \omega^{<\omega}$ such that

$$f(a_0, x) \in \bigcap\nolimits_{\xi<\omega_1} \bigcup\nolimits_{T\in\mathbb{P}_\xi} [T] \smallsetminus [a_0]_{\mathsf{E}_0}\,,$$

whenever a real $x \in \mathscr{N}_u$ is $\mathbb{C}$-generic over $\mathbf{L}[a_0]$. (Recall that $\mathscr{N}_u = \{y \in \omega^\omega : u \subset y\}$.) Let H be the canonical homomorphism of ω^ω onto $\mathscr{N}_u$. We put $f(a,x) = g(a, H(x))$ for $a \in 2^\omega$, $x \in \omega^\omega$. Then H preserves the $\mathbb{C}$-genericity, and hence

$$f(a_0, x) \in \bigcap\nolimits_{\xi<\omega_1} \bigcup\nolimits_{T\in\mathbb{P}_\xi} [T] \smallsetminus [a]_{\mathsf{E}_0}\,, \qquad (**)$$

whenever $x \in \omega^\omega$ is $\mathbb{C}$-generic over $\mathbf{L}[a_0]$. Note that f also has a Borel code $r \in \mathbf{BC}$ in $\mathbf{L}$, so that $f = F_r$.

It follows from Lemma 8.3 that there is an ordinal $\alpha < \omega_1$ and a tree $S \in \mathbb{U}_\alpha$, on which f is normalized for $\mathbb{U}_\alpha$, and which satisfies $a_0 \in [S]$. Normalization means that, in $\mathbf{L}$, there is a dense $\mathbf{G}_\delta$ set $X \subseteq \omega^\omega$ satisfying one of the two options of Definition 7.1. Consider a real $z \in \omega^\omega \cap \mathbf{L}$ (a $\mathbf{G}_\delta$-*code* for X in $\mathbf{L}$) such that $X = X_z = \bigcap_k \bigcup_{z(2^k\cdot 3^j)=1} \mathscr{N}_{w_j}$, where $2^{<\omega} = \{w_j : j < \omega\}$ is a fixed recursive enumeration of tuples.

Case 1: there are tuples $v \in \omega^{<\omega}$, $\sigma \in 2^{<\omega}$, such that $f(a,x) = \sigma\cdot a$ for all points $a \in [S]$ and $x \in \mathscr{N}_v \cap X$. In other words, it is true in $\mathbf{L}$ that

$$\forall\, a \in [S]\, \forall\, x \in \mathscr{N}_v \cap X_z\, (f(a,x) = \sigma\cdot a)\,.$$

But this formula is absolute by Shoenfield, so it is also true in $\mathbf{L}[a_0, x_0]$. Take $a = a_0$ (recall: $a_0 \in [S]$) and any real $x \in \mathscr{N}_v$, $\mathbb{C}$-generic over $\mathbf{L}[a_0]$. Then $x \in X_z$, because X_z is a dense $\mathbf{G}_\delta$ with a code even from $\mathbf{L}$. Thus $f(a_0, x) = \sigma \cdot a_0 \in [a_0]_{\mathsf{E}_0}$, which contradicts (**).

Case 2: $f(a,x) \notin \bigcup_{\sigma \in 2^{<\omega} \wedge U \in \mathbb{U}_\alpha} \sigma \cdot [U]$ for all $a \in [S]$ and $x \in X$. By the definition of $\mathbb{P}_\alpha$, this implies $f(a,x) \notin \bigcup_{T \in \mathbb{P}_\alpha} [T]$ for all $a \in [S]$ and $x \in X$, and this again contradicts (**) for $a = a_0$.

The resulting contradiction in both cases refutes the contrary assumption above and completes the proof (ii).

Finally, (iii) follows from Lemma 2.2 for $\mathbf{V} = \mathbf{L}[a_0]$, since obviously $x_0 \notin \mathbf{L}[a_0]$.

$\square$ (theorems 9.1 and 5.1)

10 Model IV: nontypical sets sans the axiom of choice

The following theorem solves Problem 1.2 in the positive.

Theorem 10.1. *There is a generic extension of the constructible universe* $\mathbf{L}$ *in which* **AC** *holds but it is true that the class* **HNT** *does not satisfy* **AC**.

We will use a forcing notion $\mathbf{P} \in \mathbf{L}$ defined in [17, § 7] in order to obtain a model with a non-empty countable **OD** set of pairwise generic reals, containing no **OD** reals. Modulo technical details, this forcing coincides with the Jensen forcing from [12] (also presented in [11, 28.A]). The crucial step in [17] was the proof that those key properties of Jensen's forcing responsible for the uniqueness and definability of generic reals, previously established for $\mathbf{P}$ and its finite products $\mathbf{P}^n$, for example, in [2], also hold for the countable product $\mathbf{P}^\omega$. This forcing and its derivates were used in [1] and recently in [5, 16, 24, 25] for various purposes. This forcing $\mathbf{P}$ has the following main properties 1°–5°, see [17].

1°. $\mathbf{P} \in \mathbf{L}$, $\mathbf{P} \subseteq \mathbf{PT}$, $\mathbf{P}$ contains the full tree $2^{<\omega}$.

2°. If $u \in T \in \mathbf{P}$, then the *portion* $T {\restriction}_u$ also belongs to $\mathbf{P}$.

3°. $\mathbf{P}$ satisfies CCC in $\mathbf{L}$: each antichain $A \subseteq \mathbf{P}$ is at most countable.

4°. The set $\mathbf{P}^{<\omega}$, that is, the *weak product*, or *product with finite support*, also satisfies CCC. To be precise, here $\mathbf{P}^{<\omega}$ consists of all functions $\tau : \mathtt{dom}\,\tau \to \mathbf{P}$, where $\mathtt{dom}\,\tau \subseteq \omega$ is finite.

5°. Forcing $\mathbf{P}^{<\omega}$ naturally adjoins a generic sequence of the form $\mathbf{a} = \langle a_n \rangle_{n<\omega}$ of $\mathbf{P}$-generic reals $a_n \in 2^\omega$ to $\mathbf{L}$. The corresponding set $W(\mathbf{a}) = \{a_n : n < \omega\}$ is a (countable) **OD**, and even Π^1_2 (without parameters) set in the generic extension $\mathbf{L}[\mathbf{a}]$.

To prove Theorem 10.1, we consider a $\mathbf{P}^{<\omega}$-generic extension $\mathbf{L}[\mathbf{a}]$ as in 5°, and the class $\mathbf{HNT}^{\mathbf{L}[\mathbf{a}]}$ in this extension. Our goal will be to prove that **AC** is false in

$\mathbf{HNT}^{\mathbf{L}[\mathbf{a}]}$. This will be a simple consequence of the last statement of the next lemma. In the remainder, if $W \subseteq \omega^\omega$ then $\mathbb{C}(W)$ will denote Cohen forcing for adding a generic $1-1$ function $f : \omega \xrightarrow{\text{onto}} W$. Thus, $\mathbb{C}(W)$ consists of all $1-1$ functions $p : \mathtt{dom}\, p \to W$, where $\mathtt{dom}\, p \subseteq \omega$ is finite.

Lemma 10.2. *Let* $\mathbf{a} = \langle a_n \rangle_{n<\omega}$ *be a* $\mathbf{P}^{<\omega}$*-generic sequence over* $\mathbf{L}$*, and* $W = W(\mathbf{a})$*. Then :*

(i) $\mathbf{L}(W) \subseteq (\mathbf{HOD})^{\mathbf{L}[\mathbf{a}]}$;

(ii) W *is not a wellorderable set in* $\mathbf{L}(W)$;

(iii) $\mathbf{a}$ *is a* $\mathbb{C}(W)$*-generic function over* $\mathbf{L}(W)$;[2]

(iv) *if* $\mathbf{b} \in W^\omega$ *is a* $\mathbb{C}(W)$*-generic function over* $\mathbf{L}(W)$ *then* $\mathbf{b}$ *is a* $\mathbf{P}^{<\omega}$*-generic sequence over* $\mathbf{L}$ *in the sense of* 5° *and* $\mathbf{L}(W)[\mathbf{b}] = \mathbf{L}[\mathbf{b}]$;

(v) *if* $\mathbf{b} \in W^\omega$*, the pair* $\langle \mathbf{a}, \mathbf{b} \rangle$ *is* $(\mathbb{C}(W) \times \mathbb{C}(W))$*-generic over* $\mathbf{L}(W)$*, and* $Z \in \mathbf{L}(W)[\mathbf{a}] \cap \mathbf{L}(W)[\mathbf{b}]$*,* $Z \subseteq \mathbf{L}(W)$*, then* $Z \in \mathbf{L}(W)$;

(vi) *if* $Z \in \mathbf{L}[\mathbf{a}]$*,* $Z \subseteq W^\omega$ *is a countable* **OD** *set in* $\mathbf{L}[\mathbf{a}]$*, then* $Z \subseteq \mathbf{L}(W)$.

Proof (Theorem 10.1 from the lemma)**.** Here we show how Theorem 10.1 follows from the lemma, and then we prove the lemma itself. It suffices to prove that the set $W = W(\mathbf{a}) = \{a_n : n < \omega\}$, which belongs to $\mathbf{HNT}^{\mathbf{L}[\mathbf{a}]}$ according to 5°, is not wellorderable in $\mathbf{HNT}^{\mathbf{L}[\mathbf{a}]}$. Suppose to the contrary that such a wellordering exists. Then there is also a bijection $f \in \mathbf{HNT}^{\mathbf{L}[\mathbf{a}]}$, $f : \omega \xrightarrow{\text{onto}} W$. By definition, such a bijection belongs to a countable **OD** set $Z \in \mathbf{L}[\mathbf{a}]$, $Z \subseteq W^\omega$, in $\mathbf{L}[\mathbf{a}]$. According to claim (vi), we have $Z \subseteq \mathbf{L}(W)$, so $f \in \mathbf{L}(W)$, *i.e.* W is wellordered in $\mathbf{L}(W)$, which gives a contradiction with claim (ii) of the lemma.

□ (Theorem 10.1 from Lemma 10.2)

Proof (Lemma 10.2)**.** To prove (i), note that W is a countable **OD** set in $\mathbf{L}[a]$ by 5°, therefore W belongs to **HNT**.

Further, (ii) is a common property of permutation models.

To prove (iii), assume towards the contrary that there is a set $D \in \mathbf{L}(W)$, $D \subseteq \mathbb{C}(W)$, dense in $\mathbb{C}(W)$, and such that no condition $q \in D$ is extended by $\mathbf{a}$. As an element of $\mathbf{L}(W)$, the set D is definable in $\mathbf{L}(W)$ in the form:

$$D = \{q \in \mathbb{C}(W) : \varphi(q, W, a_0, \ldots, a_n, x)\},$$

where $x \in \mathbf{L}$, $n < \omega$, and $a_0, \ldots, a_n$ are the initial terms of the sequence $\mathbf{a}$. There is a condition $\tau \in \mathbf{P}^{<\omega}$, which is compatible with $\mathbf{a}$ and $\mathbf{P}^{<\omega}$-forces, over $\mathbf{L}$, our

[2] It is an important point here that the same function or sequence $\mathbf{a} \in W^\omega$ can act as both a $\mathbf{P}^{<\omega}$-generic object over $\mathbf{L}$ and as a $\mathbb{C}(W)$-generic object over $\mathbf{L}(W)$. Moreover, the extensions $\mathbf{L}[\mathbf{a}]$ and $\mathbf{L}(W)[\mathbf{a}]$ coincide. Such representations of a one-step generic extension as a multi-step extension (here two-step) are well known, see, for example, [28, 8], [15, § 7], [18].

assumption. That is, if a $\mathbf{P}^{<\omega}$-generic sequence $\mathbf{b} = \langle b_n \rangle_{n<\omega}$ extends τ then the set $D(\mathbf{b})$ defined in $\mathbf{L}(W(\mathbf{b}))$ by

$$D(\mathbf{b}) = \{q \in \mathbb{C}(W(\mathbf{b})) : \varphi(q, W(\mathbf{b}), b_0, \ldots, b_n, x)\},$$

is dense in $\mathbb{C}(W(\mathbf{b}))$, but no condition $q \in D(\mathbf{b})$ is extended by $\mathbf{b}$.

We can w.l.o.g. assume that $\mathtt{dom}\, \tau = \{0, 1, \ldots, n\}$.

Now consider a condition $p \in \mathbb{C}(W)$ defined by $p(j) = a_j$ for all $j = 0, 1, \ldots, n$. As D is dense, there exists a condition $q \in D$ extending p. Then $\mathtt{dom}\, q = \{0, 1, \ldots, n\} \cup U$, where $U \subseteq \{n+1, n+2, \ldots\}$ is a finite set. If $i \in U$ then by definiton $q(i) = a_{k_i}$, where $k_i \geq n+1$ and the map $i \longmapsto k_i$ is injective.

There is a bijection $\pi : \omega \xrightarrow{\text{onto}} \omega$ satisfying $\pi(j) = j$ for all $j \leq n$, $\pi(i) = k_i$ for all $i \in U$, and $\pi(\ell) = \ell$ generally for all but finite numbers $\ell < \omega$, in particular, $\pi \in \mathbf{L}$. The sequence $\mathbf{b} = \langle b_n \rangle_{n<\omega}$, defined by $b_i = a_{\pi(i)}$ for all $i < \omega$, is $\mathbf{P}^{<\omega}$-generic by the choice of π, and obviously $W(\mathbf{b}) = W(\mathbf{a}) = W$. In addition, $b_j = a_j$ for all $j = 0, 1, \ldots, n$, thus $\mathbf{b}$ extends τ. We also have $D(\mathbf{b}) = D(\mathbf{a}) = D$, and hence the abovedefined condition q belongs to $D(\mathbf{b})$. *We finally claim that* $\mathbf{b}$ *extends* q. This contradicts the contrary assumption above and completes the proof of (iii).

To prove the extension claim, one has to check that $q(i) = b_i$ for all $i \in U$. If $i \in U$ then $b_i = a_{\pi(i)} = a_{k_i} = q(i)$ by construction, as required.

To prove claim (iv) of the lemma, suppose otherwise. This is forced by a condition $p \in \mathbb{C}(W)$, such that no function $\mathbf{b} \in W^\omega$, $\mathbb{C}(W)$-generic over $\mathbf{L}(W)$ and extending p, is $\mathbf{P}^{<\omega}$-generic over $\mathbf{L}$. Arguing as in the proof of (iii) above, we get a suitable permutation π that yields a function $\mathbf{b} \in W^\omega$, $\mathbb{C}(W)$-generic over $\mathbf{L}(W)$ and in the same time $\mathbf{P}^{<\omega}$-generic over $\mathbf{L}$ along with $\mathbf{a}$, and satisfies $W(\mathbf{b}) = W(\mathbf{a}) = W$ (as a finite permutation of $\mathbf{a}$), and extends the condition p. Therefore $\mathbf{b}$ is $\mathbb{C}(W)$-generic over $\mathbf{L}(W)$ by claim (iii) already established. This is a contradiction.

(v) This is a generally known fact, yet we add a short proof. As $Z \subseteq \mathbf{L}(W)$, there is a set $X \in \mathbf{L}(W)$ with $Z \subseteq X$. Consider $\mathbb{C}(W)$-names $s, t \in \mathbf{L}(W)$ such that $Z = s[\![\mathbf{a}]\!] = t[\![\mathbf{b}]\!]$, where $s[\![\mathbf{a}]\!]$ denotes the $\mathbf{a}$-interpretation of any given $\mathbb{C}(W)$-name s. By genericity, the equality $s[\![\mathbf{a}]\!] = t[\![\mathbf{b}]\!]$ is forced by a pair of conditions $p, q \in \mathbb{C}(W)$, *i.e.* $\mathbf{a}$ extends p, $\mathbf{b}$ extends q, and if a pair $\langle \mathbf{a}', \mathbf{b}' \rangle$ is $(\mathbb{C}(W) \times \mathbb{C}(W))$-generic over $\mathbf{L}(W)$ and $\mathbf{a}'$ extends p, $\mathbf{b}'$ extends q, then $s[\![\mathbf{a}']\!] = t[\![\mathbf{b}']\!]$. *We claim that the condition* p $\mathbb{C}(W)$*-decides over* $\mathbf{L}(W)$ *every sentence of the form* $x \in s[\![\dot{\mathbf{a}}]\!]$, where $\dot{\mathbf{a}}$ is a canonical $\mathbb{C}(W)$-name for the principal generic function in W^ω.

Indeed otherwise there exist functions $\mathbf{a}', \mathbf{a}'' \in W^\omega$, $\mathbb{C}(W)$-generic over $\mathbf{L}(W)$ and extending the condition p, and an element $x \in X$, such that $x \in s[\![\mathbf{a}']\!]$ but $x \notin s[\![\mathbf{a}'']\!]$. Consider a function $\mathbf{b}' \in W^\omega$, $\mathbb{C}(W)$-generic both over $\mathbf{L}(W)[\mathbf{a}']$ and over $\mathbf{L}(W)[\mathbf{a}'']$, and extending the condition q. Then either pair $\langle \mathbf{a}', \mathbf{b}' \rangle$, $\langle \mathbf{a}'', \mathbf{b}' \rangle$ is $(\mathbb{C}(W) \times \mathbb{C}(W))$-generic over $\mathbf{L}(W)$, but at least one of the two equalities $s[\![\mathbf{a}']\!] = t[\![\mathbf{b}']\!]$, $s[\![\mathbf{a}'']\!] = t[\![\mathbf{b}']\!]$ definitely fails, which is a contradiction.

Thus p indeed $\mathbb{C}(W)$-decides over $\mathbf{L}(W)$ every sentence $x \in s[\![\dot{\mathbf{a}}]\!]$. This implies

$$Z = \{x \in X : p \ \mathbb{C}(W)\text{-forces } x \in s[\![\dot{\mathbf{a}}]\!] \text{ in } \mathbf{L}(W)\} \in \mathbf{L}(W).$$

(vi) To prove this key claim we apply a method introduced in [19]. Consider a countable OD set $Z \subseteq W^\omega$ in $\mathbf{L}[\mathbf{a}]$. **Suppose towards the contrary that** $Z \not\subseteq \mathbf{L}(W)$.

There is a formula $\varphi(z)$ with an ordinal γ_0 as a parameter, such that $Z = \{z \in W^\omega : \varphi(z)\}$ in $\mathbf{L}[\mathbf{a}]$. There also exists a condition $p_0 \in \mathbb{C}(W)$, $p_0 \subset \mathbf{a}$, which forces our assumptions, that is

(1) p_0 $\mathbb{C}(W)$-forces, over $\mathbf{L}(W)$, that the set $\{z \in W^\omega : \varphi(z)\}$ is countable and is not included in $\mathbf{L}(W)$, or equivalently, if $\mathbf{b} \in W^\omega$ is $\mathbb{C}(W)$-generic over $\mathbf{L}(W)$ and extends p_0 then it is true in the extension $\mathbf{L}(W)[\mathbf{b}] = \mathbf{L}[\mathbf{b}]$ that the set $\Phi_\mathbf{b} = \{z \in W^\omega : \varphi(z)\}$ is countable and $\exists\, z\, (z \notin \mathbf{L}(W) \wedge \varphi(z))$. It follows from the countability that there is a map $f_\mathbf{b} : \omega \xrightarrow{\text{onto}} \Phi_\mathbf{b}$, $f_\mathbf{b} \in \mathbf{L}[\mathbf{b}]$.

Let $T \in \mathbf{L}(W)$ be a canonical $\mathbb{C}(W)$-name for $f_\mathbf{b}$, so $f_\mathbf{b} = T[\![\mathbf{b}]\!]$. Then (1) implies:

(2) p_0 $\mathbb{C}(W)$-forces $\mathtt{ran}\, T[\![\dot{\mathbf{a}}]\!] = \{T[\![\dot{\mathbf{a}}]\!](n) : n < \omega\} = \{z \in W^\omega : \varphi(z)\} \not\subseteq \mathbf{L}(W)$ over $\mathbf{L}(W)$, or equivalently, if $\mathbf{b} \in W^\omega$ $\mathbb{C}(W)$-generic over $\mathbf{L}(W)$ and $p_0 \subset \mathbf{b}$ then it is true in $\mathbf{L}[\mathbf{b}]$ that

$$\mathtt{ran}\, T[\![\mathbf{b}]\!] = \{T[\![\mathbf{b}]\!](n) : n < \omega\} = \{z \in W^\omega : \varphi(z)\} \not\subseteq \mathbf{L}(W).$$

Now our goal will be to **get a contradiction from** (2). Consider a regular uncountable cardinal $\kappa > \gamma_0$, such that the set $\mathbf{L}_\kappa$ is an elementary submodel of $\mathbf{L}$ w.r.t. a fragment of **ZFC** sufficiently large to prove the part of Lemma 10.2 already established including both (1) and (2). Then the set $\mathbf{L}_\kappa(W)$ contains γ_0 and the name T. As elements of the model $\mathbf{L}_\kappa(W) \subseteq \mathbf{L}_\kappa[\mathbf{a}]$, the sets W, T admit canonical $\mathbf{P}^{<\omega}$-names in $\mathbf{L}_\kappa$. Consider a countable elementary submodel $\mathfrak{M} \in \mathbf{L}$ of $\mathbf{L}_\kappa$, containing those names and γ_0. Then the sets W, T and the forcing notion $\mathbb{C}(W)$ belong to $\mathfrak{M}(W)$. Consider the Mostowski collapse map $\pi : \mathfrak{M}(W) \xrightarrow{\text{onto}} \mathbf{L}_\lambda(W)$ onto a transitive set of the form $\mathbf{L}_\lambda(W)$, countable in $\mathbf{L}[\mathbf{a}]$, where $\lambda < \omega_1^\mathbf{L}$. As W is countable, we have $\pi(W) = W$, $\pi(T) = T$, and hence $T \in \mathbf{L}_\lambda(W)$, $\mathbb{C}(W) \in \mathbf{L}_\lambda(W)$.

We assert that there is $\mathbf{b} \in W^\omega$ satisfying

(3) $\mathbf{L}[\mathbf{b}] = \mathbf{L}[\mathbf{a}]$, $\mathbf{b}$ is a $\mathbb{C}(W)$-generic function over $\mathbf{L}(W)$, $p_0 \subset \mathbf{b}$, and the pair $\langle \mathbf{a}, \mathbf{b} \rangle$ is $(\mathbb{C}(W) \times \mathbb{C}(W))$-generic over $\mathbf{L}_\lambda(W)$.

Indeed, as the set $\mathbf{L}_\lambda$ is countable in $\mathbf{L}$, there exists a bijection $h : \omega \xrightarrow{\text{onto}} \omega$, $h \in \mathbf{L}$, equal to the identity on the (finite) domain $\mathtt{dom}\, p_0$ of the condition $p_0 \in \mathbb{C}(W)$ (see above on p_0), and generic over $\mathbf{L}_\lambda$ in the sense of the Cohen-style forcing notion $\mathbb{B}$ which consists of all injective tuples $u \in \omega^{<\omega}$. Let $\mathbf{b}(n) = \mathbf{a}(h(n))$ for all n, *i.e.* $\mathbf{b} = \mathbf{a} \circ h$ is a superposition. Let's check that $\mathbf{b}$ satisfies (3).

Indeed, the function $\mathbf{a}$ of Lemma 10.2 is generic over $\mathbf{L}$, hence it is generic over $\mathbf{L}_\lambda[h] \in \mathbf{L}$, and hence the bijection h is $\mathbb{B}$-generic over $\mathbf{L}_\lambda[\mathbf{a}]$ by the product forcing theorem. Therefore h is generic over $\mathbf{L}_\lambda(W)$, a smaller model. However $\mathbf{a}$ is $\mathbb{C}(W)$-generic over $\mathbf{L}_\lambda(W)$ by (iii) of the lemma. It follows that the pair $\langle \mathbf{a}, h \rangle$ is

$(\mathbb{C}(W) \times \mathbb{B})$-generic over $\mathbf{L}_\lambda(W)$ still by the product forcing theorem. One easily proves then that $\langle \mathbf{a}, \mathbf{b} \rangle$ is $(\mathbb{C}(W) \times \mathbb{C}(W))$-generic over $\mathbf{L}_\lambda(W)$.

We further have $\mathbf{L}[\mathbf{b}] = \mathbf{L}[\mathbf{a}]$, because $h \in \mathbf{L}$. Moreover $\mathbf{b}$ is $\mathbb{C}(W)$-generic over $\mathbf{L}(W)$, since $h \in \mathbf{L}$ induces an order isomorphism of $\mathbb{C}(W)$ in $\mathbf{L}(W)$. Finally h is compatible with p_0 because h is the identity on $\mathtt{dom}\, p_0$ by construction. This completes the proof that $\mathbf{b} = \mathbf{a} \circ h$ satisfies (3).

In particular $W(\mathbf{b}) = W(\mathbf{a}) = W$ holds, $\mathtt{ran}\, T[\![\mathbf{a}]\!] = \mathtt{ran}\, T[\![\mathbf{b}]\!] \not\subseteq \mathbf{L}(W)$ by (2). On the other hand, the set $Z = \mathtt{ran}\, T[\![\mathbf{a}]\!] = \mathtt{ran}\, T[\![\mathbf{b}]\!]$, belongs to the intersection $\mathbf{L}_\lambda(W)[\mathbf{a}] \cap \mathbf{L}_\lambda(W)[\mathbf{b}]$ by construction. We conclude that Z belongs to $\mathbf{L}(W)$ by (v) of the lemma. (The above proof (v) is valid for $\mathbf{L}_\lambda$ instead of $\mathbf{L}$ as the ground model.) The contradiction obtained completes the proof of (vi).

$\square$ (Lemma 10.2 and Theorem 10.1)

11 Comments and questions

Coming back to the Cohen-generic extensions, recall that if a is a Cohen generic real over $\mathbf{L}$ then $\mathbf{HNT} = \mathbf{L}$ in $\mathbf{L}[a]$ by Theorem 2.1.

Problem 11.1. Is it true in generic extensions of $\mathbf{L}$ by a single Cohen real that any countable $\mathbf{OD}$ set consists of $\mathbf{OD}$ elements?

We cannot solve this even for *finite* **OD** sets. By the way it is not that obvious to expect the *positive* answer. Indeed, the problem solves in the *negative* for Sacks and some other generic extensions even for *pairs*, see [3, 4]. For instance, if a is a Sacks-generic real over $\mathbf{L}$ then it is true in $\mathbf{L}[a]$ that there is an **OD** unordered pair $\{X, Y\}$ of sets of reals $X, Y \subseteq \mathscr{P}(2^\omega)$ such that X, Y themselves are non-**OD** sets. See [3] for a proof of this rather surprising result originally by Solovay.

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