

## Continuity between Cauchy and Bolzano: issues of antecedents and priority

Jacques Bair , Piotr Błaszczyk , Elías Fuentes Guillén , Peter Heinig , Vladimir Kanovei & Mikhail G. Katz

To cite this article: Jacques Bair , Piotr Błaszczyk , Elías Fuentes Guillén , Peter Heinig , Vladimir Kanovei & Mikhail G. Katz (2020) Continuity between Cauchy and Bolzano: issues of antecedents and priority, British Journal for the History of Mathematics, 35:3, 207-224, DOI: [10.1080/26375451.2020.1770015](https://doi.org/10.1080/26375451.2020.1770015)

To link to this article: <https://doi.org/10.1080/26375451.2020.1770015>



Published online: 05 Jun 2020.



Submit your article to this journal [↗](#)



Article views: 55



View related articles [↗](#)



View Crossmark data [↗](#)



# Continuity between Cauchy and Bolzano: issues of antecedents and priority

JACQUES BAIR

*HEC-ULG, University of Liege, Liege, Belgium*

PIOTR BŁASZCZYK 

*Institute of Mathematics, Pedagogical University of Cracow, Cracow, Poland*

ELÍAS FUENTES GUILLÉN 

*Department of Mathematics, Faculty of Sciences, UNAM, Mexico City, Mexico*

PETER HEINIG

VLADIMIR KANOVEI 

*IPPI RAS, Moscow, Russia*

MIKHAIL G. KATZ 

*Department of Mathematics, Bar Ilan University, Ramat Gan, Israel*

In a paper published in 1970, Grattan-Guinness argued that Cauchy, in his 1821 *Cours d'Analyse*, may have plagiarized Bolzano's *Rein analytischer Beweis* (RB), first published in 1817. That paper was subsequently discredited in several works, but some of its assumptions still prevail today. In particular, it is usually considered that Cauchy did not develop his notion of the continuity of a function before Bolzano developed his in RB and that both notions are essentially the same. We argue that both assumptions are incorrect, and that it is implausible that Cauchy's initial insight into that notion, which eventually evolved to an approach using infinitesimals, could have been borrowed from Bolzano's work. Furthermore, we account for Bolzano's interest in that notion and focus on his discussion of a definition by Kästner (in Section 183 of his 1766 book), which the former seems to have misrepresented at least partially.

## 1. Introduction

The issue of priority for the definition of the continuity of a function was raised in (Grattan-Guinness 1970) in a way that provoked controversy. With regard to this issue, Grabiner sought to shift the focus of attention away from the Bolzano/Cauchy priority debate and broaden the discussion to include an analysis of their common predecessors, particularly Lagrange. She detected an 'immediate source of the independent Bolzano–Cauchy definitions' both in Lagrange's 1798 *Traité de la résolution des équations numériques de tous les degrés* and in his *Théorie des fonctions analytiques* (see Grabiner 1984, 113). Grabiner concluded that 'these two books are the most likely sources for both Cauchy's and Bolzano's definitions of continuous function' (op. cit., 114). Grabiner's analysis challenged Grattan-Guinness' claim that '[Bolzano's and

Cauchy's] new foundations, based on limit avoidance, certainly swept away the old foundations, founded largely on faith in the formal techniques' (Grattan-Guinness 1970, 382). For sources of Bolzano's notion of continuity in Lagrange, see also Rusnock (1999, 422).

Schubring similarly ruled out Grattan-Guinness' hypothesis, and furthermore challenged a common assumption that Bolzano's work was virtually unknown in the mathematical community during the first half of the nineteenth century (Schubring 1993). He reported on a (formerly) unknown review of Bolzano's three important papers from 1816 and 1817, written by a mathematician named J Hoffmann in 1821 and published in 1823.

As for the Bolzano–Cauchy continuity, Grattan-Guinness investigated the possibility of its antecedents, focusing on the following three sources: (1) Cauchy's work prior to 1821, (2) Legendre, and (3) Fourier; see Grattan-Guinness (1970, 286). His search reportedly did not turn up any reasonable antecedents: 'of the new ideas that were to achieve that aim – of them, to my great surprise, I could find nothing' (ibid.). His investigation led him to his well-known controversial conclusions. What he missed were the following sources: (1) Cauchy's earlier course summaries that were only discovered over a decade after Grattan-Guinness' article (see Section 2); (2) Lagrange (as argued by Grabiner); and (3) other eighteenth-century authors, such as Kästner and Karsten (see Section 4).

Some mathematicians and historians of mathematics assume that Bolzano's definition of the continuity of a function in his 1817 *Rein analytischer Beweis* preceded Cauchy's and that the latter first gave one in his 1821 textbook the *Cours d'Analyse*. Both assumptions turn out to be incorrect. Scholars commonly assume the following claims to be true:

- (CI 1) Bolzano and Cauchy gave essentially the same definition of continuity, and
- (CI 2) Bolzano gave it earlier.

We give some examples below.

- Jarník: 'Bolzano defines continuity essentially in the same way as Cauchy does a little later' (Jarník 1981, 36).
- Segre: 'This led [Bolzano], in his *Rein analytischer Beweis* (written in 1817, four years before Cauchy published his *Cours d'analyse*), to give a definition of continuity and derivative very similar to Cauchy's, etc.' (Segre 1994, 236).
- Ewald: '[Bolzano's] definition is essentially the same as that given by Cauchy in his *Cours d'analyse* in 1821; whether Cauchy knew of Bolzano's work is uncertain' (Ewald 1996, 226).
- Heuser: 'Cauchy defines continuity substantially in the same way as Bolzano: ...'<sup>1</sup>

Now claim (CI 1) is problematic since, as noted by Lützen,

Bolzano did not use infinitesimals<sup>2</sup> in his definition of continuity. Cauchy did (Lützen 2003, 175).

<sup>1</sup>In the original German: 'Stetigkeit definiert Cauchy inhaltlich so wie Bolzano' (Heuser 1991, 691). Heuser goes on to present Cauchy's first 1821 definition in terms of  $f(x + \alpha) - f(x)$  (see Section 2.2) but fails to mention the fact that Cauchy describes  $\alpha$  as an *infinitely small increment*.

<sup>2</sup>Note, however, that Bolzano did exploit infinitesimals in his later writings; see for example, Grattan-Guinness (1970, note 29, 379), Trlifajová (2018) and Fila (2020).

Lützen’s claim that Cauchy used infinitesimals in his definition of continuity is not entirely uncontroversial. While Cauchy indisputably used the term *infinitement petit*, the meaning of Cauchy’s term is subject to debate. Grabiner (1981), Gray (2015, 36), and some other historians feel that a Cauchyan infinitesimal is a sequence tending to zero. Others argue that there is a difference between null sequences and infinitesimals in Cauchy (see for example, Bair *et al.* 2019).

In sum, Cauchy’s 1821 definitions exploited infinitesimals (and/or sequences), whereas Bolzano’s definition in the *Rein analytischer Beweis* exploited the clause ‘provided  $\omega$  can be taken as small as we please’ in a way that can be interpreted as an incipient form of an  $\epsilon, \delta$  definition relying on implied alternations of quantifiers. Such manifest differences make it difficult to claim that the definitions were ‘essentially the same.’

To determine the status of claim (CI 2), we will examine the primary sources in Bolzano and Cauchy and compare their dates.

## 2. Evolution of Cauchy’s ideas documented by Guitard

Primary sources published in the 1980s suggest that an evolution took place in Cauchy’s ideas concerning continuity. On 4 March 1817, Cauchy presented an infinitesimal-free treatment of continuity in terms of variables which is procedurally identical with the modern definition of continuous functions via commutation of taking limit and evaluating the function, as we discuss in Section 2.1.

### 2.1. Continuity in 1817

In modern mathematics, a real function  $f$  is continuous at  $c \in \mathbb{R}$  if and only if for each sequence  $(x_n)$  converging to  $c$ , one has  $f(\lim_{n \rightarrow \infty} x_n) = \lim_{n \rightarrow \infty} f(x_n)$ , or briefly  $f \circ \lim = \lim \circ f$  at  $c$ .<sup>3</sup>

In 1817, Cauchy wrote (see Figure 1):

*La limite d’une fonction continue de plusieurs variables est la même fonction de leur limite. Conséquence de ce Théorème relativement à la continuité des fonctions composées qui ne dépendent que d’une seule variable.*<sup>4</sup> (Cauchy as quoted in Guitard 1986, 34; emphasis added; cf. Belhoste 1991, 255, note 6 and 309)

The Intermediate Value Theorem was proved in the same lecture. Cauchy’s treatment of continuity in 1817<sup>5</sup> contrasts with his definitions based on infinitesimals given four years later in the *Cours d’Analyse* (CdA).

<sup>3</sup>The equivalence of such a definition with the  $\epsilon, \delta$  one requires the axiom of choice.

<sup>4</sup>Translation: ‘The limit of a continuous function of several variables is [equal to] the same function of their limit. Consequences of this Theorem with regard to the continuity of composite functions dependent on a single variable.’ The reference for this particular lesson in the Archives of the Ecole Polytechnique is as follows: Le 4 Mars 1817, la leçon 20. Archives E. P., X II C7, Registre d’instruction 1816–1817.

<sup>5</sup>Belhoste places it even earlier, in 1816: ‘according to the *Registres*, Cauchy knew the modern concept of continuity as far back as March 1817, but the “invention” was anterior, as shown by the instructional program of December 1816’ Belhoste (1991, 255, note 6).

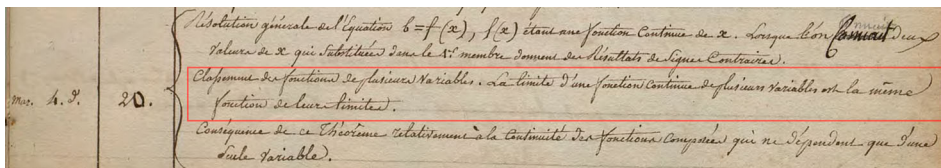


Figure 1. Cauchy's treatment of continuity dating from 4 March 1817 in the Gregorian calendar (which was a Tuesday). The 'Mar.' in the figure stands for *mardi*, Tuesday. The glyph resembling  $\partial$  to the right of the date seems to be shorthand for *ditto*, referring to the month of March mentioned on earlier lines in this *Registre de l'Instruction* for 1817.

## 2.2. Continuity in the Cours d'Analyse

In CdA, Cauchy defined continuity as follows (see Figure 2):

Among the objects related to the study of infinitely small quantities, we ought to include ideas about the continuity and the discontinuity of functions. In view of this, let us first consider functions of a single variable. Let  $f(x)$  be a function of the variable  $x$  and suppose that for each value of  $x$  between two given limits, the function always takes a unique finite value. If, beginning with a value of  $x$  contained between these limits, we add to the variable  $x$  an *infinitely small increment*  $\alpha$ , the function itself is incremented by the difference  $f(x + \alpha) - f(x)$ , which depends both on the new variable  $\alpha$  and on the value of  $x$ . Given this, the function  $f(x)$  is a *continuous* function of  $x$  between the assigned limits if, for each value of  $x$  between these limits, the numerical value of the difference  $f(x + \alpha) - f(x)$  decreases indefinitely with the numerical value of  $\alpha$ . (Cauchy as translated in (Bradley and Sandifer 2009, 26);<sup>6</sup> emphasis on 'continuous' in the original; emphasis on 'infinitely small increment' added).

This definition can be thought of as an intermediate one between the March 1817 definition purely in terms of variables and containing no mention of the infinitely small and his second 1821 definition stated purely in terms of the infinitely small (see Section 2.3).

## 2.3. Second definition of continuity in CdA

Cauchy went on to summarize the definition given above as follows (see Figure 3):

In other words, *the function  $f(x)$  is continuous with respect to  $x$  between the given limits if, between these limits, an infinitely small increment in the variable always produces an infinitely small increment in the function itself.*<sup>7</sup> (ibid.; emphasis in the original).

Since Cauchy prefaced his second definition with the words *en d'autres termes* ('in other words'), he appears to have viewed the pair of 1821 definitions

<sup>6</sup>Siegmund-Schultze (2009) writes: 'By and large, with few exceptions to be noted below, the translation is fine'.

<sup>7</sup>In the original: 'En d'autres termes, la fonction  $f(x)$  restera continue par rapport à  $x$  entre les limites données, si, entre ces limites, un accroissement infiniment petit de la variable produit toujours un accroissement infiniment petit de la fonction elle-même' (Cauchy 1821, 34–35).

Soit  $f(x)$  une fonction de la variable  $x$ , et supposons que, pour chaque valeur de  $x$  intermédiaire entre deux limites données, cette fonction admette constamment une valeur unique et finie. Si, en partant d'une valeur de  $x$  comprise entre ces limites, on attribue à la variable  $x$  un accroissement infiniment petit  $\alpha$ , la fonction elle-même recevra pour accroissement la différence

$$f(x+\alpha) - f(x),$$

qui dépendra en même temps de la nouvelle variable  $\alpha$  et de la valeur de  $x$ . Cela posé, la fonction  $f(x)$  sera, entre les deux limites assignées à la variable  $x$ , fonction *continue* de cette variable, si, pour chaque valeur de  $x$  intermédiaire entre ces limites, la valeur numérique de la différence

$$f(x+\alpha) - f(x)$$

décroit indéfiniment avec celle de  $\alpha$ . En d'autres

Figure 2. Cauchy's first 1821 definition of continuity.

décroit indéfiniment avec celle de  $\alpha$ . En d'autres termes, *la fonction  $f(x)$  restera continue par rapport à  $x$  entre les limites données, si, entre ces limites, un accroissement infiniment petit de la variable produit toujours un accroissement infiniment petit de la fonction elle-même.*

Figure 3. Cauchy's second 1821 definition of continuity.

as being equivalent. Cauchy summed up his discussion of continuity in CdA as follows:

We also say that the function  $f(x)$  is a continuous function of the variable  $x$  in a neighborhood of a particular value of the variable  $x$  whenever it is continuous between two limits of  $x$  that enclose that particular value, even if they are very close together. Finally, whenever the function  $f(x)$  ceases to be continuous in the neighborhood of a particular value of  $x$ , we say that it becomes discontinuous,

and that there is *solution*<sup>8</sup> of continuity for this particular value. (ibid.; emphasis in the original).

Note that none of the 1821 definitions exploited the notion of limit. We therefore find it puzzling to discover the contrary claim in a recent historical collection:

Cauchy gave a faultless definition of continuous function, using the notion of ‘limit’ for the first time. Following Cauchy’s idea, Weierstrass popularized the  $\varepsilon$ - $\delta$  argument in the 1870s (Dani and Papadopoulos 2019, 283).

In a related vein, Văth opines that ‘formulat[ing] properties which hold for infinitesimals (which have been use by Leibniz) in an  $\varepsilon$ - $\delta$ -type manner ... was first propagated by Cauchy’ (Văth 2007, 74). Similarly, Goldbring and Walsh claim the following:

[T]he mathematical status of [infinitesimals] was viewed as suspect and the entirety of calculus was put on firm foundations in the nineteenth century by the likes of Cauchy and Weierstrass, to name a few of the more significant figures in this well-studied part of the history of mathematics. The innovations of their ‘ $\varepsilon$ - $\delta$  method’ ... allowed one to give rigor to the naïve arguments of their predecessors (Goldbring and Walsh 2019, 843).

Presentist views of this type are, alas, not the exception, and much work is required to counter them. Recent work on Cauchy’s stance on the infinitely small and their applications includes Bair *et al.* (2017a), Błaszczyk *et al.* (2017), Bascelli *et al.* (2018), and Bair *et al.* (2020).

To summarize, in 1817, Cauchy gave a characterization of continuity in terms of variables, whereas his second 1821 definition involved only infinitesimals. Meanwhile, the first 1821 definition exploited both variables and infinitesimals.

### 3. Bolzano’s *Rein analytischer Beweis*

Could Bolzano’s *Rein analytischer Beweis* (RB) (Bolzano 1817) have influenced Cauchy’s definition of continuity? Grattan-Guinness wrote:

Bolzano had given his paper [RB] two opportunities for publication, for not only did he issue it as a pamphlet in 1817, but – with the same printing – inserted it into the 1818 volume of the Prague Academy *Abhandlungen*. That journal was available in Paris: indeed, the *Bibliothèque Impériale* (now the *Bibliothèque Nationale*) began to take it with *precisely the volume containing Bolzano’s pamphlet*. (Grattan-Guinness 1970, 396) (emphasis in the original).

Of particular interest to us is Grattan-Guinness’ reliance on the availability of RB in the Paris Imperial Library in 1818; see Section 3.1. The papers Freudenthal (1971) and Sinaceur (1973) provided evidence against Grattan-Guinness’ hypothesis. However, as noted by Jan Sebestik, their work does not rule out the possibility that

<sup>8</sup>Meaning *dissolution*, that is, absence (of continuity).

‘Cauchy could have read Bolzano’s *Rein analytischer Beweis* (or heard about it) and could have been inspired by it’ (Sebestik 1992, 109, 111). Thirty years after the Benis-Sinaceur paper, Russ wrote:

There has been *discussion* in the literature on the possibility that Cauchy might have plagiarized from Bolzano. See Grattan-Guinness (1970), Freudenthal (1971) and Sinaceur (1973). (Russ 2004, 149; emphasis added).

It is our understanding that referring to the issue as a ‘discussion’ tends to imply that the hypothesis of plagiarism has not been definitively refuted.<sup>9</sup> Arguably, therefore, the issue continues to have relevance.

### 3.1. *Grattan-Guinness’ hypothesis*

Having summarized the historical background, Grattan-Guinness proceeds to state his hypothesis:

So here is at least one plausible possibility for Cauchy to have found a copy of Bolzano’s paper, quite apart from the book-trade: he could have noticed a new journal in the library’s stock and examined it as a possible course<sup>10</sup> of interesting research. (Grattan-Guinness 1970, 396).

Grattan-Guinness specifically includes the concept of continuity in his hypothesis (op. cit., 374).

It is our understanding that, while the evidence provided in the articles Freudenthal (1971) and Sinaceur (1973) shows clear and profound differences between Cauchy and Bolzano’s stance, it does not entirely refute the aforementioned hypothesis. We will provide a refutation of a key component of Grattan-Guinness’ hypothesis concerning the concept of continuity. Our refutation is based on the facts of the chronology of the relevant works. Namely, we will show that Cauchy possessed a concept of continuity

- (1) earlier than the date of the acquisition of a journal version of RB by the Imperial Library in Paris, and
- (2) even earlier than, or at least contemporaneously with, the date of the Leipzig fair where RB was first marketed.

Note that, according to Grattan-Guinness, the Bibliothèque Impériale started to take the journal where RB appeared in the year 1818. Reading the 1818 journal version of RB could not therefore have influenced Cauchy’s treatment of continuity in 1817<sup>11</sup> (see Section 2). This refutes a key component of the plagiarism hypothesis as proposed in Grattan-Guinness (1970) with regard to the concept of continuity. The comparison of

<sup>9</sup>Similarly, in a recently published book, Rusnock and Šebestik mention that ‘there has been speculation that Cauchy may have learned a thing or two from Bolzano’ (Rusnock and Šebestik 2019, 49); see also note 3 there.

<sup>10</sup>Grattan-Guinness apparently means ‘source.’

<sup>11</sup>Cauchy had discussed continuity even earlier, in an 1814 article on complex functions (see Freudenthal 1971, 380). However, that discussion stayed at the intuitive level and cannot be described as reasonably precise.



dates establishes that Cauchy's initial insight into continuity could not have been borrowed from Bolzano's RB, though it does not rule out the possibility that Cauchy may have been acquainted with Bolzano's work before formulating the later, 1821 definitions in CdA.

Grattan-Guinness also brought broader plagiarism charges against Cauchy, which are not refuted by our comparison of dates. Notice, however, that it is implausible that Cauchy could have seen Bolzano's 1816 text *Der binomische Lehrsatz* (Bolzano 1816), where the latter also gave a definition of continuity, since there is no evidence that this text was available in France. It seems that this is why Grattan-Guinness found it necessary to speculate specifically concerning the version of Bolzano's RB available in a Paris library in 1818, so as to bolster the plausibility of the plagiarism claim. Grattan-Guinness may have had more of a point with regard to E G Björling. Apparently, in the 1850s, Cauchy may not have been transparent about the possible influence of Björling's ideas related to uniform convergence. The issue was studied in Bråting (2007). For an analysis of Cauchy's 1853 approach to uniform convergence see Bascelli *et al.* (2018).

### 3.2. Bolzano's definition in the Rein analytischer Beweis

In his RB, Bolzano criticized some proofs of the IVT for polynomials that from his stance were 'based on an incorrect concept of *continuity*,' given for example their use of 'a truth borrowed from *geometry*' or 'the introduction of the concepts of *time* and *motion*' (Bolzano 1817, 6, 8–9, 11). Instead, he defined continuity as follows:

According to a correct definition, the expression that a function  $fx$  varies according to the law of continuity for all values of  $x$  inside or outside certain limits means only that, if  $x$  is any such value the difference  $f(x + \omega) - fx$  can be made smaller than any given quantity, provided  $\omega$  can be taken as small as we please or (in the notation we introduced in §14 of *Der binomische Lehrsatz* etc., Prague, 1816)  $f(x + \omega) = fx + \Omega$ . (Bolzano as translated in Russ 2004, 149, 256.)

The dating of RB will be discussed in Section 3.3. Bolzano's definition is reasonably precise, as is Cauchy's approach. Here 'reasonably precise' means 'easily transcribable as a modern definition' (rather than merely an intuitive notion of continuity).<sup>12</sup> A modern formalization of Bolzano's 1817 definition would involve alternating quantifiers, whereas a modern formalisation of Cauchy's 1817 definition would retain almost verbatim the commutation of (a) evaluating  $f$  and (b) taking  $\lim$  (see Section 2.1). Apparently neither Jarník nor Ewald (see Section 1) was aware of Cauchy's treatment of both continuity and the IVT dating from 4 March 1817.

### 3.3. The dating of Bolzano's RB

The earliest known written record of Bolzano's RB is in a catalogue of the Easter book fair at Leipzig.

<sup>12</sup>Note that we take no position with regard to which definition was closer to a modern one, Bolzano's or Cauchy's (Bolzano's was arguably closer to the modern *Epsilontik* standard). The point we are arguing is that both were reasonably precise in the sense specified.

According to Evenhuis (2014, 4), both the catalogue (Olms 1817, 30) and the fair itself date from 27 April 1817, over a month later than the earliest written record of Cauchy's treatment of continuity. It should be noted, however, that Bolzano also gave a definition of continuity in an 1816 publication (Bolzano 1816) (see Figure 4):

For a function is called continuous if the change which occurs for a certain change in its argument, can become smaller than any given quantity, provided that the change in the argument is taken small enough.<sup>13</sup> (Bolzano as translated in Russ (2004, 184).)

This definition was immediately followed by an attempted proof of an erroneous assertion. Namely, Bolzano claimed to prove that if a function  $F$  is differentiable then its derivative,  $f$ , is continuous. This indicates that Bolzano's definition of continuity was still sufficiently ambiguous to accommodate errors, as was his  $\omega/\Omega$  notation. Recently (Fuentes Guillén and Martínez Adame 2020, abstract) have argued in *Historia Mathematica* that 'those quantities [that is, Bolzano's  $\omega$ ] are not clearly "proto-Weierstrassian".'

It is worth noting that an even earlier mention of ideas in the direction of Bolzano's definition of continuity occurs in Bolzano's mathematical diaries of early 1815: 'if therefore  $\xi$  is taken smaller than any given quantity, that is,  $= \omega$ , the value of  $f(x + \omega) - fx$  must be able to become as small as desired' (see op. cit., note 86). Insofar as Cauchy had no access either to Bolzano's diaries or the latter's 1816 work, and the former would have formulated his first definition of continuity shortly before or in any case at about the same time as the 1817 Easter book fair at Leipzig, it is implausible that Cauchy's 1817 definition could have been borrowed from Bolzano's work.

#### 4. Antecedents in Kästner, Karsten, and others

There exists a historiographic controversy with regard to the issue of continuity in the historical development of mathematics. Unguru and his disciples adopt a radical posture against such continuity. Other scholars endorse continuity at various levels and to varying extent. We adopt the latter view, to the extent that we detect continuity between, for example, the work of Kästner, on the one hand, and that of Bolzano and Cauchy, on the other. For more details see Katz (2020).

The mathematical diaries of Bolzano written during 1814–1815 also contain criticism of, for example, Carnot (1797) and Crelle (1813) because of their assumption of the law of continuity: in the first case, he stated that in such a law '[lay] the key for the resolution of the whole riddle of infinitesimal calculus' (Bolzano 1995, 152); in the latter case, he pointed out that Kästner had 'already drawn attention to the surreptitious acceptance of this law' (Bolzano 1997, 144). As we already mentioned, the first published record of a definition of continuity given by Bolzano dates from the

<sup>13</sup>In the original: 'Stetig heißt nämlich eine Function, wenn die Veränderung, die sie bey einer gewissen Veränderung ihrer Wurzel erfährt, kleiner als jede gegebene Größe zu werden vermag, wenn man nur jene klein genug nimmt' Bolzano (1816, 34). Note that Bolzano repeatedly uses *Wurzel* in the sense of 'input to a function'; see for example, footnote on page 11 of Bolzano (1817). The issue is discussed in Russ (2004, 256, note f).

nur immer will; indem die Function  $f x$  stetig ist. Stetig heißt nämlich eine Function, wenn die Veränderung, die sie bey einer gewissen Veränderung ihrer Wurzel erfährt, kleiner als jede gegebene Größe zu werden vermag, wenn man nur jene klein genug nimmt. Nun zeigt die Gleic-

Figure 4. Bolzano's definition of continuity.

following year, after which he published his reasonably precise definition included in RB.

As his later works and mathematical diaries show, Bolzano continued to be interested in that issue. Thus, in his *Theory of Functions*, written in the 1830s, he would have 'sharpened' his 1817 definition (Rusnock and Kerr-Lawson 2005, 306). Rusnock and Kerr-Lawson argue that, as early as the 1830s, Bolzano not only grasped the distinction between pointwise continuity and uniform continuity but also presented a pair of key theorems concerning the latter. Moreover, in that work Bolzano acknowledged that '[t]he concept of continuity has already been defined essentially as I do here by [other contemporary authors]' such as Cauchy and Ohm (Russ 2004, 449). However, at the same time, in that work he criticized certain specific definitions, including one by A. G. Kästner in 1766. On the one hand, Bolzano's definition surely constituted an improvement upon the definition of local continuity by Kästner in 1760 (see Figure 5). On the other hand, Bolzano seems to have misrepresented, at least partially, the relevant passage from Kästner's work of 1766.

#### 4.1. Kästner's 1760 definition

Kästner's definition, included in his volume on the analysis of finite quantities (*Analysis endlicher Größen*), or letter-algebra, and which can be found in a section entitled 'On curved lines,' runs as follows:

In a sequence<sup>14</sup> of magnitudes, their increase or decrease takes place in accordance with the law of continuity (*lege continui*), if after each term of the sequence, another one follows or precedes *the given term* that differs from it [that is, from the given term] by as little as one wishes; as a consequence,<sup>15</sup> the difference of two consecutive terms<sup>16</sup> can amount to less than any given magnitude.<sup>17</sup> (Kästner 1760, paragraph 322, 180).

<sup>14</sup>We translate *Reihe* as 'sequence', even though it is often translated as 'series', since 'series' nowadays is a standard technical term which is not appropriate here, and moreover the German term *Reihe* can mean either 'sequence' or 'series'.

<sup>15</sup>The German conjunction *so dass*, especially in Kästner's (now obsolete) spelling as two separate words, resembles the English 'such that'; in the present case, however, this is a false friend. In fact 'as a consequence' is one of several standard translations of the German conjunction *sodass*.

<sup>16</sup>Kästner's phrasing *nach einander folgender* could possibly be interpreted as the statement that the terms mentioned here are immediate successor elements, in particular since the standard technical translation

322. **Erkl.** In einer Reihe von Größen, erfolgt das Wachsthum oder das Abnehmen derselben, nach dem Gesetze der Stetigkeit (lege continui) wenn nach jedem Gliede der Reihe eines folget, oder vor ihm vorhergehen kann, das so wenig als man nur will von dem angenommenen Gliede unterschieden ist, so daß der Unterschied zweyer nach einander folgenden Glieder, weniger als jede gegebene Grösse betragen kann. Im Halbkreise

Figure 5. Kästner's 1760 definition of continuity.

#### 4.2. Kästner's influence on Bolzano

Russ notes Kästner's influence on Bolzano in the following terms:

[T]here were two authors, Wolff and Kästner, whose work, between them, dominated the century in the German-speaking regions. ... [T]hey were both committed to education and wrote highly systematic and comprehensive multivolume textbooks on mathematics that went through many editions and were very influential. Not surprisingly, they were both authors to whom Bolzano makes frequent reference in his early works. (Russ 2004, 14).

Indeed, in Bolzano's mathematical diaries there is a note from the early 1820s, entitled "On the law of continuity." Bolzano's note includes a reference to paragraph 183 of Kästner's work on mechanics (Kästner 1766) and to paragraph 235 of W J G Karsten's work on mechanics (Karsten 1769); see Bolzano (2005, 63). The formulation of both authors ultimately relied on the notion of continuity according to which '[a] continuous quantity (continuum) is that [quantity] whose parts are all connected together in such a way that where one ceases, another immediately begins, and between the end of one and the beginning of another there is nothing that does not belong to this quantity' (Russ 2004, 17); see Karsten (1767, 209): but only that of Karsten would be equivalent to IVT (Karsten 1769, 223). Interestingly, as we already mentioned, in a later work

for 'immediate successor element' is *Nachfolger*. This, however, could not be what Kästner meant to say. Kästner's phrasing (note that he does not say *Nachfolger* outright) is sufficiently vague to allow for an interpretation where he means to speak of two terms which follow *shortly* one after another, though there are other terms in between.

<sup>17</sup>In the original: 'In einer Reihe von Grössen, erfolgt das Wachsthum oder das Abnehmen derselben, nach dem Gesetze der Stetigkeit (lege continui) wenn nach jedem Gliede der Reihe eines folget, oder vor ihm vorhergehen kann, das so wenig als man nur will von dem angenommenen Gliede unterschieden ist, so daß der Unterschied zweyer nach einander folgender Glieder, weniger als jede gegebene Grösse betragen kann.' This was quoted in Spalt (2015, 283). In our translation, we try to strike a balance between literalness and readability in line with the approach taken in Blåsjö and Hogendijk (2018).

Bolzano went back to discuss the notion of continuity in that paragraph of Kästner's work. We will analyse such a reception of the latter's ideas in Section 4.3.

#### 4.3. *Bolzano misattributes a definition to Kästner*

We reviewed Kästner's 1760 definition in Section 4.1. In his *Theory of Functions*, Bolzano seems to have mistakenly attributed a different definition to Kästner in 1766, which he (Bolzano) considered to be 'too broad':

Some very respected mathematicians like *Kästner* (*höhere Mechanik*, Auflage 2, §§ 183 ff.) and *Fries* (*Naturphilosophie*, § 50) define the continuity of a function  $Fx$  as that property of it by virtue of which it does not go from a certain value  $Fa$ , to another value  $Fb$ , without first *having taken all the values lying in between*. However, it will be seen subsequently that this definition is *too wide*<sup>18</sup> if in fact the concept intended is to be equivalent to the one above. (Bolzano as translated in Russ (2004, 449); emphasis on Kästner and Fries in the original; emphasis on 'having taken all the values lying in between' and 'too wide' added.)

As we already noted, Kästner's formulation to which Bolzano refers here ultimately relied on the former's geometric notion of continuity. So, while Kästner's paragraph 183 is part of a section 'On the law of continuity' (which in turn is part of a chapter 'On the movement of solid bodies with determined magnitude and shape'), he explicitly referred to the note in definition 6 (straight and curved lines) of his book on geometry. In that note Kästner pointed out that before the curved line that goes from  $A$  to  $B$  reaches  $B$ , 'all the minor changes in between must occur' (Kästner 1758, 161).

Bolzano would seem to attribute a different definition (via the satisfaction of the Intermediate Value Theorem) to Kästner (as well as to Fries) in the particular case of that paragraph. Nonetheless, Bolzano's attribution appears to be incorrect.

In fact, Kästner's discussion of the law of continuity in his section 183 resembles, to some extent, Cauchy's definition of continuity based on infinitesimals given in Section 2.3 above (though of course Kästner's viewpoint was geometric rather than analytic):

On the Law of Continuity. 183. In the investigation which we now present, it is assumed that the speed of a body does not change instantaneously, but rather by infinitely small gradations. Just the same can be said of the direction. If one views the matter from that perspective, then a body which is being reflected does not change its direction instantaneously to the opposite direction: its speed becomes smaller and smaller in the previous direction, finally vanishes, and then transforms into a velocity having the opposite direction. This is the *Law of Continuity* (applied to these matters). To wit, by the latter law one claims that generally, no change happens suddenly, but that *every change always moves through infinitely small gradations* (of which already the movement of a point along a curve is an example; [cf. Kästner's] *Geom. 6. Erkl. Anm.*). (Kästner 1766, 350, § 183;<sup>19</sup> emphasis on 'law of continuity' on the original; emphasis on 'every change, etc.' added.)

<sup>18</sup>Perhaps a better translation is 'too broad'.

<sup>19</sup>According to Kröger (2014, Abbildung 10), there were two editions of this treatise. These are Kästner (1766, 1793). In the 1793 edition of Kästner's treatise referred to by Bolzano as *Auflage 2*, Section 183 appears on page 543.

What may have led Bolzano to claim that Kästner defined continuity based on the satisfaction of the IVT? Note that Kästner's text contains the following three sentences:

- (K1) If one views the matter from that perspective, then a body which is being reflected does not change its direction instantaneously to the opposite direction: its speed becomes smaller and smaller in the previous direction, finally vanishes, and then transforms into a velocity having the opposite direction.
- (K2) This is the Law of Continuity (applied to these matters).
- (K3) To wit, by the latter law one claims that generally, no change happens suddenly, but that every change always moves through infinitely small gradations.

Possibly, Bolzano interpreted sentence (K1) as the definition of the law of continuity mentioned in sentence (K2). Now sentence (K1) does sound like (a physical interpretation of) the IVT.

However, reading the three sentences together, it is clear that Kästner meant sentence (K3) to be the detailed formulation of the law of continuity. Meanwhile, in sentence (K2), Kästner specifically used the verb *applied*. This indicates that Kästner thought of sentence (K1) as an *application* of the law of continuity, rather than the formulation thereof. Now, in modern mathematics, it is certainly true that continuity implies the IVT: though the converse is incorrect, as Bolzano himself argued (see Russ 2004, § 84, 471–472). In his *Theory of Functions*, Bolzano outlined an idea for a function that takes every intermediate value without being continuous, as follows.

Bolzano started with an everywhere discontinuous function  $W(x)$  described in § 37, defined only on a collection of rational points, and built out of a pair of linear functions of different slope. In § 39, Bolzano asserted that the remaining infinitely many points could be used to assign the values of the function so as to 'fill in' whatever values were missing. Bolzano's argument is mentioned in Sebestik (1992, 395)<sup>20</sup> and Smoryński (2017) (see 52 and note 49 there). For a study of counterexamples to the implication 'if  $f$  satisfies IVT then  $f$  is continuous' see Oman (2014), Radcliffe (2016), and De Marco (2018).

In conclusion, Bolzano may have interpreted sentence (K1) as the formulation of continuity (rather than an application thereof). Unlike Cauchy, Bolzano seems never to have formulated a definition of continuity in terms of infinitesimals. It is possible that Kästner's sentence (K3) made no sense to Bolzano, who was therefore led to take sentence (K1) to be the formulation of continuity. Thus, while Fries may perhaps have given a different definition of continuity via the satisfaction of the IVT (as Bolzano claimed), Kästner apparently did not.

#### 4.4. *Continuity in Leibniz*

An even earlier source for local continuity may have influenced Kästner and other eighteenth-century authors. Such a source is in Leibniz's 1687 formulation of the principle of continuity:

When the difference between two instances in a given series or *that which is presupposed* can be diminished until it becomes smaller than any given quantity whatever,

<sup>20</sup>Sebestik also points out that Bolzano and Cauchy's definitions of continuity could have been 'the result of a critical reflection on the texts by Euler and Lagrange' (Sebestik (1992), 110, 81, 83).



the corresponding difference in *what is sought* or in their results must of necessity also be diminished or become less than any given quantity whatever. (Leibniz as translated by Loemker in Leibniz (1989, 351); emphasis added.)

In modern terminology, Leibnizian ‘what is sought’ is the dependent variable, while ‘that which is presupposed’ is the independent variable. What Leibniz referred to as the principle of continuity<sup>21</sup> involves, in modern terminology, the condition that a convergent sequence in the domain should get mapped to a convergent sequence in the range.<sup>22</sup>

Cauchy’s approach dating from 4 March 1817 is not the final word on continuity, but it can be described as reasonably precise (in the sense explained in Section 3.2). This is unlike many intuitive definitions given earlier<sup>23</sup> that cannot be so formalized.

Notice that Bolzano’s definition is, similarly, reasonably precise but also not without its problems. Thus, the  $\Omega$  appearing there seems to be defined as the difference  $f(x + \omega) - f(x)$ , whereas the corresponding  $\varepsilon$  in the modern definition is a  $\forall$ -quantified variable entirely unrelated to  $f$ . It is possible that this was also Bolzano’s intention, but it must be admitted that such an intention was only imperfectly expressed by Bolzano’s formula  $f(x + \omega) = fx + \Omega$  and accompanying comments; see (Fuentes Guillén and Martínez Adame 2020) for a fuller discussion.

## 5. Conclusion

We have re-examined the priority issue with regard to the concept of continuity. Course notes available at the Ecole Polytechnique indicate that Cauchy had a reasonably precise concept of continuity of a function earlier than is generally thought. In particular, Cauchy’s concept was earlier than or at least contemporaneous with, the first written record of Bolzano’s 1817 *Rein analytischer Beweis*.

In 1970, Grattan-Guinness speculated that Cauchy may have read a version of Bolzano’s *Rein analytischer Beweis* found in a Paris library in 1818, and subsequently plagiarized some of Bolzano’s insights, including continuity, when writing the 1821 *Cours d’Analyse*. Such a hypothesis is refuted by a written record of a reasonably precise treatment of continuity by Cauchy dating from March 1817, and hence anterior to the Paris library acquisition, on which, among other things, Grattan-Guinness based his hypothesis.

The proximity of the dates indicates an independence of Cauchy’s and Bolzano’s scientific insight and should contribute not only to end speculations as to possible plagiarism (with regard to the notion of continuity) on either side but also to improve our understanding of their respective developments of such a notion.

The prototypes of both Bolzano’s and Cauchy’s definitions of continuity in formulations found in eighteenth-century and early nineteenth-century works, such as those of Kästner, are yet to be explored fully.

<sup>21</sup>Not to be confused with his *law of continuity*. For a detailed discussion see Katz and Sherry (2013), Sherry and Katz (2012), Bascelli *et al.* (2016), Bair *et al.* (2017b), and Bair *et al.* (2018).

<sup>22</sup>In modern analysis, the sequence-condition is equivalent to continuity for first-countable spaces.

<sup>23</sup>Including Cauchy’s own definition in 1814, in an article on complex functions quoted by Freudenthal; cf. note 11.

## Acknowledgments

The authors thank the anonymous referees for numerous suggestions that helped improve the article. We are grateful to Olivier Azzola, archivist at the Ecole Polytechnique, for granting access to Cauchy's handwritten course summaries reproduced in [Figure 1](#).

## Disclosure statement

No potential conflict of interest was reported by the author(s).

## Funding

E. Fuentes Guillén was supported by the Postdoctoral Scholarship Program of the Dirección General de Asuntos del Personal Académico (DGAPA-UNAM). V Kanovei was supported by RFBR [grant no. 18-29-13037].

## ORCID

Piotr Błaszczyk  <http://orcid.org/0000-0002-3501-3480>

Eliás Fuentes Guillén  <http://orcid.org/0000-0001-8722-5090>

Vladimir Kanovei  <http://orcid.org/0000-0001-7415-9784>

Mikhail G. Katz  <http://orcid.org/0000-0002-3489-0158>

## References

- Bair, J; Błaszczyk, P, Ely, R, Heinig, P, and Katz, M, 'Leibniz's well-founded fictions and their interpretations', *Matematychni Studii*, 49(2) (2018), 186–224. See <http://dx.doi.org/10.15330/ms.49.2.186-224> and <https://arxiv.org/abs/1812.00226>.
- Bair, J; Błaszczyk, P, Ely, R, Henry, V, Kanovei, V, Katz, K, Katz, M, Kudryk, T, Kutateladze, S, McGaffey, T, Mormann, T, Schaps, D, and Sherry, D, 'Cauchy, infinitesimals and ghosts of departed quantifiers', *Matematychni Studii*, 47(2) (2017a), 115–144. See <http://dx.doi.org/10.15330/ms.47.2.115-144> and <https://arxiv.org/abs/1712.00226>.
- Bair, J; Błaszczyk, P, Ely, R, Henry, V, Kanovei, V, Katz, K, Katz, M, Kutateladze, S, McGaffey, T, Reeder, P, Schaps, D, Sherry, D, and Shnider, S, 'Interpreting the infinitesimal mathematics of Leibniz and Euler', *Journal for General Philosophy of Science*, 48(2) (2017b), 195–238. See <http://dx.doi.org/10.1007/s10838-016-9334-z> and <https://arxiv.org/abs/1605.00455>.
- Bair, J; Błaszczyk, P, Heinig, P, Kanovei, V, and Katz, M, '19th century real analysis, forward and backward', *Antiquitates Mathematicae*, 13 (2019), 19–49. See <http://dx.doi.org/10.14708/am.v13i1.6440> and <https://arxiv.org/abs/1907.07451>.
- Bair, J; Błaszczyk, P, Heinig, P, Kanovei, V, and Katz, M, 'Cauchy's work on integral geometry, centers of curvature, and other applications of infinitesimals', *Real Analysis Exchange*, 45 (1) (2020), 1–23.
- Bascelli, T; Błaszczyk, P, Borovik, A, Kanovei, V, Katz, K, Katz, M, Kutateladze, S, McGaffey, T, Schaps, D, and Sherry, D, 'Cauchy's infinitesimals, his sum theorem, and foundational paradigms', *Foundations of Science*, 23(2) (2018), 267–296. See <http://dx.doi.org/10.1007/s10699-017-9534-y> and <https://arxiv.org/abs/1704.07723>.
- Bascelli, T; Błaszczyk, P, Kanovei, V, Katz, K, Katz, M, Schaps, D, and Sherry, D, 'Leibniz versus Ishiguro: closing a quarter century of syncategoremania', *HOPOS: The Journal of the International Society for the History of Philosophy of Science*, 6(1) (2016), 117–147. See <http://dx.doi.org/10.1086/685645> and <https://arxiv.org/abs/1603.07209>.
- Belhoste, B, *Augustin-Louis Cauchy. A biography*, New York: Springer-Verlag, 1991. Translated from the French and with a foreword by Frank Ragland.



- Blåsjö, V, and Hogendijk, J, 'On translating mathematics', *Isis: An International Review Devoted to the History of Science and its Cultural Influences*, 109(4) (2018), 774–781. See <https://doi.org/10.1086/701258>.
- Błaszczak, P; Kanovei, V, Katz, K, Katz, M, Kutateladze, S, and Sherry, D, 'Toward a history of mathematics focused on procedures', *Foundations of Science*, 22(4) (2017), 763–783. See <http://dx.doi.org/10.1007/s10699-016-9498-3> and <https://arxiv.org/abs/1609.04531>.
- Bolzano, B, *Der binomische Lehrsatz, und als Folgerung aus ihm der polynomische, und die Reihen, die zur Berechnung der Logarithmen und Exponentialgrößen dienen, genauer als bisher erwiesen*, Prague, 1816.
- Bolzano, B, *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetztes Resultat gewähren, wenigstens eine reelle Wurzel der Gleichung liege*, Prague: Abh. Königh Böhm. Gesell. Wiss., 1817. (3) 5 (1814–1817; publ. 1818), 60 p. Translations in Ewald (1996) and Russ (2004).
- Bolzano, B, 'Miscellanea Mathematica 10' in B van Rootselaar and A van der Lugt (eds), *Bernard Bolzano Gesamtausgabe (BGA). Band 2B6/2*, Stuttgart-Bad Cannstatt: Frommann-Holzboog, 1995.
- Bolzano, B, 'Miscellanea Mathematica 12' in B van Rootselaar and A van der Lugt (eds), *Bernard Bolzano Gesamtausgabe (BGA). Band 2B7/2*, Stuttgart-Bad Cannstatt: Frommann-Holzboog, 1997.
- Bolzano, B, 'Miscellanea Mathematica 20' in van B Rootselaar and A van der Lugt (eds), *Bernard Bolzano Gesamtausgabe (BGA). Band 2B11/2*, Stuttgart-Bad Cannstatt: Frommann-Holzboog, 2005.
- Bradley, R, and Sandifer, C (eds), *Cauchy's Cours d'analyse. An annotated translation*, (Sources and Studies in the History of Mathematics and Physical Sciences), New York: Springer, 2009.
- Bråting, K, 'A new look at E. G. Björling and the Cauchy sum theorem', *Archive for History of Exact Sciences*, 61(5) (2007), 519–535.
- Carnot, L, *Réflexions sur la Métaphysique du Calcul Infinitésimal*, Paris: Duprat, 1797.
- Cauchy, A L, *Cours d'Analyse de L'École Royale Polytechnique. Première Partie. Analyse algébrique*, Paris: Imprimerie Royale, 1821. Translation in Bradley–Sandifer (Bradley and Sandifer 2009, 2009).
- Crelle, A, Versuch einer rein algebraischen und dem gegenwärtigen zustande der mathematik, etc. Vandenhoeck und Ruprecht, 1813. See <https://books.google.com.mx/books?id=hDo7AQAAIAAJ>.
- Dani, S, and Papadopoulos, A (eds), *Geometry in history*, Cham: Springer, 2019. See <http://dx.doi.org/10.1007/978-3-030-13609-3>.
- De Marco, G, 'An almost everywhere constant function surjective on every interval', *The American Mathematical Monthly*, 125(1) (2018), 75–76.
- Evenhuis, N, 'Dates of the Leipzig book fairs (1758–1860), with notes on the book catalogs', *Sherbornia*, 1(1) (2014), 1–4.
- Ewald, W (ed), *From Kant to Hilbert: a source book in the foundations of mathematics*. Vol. I, II. Compiled, edited and with introductions by William Ewald. Oxford Science Publications. New York: The Clarendon Press, Oxford University Press, 1996.
- Fila, M, 'Why and how did Bernhard Bolzano develop calculus of the infinite quantities' in *Paradoxien des Unendlichen?* In preparation, 2020.
- Freudenthal, H, 'Did Cauchy plagiarize Bolzano?', *Archive for History of Exact Sciences*, 7 (1971), 375–392.
- Fuentes Guillén, E, and Martínez Adame, C, 'The notion of variable quantities  $\omega$  in Bolzano's early works', *Historia Mathematica*, 50 (2020), 25–49. See <https://doi.org/10.1016/j.hm.2019.07.002>.
- Goldbring, I, and Walsh, S, 'An invitation to nonstandard analysis and its recent applications', *Notices of the American Mathematical Society*, 66(6) (2019), 842–851.
- Grabner, J, *The origins of Cauchy's rigorous calculus*, Cambridge, MA: MIT Press, 1981.

- Grabiner, J, 'Cauchy and Bolzano. Tradition and transformation in the history of mathematics.' in E Mendelsohn (ed), Chapter 5 in *Transformation and tradition in the sciences: essays in honour of I Bernard Cohen*. Cambridge: Cambridge University Press, 1984, 105–124. See [https://books.google.co.il/books?id=mJ\\_u4t1wrLsC](https://books.google.co.il/books?id=mJ_u4t1wrLsC).
- Grattan-Guinness, I, 'Bolzano, Cauchy and the 'new analysis' of the early nineteenth century', *Archive for History of Exact Sciences*, 6(5) (1970), 372–400.
- Gray, J, *The real and the complex: a history of analysis in the 19th century*, (Springer Undergraduate Mathematics Series), Cham: Springer, 2015.
- Guitard, T, 'La querelle des infiniment petits à l'École Polytechnique au XIXe siècle', *Historia Scientiarum*, 30 (1986), 1–61.
- Heuser, H, *Lehrbuch der analysis. Teil 2*, (12th edition, Mathematische Leitfäden), Stuttgart: B. G. Teubner, 1991.
- Jarník, V, *Bolzano and the foundations of mathematical analysis*, Praha: Society of Czechoslovak Mathematicians and Physicists, 1981, 33–42.
- Karsten, W J G, *Lehrbegriff der gesamten Mathematik: Die Rechenkunst und Geometrie. 1*, Greifswald: Röse, 1767.
- Karsten, W J G, *Lehrbegriff der gesamten Mathematik: Die Mechanik der festen Körper. 4*, Greifswald: Röse, 1769. See <https://books.google.com.mx/books?id=CNxEAAAACAAJ>.
- Kästner, A G, *Anfangsgründe der Arithmetik, Geometrie, ebenen und sphärischen Trigonometrie und Perspectiv*, Göttingen: Witwe Vandenhoeck, 1758.
- Kästner, A G, *Anfangsgründe der Analysis endlicher Größen*, Göttingen: Wittve Vandenhoeck, 1760. See [http://gdz.sub.uni-goettingen.de/no\\_cache/dms/load/img/?IDDOC=116776](http://gdz.sub.uni-goettingen.de/no_cache/dms/load/img/?IDDOC=116776).
- Kästner, A G, *Anfangsgründe der höhern Mechanik*, Göttingen: Verlag der Wittve Vandenhoeck, 1766.
- Kästner, A G, *Anfangsgründe der höhern Mechanik, Auflage 2*, Göttingen: Verlag der Wittve Vandenhoeck, 1793.
- Katz, M, 'Mathematical conquerors, Unguru polarity, and the task of history', *Journal of Humanistic Mathematics*, 10(1) (2020), 475–515. See <http://dx.doi.org/10.5642/jhummath.202001.27> and <https://arxiv.org/abs/2002.00249>.
- Katz, M, and Sherry, D, 'Leibniz's infinitesimals: their fictionality, their modern implementations, and their foes from Berkeley to Russell and beyond', *Erkenntnis*, 78(3) (2013), 571–625. See <http://dx.doi.org/10.1007/s10670-012-9370-y> and <https://arxiv.org/abs/1205.0174>.
- Kröger, D, *Abraham Gotthelf Kästner als Lehrbuchautor*. Dissertation, Bergische Universität Wuppertal, 2014. urn:nbn:de:hbz:468-20150311-103303-7.
- Leibniz, G, *Philosophical papers and letters, Synthese historical library*, Vol. 2. Leroy E. Loemker, Editor and Translator. Dordrecht–Boston–London: Kluwer Academic Publishers, 2nd edition, 1989
- Lützen, J, 'The foundation of analysis in the 19th century' in H Jahnke (ed), Chapter 6 in *A history of analysis*, (History of Mathematics, 24), Providence, RI: American Mathematical Society, 2003, 155–195.
- Olms (publisher), *Allgemeines Verzeichniß der Bücher, welche in der Frankfurter und Leipziger Ostermesse des Jahres entweder ganz neu gedruckt, oder sonst verbessert, wieder aufgelegt worden sind ...* (1817). Accessible to registered libraries at <http://www.olmsonline.de/en/dms/toc/?IDDOC=12308> and accessible also at <https://books.google.de/books?id=DlpeAAAACAAJpg=PA1> See also Evenhuis (2014).
- Oman, G, 'The converse of the intermediate value theorem: from Conway to Cantor to cosets and beyond', *Missouri Journal of Mathematical Sciences*, 26(2) (2014), 134–150.
- Radcliffe, D, 'A function that is surjective on every interval', *The American Mathematical Monthly*, 123(1) (2016), 88–89.
- Rusnock, P, 'Philosophy of mathematics: Bolzano's responses to Kant and Lagrange. Mathématique et logique chez Bolzano', *Revue d'histoire des Sciences*, 52(3–4) (1999), 399–428.

- Rusnock, P, and Kerr-Lawson, A, 'Bolzano and uniform continuity', *Historia Mathematica*, 32 (3) (2005), 303–311.
- Rusnock, P, and Šebestík, J, *Bernard Bolzano: his life and work*, Oxford: Oxford University Press, 2019.
- Russ, S, *The mathematical works of Bernard Bolzano*, Oxford: Oxford University Press, 2004.
- Schubring, G, 'Bernard Bolzano—not as unknown to his contemporaries as is commonly believed?', *Historia Mathematica*, 20(1) (1993), 45–53.
- Sebestík, J, *Logique et mathématique chez Bernard Bolzano (L'Histoire des Sciences: Textes et Études)*, Paris: Librairie Philosophique J. Vrin, 1992.
- Segre, M, 'Peano's axioms in their historical context', *Archive for History of Exact Sciences*, 48 (3–4) (1994), 201–342.
- Sherry, D, and Katz, M, 'Infinitesimals, imaginaries, ideals, and fictions', *Studia Leibnitiana*, 44 (2) (2012), 166–192. See <http://www.jstor.org/stable/43695539> and <https://arxiv.org/abs/1304.2137> (Article was published in 2014 even though the journal issue lists the year as 2012).
- Siegmund-Schultze, R, Review of Bradley–Sandifer (Bradley and Sandifer 2009) for Zentralblatt (2009). See <https://zbmath.org/?q=an%3A1189.26001>.
- Sinaceur, H, 'Cauchy et Bolzano', *Revue d'histoire des Sciences*, 26(2) (1973), 97–112.
- Smoryński, C, *MVT: a most valuable theorem*, Cham: Springer, 2017.
- Spalt, D, *Die Analysis im Wandel und Widerstreit*, Verlag Karl Auber, 2015.
- Trlifajová, K, 'Bolzano's infinite quantities', *Foundations of Science*, 23(4) (2018), 681–704.
- Väth, M, *Nonstandard analysis*, Basel: Birkhäuser Verlag, 2007.