A product forcing model in which the Russell-nontypical sets satisfy ZFC strictly between HOD and the universe*

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Abstract

A set is nontypical in the Russell sense, if it belongs to a countable ordinal definable set. The class **HNT** of all hereditarily nontypical sets satisfies all axioms of **ZF** and the double inclusion $\mathbf{HOD} \subseteq \mathbf{HNT} \subseteq \mathbf{V}$ holds. Solving a problem recently proposed by Tzouvaras, a generic extension $\mathbf{L}[a, x]$ of $\mathbf{L}$, by two reals a, x, is presented in which $\mathbf{L} = \mathbf{HOD} \subsetneqq \mathbf{L}[a] = \mathbf{HNT} \subsetneqq \mathbf{V} = \mathbf{L}[a, x]$, so that **HNT** is a model of **ZFC** strictly between **HOD** and the universe.

1 Introduction

A set x is *nontypical with a cardinal parameter* κ, for short $x \in \mathbf{NT}_\kappa$, if it belongs to an **OD** (ordinal definable) set X of cardinality $\mathtt{card}\, X < \kappa$. A set x is *hereditarily nontypical with a cardinal parameter* κ, for short $x \in \mathbf{HNT}_\kappa$, if it itself, all its elements, elements of elements, and so on, are all nontypical, in other words the *transitive closure* $\mathrm{TC}(x)$ satisfies $\mathrm{TC}(x) \subseteq \mathbf{NT}_\kappa$. These notions Tzouvaras [22, 21] connected with some philosophical and mathematical ideas of Bertrand Russell and works of van Lambalgen [19] *etc.* on the concept of randomness. They contribute to the ongoing study of important classes of sets in the set theoretic universe $\mathbf{V}$ which themselves satisfy the axioms of set theory, similarly to the Gödel class $\mathbf{L}$ of all *constructible* sets and the class **HOD** of all *hereditarily ordinal definable* sets [7].

It is clear that $\mathbf{NT}_2 = \mathbf{OD}$ and $\mathbf{HNT}_2 = \mathbf{HOD}$, thus the case $\kappa = 2$ corresponds to the ordinal definability. The classes $\mathbf{NT}_\omega$ (elements of finite ordinal definable sets) and $\mathbf{HNT}_\omega$ correspong to *algebraically definability* recently studied in [4, 5, 6]. The following classes correspond to the next cardinality level $\kappa = \omega_1$:

$$\mathbf{NT} := \mathbf{NT}_{\omega_1} \quad \text{and} \quad \mathbf{HNT} := \mathbf{HNT}_{\omega_1}.$$

Thus $x \in \mathbf{NT}$ iff x belongs to a countable **OD** set, and $x \in \mathbf{HNT}$ iff $\mathrm{TC}(x) \subseteq \mathbf{NT}$.

The class **HNT** is transitive and, as shown in [21], satisfies all axioms of **ZF** (the axiom of choice **AC** not included), and also satisfies the relation $\mathbf{HOD} \subseteq \mathbf{HNT} \subseteq \mathbf{V}$. Tzouvaras [21, 2.15] asks whether the double strict inequality $\mathbf{HOD} \subsetneqq \mathbf{HNT} \subsetneqq \mathbf{V}$

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can be realized in an appropriate model of **ZFC**. The following theorem, the main result of this paper, answers this question in the affirmative.

Theorem 1.1. *Let $\mathbb{C} = \omega^{<\omega}$ be the Cohen forcing for adding a generic real $x \in \omega^\omega$ to $\mathbf{L}$. There is a forcing notion $\mathbb{P} \in \mathbf{L}$, which consists of Silver trees, and such that if a pair of reals $\langle a, x\rangle$ is $(\mathbb{P} \times \mathbb{C})$-generic over $\mathbf{L}$ then it is true in $\mathbf{L}[a, x]$ that*

$$\mathbf{L} = \mathbf{HOD} \subsetneqq \mathbf{L}[a] = \mathbf{HNT} \subsetneqq \mathbf{V} = \mathbf{L}[a, x]\,.$$

Note that **HNT** satisfies **ZFC**, not merely **ZF**, in the model $\mathbf{L}[a, x]$ of the theorem.

Remark 1.2. This result is an essential strengthening of [17, Theorem 9.1]. Comparably to the latter, the claims that $\mathbf{L} = \mathbf{HOD}$ (instead of simply $a \notin \mathbf{HOD}$) and especially $\mathbf{L}[a] = \mathbf{HNT}$ (instead of just $x \notin \mathbf{HNT}$) are added here, w.r.t. basically the same model, which makes the research more accomplished.

To make the text of this preprint more self-contained, we decided to near-copypast some definitions and auxiliary results from [17], instead of briefly citing them as it would be more accustomed in a journal paper. □

To prove the theorem, we make use of a forcing notion $\mathbb{P}$ introduced in [10] in order to define a generic real $a \in 2^\omega$ whose E_0-equivalence class $[a]_{\mathsf{E}_0}$ is a lightface $\varPi^1_2$ (hence **OD**) set of reals with no **OD** element. This property of $\mathbb{P}$ is responsible for a $\mathbb{P}$-generic real a to belong to **HNT**, and ultimately to $\mathbf{L}[a] \subseteq \mathbf{HNT}$, in $\mathbf{L}[a, x]$. This will be based on some results on Silver trees and Borel functions in Sections 2,3,4. The construction of $\mathbb{P}$ in $\mathbf{L}$ is given in Sections 5,6. The proof that $\mathbf{L}[a] \subseteq \mathbf{HNT}$ in $\mathbf{L}[a, x]$ follows in Section 8.

The inverse inclusion $\mathbf{HNT} \subseteq \mathbf{L}[a]$ in $\mathbf{L}[a, x]$ will be proved in Section 9 on the basis of our earlier result [11] on countable **OD** sets in Cohen-generic extensions.

2 Perfect trees and Silver trees

Our results will involve forcing notions that consist of perfect trees and Silver trees. Here we introduce the relevant terminology from our earlier works [10, 12, 13].

By $2^{<\omega}$ we denote the set of all *tuples* (finite sequences) of terms $0, 1$, including the empty tuple Λ. The length of a tuple s is denoted by $\mathtt{lh}\, s$, and $2^n = \{s \in 2^{<\omega} : \mathtt{lh}\, s = n\}$ (all tuples of length n). A tree $\varnothing \neq T \subseteq 2^{<\omega}$ is *perfect*, symbolically $T \in \mathbf{PT}$, if it has no endpoints and isolated branches. In this case, the set

$$[T] = \{a \in 2^\omega : \forall\, n\, (a{\restriction}n \in T)\}$$

of all *branches* of T is a perfect set in 2^ω. Note that $[S] \cap [T] = \varnothing$ iff $S \cap T$ is finite.

- If $u \in T \in \mathbf{PT}$, then a *portion* (or a *pruned tree*) $T{\restriction}_u \in \mathbf{PT}$ is defined by $T{\restriction}_u = \{s \in T : u \subset s \vee s \subseteq u\}$.
- A tree $S \subseteq T$ is *clopen* in T iff it is equal to the union of a finite number of portions of T. This is equivalent to $[S]$ being clopen in $[T]$.

A tree $T \subseteq 2^{<\omega}$ is a *Silver tree*, symbolically $T \in \mathbf{ST}$, if there is an infinite sequence of tuples $u_k = u_k(T) \in 2^{<\omega}$, such that T consists of all tuples of the form

$$s = u_0{}^\frown i_0{}^\frown u_1{}^\frown i_1{}^\frown u_2{}^\frown i_2{}^\frown \ldots {}^\frown u_n{}^\frown i_n$$

and their sub-tuples, where $n < \omega$ and $i_k = 0, 1$. Then the *stem* $\mathtt{stem}(T) = u_0(T)$ is equal to the largest tuple $s \in T$ with $T = T\restriction_s$, and $[T]$ consists of all infinite sequences $a = u_0{}^\frown i_0{}^\frown u_1{}^\frown i_1{}^\frown u_2{}^\frown i_2{}^\frown \cdots \in 2^\omega$, where $i_k = 0, 1$, $\forall\, k$. Put

$$\mathtt{spl}_n(T) = \mathtt{lh}\, u_0 + 1 + \mathtt{lh}\, u_1 + 1 + \cdots + \mathtt{lh}\, u_{n-1} + 1 + \mathtt{lh}\, u_n\,.$$

In particular, $\mathtt{spl}_0(T) = \mathtt{lh}\, u_0$. Thus $\mathtt{spl}(T) = \{\mathtt{spl}_n(T) : n < \omega\} \subseteq \omega$ is the set of all *splitting levels* of the Silver tree T.

Action. Let $\sigma \in 2^{<\omega}$. If $v \in 2^{<\omega}$ is another tuple of length $\mathtt{lh}\, v \ge \mathtt{lh}\, \sigma$, then the tuple $v' = \sigma \bullet v$ of the same length $\mathtt{lh}\, v' = \mathtt{lh}\, v$ is defined by $v'(i) = v(i) +_2 \sigma(i)$ (addition modulo 2) for all $i < \mathtt{lh}\, \sigma$, but $v'(i) = v(i)$ whenever $\mathtt{lh}\, \sigma \le i < \mathtt{lh}\, v$. If $\mathtt{lh}\, v < \mathtt{lh}\, \sigma$, then we just define $\sigma \bullet v = (\sigma \restriction \mathtt{lh}\, v) \bullet v$.

If $a \in 2^\omega$, then similarly $a' = \sigma \bullet a \in 2^\omega$, $a'(i) = a(i) +_2 \sigma(i)$ for $i < \mathtt{lh}\, \sigma$, but $a'(i) = a(i)$ for $i \ge \mathtt{lh}\, \sigma$. If $T \subseteq 2^{<\omega}$, $X \subseteq 2^\omega$, then the sets

$$\sigma \bullet T = \{\sigma \bullet v : v \in T\} \quad \text{and} \quad \sigma \bullet X = \{\sigma \bullet a : a \in X\}$$

are *shifts* of the tree T and the set X accordingly.

Lemma 2.1 ([13], 3.4)**.** *If $n < \omega$ and $u, v \in T \cap 2^n$, then $T\restriction_u = v \bullet u \bullet (T\restriction_v)$.*
If $t \in T \in \mathbf{ST}$ and $\sigma \in 2^{<\omega}$, then $\sigma \bullet T \in \mathbf{ST}$ and $T\restriction_s \in \mathbf{ST}$. □

Definition 2.2 (refinements). Assume that $T, S \in \mathbf{ST}$, $S \subseteq T$, $n < \omega$. We define $S \subseteq_n T$ (the tree S *n-refines* T) if $S \subseteq T$ and $\mathtt{spl}_k(T) = \mathtt{spl}_k(S)$ for all $k < n$. This is equivalent to ($S \subseteq T$ and) $u_k(S) = u_k(T)$ for all $k < n$, of course. □

Then $S \subseteq_0 T$ is equivalent to $S \subseteq T$, and $S \subseteq_{n+1} T$ implies $S \subseteq_n T$ (and $S \subseteq T$), but if $n \ge 1$ then $S \subseteq_n T$ is equivalent to $\mathtt{spl}_{n-1}(T) = \mathtt{spl}_{n-1}(S)$.

Lemma 2.3. *Assume that $T, U \in \mathbf{ST}$, $n < \omega$, $h > \mathtt{spl}_{n-1}(T)$, $s_0 \in 2^h \cap T$, and $U \subseteq T\restriction_{s_0}$. Then there is a unique tree $S \in \mathbf{ST}$ such that $S \subseteq_n T$ and $S\restriction_{s_0} = U$.*
If in addition U is clopen in T then S is clopen in T as well.

Proof (sketch)**.** Define a tree S so that $S \cap 2^h = T \cap 2^h$, and if $t \in T \cap 2^h$ then, by Lemma 2.1, $S\restriction_t = (t \bullet s_0) \bullet U$; then $S\restriction_{s_0} = U$. To check that $S \in \mathbf{ST}$, we can easily compute the tuples $u_k(S)$. Namely, as $U \subseteq T\restriction_{s_0}$, we have $s_0 \subseteq u_0(U) = \mathtt{stem}(U)$, hence $\ell = \mathtt{lh}\,(u_0(U)) \ge h > m = \mathtt{spl}_{n-1}(T)$. Then $u_k(S) = u_k(T)$ for all $k < n$, $u_n(S) = u_0(U)\restriction [m, \ell)$ (thus $u_n(S) \in 2^{\ell - m}$), and $u_k(S) = u_k(U)$ for all $k > n$. □

Lemma 2.4 ([13], Lemma 4.4)**.** *Let $\ldots \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0$ be a sequence of trees in $\mathbf{ST}$. Then $T = \bigcap_n t_n \in \mathbf{ST}$.* □

Proof (sketch)**.** By definition we have $u_k(T_n) = u_k(T_{n+1})$ for all $k \le n$. Then one easily computes that $u_n(T) = u_n(T_n)$ for all n. □

3 Reduction of Borel maps to continuous ones

A classical theorem claims that in Polish spaces every Borel function is continuous on a suitable dense $\mathbf{G}_\delta$ set (Theorem 8.38 in Kechris [18]). It is also known that a Borel map defined on 2^ω is continuous on a suitable Silver tree. The next lemma combines these two results. Our interest in functions defined on $2^\omega \times \omega^\omega$ is motivated by further applications to reals in generic extensions of the form $\mathbf{L}[a, x]$, where $a \in 2^\omega$ is $\mathbb{P}$-generic real for some $\mathbb{P} \subseteq \mathbf{ST}$ while $x \in \omega^\omega$ is just Cohen generic.

In the remainder, if $v \in \omega^{<\omega}$ (a tuple of natural numbers), then we define $\mathscr{N}_v = \{x \in \omega^\omega : v \subset x\}$, a *Baire interval* or *portion* in the Baire space ω^ω.

Lemma 3.1. *Let $T \in \mathbf{ST}$ and $f : 2^\omega \times \omega^\omega \to 2^\omega$ be a Borel map. There is a Silver tree $S \subseteq T$ and a dense $\mathbf{G}_\delta$ set $D \subseteq \omega^\omega$ such that f is continuous on $[S] \times D$.*

Proof. By the abovementioned classical theorem, f is already continuous on some dense $\mathbf{G}_\delta$ set $Z \subseteq [T] \times \omega^\omega$. It remains to define a Silver tree $S \subseteq T$ and a dense $\mathbf{G}_\delta$ set $D \subseteq \omega^\omega$ such that $[S] \times D \subseteq Z$. This will be our goal.

We have $Z = \bigcap_n Z_n$, where each $Z_n \subseteq [T] \times \omega^\omega$ is open dense.

We fix a recursive enumeration $\omega \times \omega^{<\omega} = \{\langle N_k, v_k \rangle : k < \omega\}$. We will define a sequence of Silver trees S_k and tuples $w_k \in \omega^{<\omega}$ satisfying the following:

(1) $\ldots \subseteq_4 S_3 \subseteq_3 S_2 \subseteq_2 s_1 \subseteq_1 S_0 = T$, as in Lemma 2.4;

(2) if $k < \omega$ then S_{k+1} is **clopen** in S_k (see Section 2);

(3) $v_k \subseteq w_k$ and $[S_{k+1}] \times \mathscr{N}_{w_k} \subseteq Z_{N_k}$, for all k.

At step 0 we already have $S_0 = T$.

Assume that $S_k \in \mathbf{ST}$ has already been defined. Let $h = \mathtt{spl}_{k+1}(S_k)$.

Consider any tuple $t \in 2^h \cap S_k$. As Z_{N_k} is open dense, there is a tuple $u_1 \in \omega^{<\omega}$ and a Silver tree $A_1 \subseteq S_k {\restriction}_t$, clopen in S_k (for example, a portion in S_k) such that $v_k \subseteq u_1$ and $[A_1] \times \mathscr{N}_{u_1} \subseteq Z_{N_k}$. According to Lemma 2.3, there exists a Silver tree $U_1 \subseteq_{k+1} S_k$, clopen in S_k along with A, such that $U_1 {\restriction}_t = A_1$, so $[U_1 {\restriction}_t] \times \mathscr{N}_{u_1} \subseteq Z_{N_k}$ by construction.

Now take another tuple $t' \in 2^h \cap S_k$, and similarly find $u_2 \in \omega^{<\omega}$ and a Silver tree $A_2 \subseteq U_1 {\restriction}_{t'}$, clopen in U_1, such that $u_1 \subseteq u_2$ and $[A_2] \times \mathscr{N}_{u_2} \subseteq Z_{N_k}$. Once again there is a Silver tree $U_2 \subseteq_{k+1} U_1$, clopen in S_k and such that $[U_2 {\restriction}_{t'}] \times \mathscr{N}_{u_2} \subseteq Z_{N_k}$.

We iterate this construction over all tuples $t \in 2^h \cap S_k$, $\subseteq_{k+1}$-shrinking trees and extending tuples in $\omega^{<\omega}$. We get a Silver tree $U \subseteq_{k+1} S_k$, clopen in S_k, and a tuple $w \in \omega^{<\omega}$, that $v_k \subseteq w$ and $[U] \times \mathscr{N}_w \subseteq Z_{N_k}$. Take $w_k = w$, $S_{k+1} = U$. This completes the inductive step.

As a result we get a sequence $\ldots \subseteq_4 S_3 \subseteq_3 S_2 \subseteq_2 S_1 \subseteq_1 S_0 = T$ of Silver trees S_k, and tuples $w_k \in \omega^{<\omega}$ $(k < \omega)$, which satisfy (1),(2),(3).

We put $S = \bigcap_k S_k$; then $S \in \mathbf{ST}$ by (1) and Lemma 2.4, and $S \subseteq T$.

If $n < \omega$ then let $W_n = \{w_k : N_k = n\}$. We claim that $D_n = \bigcup_{w \in W_n} \mathscr{N}_w$ is an open dense set in ω^ω. Indeed, let $v \in \omega^{<\omega}$. Consider any k such that that $v_k = v$

and $N_k = n$. By construction, we have $v \subseteq w_k \in W_n$, as required. We conclude that the set $D = \bigcap_n D_n$ is dense and $\mathbf{G}_\delta$.

To check $[S] \times D \subseteq Z$, let $n < \omega$; we show that $[S] \times D \subseteq Z_n$. Let $a \in [S]$ and $x \in D$, in particular $x \in D_n$, so $x \in \mathscr{N}_{w_k}$ for some k with $N_k = n$. However, $[S_{k+1}] \times \mathscr{N}_{w_k} \subseteq Z_n$ by (3), and at the same time obviously $a \in [S_{k+1}]$. We conclude that in fact $\langle a, x \rangle \in Z_n$, as required. □ (Lemma 3.1)

Corollary 3.2. *Let $T \in \mathbf{ST}$ and $f : 2^\omega \to 2^\omega$ be a Borel map. There is a Silver tree $S \subseteq T$ such that f is continuous on $[S]$.* □

We add the following result that belongs to the folklore of the Silver forcing. See Corollary 5.4 in [12] for a proof.

Lemma 3.3. *Assume that $T \in \mathbf{ST}$ and $f : 2^\omega \to 2^\omega$ is a continuous map. Then there is a Silver tree $S \subseteq T$ such that f is either a bijection or a constant on $[S]$.* □

4 Normalization of Borel maps

Definition 4.1. A map $f : 2^\omega \times \omega^\omega \to 2^\omega$ is *normalized on $T \in \mathbf{ST}$ for $\mathbb{U} \subseteq \mathbf{ST}$* if there exists a dense $\mathbf{G}_\delta$ set $X \subseteq \omega^\omega$ such that f is continuous on $[T] \times X$ and:

- either (a) there are tuples $v \in \omega^{<\omega}$, $\sigma \in 2^{<\omega}$ such that $f(a, x) = \sigma \cdot a$ for all $a \in [T]$ and $x \in \mathscr{N}_v \cap X$, where, we remind, $\mathscr{N}_v = \{x \in \omega^\omega : v \subset x\}$;
- or (b) $f(a, x) \notin \bigcup_{\sigma \in 2^{<\omega} \wedge S \in \mathbb{U}} \sigma \cdot [S]$ for all $a \in [T]$ and $x \in X$. □

Theorem 4.2. *Let $\mathbb{U} = \{T_0, T_1, T_2, \dots\} \subseteq \mathbf{ST}$ and $f : 2^\omega \times \omega^\omega \to 2^\omega$ be a Borel map. There is a set $\mathbb{U}' = \{S_0, S_1, S_2, \dots\} \subseteq \mathbf{ST}$, such that $S_n \subseteq T_n$ for all n and f is normalized on S_0 for $\mathbb{U}'$.*

Proof. First of all, according to Lemma 3.1, there is a Silver tree $T' \subseteq T_0$ and a dense $\mathbf{G}_\delta$ set $W \subseteq \omega^\omega$ such that f is continuous on $[T'] \times W$. And since any dense $\mathbf{G}_\delta$ set $X \subseteq \omega^\omega$ is homeomorphic to ω^ω, we can w.l.o.g. assume that $W = \omega^\omega$ and $T' = T_0$. Thus, we simply suppose that f is already *continuous on* $[T_0] \times \omega^\omega$.

Assume that option (a) of the definition of 4.1 does not take place, *i.e.*

$(*)$ if $X \subseteq \omega^\omega$ is dense $\mathbf{G}_\delta$, and $v \in \omega^{<\omega}$, $\sigma \in 2^{<\omega}$, $S \in \mathbf{ST}$, $S \subseteq T_0$, then there are reals $a \in [S]$ and $x \in \mathscr{N}_v \cap X$ such that $f(a, x) \neq \sigma \cdot a$.

We'll construct Silver trees $S_n \subseteq T_n$ and a dense $\mathbf{G}_\delta$ set $X \subseteq \omega^\omega$ satisfying (b) of Definition 4.1, that is, in our case, the relation $f(a, x) \notin \bigcup_{\sigma \in 2^{<\omega} \wedge n < \omega} \sigma \cdot [S_n]$ will be fulfilled for all $a \in [S_0]$ and $x \in X$.

To maintain the construction, we fix any enumeration $\omega \times 2^{<\omega} \times \omega^{<\omega} = \{\langle N_k, \sigma_k, v_k \rangle : k < \omega\}$. Auxiliary Silver trees S^n_k $(n, k < \omega)$ and tuples $w_k \in \omega^{<\omega}$ $(k < \omega)$, satisfying the following conditions, will be defined.

(1) $\ldots \subseteq_4 S^n_3 \subseteq_3 S^n_2 \subseteq_2 S^n_1 \subseteq_1 S^n_0 = T_n$ as in Lemma 2.4, for each $n < \omega$;

(2) $S^n_{k+1} = S^n_k$ for all $n > 0,\ n \neq N_k$;

(3) $S^0_{k+1} \subseteq_{k+1} S^0_k,\ S^{N_k}_{k+1} \subseteq_{k+1} S^{N_k}_k,\ v_k \subseteq w_k$, and $f(a,x) \notin \sigma_k \cdot [S^N_{k+1}]$ for all reals $a \in [S^0_{k+1}]$ and $x \in \mathscr{N}_{w_k}$.

At step 0 of the construction, we put $S^n_0 = T_n$ for all n, by (1).

Assume that $k < \omega$ and all Silver trees $S^n_k,\ n < \omega$ are already defined. We put $S^n_{k+1} = S^n_k$ for all $n > 0,\ n \neq N_k$, by (2).

To define the trees S^0_{k+1} and $S^{N_k}_{k+1}$, we put $h = \mathtt{spl}_{k+1}(S^0_k)$, $m = \mathtt{spl}_{k+1}(S^N_k)$.

Case 1: $N_k > 0$. Take any pair of tuples $s \in 2^h \cap S^0_k$, $t \in 2^m \cap S^{N_k}_k$ and any reals $a_0 \in [S^0_k \restriction_s]$ and $x_0 \in \omega^\omega$. Consider any real $b_0 \in [S^{N_k}_k \restriction_t]$ not equal to $\sigma_k \cdot f(a_0, x_0)$. Let's say $b_0(\ell) = i \neq j = (\sigma_k \cdot f(a_0, x_0))(\ell)$, where $i, j \leq 1,\ \ell < \omega$. By the continuity of f, there is a tuple $u_1 \in \omega^{<\omega}$ and Silver tree $A \subseteq S^0_k \restriction_s$ such that $v_k \subseteq u_1 \subset x_0$, $a_0 \in [A]$, and $(\sigma_k \cdot f(a,x))(\ell) = j$ for all $x \in \mathscr{N}_{u_1}$ and $a \in [A]$. It is also clear that $B = \{\tau \in S^{N_k}_k \restriction_t : \mathtt{lh}\,\tau \leq \ell \vee \tau(\ell) = i\}$ is a Silver tree containing b_0, and $b(\ell) = i$ for all $b \in [B]$. According to Lemma 2.3, there are Silver trees $U_1 \subseteq_{k+1} S^0_k$ and $V_1 \subseteq_{k+1} S^{N_k}_k$, such that $U_1 \restriction_s = A$ and $V_1 \restriction_t = B$, hence by construction we have $\sigma_k \cdot f(a,x) \notin [V_1 \restriction_t]$ for all $a \in [U_1 \restriction_s]$ and $x \in \mathscr{N}_{u_1}$.

Now consider another pair of tuples $s \in 2^h \cap S^0_k$, $t \in 2^m \cap S^{N_k}_k$. We similarly get Silver trees $U_2 \subseteq_{k+1} U_1$ and $V_2 \subseteq_{k+1} V_1$, and a tuple $u_2 \in \omega^{<\omega}$, such that $u_1 \subseteq u_2$ and $\sigma_k \cdot f(a,x) \notin [V_2(\to t')]$ for all $a \in [U_2 \restriction_{s'}]$ and $x \in \mathscr{N}_{u_2}$. In this case, we have $V_2 \restriction_t \subseteq V_1 \restriction_t$ and $U_2 \restriction_s \subseteq U_1 \restriction_s$, so that what has already been achieved at the previous step is preserved.

We iterate through all pairs of $s \in 2^h \cap S^0_k$, $t \in 2^m \cap S^{N_k}_k$, $\subseteq_{k+1}$-shrinking trees and extending tuples in $\omega^{<\omega}$ at each step. This results in a pair of Silver trees $U \subseteq_{k+1} S^0_k$, $V \subseteq_{k+1} S^{N_k}_k$ and a tuple $w \in \omega^{<\omega}$ such that $v_k \subseteq w$ and $\sigma_k \cdot f(a,x) \notin [V]$ for all reals $a \in [U]$ and $x \in \mathscr{N}_w$. Now to fulfill (3), take $w_k = w$, $S^0_{k+1} = U$, and $S^{N_k}_{k+1} = V$. Recall that here $N_k > 0$.

Case 2: $N_k = 0$. Here the construction somewhat changes, and hypothesis $(*)$ will be used. We claim that there exist:

(4) a tuple $w_k \in \omega^{<\omega}$ and a Silver tree $S^0_{k+1} \subseteq_{k+1} S^0_k$ such that $v_k \subseteq w_k$ and $f(a,x) \notin \sigma_k \cdot [S^0_{k+1}]$ for all $a \in [S^0_{k+1}]$, $x \in \mathscr{N}_{w_k}$. (**Equivalent to** (3) **as** $N_k = 0$.)

Take any pair of tuples $s, t \in 2^h \cap S^0_k$, where $h = \mathtt{spl}_{k+1}(S^0_k)$ as above. Thus $S^0_k \restriction_t = t \cdot s \cdot (S^0_k \restriction_s)$, by Lemma 2.1. According to $(*)$, there are reals $x_0 \in \mathscr{N}_v$ and $a_0 \in [S^0_k \restriction_s]$ satisfying $f(a_0, x_0) \neq \sigma_k \cdot s \cdot t \cdot a_0$, or equivalently, $\sigma_k \cdot f(a_0, x_0) \neq s \cdot t \cdot a_0$.

Similarly to Case 1, we have $(\sigma_k \cdot f(a_0, x_0))(\ell) = i \neq j = (s \cdot t \cdot a_0)(\ell)$ for some $\ell < \omega$ and $i, j \leq 1$. By the continuity of f, there is a tuple $u_1 \in \omega^{<\omega}$ and a Silver tree $A \subseteq S^0_k \restriction_s$, clopen in S^0_k, such that $v_k \subseteq u_1 \subset x_0$, $a_0 \in [A]$, and $(\sigma_k \cdot f(a,x))(\ell) = j$ but $(s \cdot t \cdot a)(\ell) = j$ for all $x \in \mathscr{N}_{u_1}$ and $a \in [A]$. Lemma 2.3 gives us a Silver tree $U_1 \subseteq_{k+1} S^0_k$, clopen in S^0_k as well, such that $U_1 \restriction_s = A$ — and then $U_1 \restriction_t = s \cdot t \cdot A$. Therefore $\sigma_k \cdot f(a,x) \notin [U_1 \restriction_t]$ holds for all $a \in [U_1 \restriction_s]$ and $x \in \mathscr{N}_{u_1}$ by construction.

Having worked out all pairs of tuples $s, t \in 2^h \cap S^0_k$, we obtain a Silver tree $U \subseteq_{k+1} S^0_k$ and a tuple $w \in \omega^{<\omega}$, such that $v_k \subseteq w$ and $\sigma_k \cdot f(a,x) \notin [U]$ for all $a \in [U]$ and $x \in \mathscr{N}_w$. Now to fulfill (4), take $w_k = w$ and $S^0_{k+1} = U$.

To conclude, we have for each n a sequence $\ldots \subseteq_4 S^n_3 \subseteq_3 S^n_2 \subseteq_2 S^n_1 \subseteq_1 S^n_0 = T_n$ of Silver trees S^n_k, along with tuples $w_k \in \omega^{<\omega}$ $(k < \omega)$, and these sequences satisfy the requirements (1),(2),(3) (equivalent to (4) in case $N_k = 0$).

We put $S_n = \bigcap_k S^n_k$. Then $S_n \in \mathbf{ST}$ by Lemma 2.4, and $S_n \subseteq T_n$.

If $n < \omega$ and $\sigma \in 2^{<\omega}$ then let $W_{n\sigma} = \{w_k : N_k = n \wedge \sigma_k = \sigma\}$. The set $X_{n\sigma} = \bigcup_{w \in W_{n\sigma}} \mathscr{N}_w$ is then open dense in ω^ω. Indeed, if $v \in \omega^\omega$ then we take k such that $v_k = v$, $N_k = n$, $\sigma_k = \sigma$; then $v \subseteq w_k \in W_{n\sigma}$ by construction. Therefore, $X = \bigcap_{n<\omega, \sigma \in 2^{<\omega}} X_{n\sigma}$ is a dense $\mathbf{G}_\delta$ set. Now to check property (b) of Definition 4.1, consider any $n < \omega$, $\sigma \in 2^{<\omega}$, $a \in [S_0]$, $x \in X$; we claim that $f(a,x) \notin \sigma \cdot [S_n]$.

By construction, we have $x \in X_{n\sigma}$, *i.e.* $x \in \mathscr{N}_{w_k}$, where $k \in W_{n\sigma}$, so that $N_k = n$, $\sigma_k = \sigma$. Now $f(a,x) \notin \sigma \cdot [S_n]$ directly follows from (3) for this k, since $S_0 \subseteq S^0_{k+1}$ and $S_n \subseteq S^n_{k+1}$. $\square$ (Theorem 4.2)

5 The forcing notion for Theorem 1.1

Using the standard encoding of Borel sets, as *e.g.* in [20] or [9, § 1D], we fix a coding of Borel functions $f : 2^\omega \to 2^\omega$. As usual, it includes a *Π^1_1-set*[1] *of codes* $\mathbf{BC} \subseteq \omega^\omega$, and for each code $r \in \mathbf{BC}$ a certain Borel function $F_r : 2^\omega \to 2^\omega$ coded by r. We assume that each Borel function has some code, and there is a Σ^1_1 relation $\mathfrak{S}(\cdot,\cdot,\cdot)$ and a Π^1_1 relation $\mathfrak{P}(\cdot,\cdot,\cdot)$ such that for all $r \in \mathbf{BC}$ and $a, b \in 2^\omega$ it holds $F_r(a) = b \iff \mathfrak{S}(r,a,b) \iff \mathfrak{P}(r,a,b)$.

Similarly, we fix a coding of Borel functions $f : 2^\omega \times \omega^\omega \to 2^\omega$, that includes a *$\Pi^1_1$-set of codes* $\mathbf{BC_2} \subseteq \omega^\omega$, and for each code $r \in \mathbf{BC_2}$ a Borel function $F^2_r : 2^\omega \times \omega^\omega \to 2^\omega$ coded by r, such that each Borel function has some code, and there is a Σ^1_1 relation $\mathfrak{S}^2(\cdot,\cdot,\cdot,\cdot)$ and a Π^1_1 relation $\mathfrak{P}^2(\cdot,\cdot,\cdot,\cdot)$ such that for all $r \in \mathbf{BC}$, $x \in \omega^\omega$, and $a, b \in 2^\omega$ it holds $F^2_r(a,x) = b \iff \mathfrak{S}^2(r,a,x,b) \iff \mathfrak{P}^2(r,a,x,b)$.

If $\mathbb{U} \subseteq \mathbf{ST}$, then $\mathtt{Clos}(\mathbb{U})$ denotes the set of all trees of the form $\sigma \cdot (T \restriction_s)$, where $\sigma \in 2^{<\omega}$ and $s \in T \in \mathbb{U}$, *i.e.* the closure of $\mathbb{U}$ w.r.t. both shifts and portions.

The following construction is maintained in L. We define a sequence of countable sets $\mathbb{U}_\alpha \subseteq \mathbf{ST}$, $\alpha < \omega_1$ satisfying the following conditions 1°–6°.

1°. Each $\mathbb{U}_\alpha \subseteq \mathbf{ST}$ is countable, $\mathbb{U}_0$ consists of a single tree $2^{<\omega}$.

We then define $\mathbb{P}_\alpha = \mathtt{Clos}(\mathbb{U}_\alpha)$, $\mathbb{P}_{<\alpha} = \bigcup_{\xi<\alpha} \mathbb{P}_\xi$. These sets are obviously closed with respect to shifts and portions, that is $\mathtt{Clos}(\mathbb{P}_\alpha) = \mathbb{P}_\alpha$ and $\mathtt{Clos}(\mathbb{P}_{<\alpha}) = \mathbb{P}_{<\alpha}$.

2°. For every $T \in \mathbb{P}_{<\alpha}$ there is a tree $S \in \mathbb{U}_\alpha$, $S \subseteq T$.

Let $\mathbf{ZFC}^-$ be the subtheory of the theory $\mathbf{ZFC}$, containing all axioms except the power set axiom, and additionally containing an axiom asserting the existence of

[1] The letters Σ and Π denote effective (lightface) projective classes.

the power set $\mathscr{P}(\omega)$. This implies the existence of $\mathscr{P}(X)$ for any countable X, the existence of ω_1 and 2^ω, as well as the existence of continual sets like 2^ω or $\mathbf{ST}$.

By $\mathfrak{M}_\alpha$ we denote the smallest model of $\mathbf{ZFC}^-$ of the form $\mathbf{L}_\lambda$ containing the sequence $\langle \mathbb{U}_\xi \rangle_{\xi<\alpha}$, in which α and all sets $\mathbb{U}_\xi$, $\xi < \alpha$, are countable.

3°. If a set $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}_{<\alpha}$ is dense in $\mathbb{P}_{<\alpha}$, and $U \in \mathbb{U}_\alpha$, then $U \subseteq^{\mathtt{fin}} \bigcup D$, meaning that there is a finite set $D' \subseteq D$ such that $U \subseteq \bigcup D'$.

4°. If a set $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}_{<\alpha} \times \mathbb{P}_{<\alpha}$ is dense in $\mathbb{P}_{<\alpha} \times \mathbb{P}_{<\alpha}$, and $U \neq V$ belong to $\mathbb{U}_\alpha$, then $U \times V \subseteq^{\mathtt{fin}} \bigcup D$, meaning that there is a finite set $D' \subseteq D$ such that $[U] \times [V] \subseteq \bigcup_{\langle U',V' \rangle \in D'} [U'] \times [V']$.

Given that $\mathtt{Clos}(\mathbb{P}_{<\alpha}) = \mathbb{P}_{<\alpha}$, this is automatically transferred to all trees $U \in \mathbb{P}_\alpha$ as well. It follows that D remains predense in $\mathbb{P}_{<\alpha} \cup \mathbb{P}_\alpha$.

To formulate the next property, we fix an enumeration

$$\mathbf{ST} \times \mathbf{BC} \times \mathbf{BC_2} = \{\langle T_\xi, b_\xi, c_\xi \rangle : \xi < \omega_1\}$$

in $\mathbf{L}$, which 1) is definable in $\mathbf{L}_{\omega_1}$, and 2) each value in $\mathbf{ST} \times \mathbf{BC} \times \mathbf{BC_2}$ is taken uncountably many times.

5°. If $T_\alpha \in \mathbb{P}_{<\alpha}$ then there is a tree $S \in \mathbb{U}_\alpha$ such that $S \subseteq T$ and:

(a) $F^2_{b_\alpha}$ is normalized for $\mathbb{U}_\alpha$ on $[S]$ in the sense of Definition 4.1, and

(b) F_{c_α} is continuous and either a bijection or a constant on $[S]$.

6°. The sequence $\langle \mathbb{U}_\alpha \rangle_{\alpha<\omega_1}$ is $\in$-definable in $\mathbf{L}_{\omega_1}$.

The construction goes on as follows. **Arguing in L**, suppose that

(†) $\alpha < \omega_1$, the subsequence $\langle \mathbb{U}_\xi \rangle_{\xi<\alpha}$ has been defined and satisfies 1°,2° below α, and the sets $\mathbb{P}_\xi = \mathtt{Clos}(\mathbb{U}_\xi)$ (for $\xi < \alpha$), $\mathbb{P}_{<\alpha}$, $\mathfrak{M}_\alpha$ are defined as above.

See the proof of the next lemma in Section 6 below.

Lemma 5.1 ($\mathbb{U}$-extension lemma, in $\mathbf{L}$)**.** *Under the assumptions of* (†)*, there is a countable set* $\mathbb{U}_\alpha \subseteq \mathbf{ST}$ *satisfying* 2°*,* 3°*,* 4°*,* 5°*.*

To accomplish the construction, we take $\mathbb{U}_\alpha$ to be the smallest, in the sense of the Gödel wellordering of $\mathbf{L}$, of those sets that exist by Lemma 5.1. Since the whole construction is relativized to $\mathbf{L}_{\omega_1}$, the requirement 6° is also met.

We put $\mathbb{P}_\alpha = \mathtt{Clos}(\mathbb{U}_\alpha)$ for all $\alpha < \omega_1$, and $\mathbb{P} = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha$.

The following result, in part related to CCC, is a fairly standard consequence of 3° and 4°, see for example [10, 6.5], [12, 12.4], or [8, Lemma 6]; we will skip the proof.

Lemma 5.2 (in $\mathbf{L}$)**.** *The forcing notion* $\mathbb{P}$ *belongs to* $\mathbf{L}$*, satisfies* $\mathbb{P} = \mathtt{Clos}(\mathbb{P})$ *and satisfies CCC in* $\mathbf{L}$*. The product* $\mathbb{P} \times \mathbb{P}$ *satisfies CCC in* $\mathbf{L}$ *as well.* □

Lemma 5.3 (in **L**). *Assume that $T \in \mathbb{P}$. If $g : 2^\omega \to 2^\omega$ is a Borel map then there is a tree $S \in \mathbb{U}_\alpha$, $S \subseteq T$, such that g is either a bijection or a constant on $[S]$.*

If $f : 2^\omega \times \omega^\omega \to 2^\omega$ is a Borel map then there is an ordinal $\alpha < \omega_1$ and a tree $S \in \mathbb{U}_\alpha$, $S \subseteq T$, such that f is normalized for $\mathbb{U}_\alpha$ on $[S]$.

Proof. By the choice of the enumeration of triples in $\mathbf{ST} \times \mathbf{BC} \times \mathbf{BC_2}$, there is an ordinal $\alpha < \omega_1$ such that $T \in \mathbb{P}_{<\alpha}$ and $T = T_\alpha$, $f = F^2_{b_\alpha}$, $g = F_{b_\alpha}$. It remains to refer to 5°. $\square$

6 Proof of the extension lemma

This section is entirely devoted to the **proof of Lemma 5.1**.

We work in L under the assumptions of $(\dagger)$ **above.**

We first define a set $\mathbb{U} = \{U_n : n < \omega\}$ of Silver trees $U_n \subseteq 2^\omega$ satisfying 2°, 3° 4°; then further narrowing of the trees will be made to also satisfy 5°. This involves a splitting/fusion construction known from our earlier papers, see [10, § 4], [13, § 9–10], [12, § 10], [16, § 7], and to some extent from the proof of Theorem 4.2 above.

We fix enumerations

$$\mathscr{D} = \{D(j) : j < \omega\} \quad \text{and} \quad \mathscr{D}_2 = \{D_2(j) : j < \omega\}$$

of the set $\mathscr{D}$ of all sets $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}_{<\alpha}$ open-dense in $\mathbb{P}_{<\alpha}$, and the set $\mathscr{D}_2$ of all sets $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}_{<\alpha} \times \mathbb{P}_{<\alpha}$ open-dense in $\mathbb{P}_{<\alpha} \times \mathbb{P}_{<\alpha}$. We also fix a bijection $\beta : \omega \xrightarrow{\text{onto}} \omega^4$ which assumes each value $\langle j, j', M, N\rangle \in \omega^4$ infinitely many times.

The construction of the trees U_n is organized in the form $U_n = \bigcup_k U^n_k$, where the Silver trees U^n_k satisfy the following requirements:

(1) $\ldots \subseteq_4 U^n_3 \subseteq_3 U^n_2 \subseteq_2 U^n_1 \subseteq_1 U^n_0$ as in Lemma 2.4 for each $n < \omega$;

(2) if $T \in \mathbb{P}_{<\alpha}$ then $T = U^n_0$ for some n;

(3) each U^n_k is a k-collage over $\mathbb{P}_{<\alpha}$.

A Silver tree T is a *k-collage* over $\mathbb{P}_{<\alpha}$ [13, 12] when $T{\restriction}_s \in \mathbb{P}_{<\alpha}$ for each tuple $s \in T \cap 2^h$, where $h = \mathtt{spl}_k(T)$. Then 0-collages are just trees in $\mathbb{P}_{<\alpha}$, and every k-collage is a $k+1$-collage as well since $\mathtt{Clos}(\mathbb{P}_{<\alpha}) = \mathbb{P}_{<\alpha}$.

(4) if $k \geq 1$, $\beta(k) = \langle j, j', M, N\rangle$, $\mu = \mathtt{spl}_k(U^M_k)$, $\nu = \mathtt{spl}_k(U^N_k)$ (integers), $s \in U^M_k \cap 2^\mu$, $t \in U^N_k \cap 2^\nu$ (tuples of length resp. μ, ν), $M \neq N$, then the tree $U^M_k{\restriction}_s$ belongs to $D(j)$ and the pair $\langle U^M_k{\restriction}_s, U^N_k{\restriction}_t\rangle$ belongs to $D_2(j')$.

It follows that $U^M_k \subseteq^{\mathtt{fin}} \bigcup D(j)$ and $\langle U^M_k, U^N_k\rangle \subseteq^{\mathtt{fin}} \bigcup D_2(j')$ in the sense of 3° and 4° of Section 5.

To begin the inductive construction, we assign $U^n_0 \in \mathbb{P}_{<\alpha}$ so that $\{U^n_0 : n < \omega\} = \mathbb{P}_{<\alpha}$, to get (2). Now let's maintain the step $k \to k+1$. Thus suppose that $k < \omega$, and all Silver trees U^n_k, $n < \omega$ are defined and are k-collages over $\mathbb{P}_{<\alpha}$.

Let $\beta(k) = \langle j, j', M, N\rangle$. If $N = M$ then put $U^n_{k+1} = U^n_k$ for all n.

Now assume that $M \neq N$. Put $U^n_{k+1} = U^n_k$ for all $n \notin \{M, N\}$.

It takes more effort to define U^M_{k+1} and U^N_{k+1}. Let $\mu = \mathtt{spl}_{k+1}(U^M_k)$, $\nu = \mathtt{spl}_{k+1}(U^N_k)$. To begin with we put $U^M_{k+1} := U^M_k$ and $U^N_{k+1} := U^N_k$. These $k+1$-collages are the initial values for the trees U^M_{k+1} and U^N_{k+1}, to be $\subseteq_{k+1}$-shrinked in a finite number of substeps (within the step $k \to k+1$), each corresponding to a pair of tuples $s \in U^M_k \cap 2^\mu$ and $t \in U^N_k \cap 2^\nu$.

Namely let $s \in U^M_{k+1} \cap 2^\mu$, $t \in U^N_{k+1} \cap 2^\nu$ be the first such pair. The trees $U^M_{k+1}{\restriction}_s$, $U^N_{k+1}{\restriction}_t$ belong to $\mathbb{P}_{<\alpha}$ as U^M_{k+1}, U^N_{k+1} are $k+1$-collages over $\mathbb{P}_{<\alpha}$. Therefore by the open density there exist trees $A, B \in D(j)$ such that the pair $\langle U^M_{k+1}{\restriction}_s, U^N_{k+1}{\restriction}_t\rangle$ belongs to $D_2(j')$ and $A \subseteq U^M_{k+1}{\restriction}_s$, $B \subseteq U^N_{k+1}{\restriction}_t$. Now Lemma 2.3 gives us Silver trees $S \subseteq_{k+1} U^M_k$ and $T \subseteq_{k+1} U^N_k$ satisfying $S{\restriction}_s \subseteq A$, $T{\restriction}_t \subseteq B$. Moreover, by Lemma 2.1, S and T still are $k+1$-collages over $\mathbb{P}_{<\alpha}$ since $\mathbb{P}_{<\alpha}$ is closed under shifts by construction. To conclude, we have defined $k+1$-collages $S \subseteq_{k+1} U^M_{k+1}$ and $T \subseteq_{k+1} U^N_{k+1}$ over $\mathbb{P}_{<\alpha}$, satisfying $S{\restriction}_s \in D(j)$, $T{\restriction}_t \in D(j)$, and $\langle S{\restriction}_s, T{\restriction}_t\rangle \in D_2(j')$. We re-assign the "new" U^M_{k+1} and U^N_{k+1} to be equal to resp. S, T.

Applying this $\subseteq_{k+1}$-shrinking procedure consecutively for all pairs of tuples $s \in U^M_k \cap 2^\mu$ and $t \in U^N_k \cap 2^\nu$, we eventually (after finitely many substeps according to the number of all such pairs), we get a pair of $k+1$-collages $U^M_{k+1} \subseteq_{k+1} U^M_k$ and $U^N_{k+1} \subseteq_{k+1} U^N_k$ over $\mathbb{P}_{<\alpha}$, such that for every pair of tuples $s \in U^M_k \cap 2^\mu$ and $t \in U^N_k \cap 2^\nu$, we have $U^M_{k+1}{\restriction}_s \in D(j)$ and $\langle U^M_{k+1}{\restriction}_s, U^N_{k+1}{\restriction}_t\rangle \in D_2(j')$, so conditions (3) and (4) are satisfied.

Having defined, in $\mathbf{L}$, a system of Silver trees U^n_k satisfying (1),(2),(3),(4), we then put $U_n = \bigcap_k U^N_k$ for all n. Those are Silver trees by Lemma 2.4. The collection $\mathbb{U}_\alpha := \{U_n : n < \omega\}$ satisfies 2° of Section 5 by (2).

To check condition 3° of Section 5, let $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}_{<\alpha}$ be dense in $\mathbb{P}_{<\alpha}$, and $U \in \mathbb{U}_\alpha$. We can w.l.o.g. assume that D is open-dense, for if not then replace T by $D' = \{S \in \mathbb{P}_{<\alpha} : \exists T \in D\,(S \subseteq T)\}$. Then $D = D(j)$ for some j, and $U = U_M$ for some M by construction. Now consider any index k such that $\beta(k) = \langle M, N, j, j'\rangle$ for M, j as above and any N, j'. Then we have $U = U_M \subseteq U^M_k$ by construction, and $U^M_k \subseteq^{\mathtt{fin}} \bigcup D$ by (4), thus $U \subseteq^{\mathtt{fin}} \bigcup D$, as required.

Condition 4° is verified similarly.

It remains to somewhat shrink all trees U_n to also fulfill 5°. We still work in $\mathbf{L}$.

Recall that an enumeration $\mathbf{ST} \times \mathbf{BC} \times \mathbf{BC_2} = \{\langle T_\xi, b_\xi, c_\xi\rangle : \xi < \omega_1\}$, parameter-free definable in $\mathbf{L}_{\omega_1}$, is fixed in Section 5. We suppose that the tree T_α belongs to $\mathbb{P}_{<\alpha}$. (If not then we don't worry about 5°.) Consider, according to 2°, a tree $U = U_M \in \mathbb{U}_\alpha$ satisfying $T \subseteq T_\alpha$. Using Corollary 3.2, Lemma 3.3, and Theorem 4.2, we shrink each tree $U_n \in \mathbb{U}_\alpha$ to a tree $U'_n \in \mathbf{ST}$, $U' \subseteq U$, so that the function $F^2_{b_\alpha}$ is normalized on U'_M for $\mathbb{U}' = \{U'_n : n < \omega\}$ and F_{c_α} is continuous and either a bijection or a constant on $[U'_M]$. Take $\mathbb{U}'$ as the final $\mathbb{U}_\alpha$ and T' as U'_M to fulfill 5°.

□ (Lemma 5.1)

7 The model, part I

We use the product $\mathbb{P} \times \mathbb{C}$ of the forcing notion $\mathbb{P}$ defined in $\mathbf{L}$ in Section 5 and satisfying conditions 1°–6° as above, and the Cohen forcing, here in the form of $\mathbb{C} = \omega^{<\omega}$, to prove the following more detailed form of Theorem 1.1. The proof of this theorem in the next three sections is based on a combination of different ideas.

Theorem 7.1. *Let a pair of reals $\langle a_0, x_0 \rangle$ be $\mathbb{P} \times \mathbb{C}$-generic over $\mathbf{L}$. Then*

(I) *a_0 is not $\mathbf{OD}$, and moreover, $\mathbf{HOD} = \mathbf{L}$ in $\mathbf{L}[a_0, x_0]$;*

(II) *a_0 belongs to $\mathbf{HNT}$, and moreover, $\mathbf{L}[a_0] \subseteq \mathbf{HNT}$ in $\mathbf{L}[a_0, x_0]$;*

(III) *x_0 does not belong to $\mathbf{HNT}$, and moreover, $\mathbf{HNT} \subseteq \mathbf{L}[a_0]$ in $\mathbf{L}[a_0, x_0]$.*

We prove Claim (I) of the theorem in this section. The proof is based on several lemmas. According to the next lemma, it suffices to prove that $\mathbf{HOD} = \mathbf{L}$ in $\mathbf{L}[a_0]$.

Lemma 7.2. $(\mathbf{HOD})^{\mathbf{L}[a_0, x_0]} \subseteq (\mathbf{HOD})^{\mathbf{L}[a_0]}$.

Proof. By the forcing product theorem, x_0 is a Cohen generic real over $\mathbf{L}[a_0]$. It follows by a standard argument based on the full homogeneity of the Cohen forcing $\mathbb{C}$ that if $H \subseteq \mathbf{Ord}$ is $\mathbf{OD}$ in $\mathbf{L}[a_0, x_0]$ then $H \in \mathbf{L}[a_0]$ and H is $\mathbf{OD}$ in $\mathbf{L}[a_0]$.

Now prove the implication $Y \in (\mathbf{HOD})^{\mathbf{L}[a_0, x_0]} \implies Y \in \mathbf{L} \wedge Y \in (\mathbf{HOD})^{\mathbf{L}[a_0]}$ by induction on the set-theoretic rank $\mathtt{rk}\, x$ of $x \in \mathbf{L}[a_0, x_0]$. Since each set consists only of sets of strictly lower rank, it is sufficient to check that if a set $H \in \mathbf{L}[a_0, x_0]$ satisfies $H \subseteq (\mathbf{HOD})^{\mathbf{L}[a_0]}$ and $H \in \mathbf{HOD}$ in $\mathbf{L}[a_0, x_0]$ then $H \in \mathbf{L}[a_0$ and $H \in (\mathbf{OD})^{\mathbf{L}[a_0]}$. Here we can assume that in fact $H \subseteq \mathbf{Ord}$, since $\mathbf{HOD}$ allows an $\mathbf{OD}$ wellordering and hence an $\mathbf{OD}$ bijection onto $\mathbf{Ord}$. But in this case $H \in \mathbf{L}[a_0]$ and H is $\mathbf{OD}$ in $\mathbf{L}[a_0]$ by the above, as required. □

Lemma 7.3 (Lemma 7.5 in [10])**.** *a_0 is not $\mathbf{OD}$ in $\mathbf{L}[a_0]$.*

Proof. Suppose towards the contrary that a_0 is $\mathbf{OD}$ in $\mathbf{L}[a_0]$. But a_0 is a $\mathbb{P}$-generic real over $\mathbf{L}$, so the contrary assumption is forced. In other words, there is a tree $T \in \mathbb{P}$ with $a_0 \in [T]$ and a formula $\vartheta(x)$ with ordinal parameters, such that if $a \in [T]$ is $\mathbb{P}$-generic over $\mathbf{L}$ then a is the only real in $\mathbf{L}[a]$ satisfying $\vartheta(a)$. Let $s = \mathtt{stem}(()T)$. Then both $s^\frown 0$ and $s^\frown 1$ belong to T, and either $s^\frown 0 \subset a_0$ or $s^\frown 1 \subset a_0$. Let, say, $s^\frown 0 \subset a_0$. Let $n = \mathtt{lh}(s)$ and $\sigma = 0^{n\frown} 1$, so that all three strings $s^\frown 0$, $s^\frown 1$, σ belong to 2^{n+1}, and $s^\frown 0 = \sigma \cdot (s^\frown 1)$. As the forcing $\mathbb{P}$ is invariant under the action of σ, the real $a_1 = \sigma \cdot a_0$ is $\mathbb{P}$-generic over $\mathbf{L}$, and $\sigma \cdot T = T$. We conclude that it is true in $\mathbf{L}[a_1] = \mathbf{L}[a_0]$ that a_1 is still the only real in $\mathbf{L}[a_1]$ satisfying $\vartheta(a_1)$. However obviously $a_1 \neq a_0$! □

Lemma 7.4. *If $b \in \mathbf{L}[a_0] \smallsetminus \mathbf{L}$ is a real then b is not $\mathbf{OD}$ in $\mathbf{L}[a_0]$.*

Proof. It follows from Lemma 5.2 (and the countability of $\mathbb{C}$) that the forcing $\mathbb{P} \times \mathbb{C}$ preserves cardinals. We conclude that that $b = g(a_0)$ for some Borel function $g = F_r : 2^\omega \to 2^\omega$ with a code $r \in \mathbf{BC} \cap \mathbf{L}$. Now by Lemma 5.3 there is a tree $S \in \mathbb{P}$

such that $a_0 \in [S]$ and $h = g \restriction [S]$ is a bijection of a constant. If h is a bijection then $b \notin \mathbf{OD}$ in $\mathbf{L}[a_0]$ since otherwise $a_0 = h^{-1}(b) \in \mathbf{OD}$, contrary to Lemma 7.3. If h is a constant, so that there is a real $b_0 \in \mathbf{L} \cap 2^\omega$ such that $h(a) = b_0$ for all $a \in [S]$, then $b = h(a_0) = c \in \mathbf{L}$, contrary to the choice of b. $\square$

Lemma 7.5. *If $X \subseteq \mathbf{Ord}$, $X \in \mathbf{L}[a_0] \smallsetminus \mathbf{L}$, then X is not $\mathbf{OD}$ in $\mathbf{L}[a_0]$.*

Proof. Suppose to the contrary that $X \subseteq \mathbf{Ord}$, $X \in \mathbf{L}[a_0] \smallsetminus \mathbf{L}$, and X is $\mathbf{OD}$ in $\mathbf{L}[a_0]$. Let t be a $\mathbb{P}$-name for X. Then a condition $T_0 \in \mathbb{P}$ (a Silver tree) $\mathbb{P}$-forces

$$t \in \mathbf{L}[a_0] \smallsetminus \mathbf{L} \ \wedge \ t \in \mathbf{OD}$$

over $\mathbf{L}$. Say that t *splits* conditions $S, T \in \mathbb{P}$ if there is an ordinal γ suct that S forces $\gamma \in t$ but T forces $\gamma \notin t$ or vice versa; let γ_{ST} be the least such an ordinal γ.

We claim that the set

$$D = \{\langle S, T\rangle : S, T \in \mathbb{P} \wedge S \cup T \subseteq T_0 \wedge t \text{ splits } S, T\} \in \mathbf{L}$$

is dense in $\mathbb{P} \times \mathbb{P}$ above $\langle T_0, T_0\rangle$. Indeed let $S, T \in \mathbb{P}$ be subtrees of T_0. If t splits no stronger pair of trees $S' \subseteq S$, $T' \subseteq T$ in $\mathbb{P}$ then easily both S and T decide $\gamma \in t$ for every ordinal γ, a contradiction with the choice of T_0. Thus D is indeed dense.

Let, in $\mathbf{L}$, $A \subseteq D$ be a maximal antichain; A is countable in $\mathbf{L}$ by Lemma 5.2, and hence the set $W = \{\gamma_{ST} : \langle S, T\rangle \in A\} \in \mathbf{L}$ is countable in $\mathbf{L}$. We claim that

(‡) the intersection $b = X \cap W$ does not belong to $\mathbf{L}$.

Indeed otherwise there is a tree $T_1 \in \mathbb{P}$, $T_1 \subseteq T_0$, which $\mathbb{P}$-forces that $t \cap W = b$. (The sets $W, b \in \mathbf{L}$ are identified with their names.)

By the countability of A, W there is an ordinal $\alpha < \omega_1^{\mathbf{L}}$ such that $A \subseteq \mathbb{P}_{<\alpha} \times \mathbb{P}_{<\alpha}$, $T_1 \in \mathbb{P}_{<\alpha}$, and $W \subseteq \alpha$. We can w.l.o.g. assume that $A \in \mathfrak{M}_\alpha$, for if not then further increase α below $\omega_1^{\mathbf{L}}$ accordingly. Let $u = \mathtt{stem}(T_1)$. The trees $T_{10} = T_1 \restriction_{u^\frown 0}$ and $T_{11} = T_1 \restriction_{u^\frown 1}$ belong to $\mathbb{P}_{<\alpha}$ along with T_1, and hence there are trees U, $V \in \mathbb{U}_\alpha$ with $U \subseteq T_{10}$ and $V \subseteq T_{11}$. Clearly $U \neq V$, so that we have $[U] \times [V] \subseteq \bigcup_{\langle U', V'\rangle \in A'} [U'] \times [V']$ for a finite set $A' \subseteq A$ by 4° of Section 5. Now take reals $a' \in [U]$ and $a'' \in [V]$ both $\mathbb{P}$-generic over $\mathbf{L}$. Then there is a pair of trees $\langle U', V'\rangle \in A'$ such that $a' \in [U']$ and $a'' \in [V']$. The interpretations $X' = t[a']$ and $X'' = t[a'']$ are then different on the ordinal $\gamma = \gamma_{U'U''} \in W$ since $A' \subseteq A \subseteq D$. Thus the restricted sets $b' = X' \restriction W$ and $b'' = X'' \restriction W$ differ from each other. In particular at least one of b', b'' is not equal to b. But $a', a'' \in [T_1]$ by construction, hence this contradicts the choice of T_1 and completes the proof of (‡).

Recall that $b \subseteq W$, and $W \in \mathbf{L}$ is countable in $\mathbf{L}$. It follows that b can be considered as a real, so we conclude that b is not $\mathbf{OD}$ in $\mathbf{L}[a_0]$ by Lemma 7.4 and (‡).

However $b = X \cap W$, where X is $\mathbf{OD}$ and $W \in \mathbf{L}$, hence W is $\mathbf{OD}$ in $\mathbf{L}[a_0]$ and b is $\mathbf{OD}$ in $\mathbf{L}[a_0]$. The contradiction obtained ends the proof of the lemma. $\square$ (Lemma)

Now Theorem 7.1(I) immediately follows from Lemma 7.2 and Lemma 7.5.

$\square$ (Claim (II) of Theorem 7.1)

8 The model, part II

Here we establish Claim (II) of Theorem 7.1. To prove $\mathbf{L}[a_0] \subseteq \mathbf{HNT}$ it suffices to show that a_0 itself belongs to $\mathbf{HNT}$, and then make use of the fact that by Gödel every set $z \in \mathbf{L}[a_0]$ has the form $x = F(a_0)$, where F is an $\mathbf{OD}$ function.

Further, to prove $a_0 \in \mathbf{HNT}$ it suffices to check that the E_0-equivalence class[2] $[a_0]_{\mathsf{E}_0} = \{b \in 2^\omega : a_0 \mathrel{\mathsf{E}_0} b\}$ (which is a countable set) of our generic real a_0 is an $\mathbf{OD}$ set in $\mathbf{L}[a_0, x_0]$. According to 6°, it suffices to establish the equality

$$[a_0]_{\mathsf{E}_0} = \bigcap_{\xi<\omega_1} \bigcup_{T \in \mathbb{P}_\xi} [T]\,. \tag{$*$}$$

Note that every set $\mathbb{P}_\xi$ is pre-dense in $\mathbb{P}$; this follows from 3° and 5°, see, for example, Lemma 6.3 in [10]. This immediately implies $a_0 \in \bigcup_{T \in \mathbb{P}_\xi} [T]$ for each ξ. Yet all sets $\mathbb{P}_\xi$ are invariant w.r.t. shifts by construction. Thus we have $\subseteq$ in (*).

To prove the inverse inclusion, assume that a real $b \in 2^\omega$ belongs to the right-hand side of (*) in $\mathbf{L}[a_0, x_0]$. It follows from Lemma 5.2 (and the countability of $\mathbb{C}$) that the forcing $\mathbb{P} \times \mathbb{C}$ preserves cardinals. We conclude that that $b = g(a_0, x_0)$ for some Borel function $g = F_q : 2^\omega \times \omega^\omega \to 2^\omega$ with a code $q \in \mathbf{BC} \cap \mathbf{L}$.

Assume to the contrary that $b = g(a_0, x_0) \notin [a_0]_{\mathsf{E}_0}$.

Since $x_0 \in \omega^\omega$ is a $\mathbb{C}$-generic real over $\mathbf{L}[a_0]$ by the forcing product theorem, this assumption is forced, so that there is a tuple $u \in \mathbb{C} = \omega^{<\omega}$ such that

$$f(a_0, x) \in \bigcap_{\xi<\omega_1} \bigcup_{T \in \mathbb{P}_\xi} [T] \smallsetminus [a_0]_{\mathsf{E}_0}\,,$$

whenever a real $x \in \mathscr{N}_u$ is $\mathbb{C}$-generic over $\mathbf{L}[a_0]$. (Recall that $\mathscr{N}_u = \{y \in \omega^\omega : u \subset y\}$.) Let H be the canonical homomorphism of ω^ω onto $\mathscr{N}_u$. We put $f(a, x) = g(a, H(x))$ for $a \in 2^\omega$, $x \in \omega^\omega$. Then H preserves the $\mathbb{C}$-genericity, and hence

$$f(a_0, x) \in \bigcap_{\xi<\omega_1} \bigcup_{T \in \mathbb{P}_\xi} [T] \smallsetminus [a]_{\mathsf{E}_0}\,, \tag{$**$}$$

whenever $x \in \omega^\omega$ is $\mathbb{C}$-generic over $\mathbf{L}[a_0]$. Note that f also has a Borel code $r \in \mathbf{BC}$ in $\mathbf{L}$, so that $f = F_r$.

It follows from Lemma 5.3 that there is an ordinal $\alpha < \omega_1$ and a tree $S \in \mathbb{U}_\alpha$, on which f is normalized for $\mathbb{U}_\alpha$, and which satisfies $a_0 \in [S]$. Normalization means that, in $\mathbf{L}$, there is a dense $\mathbf{G}_\delta$ set $X \subseteq \omega^\omega$ satisfying one of the two options of Definition 4.1. Consider a real $z \in \omega^\omega \cap \mathbf{L}$ (a $\mathbf{G}_\delta$-*code* for X in $\mathbf{L}$) such that $X = X_z = \bigcap_k \bigcup_{z(2^k \cdot 3^j)=1} \mathscr{N}_{w_j}$, where $2^{<\omega} = \{w_j : j < \omega\}$ is a fixed recursive enumeration of tuples.

Case 1: there are tuples $v \in \omega^{<\omega}$, $\sigma \in 2^{<\omega}$, such that $f(a, x) = \sigma{\bullet}a$ for all points $a \in [S]$ and $x \in \mathscr{N}_v \cap X$. In other words, it is true in $\mathbf{L}$ that

$$\forall\, a \in [S]\, \forall\, x \in \mathscr{N}_v \cap X_z\, (f(a, x) = \sigma \bullet a)\,.$$

[2] Recall that the equivalence relation E_0 is defined on 2^ω so that $a \mathrel{\mathsf{E}_0} b$ iff the set $a \,\Delta\, b = \{k : a(k) \neq b(k)\}$ is finite. Equivalently, $a \mathrel{\mathsf{E}_0} b$ iff $a = \sigma{\bullet}b$ for some tuple $\sigma \in 2^{<\omega}$. Then $[a]_{\mathsf{E}_0} = \{b \in 2^\omega : a \mathrel{\mathsf{E}_0} b\} = \{\sigma \bullet a : \sigma \in 2^{<\omega}\}$ is the E_0-*equivalence class* of a.

But this formula is absolute by Shoenfield, so it is also true in $\mathbf{L}[a_0, x_0]$. Take $a = a_0$ (recall: $a_0 \in [S]$) and any real $x \in \mathscr{N}_v$, $\mathbb{C}$-generic over $\mathbf{L}[a_0]$. Then $x \in X_z$, because X_z is a dense $\mathbf{G}_\delta$ with a code even from $\mathbf{L}$. Thus $f(a_0, x) = \sigma \cdot a_0 \in [a_0]_{\mathsf{E}_0}$, which contradicts (**).

Case 2: $f(a,x) \notin \bigcup_{\sigma \in 2^{<\omega} \wedge U \in \mathbb{U}_\alpha} \sigma \cdot [U]$ for all $a \in [S]$ and $x \in X$. By the definition of $\mathbb{P}_\alpha$, this implies $f(a,x) \notin \bigcup_{T \in \mathbb{P}_\alpha} [T]$ for all $a \in [S]$ and $x \in X$, and this again contradicts (**) for $a = a_0$.

The resulting contradiction in both cases refutes the contrary assumption above and completes the proof.

□ (Claim (II) of Theorem 7.1)

9 The model, part III

Here we prove Claim (III) of Theorem 7.1. We make use of the following result here.

Lemma 9.1. *Let $x \in \omega^\omega$ be Cohen-generic over a set universe $\mathbf{V}$. Then it holds in $\mathbf{V}[x]$ that if $Z \subseteq 2^\omega$ is a countable $\mathbf{OD}$ set then $Z \in \mathbf{V}$. More generally if $q \in 2^\omega \cap \mathbf{V}$ then it holds in $\mathbf{V}[x]$ that if $Z \subseteq 2^\omega$ is a countable $\mathbf{OD}(q)$ set then $Z \in \mathbf{V}$.* □

Proof (sketch)**.** The pure $\mathbf{OD}$ case is Theorem 1.1 in [11].[3] The proof of the general case does not differ, q is present in the flow of arguments as a passive parameter. □

This result admits the following extension for the case $\mathbf{V} = \mathbf{L}$. Here $\mathbf{OD}(a)$ naturally means sets definable by a formula containing a_0 and ordinals as parameters

Corollary 9.2. *Assume that $a \in 2^\omega$ and $x \in \omega^\omega$ is Cohen-generic over $\mathbf{L}[a]$. Then it holds in $\mathbf{L}[a,x]$ that if $X \in \mathbf{L}[a]$ and $A \subseteq 2^X$ is a countable $\mathbf{OD}(a)$ set then $A \subseteq \mathbf{L}$.*

Proof. As the Cohen forcing is countable, there is a set $Y \subseteq X$, $Y \in \mathbf{L}[a]$, countable in $\mathbf{L}[a]$ and such that if $f \neq g$ belong to 2^X then $f(x) \neq g(x)$ for some $x \in Y$. Then Y is countable and $\mathbf{OD}(a)$ in $\mathbf{L}[a,x]$, so the *projection* $B = \{f \restriction Y : f \in A\}$ of the set A will also be countable and $\mathbf{OD}(a)$ in $\mathbf{L}[a,x]$. We have $B \in \mathbf{L}[a]$ by Lemma 9.1. (The set Y here can be identified with ω.) Hence, each $f \in B$ is $\mathbf{OD}(a)$ in $\mathbf{L}[a,x]$. However, if $f \in A$ and $w = f \restriction Y$, then by the choice of Y it holds in $\mathbf{L}[a,x]$ that f is the only element in A satisfying $f \restriction Y = w$. Therefore $f \in \mathbf{OD}(a)$ in $\mathbf{L}[a,x]$. We conclude that $f \in \mathbf{L}[a]$. □

Proof (Claim (III) of Theorem 7.1)**.** We prove an even stronger claim

$$x \in \mathbf{HNT}(a_0) \implies x \in \mathbf{L}[a_0]$$

in $\mathbf{L}[a_0, x_0]$ by induction on the set-theoretic rank $\mathtt{rk}\, x$ of sets $x \in \mathbf{L}[a_0, x_0]$. Here $\mathbf{HNT}(a_0)$ naturally means all sets hereditarily $\mathbf{NT}(a_0)$, the latter meals all elements of countable sets in $\mathbf{OD}(a_0)$.

[3] See our papers [11, 15, 14] for more on countable and Borel $\mathbf{OD}$ sets in Cohen and some other generic extensions.

Since each set consists only of sets of strictly lower rank, it is sufficient to check that if a set $H \in \mathbf{L}[a_0, x_0]$ satisfies $H \subseteq \mathbf{L}[a_0]$ and $H \in \mathbf{HNT}(a_0)$ in $\mathbf{L}[a_0, x_0]$ then $H \in \mathbf{L}[a_0]$. Here we can assume that in fact $H \subseteq \mathbf{Ord}$, since $\mathbf{L}[a_0]$ allows an $\mathbf{OD}(a_0)$ wellordering. Thus, let $H \subseteq \lambda \in \mathbf{Ord}$. Additionally, since $H \in \mathbf{HNT}(a_0)$, we have, in $\mathbf{L}[a_0, x_0]$, a countable $\mathbf{OD}(a_0)$ set $A \subseteq \mathscr{P}(\lambda)$ containing H. However, $A \in \mathbf{L}[a_0]$ by Corollary 9.2. This implies $H \in \mathbf{L}[a_0]$ as required. $\square$

$\square$ (Claim (III) and Theorem 7.1 as a whole)

$\square$ (Theorem 1.1)

10 Comments and questions

1. Recall that if x is a Cohen real over $\mathbf{L}$ then $\mathbf{HNT} = \mathbf{L}$ in $\mathbf{L}[x]$ by Lemma 9.1.

Problem 10.1. Is it true in generic extensions of $\mathbf{L}$ by a single Cohen generic real that a countable $\mathbf{OD}$ set of any kind necessarily consists only of $\mathbf{OD}$ elements?

We cannot solve this even for *finite* $\mathbf{OD}$ sets.

By the way it is not that obvious to expect the *positive* answer. Indeed, the problem solves in the *negative* for Sacks and some other generic extensions even for *pairs*, see [1, 2]. For instance, if x is a Sacks-generic real over $\mathbf{L}$ then it is true in $\mathbf{L}[x]$ that there is an $\mathbf{OD}$ unordered pair $\{X, Y\}$ of sets of reals $X, Y \subseteq \mathscr{P}(2^\omega)$ such that X, Y themselves are non-$\mathbf{OD}$ sets. See [1] for a proof of this rather surprising result originally by Solovay.

2. See Fuchs [3] (unpublished) for some other research lines related to Russell-nontypical sets with various cardinal parameters.

References

[1] Ali Enayat and Vladimir Kanovei, *An unpublished theorem of Solovay on OD partitions of reals into two non-OD parts, revisited*, J. Math. Log. **21** (2021), no. 3, 1–22, Article No. 2150014.

[2] Ali Enayat, Vladimir Kanovei, and Vassily Lyubetsky, *On effectively indiscernible projective sets and the Leibniz-Mycielski axiom*, Mathematics **9** (2021), no. 14, 1–19 (English), Article No 1670.

[3] Gunter Fuchs, *Blurry definability*, Preprint, 2021.

[4] Gunter Fuchs, Victoria Gitman, and Joel David Hamkins, *Ehrenfeucht's lemma in set theory*, Notre Dame J. Formal Logic **59** (2018), no. 3, 355–370 (English).

[5] Marcia J. Groszek and Joel David Hamkins, *The implicitly constructible universe*, J. Symb. Log. **84** (2019), no. 4, 1403–1421 (English).

[6] Joel David Hamkins and Cole Leahy, *Algebraicity and implicit definability in set theory*, Notre Dame J. Formal Logic **57** (2016), no. 3, 431–439 (English).

[7] Thomas Jech, *Set theory*, The third millennium revised and expanded ed., Springer-Verlag, Berlin-Heidelberg-New York, 2003 (English), Pages xiii + 769.

[8] Ronald Jensen, *Definable sets of minimal degree*, Math. Logic Found. Set Theory, Proc. Int. Colloqu., Jerusalem 1968 (Yehoshua Bar-Hillel, ed.), Studies in logic and the foundations of mathematics, vol. 59, North-Holland, Amsterdam-London, 1970, pp. 122–128.

[9] Vladimir Kanovei and Vassily Lyubetsky, *On some classical problems in descriptive set theory*, Russ. Math. Surv. **58** (2003), no. 5, 839–927 (Russian, English).

[10] ———, *A definable E_0 class containing no definable elements*, Arch. Math. Logic **54** (2015), no. 5-6, 711–723 (English).

[11] ———, *Countable OD sets of reals belong to the ground model*, Arch. Math. Logic **57** (2018), no. 3-4, 285–298 (English).

[12] ———, *Definable E_0 classes at arbitrary projective levels*, Ann. Pure Appl. Logic **169** (2018), no. 9, 851–871 (English).

[13] ———, *Non-uniformizable sets of second projective level with countable cross-sections in the form of Vitali classes*, Izvestiya: Mathematics **82** (2018), no. 1, 61–90.

[14] ———, *Borel OD sets of reals are OD-Borel in some simple models*, Proc. Am. Math. Soc. **147** (2019), no. 3, 1277–1282 (English).

[15] ———, *Definable elements of definable Borel sets*, Math. Notes **105** (2019), no. 5, 684–693 (English).

[16] ———, *Models of set theory in which separation theorem fails*, Izvestiya: Mathematics **85** (2021), no. 6, to appear.

[17] ———, *On Russell typicality in Set Theory*, arXiv e-prints (2021), arXiv:2111.07654.

[18] Alexander S. Kechris, *Classical descriptive set theory*, Springer-Verlag, New York, 1995. MR 96e:03057

[19] Michiel van Lambalgen, *The axiomatization of randomness*, J. Symb. Log. **55** (1990), no. 3, 1143–1167 (English).

[20] Robert M. Solovay, *A model of set-theory in which every set of reals is Lebesgue measurable*, Ann. Math. (2) **92** (1970), 1–56.

[21] Athanassios Tzouvaras, *Typicality á la Russell in set theory*, ResearchGate Preprint, May 2021, 14 pp., ResearchGate Link.

[22] ———, *Russell's typicality as another randomness notion*, Mathematical Logic Quarterly **66** (2020), no. 3, 355–365.