# Non-uniformizable sets with countable cross-sections on a given level of the projective hierarchy

by

## Vladimir Kanovei and Vassily Lyubetsky (Moscow)

**Abstract.** We present a model of set theory in which, for a given $n \geq 2$, there exists a planar non-ROD-uniformizable lightface $\varPi_n^1$ set, all of whose vertical cross-sections are countable sets and, more specifically, Vitali classes, while all planar boldface $\mathbf{\Sigma}_n^1$ sets with countable cross-sections are $\mathbf{\Delta}_{n+1}^1$-uniformizable. Thus it is true in this model that the ROD-uniformization principle for sets with countable cross-sections first fails precisely at a given projective level.

**1. Introduction.** *The uniformization problem* was introduced into descriptive set theory by Luzin in a short note [31] and in a more detailed paper [32] ($^1$). According to Luzin, a planar set $Q$ in the real number plane $\mathbb{R} \times \mathbb{R}$ is said to be *uniform* (or *single-valued*) if it intersects every vertical straight line in at most one point. If $Q \subseteq P \subseteq \mathbb{R} \times \mathbb{R}$ for a uniform set $Q$ whose projection to the first axis is equal to that of $P$, then Luzin says that the set $Q$ *uniformizes* $P$. In other words, uniformizing a given planar set $P$ means choosing a point $q_x$ in every non-empty vertical cross-section $P_x$ of $P$, and then gathering all the chosen points $q_x$, or more precisely, all the pairs of the form $\langle x, q_x \rangle$, into a single uniformizing set $Q \subseteq P$. According to Luzin,

($^1$) These notes were not published among Luzin's papers on descriptive set theory in Volume II of his collected works [34]. However its main elements were considered, partially translated, and analyzed in detail by V. A. Uspensky in [38]. In [31], Luzin gives a rather long citation from Hadamard's first letter in the well-known "Five letters" [10], which can be understood to mean that Hadamard makes a distinction between a pure Zermelo-style choice and a choice of elements in non-empty sets by means of a concrete effectively defined function. This gave Luzin an occasion to connect the uniformization problem with the name of Hadamard in the titles of [31, 32]. Uspensky argues in [38, Section 4] that the role of Hadamard is definitely exaggerated here, while the priority with regard to the uniformization problem and related notions belongs to Luzin himself.

*the uniformization problem* consists in the question: *is it possible or not to define a point set E for which we cannot name any uniformizing set E'?* (The translation is quoted from [38, p. 120], the italic text by Luzin and Uspensky.)

In modern set-theoretic terminology, there exist exact definitions for such notions of "naïve" set theory as 'to define', 'to name', 'to give an effective construction', and the like. The largest class of effectively defined sets is usually assumed to be the class ROD of *real-ordinal definable* sets, which consists of all sets definable by a formula with real numbers and ordinals as parameters of the definition. The class ROD contains the subclass OD of all *ordinal-definable* sets, namely, sets definable by a formula with ordinals (but not reals) as parameters.

There are more special subclasses of ROD and OD, namely, *projective classes* $\mathbf{\Sigma}_n^1$, $\mathbf{\Pi}_n^1$, and $\mathbf{\Delta}_n^1 = \mathbf{\Sigma}_n^1 \cap \mathbf{\Pi}_n^1$, and resp. *effective projective classes* $\Sigma_n^1$, $\Pi_n^1$, and $\Delta_n^1 = \Sigma_n^1 \cap \Pi_n^1$; here $n \geq 1$. See [35], as well as [23], [25], [26], [14], [17], for projective hierarchy. Recall that $\mathbf{\Delta}_1^1 = $ Borel sets, $\mathbf{\Sigma}_1^1 = $ Suslin, or A-sets, $\mathbf{\Pi}_1^1 = $ co-Suslin, or CA-sets, at the level $n = 1$.

The following uniformization theorem is considered to be one the most important results in classical descriptive set theory.

THEOREM 1.1 (Novikov–Kondo–Addison). *Every planar set in one of the classes $\mathbf{\Pi}_1^1$, $\Pi_1^1$, $\mathbf{\Sigma}_2^1$, $\Sigma_2^1$ can be uniformized by a set in the same class.*

The key ingredient here was P. S. Novikov's method (introduced in [33]) of effectively choosing a point in every non-empty $\mathbf{\Pi}_1^1$ set. Relying on this method, Kondo [29] obtained the result for $\mathbf{\Pi}_1^1$. Addison [2, 3] transferred it to the effective class $\Pi_1^1$. The results for classes $\mathbf{\Sigma}_2^1$, $\Sigma_2^1$ are obtained by an elementary argument. For these and other theorems on uniformization and related questions see references above, as well as [39, 36, 11, 5, 8, 7] for modern studies; see also the introductory section of [21].

For $\mathbf{\Pi}_2^1$ and higher projective classes, similar uniformization theorems are not available since there exist models of set theory in which certain $\mathbf{\Pi}_2^1$ sets are not uniformizable not only by a projective set (of any class), but even in general by a ROD set. The first such model was defined by Levy in [30, Theorem 3], where the counterexample required is a $\Pi_2^1$ set $P = \{\langle x, y \rangle \in \mathbb{R}^2 : y \notin \mathbf{L}[x]\}$, which is not uniformizable by a ROD set in the model. Recall that $\mathbf{L}[x]$ contains all sets Gödel constructible *relative to* $x$.

Note that every vertical cross-section $P_x = \mathbb{R} \smallsetminus \mathbf{L}[x]$ of the set $P$ is either empty (provided that $\mathbb{R} \subseteq \mathbf{L}[x]$), or uncountable, so that it can never be non-empty finite or countable. (In the Solovay model [37] all cross-sections $P_x$ are co-countable.) The problem of the existence of non-uniformizable $\mathbf{\Pi}_2^1$ sets with *countable* vertical cross-sections was solved in [19] by a model containing such a set. Then a more precise result was obtained:

THEOREM 1.2 (proved in [21], equal to the case $\mathtt{n} = 2$ in the next Theorem 2.1). *There exists a model of* **ZFC** *in which it is true that there is a planar $\Pi^1_2$ set $W \subseteq \mathbb{R}^2$, all of whose non-empty vertical cross-sections $W_x$ are Vitali classes ($^2$), and which is not uniformizable by a ROD set.*

The proof involves a forcing notion defined in the constructible universe **L** as an uncountable product of invariant versions of the Jensen minimal forcing [13]. (See also [12, 28A] for Jensen's forcing.) Some other results obtained by this method include a countable $\Pi^1_2$ set containing no definable elements [27], a Vitali class with the same properties [18], and a $\Pi^1_2$ Groszek–Laver pair of Vitali classes [18]. See [21, 2.6] for the interest in Vitali classes in the context of these results.

**2. The main results.** In continuation of this research line, we prove here the following theorem.

THEOREM 2.1. *Let $\mathtt{n} \geq 3$. There is a model of* **ZFC** *set theory in which the following is true*:

(i) *there is a $\Pi^1_{\mathtt{n}}$ set $P \subseteq \mathbb{R} \times \mathbb{R}$ such that all sections $P_x = \{y : \langle x, y \rangle \in P\}$ are Vitali classes, and $P$ is not uniformizable by a ROD set;*

(ii) *if $p \in \mathbb{R}$ then every $\Sigma^1_{\mathtt{n}}(p)$ set $P' \subseteq \mathbb{R} \times \mathbb{R}$ with countable vertical sections, is uniformizable by a $\Delta^1_{\mathtt{n}+1}(p)$ set, hence, by a ROD set.*

Following the modern style in descriptive set theory based on certain technical advantages, we shall consider the Cantor discontinuum $2^\omega$ with a special equivalence relation ($^3$) $\mathsf{E}_0$, instead of the real line $\mathbb{R}$ with the Vitali equivalence relation, in the substantial part of the proof. Thus the following theorem will be proved:

THEOREM 2.2. *Let $\mathtt{n} \geq 3$. There exists a model of* **ZFC** *in which the following holds*:

(i) *there exists a $\Pi^1_{\mathtt{n}}$ set $W \subseteq 2^\omega \times 2^\omega$ such that all vertical cross-sections $W_x = \{y : \langle x, y \rangle \in W\}$ are $\mathsf{E}_0$-classes, and $W$ is not uniformizable by a ROD set;*

(ii) *if $p \in \mathbb{R}$ then every $\Sigma^1_{\mathtt{n}}(p)$ set $W' \subseteq 2^\omega \times 2^\omega$, all of whose sections $W'_x = \{y : \langle x, y \rangle \in W'\}$ are countable sets, is uniformizable by a $\Delta^1_{\mathtt{n}+1}(p)$ set.*

**Theorem 2.2 implies Theorem 2.1.** The transformation of a set $W$ as in 2.2(i) into a set $P$ as in 2.1(i) is carried out by means of elementary topological arguments, close to a similar transformation in [21, §17], so

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($^2$) A *Vitali class* is any set of the form $x + \mathbb{Q}$, i.e., a shift of the set $\mathbb{Q}$ of rationals.

($^3$) The relation $\mathsf{E}_0$ is defined on $2^\omega$ so that $x \mathsf{E}_0 y$ iff the equality $x(n) = y(n)$ holds for all but finitely many $n$. If $X, Y \subseteq 2^\omega$ then $X \equiv_{\mathsf{E}_0} Y$ means that every element $a \in X$ is $\mathsf{E}_0$-equivalent to some $b \in Y$, and vice versa. See more on this e.g. in [24, 25, 17].

we skip this argument. The derivation of 2.1(ii) from 2.2(ii) is carried out by means of an effective homeomorphism between the real line $\mathbb{R}$ and the co-countable set $X = \{x \in 2^\omega : \forall\, m \,\exists\, j \geq m\, (x(j) = 0)\}$. $\qquad\square$

**3. Structure of the paper.** The proof of Theorem 2.2 is organized as follows.

The notions related to perfect trees in the set of all dyadic strings $2^{<\omega}$ are introduced in Sections 4 and 5. We consider a collection **LT** of all *large* trees—essentially those on which the relation $\mathsf{E}_0$ does not admit a Borel transversal. Every set $P \subseteq \mathbf{LT}$ closed under truncating trees at strings, and $\mathsf{E}_0$ *-invariant*, i.e., invariant relative to that action of finite strings which induces the relation $\mathsf{E}_0$ (Remark 4.1), is considered (Section 6) as a forcing notion adding a $P$-generic real $x \in 2^\omega$. In fact, as $P$ is $\mathsf{E}_0$ -invariant, an entire $\mathsf{E}_0$ -equivalence class $[x]_{\mathsf{E}_0} = \{y \in 2^\omega : x\, \mathsf{E}_0\, y\}$ of generic reals is adjoined.

Then in Section 7 we define the set **MT** of all *multitrees*, equal to the countable-support product $\mathbf{LT}^{\omega_1}$. We study multitrees (including the behavior of continuous functions on multitrees) in Sections 8–11.

Arguing in the constructible universe $\mathbf{L}$, we define a forcing notion for Theorem 2.2 in Section 15 in the form of a countable-support product $\mathbb{P} = \prod_{\xi < \omega_1} \mathbb{P}(\xi) \subseteq \mathbf{MT}$, where each factor $\mathbb{P}(\xi) \subseteq \mathbf{LT}$ has the form of a union $\mathbb{P}(\xi) = \bigcup_{\xi \leq \alpha < \omega_1} \mathbb{P}_\alpha(\xi)$, where all summands are countable $\mathsf{E}_0$ -invariant sets $\mathbb{P}_\alpha(\xi) \subseteq \mathbf{LT}$ in $\mathbf{L}$, pre-dense in $\mathbb{P}(\xi)$. $\mathbb{P}$-generic extensions of $\mathbf{L}$ will be models for Theorem 2.2. It turns out that each factor $\mathbb{P}(\xi)$ adjoins a $\mathbb{P}(\xi)$-generic real $x_\xi$, so that the whole extension is equal to $\mathbf{L}[\langle x_\xi \rangle_{\xi < \omega_1}]$. The following is the first key property of the forcing notion $\mathbb{P}$:

(1) if $\xi < \omega_1$ then the set $\mathbb{P}(\xi)$ is $\mathsf{E}_0$ -invariant.

The next principal issue in the construction of forcing notions $\mathbb{P}(\xi)$ is similar to the construction of Jensen's forcing in [13] and in some other cases. It consists in the definition of every "level" $\mathbb{P}_\alpha(\xi)$ as *generic* in some sense over the previous "levels" $\mathbb{P}_\gamma(\xi)$, $\gamma < \alpha$. This involves a fairly complex construction in Sections 12–14, based on the splitting technique for perfect trees. This implies cardinal preservation (Lemma 16.3), continuous reading of names (Lemma 17.4), as well as the following:

(2) for every index $\xi < \omega_1$, the set of all $\mathbb{P}(\xi)$-generic reals in the extension is equal to the $\mathsf{E}_0$ -class $[x_\xi]_{\mathsf{E}_0}$ of the generic real $x_\xi$, and also is equal to the intersection $Y_\xi = \bigcap_{\xi \leq \alpha < \omega_1} \bigcup_{T \in \mathbb{P}_\alpha(\xi)} [T]$.

Basically we need here only the equality $[x_\xi]_{\mathsf{E}_0} = Y_\xi$ (Theorem 18.1). The passage from a single generic real, as in Jensen, to an $\mathsf{E}_0$ -class of generic reals is implied here by the $\mathsf{E}_0$ -invariance property as in (1). As a corollary, the definability of the set $W = \{\langle \xi, y \rangle : \xi < \omega_1 \wedge y \in [x_\xi]_{\mathsf{E}_0}\}$ (the base

for a counterexample for 2.2(i)) in a $\Pi$-generic extension follows from the definability of the indexed set $\langle \mathbb{P}_\alpha(\xi) \rangle_{\xi \leq \alpha < \omega_1}$ in $\mathbf{L}$ (Section 19).

Following this idea, we proved Theorem 1.2 in [21] (= case $\mathsf{n} = 2$ in Theorem 2.2). By the way, the ROD-non-uniformizability of $W$ follows from the $\mathsf{E}_0$-invariance of each component of the forcing notion $\Pi$ by (1), both in [21] and here. The main case $\mathsf{n} \geq 3$ in Theorem 2.2 differs from the case $\mathsf{n} = 2$ in that it is necessary to prove claim (ii) of the theorem in the extension, while that claim immediately holds for $\mathsf{n} = 2$ by Theorem 1.1. We get 2.2(ii) via the following property true in $\Pi$-generic extensions:

(3) if $x \in 2^\omega$ and $X \subseteq 2^\omega$ is a countable $\Sigma^1_{\mathsf{n}}(x)$ set then $X \subseteq \mathbf{L}[x]$.

This property holds in Cohen and some other generic extensions even for $\mathrm{OD}(x)$ sets $X$ (see [20]). It also holds in $\mathbf{MT}$-generic extensions of $\mathbf{L}$, where it is implied by the permutation invariance of the forcing notion $\mathbf{MT} = \mathbf{LT}^{\omega_1}$ and by a very special feature of those extensions, namely,

(4) if $x, y \in 2^\omega$ in a $\mathbf{MT}$-generic extension $\mathbf{L}[\langle x_\xi \rangle_{\xi < \omega_1}]$, and $y \notin \mathbf{L}[x]$, then there exists an ordinal $\xi$ such that $x_\xi \in \mathbf{L}[y]$ but $x \in \mathbf{L}[\langle x_\eta \rangle_{\eta \neq \xi}]$

(compare to [16, Theorem 20] for the $\omega_1$-product of the Sacks forcing). $\Pi$-generic extensions satisfy (4) as well. (See Theorem 17.5, based on the study of continuous functions, defined on multitrees, in Section 8.) Yet this does not directly imply (3) since the forcing notion $\Pi = \prod_\xi \mathbb{P}(\xi)$ is not permutation-invariant as the components $\mathbb{P}(\xi)$ are pairwise different.

This leads to the following modification of the forcing construction. Generally, the construction of $\Pi$ can be viewed as the choice of a maximal chain in a certain partially ordered set $\mathscr{P}$ of cardinality $\aleph_1$ in $\mathbf{L}$.

(5) We require that this maximal chain intersects all sets dense in $\mathscr{P}$ which belong to the definability class $\Sigma^1_{\mathsf{n}-1}$. (See Theorem 15.4, item (ii) of which contains a property more flexible than this straightforward genericity, but also more difficult to formulate.)

Theorem 15.4 evaluates the definability of this construction. This leads to the definability class $\Pi^1_{\mathsf{n}}$ of the set $W$ (see above) in suitable generic extensions. In addition, the forcing notion $\Pi$ turns out to be enough "generic" in $\mathbf{MT}$, so that it intersects all sets of definability class $\Sigma^1_{\mathsf{n}-1}$, dense in $\mathbf{MT}$ (Lemma 16.4). This implies a degree of "similarity" of $\Pi$-generic and permutation-invariant $\mathbf{MT}$-generic extensions, up to the $\mathsf{n}$th level of the projective hierarchy. And further, by fairly complicated arguments in Sections 20–22 (which also make use of (4)), we obtain (3) in $\Pi$-generic extensions, circumventing the above-mentioned problem of the permutation noninvariance of $\Pi$ and leading to item (ii) of Theorem 2.2.

**4. Trees and large trees.** Here and in the next section, we reproduce some definitions and results from [9] related to perfect and large trees and their transformations.

**Strings.** $2^{<\omega}$ is the set of all strings (finite sequences) of numbers $0, 1$, including *the empty string* $\Lambda$. If $t \in 2^{<\omega}$ and $i = 0, 1$ then $t^{\frown}i$ is the extension of $t$ by $i$ as the rightmost term. If $s, t \in 2^{<\omega}$ then $s \subseteq t$ means that the string $t$ extends $s$ (including the case $s = t$), while $s \subset t$ means proper extension. The length of $s$ is $\mathtt{lh}(s)$, and $2^n = \{s \in 2^{<\omega} : \mathtt{lh}(s) = n\}$ (strings of length $n$).

**Action.** Every string $s \in 2^{<\omega}$ *acts* on $2^\omega$ so that if $x \in 2^\omega$ then $(s \cdot x)(k) = x(k) + s(k) \pmod 2$ for $k < \mathtt{lh}(s)$, and $(s \cdot x)(k) = x(k)$ otherwise. If $X \subseteq 2^\omega$ and $s \in 2^{<\omega}$ then let $s \cdot X = \{s \cdot x : x \in X\}$.

REMARK 4.1. This action induces the relation $\mathsf{E}_0$ (footnote 3), so that if $x, y \in 2^\omega$ then $x \, \mathsf{E}_0 \, y$ is equivalent to $y = s \cdot x$ for a string $s \in 2^{<\omega}$. □

Similarly if $s \in 2^m$, $t \in 2^n$, $m \leq n$ then define a string $s \cdot t \in 2^n$ so that $(s \cdot t)(k) = t(k) + s(k) \pmod 2$ for $k < m$, and $(s \cdot t)(k) = t(k)$ for $m \leq k < n$. But if $m > n$ then let $s \cdot t = (s \restriction n) \cdot t$. In both cases, $\mathtt{lh}(s \cdot t) = \mathtt{lh}(t)$. If $T \subseteq 2^{<\omega}$ then we let $s \cdot T = \{s \cdot t : t \in T\}$.

**Trees.** A set $T \subseteq 2^{<\omega}$ is a *tree* if for any strings $s \subset t$ in $2^{<\omega}$, $t \in T$ implies $s \in T$. If $T \subseteq 2^{<\omega}$ is a tree and $u \in T$, then define a *truncated subtree* $T \restriction_u = \{t \in T : u \subseteq t \vee t \subseteq u\}$ of $T$. Clearly if $\sigma \in 2^{<\omega}$ then $\sigma \cdot (T \restriction_u) = (\sigma \cdot T) \restriction_{\sigma \cdot u}$. A non-empty tree $T \subseteq 2^{<\omega}$ is *perfect*, in symbols $T \in \mathbf{PT}$, if it has no endnodes and no isolated branches. In this case, there is a longest string $s = \mathtt{stem}(T) \in T$ satisfying $T = T \restriction_s$ (the *stem* of $T$); then $s^{\frown}0 \in T$ and $s^{\frown}1 \in T$. If $T \in \mathbf{PT}$ then the set $[T] = \{a \in 2^\omega : \forall n \, (a \restriction n \in T)\}$ of all *branches* of $T$ is a perfect set in $2^\omega$.

**Large trees.** A tree $T \in \mathbf{PT}$ is *large*, $T \in \mathbf{LT}$, if there exists a system of strings $q_k^i = q_k^i[T] \in 2^{<\omega}$, $k < \omega$ and $i = 0, 1$, such that

(1)  $\mathtt{lh}(q_k^0) = \mathtt{lh}(q_k^1) \geq 1$ and $q_k^0(0) = 0$, $q_k^1(0) = 1$ for all $k$;
(2)  $T$ consists of all strings of the form $s = r^{\frown}q_0^{i_0}{}^{\frown}q_1^{i_1}{}^{\frown}\cdots{}^{\frown}q_n^{i_n}$ and their substrings, where $n < \omega$, $r = \mathtt{stem}(T)$, $i_k = 0, 1$ for all $k$.

It this case, the set $[T]$ consists of all infinite strings of the form $a = r^{\frown}q_0^{i_0}{}^{\frown}q_1^{i_1}{}^{\frown}q_2^{i_2}{}^{\frown}\cdots \in 2^\omega$, where $i_k = 0, 1$ for all $k$. We let

$$\mathtt{spl}_n(T) = \mathtt{lh}(r) + \mathtt{lh}(q_0^{i_0}) + \mathtt{lh}(q_1^{i_1}) + \cdots + \mathtt{lh}(q_{n-1}^{i_{n-1}})$$

(independent of the values of $i_k = 0, 1$). In particular, $\mathtt{spl}_0(T) = \mathtt{lh}(r)$. Thus $\mathtt{spl}(T) = \{\mathtt{spl}_n(T) : n < \omega\} \subseteq \omega$ is the set of all *splitting levels* of $T$.

REMARK 4.2. If $T \in \mathbf{LT}$ then the set $[T]$ is $\mathsf{E}_0$-*nonsmooth*, that is, there is no Borel map $f : [T] \to 2^\omega$ satisfying $x \, \mathsf{E}_0 \, y \Leftrightarrow f(x) = f(y)$ for all $x, y \in [T]$. Conversely, every $\mathsf{E}_0$-nonsmooth Borel set $X \subseteq 2^\omega$ contains a subset of the form $[T]$, where $T \in \mathbf{LT}$. See [6], [17, 10.9], [28, 7.1] on this category of sets. □

**5. Splitting.** *The simple splitting* of a tree $T \in \mathbf{LT}$ consists of subtrees $T(\to i) = T{\restriction}_{r^\frown i}$, $i = 0, 1$, where $r = \mathtt{stem}(T)$, so that $[T(\to i)] = \{x \in [T] : x(\mathtt{lh}(r)) = i\}$. Then $T(\to i) \in \mathbf{LT}$, $\mathtt{stem}(T(\to i)) = r^\frown q_0^i(T)$, $q_k^j(T(\to i)) = q_{k+1}^j(T)$ for all $k$ and $j = 0, 1$, and $\mathtt{spl}(T(\to i)) = \mathtt{spl}(T) \smallsetminus \{\mathtt{spl}_0(T)\}$.

Splittings can be iterated. We let $T(\to \Lambda) = T$ for the empty string $\Lambda$, and if $s \in 2^n$, $s \neq \Lambda$ then we define

$$T(\to s) = T(\to s(0))(\to s(1))(\to s(2)) \ldots (\to s(n-1)) \in \mathbf{LT}.$$

EXAMPLE 5.1. If $s \in 2^{<\omega}$ then the tree $T[s] = \{t \in 2^{<\omega} : s \subseteq t \vee t \subset s\}$ belongs to $\mathbf{LT}$, $\mathtt{stem}(T[s]) = s$, and $q_k^i(T[s]) = \langle i \rangle$. In particular $T[\Lambda] = 2^{<\omega}$ and $T[s] = (2^{<\omega})(\to s) = (2^{<\omega}){\restriction}_s$ for all $s$. □

LEMMA 5.2. *Let* $T \in \mathbf{LT}$. *If* $s \in 2^{<\omega}$ *then* $T(\to s) = T{\restriction}_{u[s]}$, *where* $u[s] = u[s, T] = \mathtt{stem}(T(\to s)) = \mathtt{stem}(T)^\frown q_0^{s(0)} {}^\frown q_1^{s(1)} {}^\frown \cdots {}^\frown q_{n-1}^{s(n-1)} \in T$. *Conversely, if* $u \in T$ *then there is a string* $s = s[u] \in 2^{<\omega}$ *such that* $T{\restriction}_u = T(\to s)$.

*Proof.* To prove the converse, we put $s(k) = u(\mathtt{spl}_k(T))$ for all $k$ such that $\mathtt{spl}_k(T) < \mathtt{lh}(u)$. ∎

LEMMA 5.3. *Let* $T \in \mathbf{LT}$, $n < \omega$, $h = \mathtt{spl}_n(T)$. *Then:*

(i) *if* $u, v \in T \cap 2^h$ *then* $T{\restriction}_u = (u \cdot v) \cdot (T{\restriction}_v)$;

(ii) *if* $s, t \in 2^n$ *then* $T(\to s) = \sigma \cdot (T(\to t))$, *where* $\sigma = u[s, T] \cdot u[t, T]$;

(iii) *if* $u, v \in T \cap 2^j$, $j < \omega$, *then* $T{\restriction}_u = \sigma \cdot (T{\restriction}_v)$ *for some* $\sigma \in 2^{<\omega}$.

*Proof.* To prove (ii) use Lemma 5.2. To prove (iii) take the least number $h \in \mathtt{spl}(T)$ with $j \leq h$. There is a unique pair of strings $u', v' \in 2^h$ satisfying $u \subseteq u'$, $v \subseteq v'$. Then $T{\restriction}_u = T{\restriction}_{u'}$, $T{\restriction}_v = T{\restriction}_{v'}$, and $T{\restriction}_{u'} = (u' \cdot v') \cdot (T{\restriction}_{v'})$. ∎

**Refinement.** If $R, T \in \mathbf{LT}$ and $n \in \omega$ then define $R \subseteq_n T$ (*refinement*) if $R(\to s) \subseteq T(\to s)$ for all $s \in 2^n$; $R \subseteq_0 T$ is equivalent to $R \subseteq T$. Clearly $R \subseteq_{n+1} T$ implies $R \subseteq_n T$ (and $R \subseteq T$). Moreover, if $n \geq 1$ then $R \subseteq_n T$ is equivalent to $\mathtt{stem}(R) = \mathtt{stem}(T)$, $q_k^i[R] = q_k^i[T]$ for all $i = 0, 1$ and $k < n - 1$, and $q_{n-1}^i[T] \subseteq q_{n-1}^i[R]$ for all $i = 0, 1$.

LEMMA 5.4. *If* $T \in \mathbf{LT}$, $s_0 \in 2^n$, *and* $U \in \mathbf{LT}$, $U \subseteq T(\to s_0)$, *then there is a unique tree* $T' \in \mathbf{LT}$ *satisfying* $T' \subseteq_n T$ *and* $T'(\to s_0) = U$. *We have*:

(i) $T'(\to s) = u[s_0, T] \cdot u[s, T] \cdot T'(\to s_0)$ *for all* $s \in 2^n$;

(ii) *if* $[U]$ *is clopen in* $[T(\to s_0)]$ *then* $[T']$ *is clopen in* $[T]$.

*Proof.* If $s \in 2^n$ then $T(\to s) = u[s_0, T] \cdot u[s, T] \cdot T(\to s_0)$ by Lemma 5.3. Put $U_s = u[s_0, T] \cdot u[s, T] \cdot U$ for all $s \in 2^n$, in particular, $U_{s_0} = U$. The tree $T' = \bigcup_{u \in 2^n} U_s$ is as required. ∎

The next lemma is a more complex version of $\subseteq_n$-refinement. For the proof see [9, Lemma 4.1(iv)].

LEMMA 5.5. *If* $T \in \mathbf{LT}$, $s_0, s_1 \in 2^n$, *and* $U, V \in \mathbf{LT}$, $U \subseteq T(\to s_0{}^\frown 0)$, $V \subseteq T(\to s_1{}^\frown 1)$, *and* $[U] \equiv_{\mathsf{E}_0} [V]$ (see footnote 3 for $\equiv_{\mathsf{E}_0}$), *then there exists a tree* $T' \in \mathbf{LT}$ *satisfying* $T' \subseteq_{n+1} T$, $T'(\to s_0{}^\frown 0) \subseteq U$, $T'(\to s_1{}^\frown 1) \subseteq V$. □

LEMMA 5.6. *Let* $\cdots \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0$ *be an infinite sequence of trees in* $\mathbf{LT}$. *Then* $T = \bigcap_n T_n \in \mathbf{LT}$ *and* $T \subseteq_{n+1} T_n$ *for all* $n$.

*Proof.* Note that $\mathtt{spl}(T) = \{\mathtt{spl}_n(T_n) : n < \omega\}$; this implies both claims. ∎

## 6. Large tree forcings

DEFINITION 6.1. Let a $\mathbf{LT}$-*forcing* be any set $P \subseteq \mathbf{LT}$ satisfying

(A) if $u \in T \in P$ then $T \restriction_u \in P$, or equivalently, if $T \in P$ and $s \in 2^{<\omega}$ then $T(\to s) \in P$;

(B) $P$ is $\mathsf{E}_0$-*invariant*, i.e., if $T \in P$ and $\sigma \in 2^{<\omega}$ then $\sigma \cdot T \in P$.

If in addition $2^{<\omega} \in P$ then $P$ is a *regular* $\mathbf{LT}$-forcing. □

Any $\mathbf{LT}$-forcing $P$ can be considered as a forcing notion (a set of forcing conditions), ordered so that if $T \subseteq T'$ then $T$ is a stronger condition. Such a forcing $P$ adjoins a real $x \in 2^\omega$. That is, if a set $G \subseteq P$ is $P$-generic over a ground model $M$, then the intersection $\bigcap_{T \in G}[T]$ contains a unique real $x = x[G] \in 2^\omega$, and this real satisfies $M[G] = M[x[G]]$ and $G = \{T \in P : x \in [T]\}$. Reals $x[G]$ of this form are called *P-generic*.

EXAMPLE 6.2. The set $\mathbf{LT}$ of all large trees is clearly a $\mathbf{LT}$-forcing. Another example of a $\mathbf{LT}$-forcing is the countable set $P_{\mathrm{coh}} = \{T[s] : s \in 2^{<\omega}\}$ of all trees $T[s]$ of Example 5.1, i.e. *Cohen's forcing*. Finally, if $\varnothing \neq Q \subseteq \mathbf{LT}$ then

$$P = \{\sigma \cdot (T \restriction_u) : u \in T \in Q \wedge \sigma \in 2^{<\omega}\} = \{\sigma \cdot (T(\to s)) : T \in Q \wedge s, \sigma \in 2^{<\omega}\}$$

is a $\mathbf{LT}$-forcing by [21, Lemma 5.4]. □

A tree $T \in \mathbf{LT}$ is an *n-collage* over a $\mathbf{LT}$-forcing $P$ if $T(\to u) \in P$ for all $u \in 2^n$. Thus a 0-collage is just a tree in $P$, and every $n$-collage is an $n+1$-collage as well.

LEMMA 6.3. *If* $T \in \mathbf{LT}$, $P$ *is a* $\mathbf{LT}$-*forcing,* $u \in 2^n$, *and* $T(\to u) \in P$, *then* $T$ *is an n-collage over* $P$. *In particular, under the assumptions of Lemma 5.4, if* $U \in P$ *then the tree* $T'$ *obtained is an n-collage over* $P$.

*Proof.* If $v \in 2^n$ then $T(\to v) = \tau \cdot T(\to u)$ for a string $\tau \in 2^{<\omega}$ by Lemma 5.3. Thus $T(\to v) \in P$ since $T(\to u) \in P$. ∎

If $T \in \mathbf{LT}$ and $D \subseteq \mathbf{LT}$ then $T \subseteq^{\mathtt{fin}} \bigcup D$ means that there is a finite set $D' \subseteq D$ satisfying $T \subseteq \bigcup D'$, or equivalently, $[T] \subseteq \bigcup_{S \in D'}[S]$.

DEFINITION 6.4 (extensions). Let $P, Q \subseteq \mathbf{LT}$ be $\mathbf{LT}$-forcings. The forcing $Q$ is an *extension* of $P$, in symbols $P \sqsubset Q$, if

(1) $Q$ is dense in $P \cup Q$: if $T \in P$ then $\exists\, S \in Q\,(S \subseteq T)$;
(2) if $S \in Q$ then $S \subseteq^{\mathtt{fin}} \bigcup P$.

If $\mathfrak{M}$ is any set, and, in addition to $P \sqsubset Q$, $S \subseteq^{\mathtt{fin}} \bigcup D$ holds for all $S \in Q$ and all sets $D \in \mathfrak{M}$, $D \subseteq P$, which are pre-dense ($^4$) in $P$, then we say that $Q$ is an $\mathfrak{M}$-*extension* of $P$, written $P \sqsubset_{\mathfrak{M}} Q$. ▫

LEMMA 6.5.

(i) *If* $Q \subseteq Q'$ *and* $S \subseteq^{\mathtt{fin}} \bigcup Q$ *for all* $S \in Q'$ *then* $Q \sqsubset Q'$;
(ii) *if* $P \sqsubset_{\mathfrak{M}} Q \sqsubset R$ (*the second relation is* $\sqsubset$, *not* $\sqsubset_{\mathfrak{M}}$!) *then* $P \sqsubset_{\mathfrak{M}} R$;
(iii) *if* $\langle P_\alpha \rangle_{\alpha < \lambda}$ *is an* $\sqsubset$-*increasing sequence of* $\mathbf{LT}$-*forcings and* $0 \le \mu < \lambda$ *then the set* $P_\mu$, *is pre-dense in* $P = \bigcup_{\alpha < \lambda} P_\alpha$.

*Proof.* (ii) $P \sqsubset R$ is clear. Assume that a set $D \in \mathfrak{M}$, $D \subseteq P$, is pre-dense in $P$, and $S \in R$. Then $S \subseteq^{\mathtt{fin}} \bigcup Q$ (since $Q \sqsubset R$), thus $S \subseteq T_1 \cup \dots \cup T_n$, where $T_1, \dots, T_n \in Q$. Now $T_i \subseteq^{\mathtt{fin}} \bigcup D$, $i = 1, \dots, n$, since $P \sqsubset_{\mathfrak{M}} Q$. It follows that $S \subseteq^{\mathtt{fin}} \bigcup D$ holds as well.

(iii) Let $S \in P_\alpha$. If $\alpha \le \mu$ then $T \in P_\mu$, $T \subseteq S$ holds by 6.4(1). If $\mu < \alpha$ then $S \subseteq T_1 \cup \dots \cup T_n$, where $T_1, \dots, T_n \in P_\mu$. Then $S{\restriction}_t \subseteq T_i$ for some $t \in S$ and $i$. But $S' = S{\restriction}_t \in P_\alpha$. ∎

**7. Multitrees.** Let a *multitree* be any function $\mathbf{T} : |\mathbf{T}| \to \mathbf{LT}$, where $|\mathbf{T}| = \mathtt{dom}\,\mathbf{T} \subseteq \omega_1$ is at most countable and every value $\mathbf{T}(\xi)$, $\xi \in |\mathbf{T}|$, is a tree in $\mathbf{LT}$. Let $\mathbf{MT}$ denote the set of all multitrees. If $\mathbf{T} \in \mathbf{MT}$ then we define a brick in $(2^\omega)^{|\mathbf{T}|}$,

$$[\mathbf{T}] = \{x \in (2^\omega)^{|\mathbf{T}|} : \forall \xi \in |\mathbf{T}|\,(x(\xi) \in [\mathbf{T}(\xi)])\}$$
$$= \{x \in (2^\omega)^{|\mathbf{T}|} : \forall \xi\, \forall m\,(x(\xi){\restriction} m \in \mathbf{T}(\xi))\},$$

naturally identified with the cartesian product $\prod_{\xi \in |\mathbf{T}|} [\mathbf{T}(\xi)]$.

If $B \subseteq \omega_1$ is at most countable then let $\mathbf{MT}_B = \{\mathbf{T} \in \mathbf{MT} : |\mathbf{T}| = B\}$.

The set $\mathbf{MT}$ is ordered componentwise: $\mathbf{T} \le \mathbf{S}$ ($\mathbf{T}$ is a *stronger* multitree) whenever $|\mathbf{S}| \subseteq |\mathbf{T}|$ and $\mathbf{T}(\xi) \subseteq \mathbf{S}(\xi)$ for all $\xi \in |\mathbf{S}|$. Thus the ordering of multitrees corresponds to componentwise inclusion. The weakest (the largest

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($^4$) Pre-density means that every tree $T \in P$ is compatible in $P$ with some $S \in D$, i.e. there is a tree $R \in P$ satisfying $R \subseteq T$ and $R \subseteq S$.

in the sense of $\leqslant$) condition in $\mathbf{MT}$ is the empty multitree $\mathbf{\Lambda}$, satisfying $|\mathbf{\Lambda}| = \varnothing$.

It takes some effort to get right versions of definitions and results of Section 5 in the context of multitrees.

DEFINITION 7.1. If $\mathbf{T} \in \mathbf{MT}_B$ and $C \subseteq B$, then $\mathbf{T} \restriction C \in \mathbf{MT}_C$ is the ordinary restriction. But if $B \subseteq C$ then a multitree $\mathbf{T} \upuparrows C \in \mathbf{MT}_C$ is defined by $(\mathbf{T} \upuparrows C)(\xi) = \mathbf{T}(\xi)$ for $\xi \in B$, and $(\mathbf{T} \upuparrows C)(\xi) = 2^{<\omega}$ for $\xi \in C \smallsetminus B$. □

DEFINITION 7.2. If $\mathbf{U}$ is a multitree and $\mathbf{D}$ is a set of multitrees, then $\mathbf{U} \subseteq^{\mathtt{fin}} \bigvee \mathbf{D}$ means that there is a finite set $\mathbf{D}' \subseteq \mathbf{D}$ such that (1) $|\mathbf{V}| \subseteq C = |\mathbf{U}|$ for all $\mathbf{V} \in \mathbf{D}'$, and (2) $[\mathbf{U}] \subseteq \bigcup_{\mathbf{V} \in \mathbf{D}'}[\mathbf{V} \upuparrows C]$ (see Definition 7.1 for $\upuparrows$). If in addition (3) $[\mathbf{V} \upuparrows C] \cap [\mathbf{V}' \upuparrows C] = \varnothing$ for all $\mathbf{V} \neq \mathbf{V}'$ in $\mathbf{D}'$, then we write $\mathbf{U} \subseteq^{\mathtt{fd}} \bigvee \mathbf{D}$. □

DEFINITION 7.3. Let $B \subseteq \omega_1$ be finite or countable. Fix a function $\phi : \omega \xrightarrow{\text{onto}} B$ that takes each value infinitely many times, so that if $\xi \in B$ then

$$\phi^{-1}(\xi) = \{k : \phi(k) = \xi\} = \{\mathbf{k}_{0\xi} < \mathbf{k}_{1\xi} < \mathbf{k}_{2\xi} < \cdots\}$$

is an infinite set. Such a function will be called $B$-complete. If $m < \omega$ then let $\boldsymbol{\nu}_{m\xi}$ be equal to the number of indices $k < m$, $k \in \phi^{-1}(\xi)$. Then $\sum_{\xi \in B} \boldsymbol{\nu}_{m\xi} = m$, and $\boldsymbol{\nu}_{m\xi} > 0$ holds for all $\xi \in \phi"m = \{\phi(k) : k < m\}$.

Let $m < \omega$ and $\sigma \in 2^m$. If $\xi \in \phi"m$ then the set $\phi^{-1}(\xi)$ cuts a substring $\sigma \restriction \xi \in 2^{\boldsymbol{\nu}_{m\xi}}$ of length $\mathtt{lh}(\sigma \restriction \xi) = \boldsymbol{\nu}_{m\xi}$ off $\sigma$, defined by $\sigma \restriction \xi(j) = \sigma(\mathbf{k}_{j\xi})$ for all $j < \boldsymbol{\nu}_{m\xi}$. Thus the string $\sigma \in 2^m$ splits into a system of strings $\sigma \restriction \xi \in 2^{\boldsymbol{\nu}_{m\xi}}$ ($\xi \in \phi"m$) of total length $\sum_{\xi \in \phi"m} \boldsymbol{\nu}_{m\xi} = m$.

If $\mathbf{T} \in \mathbf{MT}_B$ then define a multitree $\mathbf{T}(\Rightarrow\sigma) \in \mathbf{MT}_B$ so that $\mathbf{T}(\Rightarrow\sigma)(\xi) = \mathbf{T}(\xi)(\rightarrow\sigma \restriction \xi)$ for all $\xi \in B$. In particular, if $\xi \in B \smallsetminus \phi"m$ then $\mathbf{T}(\Rightarrow\sigma)(\xi) = \mathbf{T}(\xi)$, where $m = \mathtt{lh}(\sigma)$, because $\mathtt{lh}(\sigma \restriction \xi) = \boldsymbol{\nu}_{m\xi} = 0$ holds for $\xi \notin \phi"m$.

If $\sigma, \tau \in 2^m$, $m < \omega$, then put $D[\sigma, \tau] = B \smallsetminus \{\phi(i) : i < m \wedge \sigma(i) \neq \tau(i)\}$.

Let $\mathbf{T}, \mathbf{S} \in \mathbf{MT}_B$. Define $\mathbf{T} \leqslant_m \mathbf{S}$ if $\mathbf{T}(\xi) \subseteq_{\boldsymbol{\nu}_{m\xi}} \mathbf{S}(\xi)$ for all $\xi \in B$. This is equivalent to $\mathbf{T}(\Rightarrow\sigma) \subseteq \mathbf{S}(\Rightarrow\sigma)$ for all $\sigma \in 2^n$. □

LEMMA 7.4. *In the notation of Definition 7.3, let $\mathbf{T} \in \mathbf{MT}_B$. Then:*

(i) *if $\sigma \in 2^{<\omega}$ then $\mathbf{T}(\Rightarrow\sigma) \in \mathbf{MT}_B$ and the set $[\mathbf{T}(\Rightarrow\sigma)]$ is clopen in $[\mathbf{T}]$;*

(ii) *if $m < \omega$ and $\sigma, \tau \in 2^m$ then $\mathbf{T}(\Rightarrow\sigma) \restriction D[\sigma, \tau] = \mathbf{T}(\Rightarrow\tau) \restriction D[\sigma, \tau]$;*

(iii) *if $x \in [\mathbf{T}]$, and $U$ is an open nbhd of $x$, then there exists a string $\sigma \in 2^m$ satisfying $x \in [\mathbf{T}(\Rightarrow\sigma)] \subseteq U$;*

(iv) *if $m < \omega$, $\sigma \in 2^m$, and $\mathbf{U} \in \mathbf{MT}_B$, $\mathbf{U} \leqslant \mathbf{T}(\Rightarrow\sigma)$, then there exists a unique multitree $\mathbf{S} \in \mathbf{MT}_B$ such that $\mathbf{S} \leqslant_m \mathbf{T}$ and $\mathbf{S}(\Rightarrow\sigma) = \mathbf{U}$, and then if $[\mathbf{U}]$ is clopen in $[\mathbf{T}(\Rightarrow\sigma)]$ then $[\mathbf{S}]$ is clopen in $[\mathbf{T}]$;*

(v) *if $\mathbf{D}$ is a set of multitrees and $\mathbf{T} \subseteq^{\mathtt{fin}} \bigvee \mathbf{D}$, then there is a string $\sigma \in 2^{<\omega}$ and a multitree $\mathbf{S} \in \mathbf{D}$ such that $\mathbf{T}(\Rightarrow\sigma) \leqslant \mathbf{S}$.*

*Proof.* (i) is clear. (iii) We have $\{x\} = \bigcap_m [\mathbf{T}(\Rightarrow a \upharpoonright m)]$ for a suitable sequence $a \in 2^\omega$. By compactness, there is $m$ such that $\mathbf{T}(\Rightarrow a \upharpoonright m) \subseteq U$.

(iv) If $\xi \in B$ then $\mathbf{U}(\xi) \subseteq \mathbf{T}(\Rightarrow \sigma)(\xi) = \mathbf{T}(\xi)(\to s)$, where $s = \sigma \upharpoonright \xi$. By Lemma 5.4 there is a tree $S_\xi \in \mathbf{LT}$ satisfying $S_\xi \subseteq_n \mathbf{T}(\xi)$, where $n = \boldsymbol{\nu}_{m\xi} = \mathtt{lh}(s)$, and $S_\xi(\to s) = \mathbf{U}(\xi)$. Let $\mathbf{S}(\xi) = S_\xi$ for all $\xi$.

(v) There is a multitree $\mathbf{S} \in \mathbf{D}$ such that $|\mathbf{S}| \subseteq B = |\mathbf{T}|$ and the intersection $U = [\mathbf{T}] \cap [\mathbf{S} \upharpoonright B]$ has a non-empty interior in $[\mathbf{T}]$. It remains to refer to (iii). ■

LEMMA 7.5. *In the notation of Definition 7.3, let $\cdots \leqslant_3 \mathbf{T}_2 \leqslant_2 \mathbf{T}_1 \leqslant_1 \mathbf{T}_0$ be a sequence of multitrees in $\mathbf{MT}_B$. Then the multitree $\mathbf{T} = \bigwedge_n \mathbf{T}_n$, defined by $\mathbf{T}(\xi) = \bigcap_n \mathbf{T}_n(\xi)$ for all $\xi \in B$, belongs to $\mathbf{MT}_B$ and $\mathbf{T} \leqslant_{n+1} \mathbf{T}_n$ for all $n$.*

*Proof.* Apply Lemma 5.6 componentwise. ■

**8. Continuous maps and reducibility.** We consider here some details related to continuous maps defined on bricks coming from multitrees, similar to some results obtained in [15, 16] in the context of perfect sets and trees. Assume that a set $B \subseteq \omega_1$ is countable, $\mathbf{T} \in \mathbf{MT}_B$, and maps $f, g : [\mathbf{T}] \to \omega^\omega$ are continuous. We say that:

- $f$ *is reduced to* $C \subseteq B$ *on* $[\mathbf{T}]$ if $f(x) = f(y)$ holds whenever $x, y \in [\mathbf{T}]$ and $x \upharpoonright C = y \upharpoonright C$;
- $f$ *is reduced to* $g$ *on* $[\mathbf{T}]$ if $f(x) = f(y)$ holds whenever $x, y \in [\mathbf{T}]$ and $g(x) = g(y)$;
- $f$ *captures* $\alpha \in B$ *on* $[\mathbf{T}]$ if the *co-ordinate map* $c_\alpha(x) = x(\alpha)$ is reduced to $f$, so that $x(\alpha) = y(\alpha)$ holds whenever $x, y \in [\mathbf{T}]$ and $f(x) = f(y)$.

LEMMA 8.1. *If $\mathbf{T} \in \mathbf{MT}$, $C_0, C_1, \ldots \subseteq B = |\mathbf{T}|$, $f : [\mathbf{T}] \to \omega^\omega$ is continuous and reduced to every $C_k$ on $[\mathbf{T}]$, then $f$ is reduced to $\bigcap_k C_k$ on $[\mathbf{T}]$.*

*Proof.* For just two sets, if $C = C_0 \cap C_1$ and $x, y \in [\mathbf{T}]$, $x \upharpoonright C = y \upharpoonright C$, then, using the product structure, find a point $z \in [\mathbf{T}]$ with $z \upharpoonright C_0 = x \upharpoonright C_0$ and $z \upharpoonright C_1 = y \upharpoonright C_1$. Then $f(x) = f(z) = f(y)$. The case of finitely many sets follows by simple induction. As for the general case, we can assume that $C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$, by the above. Let $C = \bigcap_k C_k$, $x, y \in [\mathbf{T}]$, $x \upharpoonright C = y \upharpoonright C$. There is a sequence of points $x_k \in [\mathbf{T}]$ satisfying $x_k \upharpoonright C_k = x \upharpoonright C_k$ and $x_k \upharpoonright (B \setminus C_k) = y \upharpoonright (B \setminus C_k)$. Then immediately $f(x_k) = f(x)$ for all $k$. On the other hand, clearly $x_k \to y$, hence $f(x_k) \to f(y)$ as $f$ is continuous. Thus $f(x) = f(y)$. ■

THEOREM 8.2. *Let $\mathbf{T} \in \mathbf{MT}_B$ where $B \subseteq \omega_1$ is at most countable, and let $f, g : [\mathbf{T}] \to \omega^\omega$ be continuous. Then there is a multitree $\mathbf{S} \in \mathbf{MT}_B$, $\mathbf{S} \leqslant \mathbf{T}$, satisfying either* (i) *$f$ is reduced to $g$ on $[\mathbf{S}]$, or* (ii) *there is an ordinal $\eta \in B$ such that $f$ captures $\eta$ on $[\mathbf{S}]$ while $g$ is reduced to $B \setminus \{\eta\}$ on $[\mathbf{S}]$.*

The co-ordinate map $c_\eta(x) = x(\eta)$ is obviously not reducible to $B \smallsetminus \{\eta\}$. Thus the theorem essentially says that the **non**-reducibility of $f$ to $g$ is detected via co-ordinate maps.

*Proof.* We argue in terms of Definition 7.3. The plan is to define a sequence of multitrees as in Lemma 7.5, with some extra properties. Let $m < \omega$. A multitree $\mathbf{R} \in \mathbf{MT}_B$ is $m$-*good* if $\mathbf{R} \leqslant \mathbf{T}$ and in addition

$(1)_f$   if $\sigma \in 2^m$ then either $f$ is reduced to $B \smallsetminus \{\phi(m)\}$ on $[\mathbf{R}(\Rightarrow \sigma)]$, or there is no $\mathbf{R}' \in \mathbf{MT}_B$, $\mathbf{R}' \leqslant \mathbf{R}(\Rightarrow \sigma)$, such that $f$ is reduced to $B \smallsetminus \{\phi(m)\}$ on $[\mathbf{R}']$;

$(1)_g$   the same for $g$;

$(2)_f$   if $\sigma, \tau \in 2^m$, then either (i) $f$ is reduced to $D[\sigma, \tau] = B \smallsetminus \{\phi(i) : i < m \wedge \sigma(i) \neq \tau(i)\}$ on $[\mathbf{R}(\Rightarrow \sigma)] \cup [\mathbf{R}(\Rightarrow \tau)]$, or (ii) $f"[\mathbf{R}(\Rightarrow \sigma)] \cap f"[\mathbf{R}(\Rightarrow \tau)] = \varnothing$;

$(2)_g$   the same for $g$.

LEMMA 8.3. *If $m < \omega$ and a multitree $\mathbf{R} \in \mathbf{MT}_B$, $\mathbf{R} \leqslant \mathbf{T}$, is $m$-good, then there is an $m+1$-good multitree $\mathbf{Q} \in \mathbf{MT}_B$ such that $\mathbf{Q} \leqslant_{m+1} \mathbf{R}$.*

*Proof of Lemma.* Consider a string $\sigma' \in 2^{m+1}$, and first define a multitree $\mathbf{S} \in \mathbf{MT}_B$, $\mathbf{S} \leqslant_{m+1} \mathbf{R}$, satisfying $(1)_f$ relative to this string only. Let $\alpha = \phi(m+1)$. If there exists a multitree $\mathbf{R}' \in \mathbf{MT}_B$, $\mathbf{R}' \leqslant \mathbf{R}(\Rightarrow \sigma')$, such that $f$ is reduced to $B \smallsetminus \{\alpha\}$ on $[\mathbf{R}']$ then let $\mathbf{U}$ be one. If there is no such $\mathbf{R}'$ then simply put $\mathbf{U} = \mathbf{R}(\Rightarrow \sigma')$. By Lemma 7.4(iv), there is a multitree $\mathbf{S} \in \mathbf{MT}_B$ such that $\mathbf{S} \leqslant_{m+1} \mathbf{R}$ and $\mathbf{S}(\Rightarrow \sigma') = \mathbf{U}$. Thus the multitree $\mathbf{S}$ satisfies $(1)_f$ with respect to $\sigma'$. Now we take $\mathbf{S}$ as the "new" multitree $\mathbf{R}$, consider another string $\sigma' \in 2^{m+1}$, and do the same. Treat all strings in $2^{m+1}$ consecutively, with the same procedure. This ends with a multitree $\mathbf{S} \in \mathbf{MT}_B$, $\mathbf{S} \leqslant_{m+1} \mathbf{R}$, satisfying $(1)_f$ for all strings in $2^{m+1}$.

Now we take care of $(2)_f$. Let $\eta_0 = \phi(m)$ and $B' = B \smallsetminus \{\eta_0\}$.

STEP 1. We fulfill $(2)_f$ for one particular pair $\sigma' = \sigma {}^\frown 0, \tau' = \sigma {}^\frown 1$, where $\sigma \in 2^m$. Then $D[\sigma', \tau'] = B'$. The goal is to define a multitree $\mathbf{Q} \in \mathbf{MT}_B$, $\mathbf{Q} \leqslant_{m+1} \mathbf{S}$, satisfying $(2)_f$ relative to this pair.

If $f$ is reduced to $B'$ on $[\mathbf{S}(\Rightarrow \sigma)]$ then $f$ is reduced to $B'$ on $[\mathbf{S}(\Rightarrow \sigma')] \cup [\mathbf{S}(\Rightarrow \tau')] \subseteq \mathbf{S}(\Rightarrow \sigma)$, and we are done. Thus, by $(1)_f$ for $\mathbf{S}(\Rightarrow \sigma)$, we can assume that *there is no $\mathbf{S}' \in \mathbf{MT}_B$, $\mathbf{S}' \leqslant \mathbf{S}(\Rightarrow \sigma)$, such that $f$ is reduced to $B'$ on $[\mathbf{S}']$.*

In particular, $f$ is **not** reduced to $B'$ on $[\mathbf{S}(\Rightarrow \sigma')]$. However $\mathbf{S}(\Rightarrow \sigma') {\upharpoonright} B' = \mathbf{S}(\Rightarrow \tau') {\upharpoonright} B'$ since $B' = D[\sigma', \tau'] = B \smallsetminus \{\eta_0\}$. It follows that there are points $x_0 \in [\mathbf{S}(\Rightarrow \sigma')]$ and $y_0 \in [\mathbf{S}(\Rightarrow \tau')]$ such that $x_0 {\upharpoonright} B' = y_0 {\upharpoonright} B'$ and $f(x_0) \neq f(y_0)$, that is, $f(x_0)(k) = p \neq q = f(y_0)(k)$ for some $k$ and $p, q = 0, 1, p \neq q$.

As $f$ is continuous, there are strings $u, v \in 2^{<\omega}$ of equal length such that $\sigma' \subset u, \tau' \subset v, x_0 \in X = [\mathbf{S}(\Rightarrow u)], y_0 \in Y = [\mathbf{S}(\Rightarrow v)]$, and $f(x)(k) = p$,

$f(y)(k) = q$ for all $x \in X$, $y \in Y$. We are going to define a multitree $\mathbf{Q} \leqslant_{n+1} \mathbf{S}$ such that $[\mathbf{Q}(\Rightarrow\sigma')] \subseteq X$ and $[\mathbf{Q}(\Rightarrow\tau')] \subseteq Y$. Then $f''[\mathbf{Q}(\Rightarrow\sigma')] \cap f''[\mathbf{Q}(\Rightarrow\tau')] = \varnothing$ by construction, as required.

If $\eta \in B$ then we let $r_\eta = \sigma \restriction \eta$, $s_\eta = u \restriction \eta$, $t_\eta = v \restriction \eta$, $\nu_\eta = \boldsymbol{\nu}_{m\eta} = \mathrm{lh}(r_\eta)$, and also, as usual, $X(\eta) = \{x(\eta) : x \in X\}$, $Y(\eta) = \{y(\eta) : y \in Y\}$.

Consider any index $\eta \neq \eta_0$. Then $x_0(\eta) = y_0(\eta)$ (as $x_0 \restriction B' = y_0 \restriction B'$), and then clearly $r_\eta \subset s_\eta = t_\eta$. It follows that the tree $U_\eta = \mathbf{S}(\eta)(\to s_\eta)$ belongs to $\mathbf{LT}$ and satisfies $U_\eta \subseteq \mathbf{S}(\eta)(\to r_\eta)$ and $[U_\eta] = X(\eta) = Y(\eta)$. By Lemma 5.4, there is a tree $Q_\eta \in \mathbf{LT}$ satisfying $Q_\eta \subseteq_{\nu_\eta} \mathbf{S}(\eta)$ and $Q_\eta(\to r_\eta) = U_\eta$.

Now consider the index $\eta_0$ itself. Let $H = \mathbf{S}(\eta_0)$. The strings $s_{\eta_0}$ and $t_{\eta_0}$ are different (of the same length), but still satisfy $r_{\eta_0}{}^\frown 0 = \sigma' \restriction \eta_0 \subseteq s_{\eta_0}$, $r_{\eta_0}{}^\frown 1 = \tau' \restriction \eta_0 \subseteq t_{\eta_0}$. Then the trees $U_{\eta_0} = H(\to s_{\eta_0}) \subseteq H(\to r_{\eta_0}{}^\frown 0)$, $V_{\eta_0} = H(\to t_{\eta_0}) \subseteq H(\to r_{\eta_0}{}^\frown 1)$ belong to $\mathbf{LT}$ and satisfy $[U_{\eta_0}] = X(\eta_0)$, $[V_{\eta_0}] = Y(\eta_0)$. And moreover $[U_{\eta_0}] \equiv_{\mathsf{E}_0} [V_{\eta_0}]$ holds by Lemma 5.3(ii). Lemma 5.5 yields a tree $H' \in \mathbf{LT}$ such that $H' \subseteq_{\nu_{\eta_0}+1} H$, and $H'(\to s{}^\frown 0) \subseteq U_{\eta_0}$, $H'(\to s{}^\frown 1) \subseteq V_{\eta_0}$.

Now define a multitree $\mathbf{Q}$ such that $\mathbf{Q}(\eta_0) = H'$ and $\mathbf{Q}(\eta) = Q_\eta$ for all $\eta \neq \eta_0$. Then by construction $\mathbf{Q} \leqslant_{m+1} \mathbf{S}$, $\mathbf{Q}(\Rightarrow\sigma') \subseteq X$, and $\mathbf{Q}(\Rightarrow\tau') \subseteq Y$, as required.

STEP 2. Iterating the construction at Step 1, we obtain a multitree $\mathbf{Q} \in \mathbf{MT}_B$, $\mathbf{Q} \leqslant_{m+1} \mathbf{S}$, satisfying $(2)_f$ for all pairs $\sigma' = \sigma{}^\frown 0, \tau' = \sigma{}^\frown 1 \in 2^{m+1}$, where $\sigma \in 2^m$.

STEP 3. We claim that $\mathbf{Q}$ satisfies $(2)_f$ for all pairs $\sigma', \tau' \in 2^{m+1}$ of any form. Indeed, let $\sigma' = \sigma{}^\frown i$, $\tau' = \tau{}^\frown j$ be any pair in $2^{m+1}$, where $\sigma, \tau \in 2^m$ and $i, j \in \{0, 1\}$. By $(2)_f$ for the pair $\sigma, \tau$, either $f$ is reduced to $U = D[\sigma, \tau]$ on $[\mathbf{Q}(\Rightarrow\sigma)] \cup [\mathbf{Q}(\Rightarrow\tau)]$, or $f''[\mathbf{Q}(\Rightarrow\sigma)] \cap f''[\mathbf{Q}(\Rightarrow\tau)] = \varnothing$. In the second case, $f''[\mathbf{Q}(\Rightarrow\sigma')] \cap f''[\mathbf{Q}(\Rightarrow\tau')] = \varnothing$. Thus we can assume that ($\dagger$) $f$ is *reduced to $U$ on* $[\mathbf{Q}(\Rightarrow\sigma)] \cup [\mathbf{Q}(\Rightarrow\tau)]$. Let $U' = D[\sigma', \tau']$. If $i = j$ or $\eta_0 \notin U$ then $U = U'$, so that $(2)_f$ relative to $\sigma', \tau'$ follows from $(2)_f$ relative to $\sigma, \tau$. Thus we can also assume that $\sigma' = \sigma{}^\frown 0, \tau' = \tau{}^\frown 1$, **and** $\eta_0 \in U$. Then $U' = U \smallsetminus \{\eta_0\} = U \cap B'$.

Because of the result at Step 2, we have two cases.

CASE 3.1: $f$ is reduced to $B'$ on $[\mathbf{Q}(\Rightarrow\sigma')] \cup [\mathbf{Q}(\Rightarrow\sigma'_1)]$, where $\sigma'_1 = \sigma{}^\frown 1$. We prove that $f$ is reduced to $U'$ on $[\mathbf{Q}(\Rightarrow\sigma')] \cup [\mathbf{Q}(\Rightarrow\tau')]$, so that $(2)_f$(i) holds for $\sigma', \tau'$. Indeed, assume that $x \in [\mathbf{Q}(\Rightarrow\sigma')]$, $y \in [\mathbf{Q}(\Rightarrow\tau')]$, $x \restriction U' = y \restriction U'$. Let $x' \in (2^\omega)^\omega$ be defined so that $x' \restriction B' = x \restriction B'$ but $x'(\eta_0) = y(\eta_0)$. Thus if $\eta \neq \eta_0$ then $x'(\eta) = x(\eta) \in [\mathbf{Q}(\Rightarrow\sigma')(\eta)] = [\mathbf{Q}(\Rightarrow\sigma'_1)(\eta)]$ (because $\mathbf{Q}(\Rightarrow\sigma') \restriction B' = \mathbf{Q}(\Rightarrow\sigma'_1) \restriction B'$). While for $\eta_0$ itself we have $x'(\eta_0) = y(\eta_0) \in [\mathbf{Q}(\Rightarrow\tau')] = [\mathbf{Q}(\Rightarrow\sigma'_1)]$ (because now we have $\eta_0 \in U = D[\tau', \sigma'_1]$). It follows

that $x' \in [\mathbf{Q}(\Rightarrow \sigma'_1)]$. Therefore, by the Case 3.1 hypothesis, we have $f(x) = f(x')$. On the other hand, $x' \upharpoonright U = y \upharpoonright U$, therefore $f(y) = f(x')$ by the assumption (†) above. Thus $f(x) = f(y)$, as required.

CASE 3.2: $f"[\mathbf{Q}(\Rightarrow \sigma')] \cap f"[\mathbf{Q}(\Rightarrow \sigma'_1)] = \varnothing$. However $f$ is reduced to $U = D[\sigma, \tau]$ on $[\mathbf{Q}(\Rightarrow \sigma)] \cup [\mathbf{Q}(\Rightarrow \tau)]$ by the assumption (†) above, hence on the smaller set $[\mathbf{Q}(\Rightarrow \sigma'_1)] \cup [\mathbf{Q}(\Rightarrow \tau')]$ as well, while $\mathbf{Q}(\Rightarrow \sigma'_1) \upharpoonright U = \mathbf{Q}(\Rightarrow \tau') \upharpoonright U$ (since $U = D[\sigma'_1, \tau'] = D[\sigma, \tau]$). Now we conclude that $f"[\mathbf{Q}(\Rightarrow \sigma'_1)] = f"[\mathbf{Q}(\Rightarrow \tau')]$. It follows that $f"[\mathbf{Q}(\Rightarrow \sigma')] \cap f"[\mathbf{Q}(\Rightarrow \tau')] = \varnothing$, hence $\mathbf{Q}$ satisfies $(2)_f$(ii) for $\sigma', \tau'$.

*Conclusion.* Thus indeed we have got a multitree $\mathbf{Q} \in \mathbf{MT}_B$ satisfying $\mathbf{Q} \leqslant_{m+1} \mathbf{S}$ and $(2)_f$ for all $\sigma', \tau' \in 2^{m+1}$ (and still satisfying $(1)_f$).

It remains to repeat the same procedure for $g$. ∎$_{\text{Lemma}}$

We come back to the proof of Theorem 8.2. Lemma 8.3 yields an infinite sequence $\cdots \leqslant_3 \mathbf{S}_2 \leqslant_2 \mathbf{S}_1 \leqslant_1 \mathbf{S}_0 = \mathbf{T}$ of multitrees $\mathbf{S}_m \in \mathbf{MT}_B$, such that each $\mathbf{S}_m$ is a $m$-good. The limit multitree $\mathbf{S} = \bigwedge_m \mathbf{S}_m \in \mathbf{MT}_B$ satisfies $\mathbf{S} \leqslant_{m+1} \mathbf{S}_m$ for all $m$ by Lemma 7.5. Therefore $\mathbf{S}$ is $m$-good for every $m$, hence we can freely use $(1)_{f,g}$ and $(2)_{f,g}$ in the following final argument.

CASE 1: if $m < \omega$, $\sigma, \tau \in 2^m$, and $f"[\mathbf{S}(\Rightarrow \sigma)] \cap f"[\mathbf{S}(\Rightarrow \tau)] = \varnothing$, then $g"[\mathbf{S}(\Rightarrow \sigma)] \cap g"[\mathbf{S}(\Rightarrow \tau)] = \varnothing$. We prove that $f$ is reduced to $g$ on $[\mathbf{S}]$ in this case, as required by (i) of the theorem. Let $x, y \in [\mathbf{S}]$ and $f(x) \neq f(y)$; we show that $g(x) \neq g(y)$. Pick $a, b \in 2^\omega$ satisfying $\{x\} = \bigcap_m [\mathbf{S}(\Rightarrow a \upharpoonright m)]$ and $\{y\} = \bigcap_m [\mathbf{S}(\Rightarrow b \upharpoonright m)]$. As $x \neq y$, we have $f"[\mathbf{S}(\Rightarrow a \upharpoonright m)] \cap f"[\mathbf{S}(\Rightarrow b \upharpoonright m)] = \varnothing$ for some $m$ by continuity and compactness. Then by the Case 1 assumption, $g"[\mathbf{S}(\Rightarrow a \upharpoonright m)] \cap g"[\mathbf{S}(\Rightarrow b \upharpoonright m)] = \varnothing$ holds, hence $g(x) \neq g(y)$.

CASE 2: not Case 1. Then there is a number $m < \omega$ and a pair of strings $\sigma' = \sigma^\frown i$, $\tau' = \tau^\frown k \in 2^{m+1}$ such that $f"[\mathbf{S}(\Rightarrow \sigma')] \cap f"[\mathbf{S}(\Rightarrow \tau')] = \varnothing$ but $g"[\mathbf{S}(\Rightarrow \sigma')] \cap g"[\mathbf{S}(\Rightarrow \tau')] \neq \varnothing$, hence $g$ is reduced to $U' = D[\sigma', \tau']$ on $Z' = [\mathbf{S}(\Rightarrow \sigma')] \cup [\mathbf{S}(\Rightarrow \tau')]$ by $(2)_g$. Assume that $m$ is the least possible for this case. We shall prove that the multitree $\mathbf{S}(\Rightarrow \sigma)$ satisfies (ii) of Theorem 8.2, with the ordinal $\eta_0 = \phi(m)$, that is, (*) $g$ is reduced to $B' = B \smallsetminus \{\eta_0\}$ on $[\mathbf{S}(\Rightarrow \sigma)]$, and (**) $f$ captures $\eta_0$ on $[\mathbf{S}(\Rightarrow \sigma)]$.

LEMMA 8.4. *The map $f$ is:*

(A) *reduced to $U = D[\sigma, \tau]$ on the set $Z = [\mathbf{S}(\Rightarrow \sigma)] \cup [\mathbf{S}(\Rightarrow \tau)]$,*
(B) **not** *reduced to $U' = D[\sigma', \tau']$ on $Z' = [\mathbf{S}(\Rightarrow \sigma')] \cup [\mathbf{S}(\Rightarrow \tau')]$,*
(C) **not** *reduced to $B' = B \smallsetminus \{\eta_0\}$ on any multitree $\mathbf{S}' \subseteq \mathbf{S}(\Rightarrow \sigma)$.*

*In addition,* (D) $U \neq U'$, *hence* $\eta_0 \in U$ *and* $U' = U \smallsetminus \{\eta_0\}$.

*Proof.* (A) Otherwise $f"[\mathbf{S}(\Rightarrow \sigma)] \cap f"[\mathbf{S}(\Rightarrow \tau)] = \varnothing$ by $(2)_f$, hence, by the minimality of $m$, $g"[\mathbf{S}(\Rightarrow \sigma)] \cap g"[\mathbf{S}(\Rightarrow \tau)] = \varnothing$, so $g"[\mathbf{S}(\Rightarrow \sigma')] \cap$

$g"[\mathbf{S}(\Rightarrow\tau')] = \varnothing$ as well, contrary to the fact that $g$ is reduced to $U'$ on $Z' = [\mathbf{S}(\Rightarrow\sigma')] \cup [\mathbf{S}(\Rightarrow\tau')]$, because $\mathbf{S}(\Rightarrow\sigma')\!\restriction\! U' = \mathbf{S}(\Rightarrow\tau')\!\restriction\! U'$ by Lemma 7.4(ii).

(B) The otherwise assumption contradicts the equality $f"[\mathbf{S}(\Rightarrow\sigma')] \cap f"[\mathbf{S}(\Rightarrow\tau')] = \varnothing$.

(D) follows from (A) and (B).

(C) Otherwise the map $f$ is reduced to $B'$ on $\mathbf{S}(\Rightarrow\sigma)$ by $(1)_f$. Then $f$ is reduced to $U'$ on $[\mathbf{S}(\Rightarrow\sigma)]$ by Lemma 8.1 since $U' = U \smallsetminus \{\eta_0\}$ by (D). It follows that $f$ is reduced to $U'$ on $Z$ ([5]), hence on $Z' \subseteq Z$ as well. But this contradicts to (B). $\blacksquare$ Lemma

Now, as $U' = U \smallsetminus \{\eta_0\} \subseteq B' = B \smallsetminus \{\eta_0\}$ by (D), the multitree $\mathbf{S}(\Rightarrow\sigma')$ witnesses that $g$ is reduced to $B'$ on $[\mathbf{S}(\Rightarrow\sigma)]$ by $(1)_g$. Thus we have (*).

To check (**), let $x, y \in [\mathbf{S}(\Rightarrow\sigma)]$ and $f(x) = f(y)$; we prove $x(\eta_0) = y(\eta_0)$. Indeed, $\{x\} = \bigcap_n[\mathbf{S}(\Rightarrow a\!\restriction\! n)]$ and $\{y\} = \bigcap_n[\mathbf{S}(\Rightarrow b\!\restriction\! n)]$, where $a, b \in 2^\omega$, $\sigma \subset a$, $\sigma \subset b$. Let $U = \bigcap_n D[a\!\restriction\! n, b\!\restriction\! n]$. Then $x\!\restriction\! U = y\!\restriction\! U$, since $\mathbf{S}(\Rightarrow a\!\restriction\! n)\!\restriction\! D[a\!\restriction\! n, b\!\restriction\! n] = \mathbf{S}(\Rightarrow b\!\restriction\! n)\!\restriction\! D[a\!\restriction\! n, b\!\restriction\! n]$ for all $n$. Thus it suffices to check $\eta_0 \in D[a\!\restriction\! n, b\!\restriction\! n]$ for all $n$.

Suppose towards the contrary that $\eta_0 = \phi(m) \notin D[a\!\restriction\! n, b\!\restriction\! n]$ for some $n$. Then $n > m$ because $a\!\restriction\! m = b\!\restriction\! m = \sigma$. However $f$ is reduced to $D[a\!\restriction\! n, b\!\restriction\! n]$ on $[\mathbf{S}(\Rightarrow a\!\restriction\! n)]$ by $(2)_f$, since $f(x) = f(y)$. Yet we have $\eta_0 \notin D[a\!\restriction\! n, b\!\restriction\! n]$, therefore $D[a\!\restriction\! n, b\!\restriction\! n] \subseteq B' = B \smallsetminus \{\eta_0\}$. It follows that $f$ is reduced to $B'$ on $[\mathbf{S}(\Rightarrow a\!\restriction\! n)]$. But this contradicts Lemma 8.4(C) with $\mathbf{S}' = \mathbf{S}(\Rightarrow a\!\restriction\! n)$.

To conclude Case 2, we have checked (*) and (**). $\blacksquare$ Theorem 8.2

**9. Multiforcings and submultiforcings.** Let a *multiforcing* be any function $\mathbf{P}$ such that $|\mathbf{P}| = \mathrm{dom}\,\mathbf{P} \subseteq \omega_1$ and every value $\mathbf{P}(\xi)$, $\xi \in |\mathbf{P}|$, is a **LT**-forcing. Thus a multiforcing is a partial $\omega_1$-sequence of **LT**-forcings. A multiforcing $\mathbf{P}$ is *small* if the base $|\mathbf{P}|$ and each forcing $\mathbf{P}(\xi)$, $\xi \in |\mathbf{P}|$, are at most countable sets, and *regular* if $2^{<\omega} \in \mathbf{P}(\xi)$ for all $\xi \in |\mathbf{P}|$.

If $\mathbf{P}$ is a multiforcing then let $\mathbf{MT}(\mathbf{P})$ denote the set of all multitrees $\mathbf{T}$ such that $|\mathbf{T}| \subseteq |\mathbf{P}|$ and $\mathbf{T}(\xi) \in \mathbf{P}(\xi)$ for all $\xi \in |\mathbf{P}|$. The set $\mathbf{MT}(\mathbf{P})$ can be identified with the *countable base product* $\prod_{\xi \in |\mathbf{P}|} \mathbf{P}(\xi)$.

The next definition introduces a type of sets containing multitrees and satisfying some minimal closure conditions.

DEFINITION 9.1. Let $\mathbf{P}$ be a regular multiforcing. A set $\mathfrak{S} \subseteq \mathbf{MT}(\mathbf{P})$ is a *submultiforcing* if it satisfies the following:

(I) if $\mathbf{T} \in \mathfrak{S}$, $\xi \in |\mathbf{T}|$, and $T \in \mathbf{P}(\xi)$, then the multitree $\mathbf{S}$ defined by $|\mathbf{S}| = |\mathbf{T}|$, $\mathbf{S}(\xi) = T$, and $\mathbf{S}(\eta) = \mathbf{T}(\eta)$ for $\eta \neq \xi$, also belongs to $\mathfrak{S}$;

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([5]) Let $x, y \in Z = [\mathbf{S}(\Rightarrow\sigma)] \cup [\mathbf{S}(\Rightarrow\tau)]$ and $x\!\restriction\! U' = y\!\restriction\! U'$. As $\mathbf{S}(\Rightarrow\sigma)\!\restriction\! U = \mathbf{S}(\Rightarrow\tau)\!\restriction\! U$ by Lemma 7.4(ii), there are points $x', y' \in [\mathbf{S}(\Rightarrow\sigma)]$ with $x\!\restriction\! U = x'\!\restriction\! U$ and $y\!\restriction\! U = y'\!\restriction\! U$. We have $f(x) = f(x')$ and $f(y) = f(y')$ by (A), and $f(x') = f(y')$ since $f$ is reduced to $U'$ on $[\mathbf{S}(\Rightarrow\sigma)]$. We conclude that $f(x) = f(y)$.

(II) if $\mathbf{T} \in \mathfrak{S}$, $\xi \in |\mathbf{P}| \smallsetminus |\mathbf{T}|$, and $T \in \mathbf{P}(\xi)$, then the multitree $\mathbf{S}$ defined by $|\mathbf{S}| = |\mathbf{T}| \cup \{\xi\}$, $\mathbf{S}(\xi) = T$, and $\mathbf{S}{\restriction}|\mathbf{T}| = \mathbf{T}$, also belongs to $\mathfrak{S}$;

(III) if $\mathbf{T}, \mathbf{S} \in \mathfrak{S}$ then the multitree $\mathbf{T}' = \mathbf{T}{\uparrow}(|\mathbf{T}| \cup |\mathbf{S}|)$ defined by $|\mathbf{T}'| = |\mathbf{T}| \cup |\mathbf{S}|$, $\mathbf{T}'(\xi) = \mathbf{T}(\xi)$ for $\xi \in |\mathbf{T}|$, and $\mathbf{T}'(\xi) = 2^{<\omega}$ for $\xi \in |\mathbf{S}| \smallsetminus |\mathbf{T}|$, also belongs to $\mathfrak{S}$. □

EXAMPLE 9.2. Let $\mathbf{P}$ be a regular multiforcing, and $B = |\mathbf{P}|$. Then $\mathbf{MT}(\mathbf{P})$ is the largest submultiforcing in $\mathbf{MT}(\mathbf{P})$, while the smallest submultiforcing in $\mathbf{MT}(\mathbf{P})$ is the countable set $\mathfrak{S}_{\mathrm{coh}}^{B}$ of all multitrees $\mathbf{T} \in \mathbf{MT}(\mathbf{P})$ such that $|\mathbf{T}| \subseteq B$ is finite and $\mathbf{T}(\xi) \in P_{\mathrm{coh}}$ (Example 6.2) for all $\xi \in |\mathbf{T}|$, *Cohen's forcing* in $(2^{\omega})^{B}$. □

Multitrees $\mathbf{T}, \mathbf{S}$ in a submultiforcing $\mathfrak{S} \subseteq \mathbf{MT}$ are *compatible in* $\mathfrak{S}$ if there is a multitree $\mathbf{U} \in \mathfrak{S}$ satisfying $\mathbf{U} \leqslant \mathbf{T}$ and $\mathbf{U} \leqslant \mathbf{S}$. A set $\mathbf{D} \subseteq \mathfrak{S}$ is:

- *dense in* $\mathfrak{S}$ when $\forall\, \mathbf{T} \in \mathfrak{S} \,\exists\, \mathbf{S} \in \mathbf{D}\, (\mathbf{S} \leqslant \mathbf{T})$;
- *open dense in* $\mathfrak{S}$ if in addition $\forall\, \mathbf{T}, \mathbf{S} \in \mathfrak{S}\, (\mathbf{T} \leqslant \mathbf{S} \in \mathbf{D} \Rightarrow \mathbf{T} \in \mathbf{D})$;
- *pre-dense in* $\mathfrak{S}$ if the set $\mathbf{D}^{+} = \{\mathbf{T} \in \mathfrak{S} : \exists\, \mathbf{S} \in \mathbf{D}\, (\mathbf{T} \leqslant \mathbf{S})\}$ is dense in $\mathfrak{S}$.

In the context of Definition 7.3, a multitree $\mathbf{T}$ (not necessarily $\mathbf{T} \in \mathfrak{S}$!) is called an *m-collage over* $\mathfrak{S}$ if $\mathbf{T}(\Rightarrow u) \in \mathfrak{S}$ for all strings $u \in 2^{m}$. Thus a 0-collage is any multitree in $\mathfrak{S}$, while every $m$-collage is an $m + 1$-collage as well by the closure properties in Definition 9.1.

LEMMA 9.3. *Let* $\mathbf{P}$ *be a multiforcing,* $\mathfrak{S} \subseteq \mathbf{MT}(\mathbf{P})$ *be a submultiforcing, and* $\mathbf{T} \in \mathbf{MT}_{B}$. *Then, in terms of Definition 7.3, the following holds*:

(i) *if* $\sigma \in 2^{<\omega}$ *and* $\mathbf{T} \in \mathfrak{S}$ *then* $\mathbf{T}(\Rightarrow\sigma) \in \mathfrak{S}$;

(ii) *if* $\sigma \in 2^{n}$ *and* $\mathbf{T}(\Rightarrow\sigma) \in \mathfrak{S}$, *then* $\mathbf{T}$ *is an n-collage over* $\mathfrak{S}$;

(iii) *if* $\mathbf{T}$ *is an m-collage over* $\mathfrak{S}$, *and* $\mathbf{D} \subseteq \mathfrak{S}$ *is clopen in* $\mathfrak{S}$, *then there is a multitree* $\mathbf{S} \in \mathbf{MT}_{B}$ *which is an m-collage over* $\mathfrak{S}$ *and satisfies* $\mathbf{S} \leqslant_{m} \mathbf{T}$ *and* $\mathbf{S}(\Rightarrow\sigma) \in \mathbf{D}$ *for all* $\sigma \in 2^{m}$;

(iv) *if* $U \subseteq [\mathbf{T}]$ *is a nbhd of* $x_{0} \in [\mathbf{T}]$ *in* $[\mathbf{T}]$ *then there is a multitree* $\mathbf{S} \in \mathfrak{S}$ *such that* $|\mathbf{S}| = B$, $x_{0} \in [\mathbf{S}] \subseteq U$, *and* $\mathbf{S} \leqslant \mathbf{T}$.

*Proof.* (i) Use property 6.1(A) of $\mathbf{LT}$-forcings with the closure properties of Definition 9.1. Further, splitting the operation $(\Rightarrow\sigma)$ to components as in Definition 7.3 immediately reduces (ii) to Lemma 6.3.

(iii) If $\sigma \in 2^{m}$ then by Lemma 7.4(iv) there exists a multitree $\mathbf{S} \in \mathbf{MT}_{B}$, $\mathbf{S} \leqslant_{m} \mathbf{T}$, satisfying $\mathbf{S}(\Rightarrow\sigma) \in \mathbf{D}$ for this $\sigma$. And $\mathbf{S}$ is still an $m$-collage over $\mathfrak{S}$ by (ii). Iterate this procedure, going over all strings in $\sigma \in 2^{m}$.

(iv) We refer to (i) and Lemma 7.4(iii). ■

**10. On subsets with the Baire property.** This and the next section present two applications of Lemma 7.5 to the construction of multitrees with certain properties. Compared to Theorem 8.2, where Lemma 7.5 was also

used in the course of the proof, here of necessity we shall have to consider intermediate multitrees related to some multiforcing.

LEMMA 10.1. *Let* $\mathbf{T} \in \mathbf{MT}$ *and* $B = |\mathbf{T}|$. *If the set* $X \subseteq [\mathbf{T}]$ *has the Baire property inside* $[\mathbf{T}]$ *then there is a multitree* $\mathbf{S} \in \mathbf{MT}_B$ *such that* $[\mathbf{S}] \subseteq X$ *or* $[\mathbf{S}] \subseteq [\mathbf{T}] \smallsetminus X$.

*Proof.* Fix a $B$-complete function $\phi : \omega \xrightarrow{\text{onto}} B$. In our assumptions, $X$ or $[\mathbf{T}] \smallsetminus X$ is co-meager on a non-empty clopen $U \subseteq [\mathbf{T}]$. The cases are symmetric, hence we can assume that $X$ is co-meager on $U$. Note that $[\mathbf{T}(\Rightarrow\sigma)] \subseteq U$ for some $\sigma \in 2^{<\omega}$ by Lemma 7.4(iii). Yet the set $[\mathbf{T}(\Rightarrow\sigma)]$ itself is clopen in $[\mathbf{T}]$, and $X' = X \cap [\mathbf{T}(\Rightarrow\sigma)]$ is co-meager in $[\mathbf{T}(\Rightarrow\sigma)]$. Thus the task is reduced to the case when the set $X$ is co-meager in $[\mathbf{T}]$, and this will be assumed below. In this assumption, we can further suppose that $X = \bigcap_n U_n$, where every set $U_n \subseteq [\mathbf{T}]$ is topologically open and dense in $[\mathbf{T}]$.

CASE 1: there exists a multitree $\mathbf{S} \in \mathbf{MT}_B$ such that $\mathbf{S} \leqslant \mathbf{T}$ and $[\mathbf{S}] \cap U_n = \varnothing$ for some $n$. Then $[\mathbf{S}] \subseteq [\mathbf{T}] \smallsetminus X$, as required.

CASE 2: if $\mathbf{S} \in \mathbf{MT}_B$ and $\mathbf{S} \leqslant \mathbf{T}$ then $[\mathbf{S}] \cap U_n \neq \varnothing$ for all $n$. Define a regular multiforcing $\mathbf{P}$ such that $|\mathbf{P}| = B$ and if $\xi \in B$ then

$$\mathbf{P}(\xi) = \{s \cdot (\mathbf{T}(\xi)(\to t)) : s \in 2^{<\omega} \wedge t \in \mathbf{T}(\xi)\} \cup P_{\text{coh}} \qquad \text{(see Definition 6.2).}$$

Consider the submultiforcing $\mathfrak{S} = \{\mathbf{S} \in \mathbf{MT}(\mathbf{P}) : |\mathbf{S}| = B\}$; then $\mathbf{T} \in \mathfrak{S}$. We claim that for every $m$ the set

$$\mathbf{D}_m = \{\mathbf{S} \in \mathfrak{S} : [\mathbf{S}] \cap [\mathbf{T}] = \varnothing \text{ or } \mathbf{S} \leqslant \mathbf{T} \wedge [\mathbf{S}] \subseteq U_m\}$$

is open dense in $\mathfrak{S}$ (in the sense of Section 9). The openness is obvious. To prove the density let $\mathbf{T}' \in \mathfrak{S}$. If $[\mathbf{T}'] \not\subseteq [\mathbf{T}]$ then $U = [\mathbf{T}'] \smallsetminus [\mathbf{T}]$ is topologically open in $[\mathbf{T}']$ and non-empty. By Lemma 7.4(iii), there exists a multitree $\mathbf{S} \in \mathfrak{S}$ such that $[\mathbf{S}] \subseteq U$, i.e., $\mathbf{S} \leqslant \mathbf{T}'$ and $\mathbf{S} \in \mathbf{D}_m$. Thus assume that $\mathbf{T}' \leqslant \mathbf{T}$. Then $[\mathbf{T}'] \cap U_m \neq \varnothing$ by the Case 2 assumption. Applying Lemma 7.4(iii), we find a multitree $\mathbf{S} \in \mathfrak{S}$ satisfying $[\mathbf{S}] \subseteq U_m$, that is, $\mathbf{S} \in \mathbf{D}_m$. The density is proved.

Now Lemma 9.3(iii) provides a sequence $\cdots \leqslant_3 \mathbf{T}_2 \leqslant_2 \mathbf{T}_1 \leqslant_1 \mathbf{T}_0 \leqslant \mathbf{T}$ of multitrees $\mathbf{T}_m \in \mathbf{MT}_B$ with $\mathbf{T}_m(\Rightarrow\sigma) \in \mathbf{D}_m$ for all $m$ and $\sigma \in 2^m$. The multitree $\mathbf{S} = \bigwedge_m \mathbf{T}_m$ (Lemma 7.5) then satisfies $[\mathbf{S}] \subseteq U_m$ for all $m$, hence $[\mathbf{S}] \subseteq X$. ∎

**11. Separating image from preimage.** If $x_0 \in X \subseteq 2^\omega$, $f : X \to 2^\omega$ is continuous, and $f(x_0) \neq x_0$, then there exists a nbhd $U$ of $x_0$ in $X$ whose $f$-image $f''U$ does not intersect $U$. The next theorem is a version of this claim in the context of multitrees.

DEFINITION 11.1. Let $\mathbf{T} \in \mathbf{MT}_B$ and $\xi \in B$. A continuous map $f : [\mathbf{T}] \to 2^\omega$ is called *simple on* $[\mathbf{T}]$ *for* $\xi$ if there exists a string $\sigma \in 2^{<\omega}$ such that $f(x) = \sigma \cdot x(\xi)$ holds for all $x \in [\mathbf{T}]$. □

THEOREM 11.2. *Under the conditions of Definition 7.3, suppose* $\xi \in B = |\mathbf{P}|$, $\mathfrak{S} \subseteq \mathbf{MT}(\mathbf{P})$ *is a submultiforcing,* $m, n < \omega$, $\mathbf{T} \in \mathbf{MT}_B$ *is an* $m$*-collage over* $\mathfrak{S}$*, and* $f : [\mathbf{T}] \to 2^\omega$ *is continuous. Then:*

(i) *if* $U \in \mathbf{LT}$ *is an* $n$*-collage over a* $\mathbf{LT}$*-forcing* $P$*, then there exists a multitree* $\mathbf{T}' \in \mathbf{MT}_B$ *and a tree* $U' \in \mathbf{LT}$ *such that* $\mathbf{T}' \leqslant_m \mathbf{T}$, $U' \subseteq_n U$, $\mathbf{T}'$ *is an* $m$*-collage over* $\mathfrak{S}$*,* $U'$ *is a* $n$*-collage over* $P$*, and* $[U'] \cap f"[\mathbf{T}'] = \varnothing$;

(ii) *if* $\xi \in B = |\mathbf{P}|$*, and, for all* $r \in 2^{<\omega}$*,* $f$ *is not simple for* $\xi$ *on* $\mathbf{T}(\Rightarrow r)$*, then there is a multitree* $\mathbf{T}' \in \mathbf{MT}_B$ *such that* $\mathbf{T}' \leqslant_m \mathbf{T}$*,* $\mathbf{T}'$ *is an* $m$*-collage over* $\mathfrak{S}$*, and* $[\mathbf{T}'(\xi)] \cap f"[\mathbf{T}'] = \varnothing$.

*Proof.* (i) To begin, consider a pair of strings $u \in 2^m$, $s \in 2^n$. Let $x_0 \in [\mathbf{T}(\Rightarrow u)]$. Pick $y_0 \in [U(\to s)]$, $y_0 \neq f(x_0)$. As $f$ is continuous, there exists an open nbhd $G \subseteq [\mathbf{T}]$ of $x_0$ in $\mathbf{T}(\Rightarrow u)$ and a string $t \in U(\to s)$, satisfying $t \subset y_0$, and $t \not\subset x(\xi)$ for all $x \in G$. Put $V = U\restriction_t$. Then $V \in P$ and $V \subseteq U(\to s)$. By Lemma 5.4, there exists a tree $U' \in \mathbf{LT}$ such that $U' \subseteq_n U$ and $U'(\to s) = V$. Note that $U'$ is an $n$-collage over $P$ by Lemma 6.3.

On the other hand, by Lemma 9.3(iv), there is a multitree $\mathbf{S} \in \mathfrak{S}$ such that $|\mathbf{S}| = B$ and $[\mathbf{S}] \subseteq G$. By Lemma 7.4(iv), there is a multitree $\mathbf{T}' \in \mathbf{MT}_B$ satisfying $\mathbf{T}' \leqslant_m \mathbf{T}$ and $\mathbf{T}'(\Rightarrow u) = \mathbf{S}$. Note that $\mathbf{T}'$ is an $m$-collage over $\mathfrak{S}$ by Lemma 9.3(ii). Thus $\mathbf{T}'$ and $U'$ ensure that (i) holds at least partially: $[U'(\to s)] \cap f"[\mathbf{T}'(\Rightarrow u)] = \varnothing$ holds, but not yet $[U'] \cap f"[\mathbf{T}'] = \varnothing$. However, this procedure can be iterated, by going over all pairs of strings $u \in 2^m$, $s \in 2^n$. This leads to the result required.

(ii) As in the first part, it suffices, given a pair of strings $r, s \in 2^m$ (possibly $r = s$), to find an $m$-collage $\mathbf{T}' \in \mathbf{MT}_B$ over $\mathfrak{S}$ satisfying $\mathbf{T}' \leqslant_m \mathbf{T}$ and $[\mathbf{T}'(\Rightarrow s)(\xi)] \cap f"[\mathbf{T}'(\Rightarrow r)] = \varnothing$. The tree $T = \mathbf{T}(\xi)$ belongs to $\mathbf{P}(\xi) \subseteq \mathbf{LT}$, and $\mathbf{T}(\Rightarrow s)(\xi) = T(\to s')$, $\mathbf{T}(\Rightarrow r)(\xi) = T(\to r')$, where $s' = s \restriction \xi$, $t' = t \restriction \xi$ are strings of length $n = \boldsymbol{\nu}_{m\xi}$ (see Definition 7.3). Now $T(\to s') = \tau \cdot T(\to r')$ by Lemma 5.3, where $\tau = u[s', T] \cdot u[r', T]$. But $f$ is not simple on $\mathbf{T}(\Rightarrow r)$, hence there exists a point $x_0 \in \mathbf{T}(\Rightarrow r)$ such that $f(x_0) \neq \tau \cdot x_0(\xi)$. We have two strings $v \neq w$ in $2^{<\omega}$ of equal length $\mathtt{lh}(v) = \mathtt{lh}(w) > \mathtt{lh}(\tau)$, satisfying $v \subset f(x_0)$ and $w \subset \tau \cdot x_0(\xi)$. We put $w' = \tau \cdot w$; then $w' \subset x_0(\xi)$.

But $f$ is continuous, hence using Lemma 9.3 as above, we find a multitree $\mathbf{S} \in \mathfrak{S}$ such that $|\mathbf{S}| = B$, $\mathbf{S} \leqslant \mathbf{T}(\Rightarrow r)$, and if $x \in [\mathbf{S}]$ then $v \subset f(x)$, $w \subset \tau \cdot x(\xi)$, $w' \subset x(\xi)$. And further we find a multitree $\mathbf{T}' \in \mathbf{MT}_B$ satisfying $\mathbf{T}' \leqslant_m \mathbf{T}$ and $\mathbf{T}'(\Rightarrow r) = \mathbf{S}$, and being an $m$-collage over $\mathfrak{S}$.

We claim that $[\mathbf{T}'(\Rightarrow s)(\xi)] \cap f"[\mathbf{T}'(\Rightarrow r)] = \varnothing$. Indeed, by construction if $x \in [\mathbf{S}] = [\mathbf{T}'(\Rightarrow r)]$ then $v \subset f(x)$. Thus it remains to check that $w \subset b$

for all $b \in [\mathbf{T}'(\Rightarrow s)(\xi)]$. Note that $\mathbf{T}'(\Rightarrow s)(\xi) = T'(\to s')$ and $\mathbf{T}'(\Rightarrow r)(\xi) = T'(\to r')$, where $T' = \mathbf{T}'(\xi) \in \mathbf{P}(\xi)$. On the other hand, $T'$ is a tree in $\mathbf{LT}$ and $T' \subseteq_n T$, hence $T'(\to s') = \tau \cdot T'(\to r')$ by Lemma 5.4. Thus if $b \in [\mathbf{T}'(\Rightarrow s)(\xi)]$ then $a = \tau \cdot b \in [\mathbf{T}'(\Rightarrow r)(\xi)] = [T'(\to r')]$. It follows that $w' \subset a$ by the choice of $\mathbf{S} = \mathbf{T}'(\Rightarrow r)$. Then $w \subset b = \tau \cdot a$ (since $w = \tau \cdot w'$), as required. ∎

**12. Extension of multiforcings.** The forcing notion for the proof of Theorem 2.2 will be defined as an $\omega_1$-union of an increasing $\omega_1$-sequence of multiforcings. Definition 12.3 below contains conditions which every step of the construction will have to obey. We begin with the following definition.

DEFINITION 12.1 (coding continuous maps). Let $B \subseteq \omega_1$ be at most countable. A *code of a continuous map* $(2^\omega)^B \to 2^\omega$ is an indexed family $\mathbf{c} = \langle U_i^{\mathbf{c}}(k) \rangle_{k < \omega, \, i=0,1}$ of finite sets $U_i^{\mathbf{c}}(k) \subseteq \mathfrak{S}_{\mathrm{coh}}^B$ (see Example 9.2) such that for all $k$:

(1) if $\mathbf{T} \in U_0^{\mathbf{c}}(k)$ and $\mathbf{S} \in U_0^{\mathbf{c}}(k)$ then $[\mathbf{T}{\upharpoonright}B] \cap [\mathbf{S}{\upharpoonright}B] = \varnothing$, and
(2) $\bigcup_{k<\omega, \, i=0,1} \bigcup_{\mathbf{T} \in U_i^{\mathbf{c}}(k)} [\mathbf{T}{\upharpoonright}B] = (2^\omega)^B$.

Let $\mathrm{CCF}_B$ denote the set of all such codes.

We set $\mathrm{CCF} = \bigcup_{B \subseteq \omega_1, \, \mathrm{card}\, B \leq \aleph_0} \mathrm{CCF}_B$, and if $\mathbf{c} \in \mathrm{CCF}_B$ then $|\mathbf{c}| = B$.

The coded map $f = f^{\mathbf{c}} : (2^\omega)^B \to 2^\omega$ itself is defined as follows in this case: $f^{\mathbf{c}}(x)(k) = i$ if there is a multitree $\mathbf{T} \in U_i^{\mathbf{c}}(k)$ such that $x \in [\mathbf{T}{\upharpoonright}B]$. Make use of (1) to show that the definition is sound. □

We skip a routine proof of the following lemma, based on the compactness of the spaces considered.

LEMMA 12.2. *If $B \subseteq \omega_1$ is countable, $X \subseteq (2^\omega)^B$ closed, and a map $f : X \to 2^\omega$ is continuous, then there is a code $\mathbf{c} \in \mathrm{CCF}_B$ such that $f = f^{\mathbf{c}}{\upharpoonright}X$.* ∎

DEFINITION 12.3 (in $\mathbf{L}$). Let $\mathfrak{M}$ be a countable transitive model of $\mathbf{ZFC}'$, which includes all $\mathbf{ZFC}$ axioms except for the power set axiom, but with the axiom which claims the existence of $\mathscr{P}(\omega)$. (This implies the existence of the ordinal $\omega_1$ and sets like $2^\omega$, $\mathbf{PT}$, $\mathbf{LT}$ of cardinality $\mathfrak{c} = 2^{\aleph_0}$.)

Recall that $\mathbf{L}_\alpha$ is the $\alpha$th level of the Gödel constructible hierarchy.

Let $\mathbf{P} \in \mathfrak{M}$ be a regular (small) multiforcing. Then $|\mathbf{P}| = B \in \mathfrak{M}$ and $\alpha = \sup B = \bigcup B < \omega_1$. We let $\mathfrak{S}(\mathbf{P})$ denote the closure of $\mathbf{MT}(\mathbf{P}) \cap \mathbf{L}_\alpha$ in $\mathbf{MT}(\mathbf{P})$ with respect to the three operations of Definition 9.1. Thus $\mathfrak{S}(\mathbf{P}) \in \mathfrak{M}$, $\mathfrak{S}(\mathbf{P}) \subseteq \mathbf{MT}(\mathbf{P})$, and $\mathfrak{S}(\mathbf{P})$ is a countable submultiforcing.

Note that $\mathfrak{S}(\mathbf{P})$ does not depend on $\mathfrak{M}$.

A multiforcing $\mathbf{Q}$ (not necessarily in $\mathfrak{M}$) is an $\mathfrak{M}$-*extension* of $\mathbf{P}$, in symbols $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$, if the following holds:

(A) $|\mathbf{Q}| = |\mathbf{P}|$ and $\mathbf{Q}$ is a small multiforcing;

(B) if $\xi \in |\mathbf{P}|$ then $\mathbf{P}(\xi) \sqsubset_{\mathfrak{M}} \mathbf{Q}(\xi)$ in the sense of Definition 6.4;

(C) if $\mathbf{T} \in \mathfrak{S}(\mathbf{P})$ then there is a multitree $\mathbf{S} \in \mathbf{MT}(\mathbf{Q})$ satisfying $\mathbf{S} \leqslant \mathbf{T}$ and $\mathbf{S} \subseteq^{\mathtt{fd}} \bigvee \mathbf{D}$ for all open dense sets $\mathbf{D} \subseteq \mathfrak{S}(\mathbf{P})$, $\mathbf{D} \in \mathfrak{M}$;

(D) if $\mathbf{T} \in \mathfrak{S}(\mathbf{P})$, $\xi \in |\mathbf{T}|$, a map $f : (2^{\omega})^{|\mathbf{T}|} \to 2^{\omega}$ is continuous and has a code in $\mathrm{CCF}_{|\mathbf{T}|} \cap \mathfrak{M}$, then there exists a multitree $\mathbf{S} \in \mathbf{MT}(\mathbf{Q})$ such that $|\mathbf{S}| = |\mathbf{T}|$, $\mathbf{S} \leqslant \mathbf{T}$, and either (i) there is a string $\sigma \in 2^{<\omega}$ such that $f(x) = \sigma \cdot x(\xi)$ for all $x \in [\mathbf{S}]$, or (ii) $f(x) \notin [U]$ for all $x \in [\mathbf{S}]$ and $U \in \mathbf{Q}(\xi)$. $\square$

THEOREM 12.4 (in $\mathbf{L}$). *Let $\mathfrak{M}$ be a countable transitive model of $\mathbf{ZFC}'$, and $\mathbf{P} \in \mathfrak{M}$ be a regular (small) multiforcing. Then there is an $\mathfrak{M}$-extension $\mathbf{Q}$ of $\mathbf{P}$.*

The proof of the theorem is given in the next two sections. The construction of $\mathbf{Q}$ is presented in Section 13, and the proof of its properties follows in Section 14.

**13. Construction of extending multiforcing.** The following definitions formalize the construction of generic multitrees for the proof of Theorem 12.4, by means of Lemma 7.5.

- *Arguing under the assumptions of Theorem 12.4, we let $B = |\mathbf{P}|$ and $\mathfrak{S} = \mathfrak{S}(\mathbf{P})$, so that $\sup B < \omega_1$ and $\mathfrak{S} \subseteq \mathbf{MT}(\mathbf{P})$ is a countable submultiforcing.*

- During the proof of Theorem 12.4, i.e., until the end of Section 14, we *fix a $B$-complete function $\phi : \omega \xrightarrow{\text{onto}} B$. This allows us to use the notation of Definition 7.3.*

To begin with, we reduce all multitrees $\mathbf{T} \in \mathfrak{S}$ to the domain $B$, substituting each of them by its copy $\mathbf{T}^{\uparrow} = \mathbf{T} \restriction B$ (see Definition 7.1). Thus, by the regularity of $\mathbf{P}$, we have $\mathbf{T}^{\uparrow} \in \mathbf{MT}(\mathbf{P})$ and $|\mathbf{T}^{\uparrow}| = B$, and by definition $\mathbf{T}^{\uparrow}(\xi) = \mathbf{T}(\xi)$ for $\xi \in |\mathbf{T}|$, but $\mathbf{T}^{\uparrow}(\xi) = 2^{<\omega}$ for $\xi \in B \smallsetminus |\mathbf{T}|$. We put $\mathfrak{S}^{\uparrow} = \{\mathbf{T}^{\uparrow} : \mathbf{T} \in \mathfrak{S}\}$; this is a submultiforcing, too.

DEFINITION 13.1. A *system (over $\mathfrak{S}^{\uparrow}$)* is any function $\varphi : \mathtt{dom}\,\varphi \to \mathbf{MT}_B$ where $\mathtt{dom}\,\varphi \subseteq \omega \times \omega$ is finite, and if $\langle k, m \rangle \in \mathtt{dom}\,\varphi$ then

(1) if $n < m$ then $\langle k, n \rangle$ also belongs to $\mathtt{dom}\,\varphi$;

(2) $\varphi(k, m)$ is a tree in $\mathbf{MT}_B$ and an $m$-collage over $\mathfrak{S}^{\uparrow}$, and $|\varphi(k, m)| = B$;

(3) if $m > 0$ then $\varphi(k, m) \leqslant_m \varphi(k, m-1)$.

In this case, let $\nu_k^{\varphi}$ denote the largest number $m$ satisfying $\langle k, m \rangle \in \mathtt{dom}\,\varphi$, but $\nu_k^{\varphi} = -1$ if there is no such $m$. Let $|\varphi| = \{k : \nu_k^{\varphi} \geq 0\}$, a finite set.

Let $\mathbf{Sys}(\mathfrak{S}^{\uparrow})$ denote the set of all systems.

A system $\varphi$ *extends* a system $\psi$, in symbols $\psi \subseteq \varphi$, if $\operatorname{dom}\psi \subseteq \operatorname{dom}\varphi$ and $\psi = \varphi\restriction\operatorname{dom}\psi$; while $\psi \subset \varphi$ will denote strict extension. □

LEMMA 13.2 (elementary). *Suppose that* $\varphi \in \mathbf{Sys}(\mathfrak{S}^{\uparrow})$. *Then*

(i) *if* $k \in |\varphi|$ *and* $m = \nu_k^{\varphi}$ *then the extension* $\varphi'$ *of the system* $\varphi$ *by* $\nu_k^{\varphi'} = m + 1$ *and* $\varphi'(k, m+1) = \varphi'(k, m)$ *is a system extending* $\varphi$;

(ii) *if* $k \notin |\varphi|$ *and* $\mathbf{T} \in \mathfrak{S}^{\uparrow}$, *then the extension* $\varphi'$ *of the system* $\varphi$ *by* $\operatorname{dom}\varphi' = \operatorname{dom}\varphi \cup \{\langle k, 0\rangle\}$ *and* $\varphi'(k, 0) = \mathbf{T}$ *is a system extending* $\varphi$. ∎

DEFINITION 13.3. (A) Let DEF denote the set of all sets $X \subseteq \mathrm{HC}$ definable in HC (= all hereditarily countable sets) by $\in$-formulas with parameters in $\mathfrak{M} \cup \{\mathfrak{M}, \phi\}$. As DEF is countable, Lemma 13.2 allows one to define an infinite system $\Phi : \omega \times \omega \to \mathbf{MT}_B$ satisfying the requirements (2) and (3) of Definition 13.1 on the whole domain $k, m < \omega$, and also satisfying the following *genericity condition*: every set $\Delta \in \mathrm{DEF}$ is *blocked* by one of the systems $\varphi \in \mathbf{Sys}(\mathfrak{S}^{\uparrow})$, $\varphi \subset \Phi$, in the sense that either

(I) $\varphi \in \Delta$, or

(II) there is no system $\psi \in \mathbf{Sys}(\mathfrak{S}^{\uparrow}) \cap \Delta$ extending $\varphi$.

We let $\mathbf{T}_m^k = \Phi(k, m)$ for all $k, m < \omega$.

(B) The *limit trees* $\mathbf{L}^k = \bigwedge_m \mathbf{T}_m^k$, defined by $|\mathbf{L}^k| = B$ and $\mathbf{L}^k(\xi) = \bigcap_m \mathbf{T}_m^k(\xi)$ for all $\xi \in B$, belong to $\mathbf{MT}_B$ and satisfy $\mathbf{L}^k \leqslant_{m+1} \mathbf{T}_m^k$ for all $k, m$ by Lemma 7.5. Accordingly if $\xi \in B$ then $\mathbf{L}^k(\xi) \in \mathbf{LT}$ and $\mathbf{L}^k(\xi) \subseteq_n \mathbf{T}_m^k(\xi)$ for all $m$, where $n = \boldsymbol{\nu}_{m\xi}$ (Definition 7.3). This means $\mathbf{L}^k(\xi)(\to s) \subseteq \mathbf{T}_m^k(\xi)(\to s)$ for all $s \in 2^n$.

(C) If $\xi \in B$ then the set $Q_\xi = \{\sigma \cdot \mathbf{L}^k(\xi)(\to s) : k < \omega \wedge \sigma, s \in 2^{<\omega}\}$ is a countable $\mathbf{LT}$-forcing (see Example 6.2). We define a small multiforcing $\mathbf{Q}$ by $|\mathbf{Q}| = B$ and $\mathbf{Q}(\xi) = Q_\xi$ for all $\xi \in B$. □

We shall check that the multiforcing $\mathbf{Q}$ satisfies all conditions of Definition 12.3. Note that 12.3(A) directly holds by construction. The following lemma is obvious since option (II) of Definition 13.3(A) is impossible for dense sets $\Delta$. It will be a key ingredient in the verification of other conditions below.

LEMMA 13.4. *Let a set* $\Delta \in \mathrm{DEF}$, $\Delta \subseteq \mathbf{Sys}(\mathfrak{S}^{\uparrow})$, *be dense in* $\mathbf{Sys}(\mathfrak{S}^{\uparrow})$, *that is, every system in* $\mathbf{Sys}(\mathfrak{S}^{\uparrow})$ *is extendable to a system in* $\Delta$. *Then there exists a system* $\varphi \in \Delta$ *satisfying* $\varphi \subset \Phi$. □

COROLLARY 13.5. *If* $\mathbf{T} \in \mathfrak{S}^{\uparrow}$ *then there is an index* $k$ *such that* $\mathbf{L}^k \leqslant \mathbf{T}_0^k = \mathbf{T}$. *If* $\xi \in B$ *and* $T \in \mathbf{P}(\xi)$ *then there is an index* $k$ *such that* $\mathbf{L}^k(\xi) \subseteq \mathbf{T}_0^k(\xi) = T$.

*Proof.* Consider the set $\Delta$ of all systems $\varphi \in \mathbf{Sys}(\mathfrak{S}^{\uparrow})$ such that $\varphi(k,0) = \mathbf{T}$ holds for at least one $k \in |\varphi|$. As $\mathbf{T} \in \mathfrak{S}^{\uparrow} \in \mathfrak{M}$, the set $\Delta$ belongs to DEF. We claim that $\Delta$ is dense in $\mathbf{Sys}(\mathfrak{S}^{\uparrow})$. Indeed let $\varphi \in \mathbf{Sys}(\mathfrak{S}^{\uparrow})$. Take any $k \notin |\varphi|$. By Lemma 13.2(ii) there is a system $\psi \in \mathbf{Sys}(\mathfrak{S}^{\uparrow})$ extending $\varphi$ and satisfying $\langle k,0 \rangle \in \operatorname{dom} \psi$ and $\psi(k,0) = \mathbf{T}$. Thus $\psi \in \Delta$, and the density is proved.

By Lemma 13.4, there is a system $\varphi \in \Delta$, $\varphi \subset \Phi$. Then $\mathbf{T}_0^k = \varphi(k,0) = \mathbf{T}$ for some $k$. But $\mathbf{L}^k$ satisfies $\mathbf{L}^k \leqslant \mathbf{T}_0^k$ by 13.3(B), as required.

To reduce the second claim to the first one, note that if $\xi \in B$ and $T \in \mathbf{P}(\xi)$ then by definition there is a multitree $\mathbf{T} \in \mathfrak{S}^{\uparrow}$ with $\mathbf{T}(\xi) = T$. ∎

**14. Verification of requirements.** We check the conditions of Definition 12.3 for $\mathbf{Q}$ in the context of Section 13.

**Verification of 12.3(B).** Fix $\xi \in B$. To check (1) of Definition 6.4 (the density of $\mathbf{Q}(\xi)$ in $\mathbf{Q}(\xi) \cup \mathbf{P}(\xi)$), let $T \in \mathbf{P}(\xi)$. Then $\mathbf{L}^k(\xi) \subseteq T$ for some $k$ by Corollary 13.5. But the tree $S = \mathbf{L}^k(\xi)$ belongs to $Q_\xi = \mathbf{Q}(\xi)$ by 13.3(C), as required.

Now assume that $\xi \in B$, a set $D \in \mathfrak{M}$, $D \subseteq \mathbf{P}(\xi)$ is pre-dense in $\mathbf{P}(\xi)$, and $U \in \mathbf{Q}(\xi)$. We prove $U \subseteq^{\mathtt{fin}} \bigcup D$. By definition, $U = \sigma \cdot \mathbf{L}^k(\xi)(\to s)$, where $k < \omega$, $\xi \in B$, and $s, \sigma \in 2^{<\omega}$. We can assume that $\sigma = \Lambda$, i.e., in fact just $U = \mathbf{L}^k(\xi)(\to s)$. (The general case is reduced to $U = \mathbf{L}^k(\xi)(\to s)$ by the substitution of $\sigma \cdot D$ for $D$.) Furthermore, we can assume that $s = \Lambda$, i.e., $U = \mathbf{L}^k(\xi)$, because $\mathbf{L}^k(\xi)(\to s) \subseteq \mathbf{L}^k(\xi)$. Thus let $U = \mathbf{L}^k(\xi)$. The index $k$ will be fixed.

It follows from the pre-density of $D$ and the property 9.1(I) of the sub-multiforcing $\mathfrak{S}^{\uparrow}$ that the set $\mathbf{D} \in \mathfrak{M}$ of all multitrees $\mathbf{T} \in \mathfrak{S}^{\uparrow}$ satisfying $\mathbf{T}(\xi) \subseteq V$ for some $V \in D$, is itself open dense in $\mathfrak{S}^{\uparrow}$.

We claim that the set $\Delta \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{Sys}(\mathfrak{S}^{\uparrow})$ such that $k \in |\varphi|$, and for every string $t \in 2^n$, where $n = \nu_k^{\varphi}$, the multitree $\varphi(k,n)(\Rightarrow t)$ belongs to $\mathbf{D}$, is dense in $\mathbf{Sys}(\mathfrak{S}^{\uparrow})$. Indeed, let $\varphi \in \mathbf{Sys}(\mathfrak{S}^{\uparrow})$. By Lemma 13.2(ii), we assume that $k \in |\varphi|$, i.e., $n' = \nu_k^{\varphi} \geq 0$. By definition the multitree $\mathbf{T} = \varphi(k,n')$ is an $n'$-collage over $\mathfrak{S}^{\uparrow}$, and then, by Lemma 9.3(i), an $n$-collage too, where $n = n' + 1$. Then by Lemma 9.3(iii) there is a multitree $\mathbf{T}' \in \mathbf{MT}_B$ which is an $n$-collage over $\mathfrak{S}^{\uparrow}$ and satisfies $\mathbf{T}' \leqslant_n \mathbf{T}$ and $\mathbf{T}'(\Rightarrow t) \in \mathbf{D}$ for all $t \in 2^n$. Extend $\varphi$ to a system $\psi$ by $\operatorname{dom} \psi = \operatorname{dom} \varphi \cup \{\langle k,n \rangle\}$ and $\psi(k,n) = \mathbf{T}'$; we have $\psi \in \Delta$.

Now by Lemma 13.4 there is a system $\varphi \in \Delta$ satisfying $\varphi \subset \Phi$. Then $\varphi(k,n)(\Rightarrow t) = \mathbf{T}_n^k(\Rightarrow t) \in \mathbf{D}$ for all $t \in 2^n$, where $n = \nu_k^{\varphi}$, thus $\mathbf{T}_n^k \subseteq^{\mathtt{fd}} \bigvee \mathbf{D}$, hence $\mathbf{L}^k \subseteq^{\mathtt{fd}} \bigvee \mathbf{D}$. Therefore $U = \mathbf{L}^k(\xi) \subseteq^{\mathtt{fin}} \bigcup D$ by the definition of $\mathbf{D}$.

**Verification of 12.3(C).** Assume that $\mathbf{D} \in \mathfrak{M}$, $\mathbf{D} \subseteq \mathfrak{S}$ is open dense in $\mathfrak{S}$. Accordingly the set $\mathbf{D}^{\uparrow} = \{\mathbf{T}^{\uparrow} : \mathbf{T} \in \mathbf{D}\} \subseteq \mathfrak{S}^{\uparrow}$ is dense in $\mathfrak{S}^{\uparrow}$ ([6]). By Corollary 13.5, it suffices to prove that $\mathbf{L}^k \subseteq^{\mathtt{fd}} \bigvee \mathbf{D}^{\uparrow}$ for all $k < \omega$.

By the open-density of $\mathbf{D}^{\uparrow}$, the set $\Delta_k \in \mathfrak{M}$ of all systems $\varphi \in \mathbf{Sys}(\mathfrak{S}^{\uparrow})$ such that $k \in |\varphi|$, and for every string $t \in 2^n$, where $n = \nu_k^{\varphi}$, the multitree $\varphi(k, n)(\Rightarrow t)$ belongs to $\mathbf{D}^{\uparrow}$, is dense in $\mathbf{Sys}(\mathfrak{S}^{\uparrow})$. (See the verification of 12.3(B) above.) By Lemma 13.4 there exists a system $\varphi \in \Delta_k$ satisfying $\varphi \subset \Phi$. Then $\varphi(k, n)(\Rightarrow t) = \mathbf{T}_n^k(\Rightarrow t) \in \mathbf{D}$ for all $t \in 2^n$, where $n = \nu_k^{\varphi}$, that is, $\mathbf{T}_n^k \subseteq^{\mathtt{fd}} \bigvee \mathbf{D}^{\uparrow}$ holds, hence $\mathbf{L}^k \subseteq^{\mathtt{fd}} \bigvee \mathbf{D}^{\uparrow}$, as required.

**Verification of 12.3(D).** Let $\mathbf{T} \in \mathfrak{S}$, $\xi \in C = |\mathbf{T}|$, $\mathbf{c} \in \mathrm{CCF}_C \cap \mathfrak{M}$, and $f = f^{\mathbf{c}}$ (a continuous map $(2^{\omega})^C \to 2^{\omega}$). The multitree $\mathbf{T}^{\uparrow} = \mathbf{T} {\upharpoonright} B$ belongs to $\mathfrak{S}^{\uparrow}$, and the map $f^{\uparrow}(x) = f(x {\upharpoonright} C) : (2^{\omega})^B \to 2^{\omega}$ is continuous. In terms of Section 11, we can assume that $(*)$ there is no multitree $\mathbf{T}' \in \mathfrak{S}^{\uparrow}$, $\mathbf{T}' \leqslant \mathbf{T}^{\uparrow}$, such that $f^{\uparrow}$ is simple for $\xi$ on $\mathbf{T}'$. Indeed, otherwise using Corollary 13.5 we get a multitree $\mathbf{S}$ of the form $\mathbf{L}^k$, satisfying $\mathbf{L}^k \leqslant \mathbf{T}'$, and hence (i) of 12.3(D).

Now assuming $(*)$ we accordingly prove that any multitree $\mathbf{S} = \mathbf{L}^k$ with $\mathbf{L}^k \leqslant \mathbf{T}_0^k = \mathbf{T}^{\uparrow}$ satisfies (ii) of 12.3(D). Let $U \in \mathbf{Q}(\xi) = Q_{\xi}$, and we have to prove that $f^{\uparrow}(x) \notin [U]$ for all $x \in [\mathbf{L}^k]$. By definition, $U = \tau \cdot \mathbf{L}^{\ell}(\xi)(\Rightarrow s)$, where $\tau, s \in 2^{<\omega}$ and $\ell < \omega$. Now, as $\mathbf{L}^{\ell}(\xi)(\Rightarrow s) \subseteq \mathbf{L}^{\ell}(\xi)$, we can assume that $s = \Lambda$, that is, $U = \tau \cdot \mathbf{L}^{\ell}(\xi)$. Moreover we can assume that $\tau = \Lambda$, i.e., $U = \mathbf{L}^{\ell}(\xi)$; otherwise consider the map $f'(x) = \tau \cdot f^{\uparrow}(x)$ instead of $f^{\uparrow}$.

Thus we fix an index $\ell < \omega$ and prove that $[\mathbf{L}^{\ell}(\xi)] \cap f^{\uparrow}{}''[\mathbf{L}^k] = \varnothing$.

CASE 1: $\ell \neq k$. Consider the set $\Delta$ of all systems $\varphi \in \mathbf{Sys}(\mathfrak{S}^{\uparrow})$ such that $k, \ell \in |\varphi|$, that is, $m = \nu_k^{\varphi} \geq 0$ and $n = \nu_{\ell}^{\varphi} \geq 0$, and $[\varphi(\ell, n)(\xi)] \cap f^{\uparrow}{}''[\varphi(k, m)] = \varnothing$.

LEMMA 14.1. *The set $\Delta$ is dense in $\mathbf{Sys}(\mathfrak{S}^{\uparrow})$.*

*Proof of Lemma.* Let $\varphi \in \mathbf{Sys}(\mathfrak{S}^{\uparrow})$. By Lemma 13.2(ii), we can assume that $k, \ell \in |\varphi|$, that is, $n' = \nu_{\ell}^{\varphi} \geq 0$ and $m' = \nu_k^{\varphi} \geq 0$. By definition, the multitree $\mathbf{R}' = \varphi(k, m')$ is an $m'$-collage over $\mathfrak{S}^{\uparrow}$, and so an $m$-collage too, by Lemma 9.3(i), where $m = m' + 1$.

Further, we can assume that $\phi(n') = \xi$, for if not then take the least number $n'' > n'$ satisfying $\phi(n'') = \xi$, and trivially extend the system $\varphi$ by $\varphi(\ell, j) = \varphi(\ell, n')$ for all $\ell$ with $n' < \ell \leq n''$. As above, the multitree $\mathbf{Z}' = \varphi(\ell, n')$ is an $n'$-collage over $\mathfrak{S}^{\uparrow}$, and hence an $n$-collage, where $n = n' + 1$. It follows that $\mathbf{Z}'(\Rightarrow \sigma) \in \mathfrak{S}^{\uparrow}$ for all $\sigma \in 2^n$. In particular $\mathbf{Z}'(\Rightarrow \sigma)(\xi) \in \mathbf{P}(\xi)$ for

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([6]) To prove the openness let $\mathbf{T} \in \mathbf{D}$. Then $\mathbf{T}^{\uparrow} \in \mathbf{D}^{\uparrow}$, $\mathbf{S} \in \mathfrak{S}$, and $\mathbf{S}^{\uparrow} \leqslant \mathbf{T}^{\uparrow}$. We cannot assert directly that $\mathbf{S} \leqslant \mathbf{T}$. However, the multitree $\mathbf{S}' = \mathbf{S} {\upharpoonright} (|\mathbf{T}| \cup |\mathbf{S}|)$ also belongs to $\mathfrak{S}$ by Definition 9.1(III). Note that $\mathbf{S}^{\uparrow} \leqslant \mathbf{T}^{\uparrow}$ easily implies $\mathbf{S}' \leqslant \mathbf{T}$. Therefore $\mathbf{S}' \in \mathbf{D}$, since $\mathbf{D}$ is open. We conclude that $\mathbf{S}^{\uparrow} = \mathbf{S}'^{\uparrow} \in \mathbf{D}^{\uparrow}$.

$\sigma \in 2^n$. Yet $\mathbf{Z}'(\Rightarrow\sigma)(\xi) = \mathbf{Z}'(\xi)(\rightarrow\sigma\restriction\xi)$ by Definition 7.3, where $\sigma\restriction\xi \in 2^\nu$ and $\nu = \boldsymbol{\nu}_{m\xi}$. Therefore the tree $Z' = \mathbf{Z}'(\xi)$ is a $\nu$-collage over $\mathbf{P}(\xi)$.

By Theorem 11.2(i), there exist a multitree $\mathbf{R} \in \mathbf{MT}_B$ and a tree $Z \in \mathbf{LT}$ such that $\mathbf{R} \leqslant_m \mathbf{R}'$, $Z \subseteq_\nu Z'$, $\mathbf{R}$ is an $m$-collage over $\mathfrak{S}^\uparrow$, $Z$ is an $\nu$-collage over $\mathfrak{S}(\xi)$, and $[Z] \cap f^{\uparrow}\text{''}[\mathbf{R}] = \varnothing$. Define a multitree $\mathbf{Z} \in \mathbf{MT}_B$ so that $\mathbf{Z}(\xi) = Z$ and $\mathbf{Z}(\eta) = \mathbf{Z}'(\eta)$ for all $\eta \in B$, $\eta \neq \xi$.

SUBLEMMA 14.2. $\mathbf{Z}$ *is an $n$-collage over* $\mathfrak{S}^\uparrow$ *and* $\mathbf{Z} \leqslant_n \mathbf{Z}'$.

*Proof.* Let $\sigma = \tau^\frown i \in 2^n$, where $\tau \in 2^{n'}$ and $i = 0, 1$. The strings $\sigma\restriction\eta \in 2^{\boldsymbol{\nu}_{n\eta}}$ and $\tau\restriction\eta \in 2^{\boldsymbol{\nu}_{n'\eta}}$ (Definition 7.3) are related: $\sigma\restriction\eta = \tau\restriction\eta$ for $\eta \neq \xi$, but $\sigma\restriction\xi = (\tau\restriction\xi)^\frown i$, since $\phi(n') = \xi$ and $n = n' + 1$. It follows that

$$\mathbf{Z}(\Rightarrow\sigma)(\eta) = \mathbf{Z}(\eta)(\rightarrow\sigma\restriction\eta) = \mathbf{Z}'(\eta)(\rightarrow\sigma\restriction\eta) = \mathbf{Z}'(\Rightarrow\sigma)(\eta)$$

for $\eta \neq \xi$, that is, $\mathbf{Z}(\Rightarrow\sigma)\restriction(B \smallsetminus \{\xi\}) = \mathbf{Z}'(\Rightarrow\sigma)\restriction(B \smallsetminus \{\xi\})$. Further, $\mathbf{Z}(\Rightarrow\sigma)(\xi) = \mathbf{Z}(\xi)(\rightarrow\sigma\restriction\xi) = Z(\rightarrow\sigma\restriction\xi) = Z(\rightarrow\tau\restriction\xi)(\rightarrow i) \in \mathbf{P}(\xi)$, since $Z$ is a $\nu$-collage over $\mathfrak{S}(\xi)$. This implies $\mathbf{Z}(\Rightarrow\sigma) \in \mathfrak{S}^\uparrow$ by the property 9.1(I) of submultiforcings. As $\sigma \in 2^n$ is arbitrary, $\mathbf{Z}$ is an $n$-collage over $\mathfrak{S}^\uparrow$.

To establish $\mathbf{Z} \leqslant_n \mathbf{Z}'$, we need (in the same notation) to prove $\mathbf{Z}(\Rightarrow\sigma) \leqslant \mathbf{Z}'(\Rightarrow\sigma)$ for all $\sigma \in 2^n$, that is, $\mathbf{Z}(\Rightarrow\sigma)(\eta) \subseteq \mathbf{Z}'(\Rightarrow\sigma)(\eta)$ for all $\eta \in B$. If $\eta \neq \xi$ then simply $\mathbf{Z}(\Rightarrow\sigma)(\eta) \subseteq \mathbf{Z}'(\Rightarrow\sigma)(\eta)$, as above. Further, we have $\mathbf{Z}(\Rightarrow\sigma)(\xi) = Z(\rightarrow s)$ and $\mathbf{Z}'(\Rightarrow\sigma)(\xi) = Z'(\rightarrow s)$, where $s = \sigma\restriction\xi \in 2^\nu$, $\nu = \boldsymbol{\nu}_{m\xi}$. But $Z \subseteq_\nu Z'$ by construction, hence $Z(\rightarrow s) \subseteq Z'(\rightarrow s)$, or equivalently, $\mathbf{Z}(\Rightarrow\sigma)(\xi) \subseteq \mathbf{Z}'(\Rightarrow\sigma)(\xi)$. Thus $\mathbf{Z}(\Rightarrow\sigma)(\eta) \subseteq \mathbf{Z}'(\Rightarrow\sigma)(\eta)$ for all $\eta \in B$, that is, $\mathbf{Z}(\Rightarrow\sigma) \leqslant \mathbf{Z}'(\Rightarrow\sigma)$, as required. $\blacksquare$ Sublemma

Coming back to the lemma, we extend $\varphi$ to a system $\psi$ with $\operatorname{dom}\psi = \operatorname{dom}\varphi$, $\nu_k^\varphi = m$, $\nu_\ell^\varphi = n$, $\psi(k, m) = \mathbf{R}$, and $\psi(\ell, n) = \mathbf{Z}$ (just two new values). Thus $\psi$ is a system in $\mathbf{Sys}(\mathfrak{S}^\uparrow)$. Indeed, $\psi(k, m) = \mathbf{R}$, one of the two new terms relative to $\varphi$, is an $m$-collage over $\mathfrak{S}^\uparrow$, and $\mathbf{R} \leqslant_m \mathbf{R}' = \varphi(k, m')$, where $m = m' + 1$, as required by 13.1(3). Similarly for $\psi(\ell, n) = \mathbf{Z}$, the other new term. Thus $\psi \in \mathbf{Sys}(\mathfrak{S}^\uparrow)$ and clearly $\varphi \prec \psi$. Finally, $[Z] \cap f^{\uparrow}\text{''}[\mathbf{R}] = \varnothing$ by construction, hence $\psi \in \Delta$. This ends the proof of the density of $\Delta$. $\blacksquare$ Lemma

Now Corollary 13.4 yields a system $\varphi \in \mathbf{Sys}(\mathfrak{S}^\uparrow)$, $\varphi \subset \Phi$. Then $k, \ell \in |\varphi|$, hence $m = \nu_k^\varphi \geq 0$ and $n = \nu_\ell^\varphi \geq 0$, and multitrees $\mathbf{T}_m^k = \varphi(k, m)$, $\mathbf{T}_n^\ell = \varphi(\ell, n)$ satisfy $[\mathbf{T}_n^\ell(\xi)] \cap f^{\uparrow}\text{''}[\mathbf{T}_m^k] = \varnothing$ by the definition of $\Delta$, therefore $[\mathbf{L}^k(\xi)] \cap f^{\uparrow}\text{''}[\mathbf{L}^k] = \varnothing$, because $\mathbf{L}^k \subseteq \mathbf{T}_m^k$, as required.

CASE 2: $\ell = k$. Consider the set $\Delta$ of all systems $\varphi \in \mathbf{Sys}(\mathfrak{S}^\uparrow)$ such that $k \in |\varphi|$ (and then $m = \nu_k^\varphi \geq 0$) and $[\varphi(k, m)(\xi)] \cap f^{\uparrow}\text{''}[\varphi(k, m)] = \varnothing$. We do not claim that $\Delta$ is dense. However, by Definition 13.3 there is a system $\varphi \in \mathbf{Sys}(\mathfrak{S}^\uparrow)$, $\varphi \subset \Phi$, *blocking* $\Delta$ in the sense of 13.3(A), (I)$\vee$(II).

We now assert that 13.3(A)(II) is impossible for $\varphi$. Indeed, let $m' = \nu_k^\varphi$ and $\mathbf{R}' = \varphi(k, m') = \Phi(k, m') = \mathbf{T}_{m'}^k$. Then $\mathbf{R} \subseteq \mathfrak{S}^\uparrow = \mathbf{T}_0^k$, and hence by $(*)$

(at the beginning of verification of 12.3(D)), if $\mathbf{T}' \in \mathfrak{S}^\uparrow$, $\mathbf{T}' \leqslant \mathbf{R}'$, then the map $f^\uparrow$ is not simple for $\xi$ on $\mathbf{T}'$. Therefore by Theorem 11.2(ii) there is a multitree $\mathbf{R} \in \mathbf{MT}(\mathbf{P})$ which is an $m$-collage over $\mathfrak{S}^\uparrow$, where $m = m' + 1$, and satisfies $\mathbf{R} \leqslant_m \mathbf{R}'$ and $[\mathbf{R}(\xi)] \cap f^{\uparrow}{}''[\mathbf{R}] = \varnothing$. As in Case 1, we can extend $\varphi$ to a system $\psi \in \mathbf{Sys}(\mathfrak{S}^\uparrow)$ with the only one new term $\psi(k, m) = \mathbf{R}$, and then $\psi \in \Delta$ by the choice of $\mathbf{R}$. This proves that 13.3(A)(II) cannot happen for $\varphi$.

Thus 13.3(A)(I) takes place, that is, $\varphi \in \Delta$. It follows that $[\varphi(k,m)(\xi)] \cap f^{\uparrow}{}''[\varphi(k,m)] = \varnothing$, hence $[\mathbf{T}_m^k(\xi)] \cap f^{\uparrow}{}''[\mathbf{T}_m^k] = \varnothing$. This implies $[\mathbf{L}^k(\xi)] \cap f^{\uparrow}{}''[\mathbf{L}^k] = \varnothing$, since $\mathbf{L}^k \leqslant \mathbf{T}_m^k$, as required. $\blacksquare$ Theorem 12.4

**15. The forcing.** *We argue in the constructible universe* $\mathbf{L}$ *in this section.*

We begin with some definitions related to sequences of multiforcings.

First of all, we somewhat generalize the definition of $\sqsubset_{\mathfrak{M}}$ in 12.3. Given small multiforcings $\mathbf{P}, \mathbf{Q}$ and a model $\mathfrak{M}$, we write $\mathbf{P} \sqsubset_{\mathfrak{M}}^+ \mathbf{Q}$ when $|\mathbf{P}| \subseteq |\mathbf{Q}|$ and $\mathbf{P} \sqsubset_{\mathfrak{M}} (\mathbf{Q} {\restriction} |\mathbf{P}|)$ in the sense of 12.3. If $\vec{\mathbf{P}} = \langle \mathbf{P}_\alpha \rangle_{\alpha < \lambda}$ ($\lambda < \omega_1$) is a sequence of small multiforcings $\mathbf{P}_\alpha$ then:

(a) $\mathfrak{M}(\vec{\mathbf{P}})$ will denote the least transitive model of $\mathbf{ZFC}'$ (see Definition 12.3) of the form $\mathbf{L}_\gamma$, containing $\vec{\mathbf{P}}$ (and then all multiforcings $\mathbf{P}_\nu$), in which $\lambda$ and all sets $|\mathbf{P}_\nu|$ and forcings $\mathbf{P}_\nu(\xi)$ ($\xi \in |\mathbf{P}_\nu|$) are at most countable,

(b) a multiforcing $\mathbf{P} = \bigcup^{\mathtt{cw}} \vec{\mathbf{P}} = \bigcup_{\nu < \lambda}^{\mathtt{cw}} \mathbf{P}_\nu$ (componentwise union) is defined by $|\mathbf{P}| = \bigcup_{\nu < \lambda} |\mathbf{P}_\nu|$ and $\mathbf{P}(\xi) = \bigcup_{\xi < \nu < \lambda, \xi \in |\mathbf{P}_\nu|} \mathbf{P}_\nu(\xi)$ for all $\xi \in |\mathbf{P}|$.

DEFINITION 15.1 (in $\mathbf{L}$). Let $\lambda \leq \omega_1$. $\overrightarrow{\mathbf{MF}}_\lambda$ is the set of all $\lambda$-sequences $\vec{\mathbf{P}} = \langle \mathbf{P}_\nu \rangle_{\nu < \lambda}$ of small multiforcings $\mathbf{P}_\nu$ such that for each $\nu < \lambda$:

(1) $|\mathbf{P}_\nu| = \nu + 1$,

(2) $\mathbf{P}_\nu(\nu)$ contains the tree $2^{<\omega}$ (regularity), and

(3) $\bigcup_{\mu < \nu}^{\mathtt{cw}} \mathbf{P}_\mu \sqsubset_{\mathfrak{M}(\vec{\mathbf{P}} {\restriction} \nu)}^+ \mathbf{P}_\nu$.

We put $\overrightarrow{\mathbf{MF}} = \bigcup_{\lambda < \omega_1} \overrightarrow{\mathbf{MF}}_\lambda$. $\square$

The set $\overrightarrow{\mathbf{MF}} \cup \overrightarrow{\mathbf{MF}}_{\omega_1}$ is ordered by the extension relations $\subset, \subseteq$.

LEMMA 15.2 (in $\mathbf{L}$). *Assume that* $\kappa < \lambda < \omega_1$, *and* $\vec{\mathbf{P}} = \langle \mathbf{P}_\nu \rangle_{\nu < \kappa}$ *is a sequence in* $\overrightarrow{\mathbf{MF}}_\kappa$. *Then*:

(i) $\mathbf{P} = \bigcup^{\mathtt{cw}} \vec{\mathbf{P}}$ *is a small regular multiforcing and* $|\mathbf{P}| = \kappa$;

(ii) *there is a sequence* $\vec{\mathbf{Q}} \in \overrightarrow{\mathbf{MF}}$ *such that* $\operatorname{dom} \vec{\mathbf{Q}} = \lambda$ *and* $\vec{\mathbf{P}} \subset \vec{\mathbf{Q}}$.

*Proof.* (i) By definition, $\mathbf{P}(\xi) = \bigcup_{\xi \leq \nu < \kappa} \mathbf{P}_\nu(\xi)$. The first term $\mathbf{P}_\xi(\xi)$ in the union contains $2^{<\omega}$, so that the regularity follows.

(ii) We define multiforcings $\mathbf{P}_\alpha$, $\kappa \leq \alpha < \lambda$, by induction on $\alpha$. Assume that all terms $\mathbf{P}_\nu$, $\kappa \leq \nu < \alpha$, are defined, and the resulting sequence

$\vec{\mathbf{Q}} = \langle \mathbf{P}_\mu \rangle_{\mu < \alpha}$ belongs to $\overrightarrow{\mathbf{MF}}_\alpha$. Then $\mathbf{P}' = \bigcup^{\mathrm{cw}} \vec{\mathbf{Q}} = \bigcup_{\mu < \alpha}^{\mathrm{cw}} \mathbf{P}_\mu$ is a small regular multiforcing with $|\mathbf{P}'| = \alpha$ by (i), and $\mathbf{P}' \in \mathfrak{M} = \mathfrak{M}(\vec{\mathbf{Q}})$. Theorem 12.4 gives a small multiforcing $\mathbf{Q}$ satisfying $|\mathbf{Q}| = \alpha$ and $\mathbf{P}' \sqsubset_\mathfrak{M} \mathbf{Q}$. Define a small multiforcing $\mathbf{P}_\alpha$ so that $|\mathbf{P}_\alpha| = \alpha + 1$, $\mathbf{P}_\alpha(\xi) = \mathbf{Q}(\xi)$ for all $\xi < \alpha$, and, to fix the regularity, $\mathbf{P}_\alpha(\alpha) = P_{\mathrm{coh}}$ (see Example 6.2), hence $2^{<\omega} \in \mathbf{P}_\alpha(\alpha)$. ∎

Definition 15.3 (key definition). A sequence $\vec{\mathbf{P}} \in \overrightarrow{\mathbf{MF}}$ *blocks* a set $W \subseteq \overrightarrow{\mathbf{MF}}$ if either $\vec{\mathbf{P}} \in W$ (positive block) or there is no sequence $\vec{\mathbf{Q}} \in W$ with $\vec{\mathbf{P}} \subseteq \vec{\mathbf{Q}}$ (negative block). □

Approaching the blocking sequence theorem, we recall that HC is the set of all *hereditarily countable* sets, so that $\mathrm{HC} = \mathbf{L}_{\omega_1}$ in $\mathbf{L}$. See [4, Part 2, Chapter 5.4] for the definability classes $\Sigma_n^X, \Pi_n^X, \Delta_n^X$ for any set $X$, in particular, $\Sigma_n^{\mathrm{HC}}, \Pi_n^{\mathrm{HC}}, \Delta_n^{\mathrm{HC}}$ for $X = \mathrm{HC}$ in [14, Sections 8, 9] or elsewhere.

Theorem 15.4 (in $\mathbf{L}$). *If $\mathbb{n} \geq 3$ then there is a sequence $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha \rangle_{\alpha < \omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ satisfying the following two conditions*:

(i) $\vec{\mathbb{P}}$ *itself, as a set of pairs $\langle \alpha, \mathbb{P}_\alpha \rangle$, belongs to $\Delta_{\mathbb{n}-1}^{\mathrm{HC}}$;*

(ii) (*genericity of $\vec{\mathbb{P}}$ with respect to $\Sigma_{\mathbb{n}-2}^{\mathrm{HC}}(\mathrm{HC})$ sets*) *if $W \subseteq \overrightarrow{\mathbf{MF}}$ is a $\Sigma_{\mathbb{n}-2}^{\mathrm{HC}}(\mathrm{HC})$ set (i.e., parameters in HC are admitted in the defining formula), then there is $\gamma < \omega_1$ such that the restricted sequence $\vec{\mathbb{P}} \restriction \gamma = \langle \mathbb{P}_\alpha \rangle_{\alpha < \gamma} \in \overrightarrow{\mathbf{MF}}$ blocks $W$.*

*Proof.* Let $\leqslant_\mathbf{L}$ denote a canonical well-ordering of $\mathbf{L}$; its restriction to $\mathrm{HC} = \mathbf{L}_{\omega_1}$ is a $\Delta_1^{\mathrm{HC}}$ relation. There exists a universal $\Sigma_{\mathbb{n}-2}^{\mathrm{HC}}$ set $\mathfrak{U} \subseteq \omega_1 \times \mathrm{HC}$. Thus $\mathfrak{U}$ belongs to $\Sigma_{\mathbb{n}-2}^{\mathrm{HC}}$ (parameter-free $\Sigma_{\mathbb{n}-2}$ definability in HC), and for any $\Sigma_{\mathbb{n}-2}^{\mathrm{HC}}(\mathrm{HC})$ set $X \subseteq \mathrm{HC}$ (definable in HC by a $\Sigma_{\mathbb{n}-2}$ formula with parameters in HC) there is an ordinal $\alpha < \omega_1$ satisfying $X = \mathfrak{U}_\alpha$, where $\mathfrak{U}_\alpha = \{x : \langle \alpha, x \rangle \in \mathfrak{U}\}$. The choice of $\omega_1$ as the domain of parameters is validated by the hypothesis $\mathbf{V} = \mathbf{L}$, which is accepted in this section and implies the existence of a $\Delta_1^{\mathrm{HC}}$ surjection $\omega_1 \xrightarrow{\mathrm{onto}} \mathrm{HC}$.

Coming back to Definition 15.3, note that if $\vec{\mathbf{P}} \in \overrightarrow{\mathbf{MF}}$ and $W \subseteq \overrightarrow{\mathbf{MF}}$ is any set then there is a sequence $\vec{\mathbf{Q}} \in \overrightarrow{\mathbf{MF}}$ satisfying $\vec{\mathbf{P}} \subset \vec{\mathbf{Q}}$ and blocking $W$. We define $\vec{\mathbf{Q}}_\alpha \in \overrightarrow{\mathbf{MF}}$ by induction on $\alpha < \omega_1$ so that $\vec{\mathbf{Q}}_0 = \varnothing$, $\vec{\mathbf{Q}}_\lambda = \bigcup_{\alpha < \lambda} \vec{\mathbf{Q}}_\alpha$ for limit $\lambda$, and each $\vec{\mathbf{Q}}_{\alpha+1}$ is the $\leqslant_\mathbf{L}$-least sequence $\vec{\mathbf{Q}} \in \overrightarrow{\mathbf{MF}}$ satisfying $\vec{\mathbf{P}} \subset \vec{\mathbf{Q}}$ and blocking $\mathfrak{U}_\alpha$. Then $\vec{\mathbb{P}} = \bigcup_{\alpha < \omega_1} \vec{\mathbf{Q}}_\alpha \in \overrightarrow{\mathbf{MF}}_{\omega_1}$.

Now (ii) holds by construction, while (i) admits a routine verification based on the fact that $\overrightarrow{\mathbf{MF}} \in \Delta_1^{\mathrm{HC}}$. ∎

Definition 15.5 (in $\mathbf{L}$). Fix a number $\mathbb{n} \geq 3$ for which Theorem 2.2 is to be proved. Fix a sequence $\vec{\mathbb{P}} = \langle \mathbb{P}_\alpha \rangle_{\alpha < \omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ provided by Theorem 15.4 for this $\mathbb{n}$.

We put $\mathbb{P} = \bigcup_{\alpha<\omega_1}^{\mathbf{cw}} \mathbb{P}_\alpha$. Thus $\mathbb{P}$ is a multiforcing, $|\mathbb{P}| = \omega_1$, and $\mathbb{P}(\xi) = \bigcup_{\xi\le\alpha<\omega_1} \mathbb{P}_\alpha(\xi)$ for all $\xi < \omega_1$. By construction, each set $\mathbb{P}_\alpha$ is a small multiforcing satisfying $|\mathbb{P}_\alpha| = \alpha + 1$, while each component $\mathbb{P}_\alpha(\xi)$ ($\xi \le \alpha < \omega_1$) is a countable **LT**-forcing. It follows that if $\alpha < \omega_1$ then the multiforcing $\mathbb{P}_{<\alpha} = \bigcup_{\nu<\alpha}^{\mathbf{cw}} \mathbb{P}_\nu$ satisfies $|\mathbb{P}_{<\alpha}| = \alpha$. In addition, since $\overrightarrow{\mathbb{P}} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$, we have

(∗)  $\mathbb{P}_{<\alpha} \sqsubset^+_{\mathfrak{M}_\alpha} \mathbb{P}_\alpha$, that is, $\mathbb{P}_{<\alpha} \sqsubset_{\mathfrak{M}_\alpha} \mathbb{P}_\alpha{\restriction}\alpha$, for all $\alpha$,

where $\mathfrak{M}_\alpha = \mathfrak{M}(\overrightarrow{\mathbb{P}}{\restriction}\alpha)$. The submultiforcing $\mathfrak{S}_\alpha = \mathfrak{S}(\mathbb{P}_{<\alpha})$ in $\mathbf{MT}(\mathbb{P}_{<\alpha})$ (see Definition 12.3) will also be considered. □

The set $\mathbb{\Pi} = \mathbf{MT}(\mathbb{P})$ will be used in the proof of Theorem 2.2 as a forcing notion. It is naturally identified with the countable-support product $\prod_{\xi<\omega_1} \mathbb{P}(\xi)$ (in **L**). The sets $\mathbb{P}$ and $\mathbb{\Pi}$ belong to **L** by construction.

The next theorem shows that $\mathbb{\Pi}$-generic extensions of **L** are models for Theorem 2.2. Therefore Theorem 15.6 implies Theorem 2.2 (and Theorem 2.1 as well).

THEOREM 15.6. *Under the conditions of Definition* 15.5, *let* $\mathbb{G} \subseteq \mathbb{\Pi}$ *be a generic filter over* **L**. *Then the following holds in* **L**[$\mathbb{G}$]:

(i) *condition* (i) *of Theorem* 2.2;
(ii) *condition* (ii) *of Theorem* 2.2.

To prove Theorem 15.6, we explore properties of the forcing notion $\mathbb{\Pi}$ and related generic extensions in Sections 16–18, then establish (i) of Theorem 15.6 in Section 19, and finally (ii) in Section 22 with the aid of a special approximating forcing relation `forc`.

**16. Key forcing properties.** Here we study $\mathbb{\Pi}$ as the forcing notion. *We argue under the conditions and notation of Definition* 15.5.

DEFINITION 16.1 (in **L**). If $C \subseteq \omega_1$ then we define the subproduct $\mathbb{\Pi}{\restriction}C = \mathbf{MT}(\mathbb{P}{\restriction}C) = \{\mathbf{T} \in \mathbb{\Pi} : |\mathbf{T}| \subseteq C\} = \prod_{\xi\in C} \mathbb{P}(\xi)$ with countable support. Then $\mathbb{\Pi}$ can be identified with $(\mathbb{\Pi}{\restriction}C) \times (\mathbb{\Pi}{\restriction}(\omega_1^{\mathbf{L}} \smallsetminus C))$.

If $C \subseteq \omega_1$ is at most countable (in **L**), then by the regularity of $\mathbb{P}$ the set $\mathbb{\Pi}{\restriction}C$ can be identified with $\mathbb{\Pi}_C = \{\mathbf{T} \in \mathbb{\Pi} : |\mathbf{T}| = C\}$.

If $C = \{\xi\}$, $\xi < \omega_1^{\mathbf{L}}$, then $\mathbb{\Pi}{\restriction}\{\xi\}$ is naturally identified with $\mathbb{P}(\xi)$, and then $\mathbb{\Pi}$ is identified with $\mathbb{P}(\xi) \times \mathbb{\Pi}{\restriction}C_{\ne\xi}$, where $C_{\ne\xi} = \omega_1^{\mathbf{L}} \smallsetminus \{\xi\}$. □

LEMMA 16.2. *If* $\xi \le \alpha < \gamma < \omega_1$ *then* $\mathbb{P}_\alpha(\xi) \sqsubset \mathbb{P}_\gamma(\xi)$ *in the sense of* 6.4. *Therefore each* $\mathbb{P}_\alpha(\xi)$ *is pre-dense in* $\mathbb{P}(\xi) = \bigcup_{\alpha\ge\xi} \mathbb{P}_\alpha(\xi)$ *by Lemma* 6.5(iii).

*Proof.* Arguing by induction, suppose that $\mathbb{P}_\mu(\xi) \sqsubset \mathbb{P}_\nu(\xi)$ is established for all $\xi \le \mu < \nu < \gamma$. Lemma 6.5(iii) implies that the set $\mathbb{P}_\alpha(\xi)$ is pre-dense in $\bigcup_{\xi\le\nu<\gamma} \mathbb{P}_\nu(\xi)$. The multiforcing $\mathbf{Q} = \mathbb{P}_\gamma{\restriction}\gamma$ satisfies $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbf{Q}$ by 15.5(∗). By Definition 12.3, this includes the condition $\mathbb{P}_{<\gamma}(\xi) \sqsubset_{\mathfrak{M}_\gamma} \mathbf{Q}(\xi)$.

Then clearly $\mathbf{Q}(\xi)$ is dense in $\mathbb{P}_{<\gamma}(\xi) \cup \mathbf{Q}(\xi)$. However, $\mathbf{Q}(\xi) = \mathbb{P}_\gamma(\xi)$ while $\mathbb{P}_{<\gamma}(\xi) = \bigcup_{\xi \le \nu < \gamma} \mathbb{P}_\nu(\xi)$. Therefore, first, $\mathbb{P}_\gamma(\xi)$ is dense in $\mathbb{P}_\alpha(\xi) \cup \mathbb{P}_\gamma(\xi)$, thus we have (1) of Definition 6.4. And second, as the set $\mathbb{P}_\alpha(\xi)$ is dense in $\mathbb{P}_{<\gamma}(\xi)$ by the above, and clearly $\mathbb{P}_\alpha(\xi) \in \mathfrak{M}_\gamma$, we obtain $S \subseteq^{\mathtt{fin}} \bigvee \mathbb{P}_\alpha(\xi)$ for each tree $S \in \mathbf{Q}(\xi) = \mathbb{P}_\gamma(\xi)$, thus we have (2) of Definition 6.4. ∎

LEMMA 16.3 (in $\mathbf{L}$). *Assume that, for each $n$, $\mathbf{D}_n \subseteq \mathbb{\Pi}$ is open dense in $\mathbb{\Pi}$, and let $\mathbf{T} \in \mathbb{\Pi}$. There is a multitree $\mathbf{S} \in \mathbb{\Pi}$ satisfying $\mathbf{S} \le \mathbf{T}$ and $\mathbf{S} \subseteq^{\mathtt{fd}} \bigvee \mathbf{D}_n$ for all $n$. Therefore $\mathbb{\Pi}$-generic extensions of $\mathbf{L}$ preserve $\omega_1^{\mathbf{L}}$.*

*Proof.* There is a countable elementary submodel $M$ of $\langle \mathbf{L}_{\omega_2} ; \in \rangle$, containing $\mathbf{T}$ and all sets $\mathbf{D}_n$. Then $M$ also contains $\omega_1$, as it is a definable set, and contains the sequence $\vec{\mathbb{P}}$ along with the derived sets $\mathbb{P} = \bigcup^{\mathtt{cw}} \vec{\mathbb{P}}$, $\mathbb{\Pi} = \mathbf{MT}(\mathbb{P})$, for the same reason. The set $M \cap \mathbf{L}_{\omega_1}$ is transitive. Indeed, if $X \in M \cap \mathbf{L}_{\omega_1}$ then $X$ is at most countable, hence there exist functions $f : \omega \xrightarrow{\mathrm{onto}} X$. Let $f_X$ be the least of them in the sense of the Gödel well-ordering $\le_{\mathbf{L}}$ of $\mathbf{L}$. Then $f_X \in M$ since $X \in M$ and the ordering $\le_{\mathbf{L}} \upharpoonright \mathbf{L}_{\omega_2}$ is definable in $\mathbf{L}_{\omega_2}$. It follows that each $x \in X$ belongs to $M$ because $x = f_X(k)$ for some $k$.

Let $\phi : M \xrightarrow{\mathrm{onto}} \mathbf{L}_\lambda$ be the Mostowski collapse function, and $\alpha = \phi(\omega_1)$. Then $\alpha < \lambda < \omega_1$ and, by the transitivity, it holds (*) $\phi(x) = x$ for all $x \in M \cap \mathbf{L}_{\omega_1}$. Thus $\phi(\xi) = \xi$, $\phi(T) = T$, $\phi(\mathbf{S}) = \mathbf{S}$ for each ordinal $\xi \in M \cap \omega_1$, each tree $T \in M \cap \mathbf{LT}$, and each multitree $\mathbf{S} \in M \cap \mathbf{MT}$. We conclude that $\phi(\vec{\mathbb{P}}) = \vec{\mathbb{P}} \cap \mathbf{L}_\alpha = \vec{\mathbb{P}} \upharpoonright \alpha$, $\phi(\mathbb{P}) = \mathbb{P}_{<\alpha} = \bigcup^{\mathtt{cw}}_{\gamma < \alpha} \mathbb{P}_\alpha$ (a multiforcing with $|\mathbb{P}_{<\alpha}| = \alpha$), and $\phi(\mathbb{\Pi}) = \mathbb{\Pi} \cap \mathbf{L}_\alpha = \mathbf{MT}(\mathbb{P}_{<\alpha}) \cap \mathbf{L}_\alpha$.

We assert that moreover $\phi(\mathbb{\Pi}) = \mathfrak{S}_\alpha$, where, we recall, $\mathfrak{S}_\alpha = \mathfrak{S}(\mathbb{P}_{<\alpha})$. Indeed, by Definition 12.3, $\mathfrak{S}(\mathbb{P}_{<\alpha})$ is equal to the closure of $\mathbf{MT}(\mathbb{P}_{<\alpha}) \cap \mathbf{L}_\alpha$ relative to the three operations of Definition 9.1. But $\varphi(\mathbb{\Pi}) = \mathbf{MT}(\mathbb{P}_{<\alpha}) \cap \mathbf{L}_\alpha$, thus $\mathbf{MT}(\mathbb{P}_{<\alpha}) \cap \mathbf{L}_\alpha$ is already closed under the operations, since so is $\mathbb{\Pi} = \mathbf{MT}(\mathbb{P})$. We conclude that $\mathfrak{S}(\mathbb{P}_{<\alpha}) = \mathbf{MT}(\mathbb{P}_{<\alpha}) \cap \mathbf{L}_\alpha$.

Furthermore, a similar argument allows one to prove that if $n < \omega$ then the set $\phi(\mathbf{D}_n) = \mathbf{D}_n \cap \mathbf{L}_\alpha = \mathbf{D}_n \cap \mathfrak{S}_\alpha \in \mathbf{L}_\lambda$ is open dense in $\mathfrak{S}(\mathbb{P}_{<\alpha})$. In addition, $\phi(\mathbf{T}) = \mathbf{T} \in \mathfrak{S}_\alpha$. On the other hand, by elementarity, the ordinal $\alpha$ is uncountable in $\mathbf{L}_\lambda$. It follows that $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\alpha$. However, we have $\mathbb{P}_{<\alpha} \sqsubset_{\mathfrak{M}_\alpha} \mathbb{P}_\alpha \upharpoonright \alpha$ by 15.5(*), and also $\mathbf{T} \in \mathfrak{S}_\alpha = \mathfrak{S}(\mathbb{P}_{<\alpha})$. Therefore, by Definition 12.3(C), there exists a multitree $\mathbf{S} \in \mathbf{MT}(\mathbb{P}_\alpha)$ satisfying $\mathbf{S} \le \mathbf{T}$ and $\mathbf{S} \subseteq^{\mathtt{fd}} \bigvee \phi(\mathbf{D}_n)$ for all $n$. Finally, $\mathbf{MT}(\mathbb{P}_\alpha) \subseteq \mathbb{\Pi}$ and $\phi(\mathbf{D}_n) \subseteq \mathbf{D}_n$. This ends the proof of the first claim.

To prove the second claim of the lemma, suppose towards a contradiction that $\dot{f}$ is a name of a function from $\omega$ to $\omega_1^{\mathbf{L}}$, and some $\mathbf{T} \in \mathbb{\Pi}$ forces $\mathtt{ran}\, \dot{f} = \omega_1^{\mathbf{L}}$. Let $\mathbf{D}_{n\alpha}$ be the set of all multitrees $\mathbf{R} \in \mathbb{\Pi}$ that either (1) are incompatible with $\mathbf{T}$ in $\mathbb{\Pi}$, or (2) satisfy $\mathbf{R} \le \mathbf{T}$ and $\mathbb{\Pi}$-force $\dot{f}(n) = \alpha$. A simple argument shows that every set $\mathbf{D}_n = \bigcup_\alpha \mathbf{D}_{n\alpha}$ is dense in $\mathbb{\Pi}$. By the

first claim of the lemma, there exists a multitree $\mathbf{S} \in \mathbb{\Pi}$ satisfying $\mathbf{S} \leqslant \mathbf{T}$ and $\mathbf{S} \subseteq^{\mathtt{fd}} \bigvee \mathbf{D}_n$ for all $n$. Let the relations $\mathbf{S} \subseteq^{\mathtt{fd}} \bigvee \mathbf{D}_n$ be witnessed by finite sets $\mathbf{D}'_n \subseteq \mathbf{D}_n$. Accordingly, the sets $A_n = \{\alpha : \mathbf{D}'_n \cap \mathbf{D}_{n\alpha} \neq \varnothing\}$ are finite, hence the union $A = \bigcup_n A_n$ is countable in $\mathbf{L}$, i.e., $\omega_1^{\mathbf{L}} \not\subseteq A$. On the other hand, we assert that $\mathbf{S}$ forces $\dot{f}(n) \in A_n$, for each $n$. This implies a contradiction and accomplishes the proof.

To finally prove that $\mathbf{S}$ forces $\dot{f}(n) \in A_n$, suppose to the contrary that $\mathbf{R} \in \mathbb{\Pi}$, $\mathbf{R} \leqslant \mathbf{S}$, and $\mathbf{R}$ forces $\dot{f}(n) = \alpha$, where $\alpha < \omega_1^{\mathbf{L}}$, $\alpha \notin A_n$. Then $\mathbf{R} \subseteq^{\mathtt{fd}} \bigvee \mathbf{D}_n$ by means of the same finite set $\mathbf{D}'_n \subseteq \mathbf{D}_n$. Lemma 7.4(v) provides a string $\sigma \in 2^{<\omega}$ and a multitree $\mathbf{U} \in \mathbf{D}'_n$ such that $\mathbf{R}' = \mathbf{R}(\Rightarrow\sigma) \leqslant \mathbf{U}$. Note that $\mathbf{R}' \in \mathbb{\Pi}$ by Lemma 9.3(i). Thus the multitrees $\mathbf{R}$ and $\mathbf{U}$ are compatible in $\mathbb{\Pi}$. Finally, $\mathbf{U} \in \mathbf{D}'_n \subseteq \mathbf{D}_n$, therefore $\mathbf{U} \in \mathbf{D}_{n\gamma}$ for some $\gamma$. Then by definition $\mathbf{U}$ forces $\dot{f}(n) = \gamma$, where $\gamma \in A_n$, that is, $\gamma \neq \alpha$. However, $\mathbf{R}$ forces $\dot{f}(n) = \alpha$, where $\alpha \notin A_n$, which is a contradiction. ∎

LEMMA 16.4 (in $\mathbf{L}$). *If a set $Q \subseteq \mathbf{MT}$ of multitrees belongs to $\Sigma^{\mathrm{HC}}_{\mathrm{n}-2}(\mathrm{HC})$ and $Q^- = \{\mathbf{T} \in \mathbf{MT} : \neg\,\exists\, \mathbf{S} \in Q\,(\mathbf{S} \leqslant \mathbf{T})\}$, then the set $\mathbb{\Pi} \cap (Q \cup Q^-)$ is dense in $\mathbb{\Pi}$. In particular if $Q$ is dense in $\mathbf{MT}$ then $Q \cap \mathbb{\Pi}$ is dense in $\mathbb{\Pi}$.*

*Proof.* Consider a multitree $\mathbf{T}_0 \in \mathbb{\Pi} = \mathbf{MT}(\mathbb{P})$, thus $\mathbf{T}_0 \in \mathbf{MT}(\mathbb{P}_{<\alpha_0})$ for some $\alpha_0 < \omega_1$. The set $\Delta$ of all sequences $\vec{\mathbf{P}} \in \overrightarrow{\mathbf{MF}}$, such that $\vec{\mathbb{P}} \restriction \alpha_0 \subseteq \vec{\mathbf{P}}$ and $\exists\, \mathbf{T} \in Q \cap \mathbf{MT}(\bigcup^{\mathtt{cw}} \vec{\mathbf{P}})\,(\mathbf{T} \leqslant \mathbf{T}_0)$ belongs to $\Sigma^{\mathrm{HC}}_{\mathrm{n}-2}(\mathrm{HC})$ as so does $Q$. We conclude that there exists an ordinal $\alpha < \omega_1$ such that the sequence $\vec{\mathbb{P}} \restriction \alpha$ blocks $\Delta$.

CASE 1: $\vec{\mathbb{P}} \restriction \alpha \in \Delta$; let this be witnessed by $\mathbf{T} \in Q \cap \mathbf{MT}(\bigcup^{\mathtt{cw}}(\vec{\mathbb{P}} \restriction \alpha))$. Then $\alpha_0 \leq \alpha$ and the multitree $\mathbf{T}$ belongs to $Q \cap \mathbb{\Pi}$ and satisfies $\mathbf{T} \leqslant \mathbf{T}_0$.

CASE 2: no sequence in $\Delta$ extends $\vec{\mathbb{P}} \restriction \alpha$. Let $\gamma = \max\{\alpha, \alpha_0\}$. Then $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbb{P}_\gamma \restriction \gamma$ by 15.5(∗). As $\alpha_0 \leq \gamma$, there exists a multitree $\mathbf{T} \in \mathbf{MT}(\mathbb{P}_\gamma)$, $\mathbf{T} \leqslant \mathbf{T}_0$. We can assume that $|\mathbf{T}| = |\mathbb{P}_\gamma|$, that is, $= \gamma+1$. Then $\mathbf{T}(\xi) \in \mathbb{P}_\gamma(\xi)$ for all $\xi \leq \gamma$. It remains to prove that $\mathbf{T} \in Q^-$.

Suppose to the contrary that $\mathbf{T} \notin Q^-$. By definition there is a multitree $\mathbf{S} \in Q$, $\mathbf{S} \leqslant \mathbf{T}$. Then $\gamma + 1 = |\mathbf{T}| \subseteq |\mathbf{S}|$. We can assume that $|\mathbf{S}| = \lambda < \omega_1$, $\lambda \geq \gamma + 1$. We are going to define a sequence $\vec{\mathbf{P}} = \langle \mathbf{P}_\alpha \rangle_{\alpha < \lambda} \in \overrightarrow{\mathbf{MF}}$ which extends $\vec{\mathbb{P}} \restriction \gamma$, that is, $\mathbf{P}_\alpha = \mathbb{P}_\alpha$ for all $\alpha < \gamma$, and satisfies $\mathbf{S} \in \mathbf{MT}(\bigcup^{\mathtt{cw}} \vec{\mathbf{P}})$. This implies $\vec{\mathbf{P}} \in \Delta$ by the choice of $\mathbf{S}$, which contradicts the Case 2 hypothesis and completes the proof of $\mathbf{T} \in Q^-$ and the proof of the lemma.

Thus we have to appropriately define multiforcings $\mathbf{P}_\alpha$, $\gamma \leq \alpha < \lambda$. We begin with $\mathbf{P}_\gamma$. This is based on the multiforcing $\mathbb{P}_\gamma$. Note that $\mathbf{S}(\xi) \subseteq \mathbf{T}(\xi) \in \mathbb{P}_\gamma(\xi)$ for all $\xi \leq \gamma$. We put $\mathbf{P}_\gamma(\xi) = \mathbb{P}_\gamma(\xi) \cup \{\sigma \cdot (\mathbf{S}(\xi)(\to t)) : t, \sigma \in 2^{<\omega}\}$ for all $\xi \leq \gamma$. Every "new" tree $S = \sigma \cdot (\mathbf{S}(\xi)(\to t))$ satisfies

$S \subseteq \sigma \cdot \mathbf{T}(\xi)$, where $\sigma \cdot \mathbf{T}(\xi) \in \mathbb{P}_\gamma(\xi)$. However $\mathbb{P}_{<\gamma} \sqsubseteq^+_{\mathfrak{M}_\gamma} \mathbb{P}_\gamma$ by Definition 15.5(∗). Therefore $\mathbb{P}_{<\gamma} \sqsubseteq^+_{\mathfrak{M}_\gamma} \mathbf{P}_\gamma$ as well. Thus the term $\mathbf{P}_\gamma$ extends the system $\vec{\mathbb{P}} \restriction \gamma = \langle \mathbb{P}_\alpha \rangle_{\alpha < \gamma} = \langle \mathbf{P}_\alpha \rangle_{\alpha < \gamma} \in \vec{\mathbf{MF}}_\gamma$ to a system in $\vec{\mathbf{MF}}_{\gamma+1}$, and we have $\mathbf{S}(\xi) \in \mathbf{P}_\gamma(\xi)$ for all $\xi \leq \gamma$. The extended system can be further extended to a system in $\vec{\mathbf{MF}}_\lambda$ by terms $\mathbf{P}_\alpha$, $\gamma < \alpha < \lambda$, by induction as in the proof of 15.2(ii), with the amendment that $\mathbf{P}_\alpha(\alpha) = P_{\mathrm{coh}} \cup \{\sigma \cdot (\mathbf{S}(\alpha)(\to t)) : t, \sigma \in 2^{<\omega}\}$, rather than just $\mathbf{P}_\alpha(\alpha) = P_{\mathrm{coh}}$, for all $\alpha$. ∎

**17. Generic extension.** Here we study $\Pi$-generic extensions $\mathbf{L}[\mathbb{G}]$ of $\mathbf{L}$ obtained by adjoining $\Pi$-generic sets $\mathbb{G} \subseteq \Pi$ to $\mathbf{L}$. We will use the forcing notion $\Pi = \mathbf{MT}(\mathbb{P}) \in \mathbf{L}$ and other notation of Definition 15.5, with the difference that the reasoning will not be relativized to $\mathbf{L}$ by default, and accordingly the first uncountable cardinal in $\mathbf{L}$ will be denoted by $\omega_1^{\mathbf{L}}$ instead of $\omega_1$.

DEFINITION 17.1 (generic reals). Let a set $\mathbb{G} \subseteq \Pi$ be $\Pi$-generic over $\mathbf{L}$. Note that $\omega_1^{\mathbf{L}[\mathbb{G}]} = \omega_1^{\mathbf{L}}$ by Lemma 16.3.

If $\xi < \omega_1^{\mathbf{L}}$ then $\mathbb{G}(\xi) = \{\mathbf{T}(\xi) : \xi \in |\mathbf{T}| \wedge \mathbf{T} \in \mathbb{G}\}$ is a set $\mathbb{P}(\xi)$-generic over $\mathbf{L}$, the intersection $X_\xi = \bigcap_{T \in \mathbb{G}(\xi)}[T]$ contains a single real $\boldsymbol{x}_\xi = \boldsymbol{x}_\xi[\mathbb{G}] \in 2^\omega$, and this real is $\mathbb{P}(\xi)$-generic over $\mathbf{L}$. These reals are assembled into a "multireal" $\boldsymbol{x}[\mathbb{G}] = \langle \boldsymbol{x}_\xi[\mathbb{G}] \rangle_{\xi < \omega_1^{\mathbf{L}}} \in (2^\omega)^{\omega_1^{\mathbf{L}}}$. □

COROLLARY 17.2 (of 16.1 and the product forcing theorem). *If $B \in \mathbf{L}$, $B \subseteq \omega_1^{\mathbf{L}}$ is at most countable in $\mathbf{L}$, and $\mathbb{G} \subseteq \Pi$ is $\Pi$-generic over $\mathbf{L}$, then the set $\mathbb{G}_B = \{\mathbf{T} \in \mathbb{G} : |\mathbf{T}| = B\}$ is $\Pi_B$-generic over $\mathbf{L}$.* ∎

Recall that $C_{\neq\xi} = \omega_1^{\mathbf{L}} \smallsetminus \{\xi\}$.

PROPOSITION 17.3 (in terms of Definition 17.1). *If $\xi < \omega_1^{\mathbf{L}}$ then the real $\boldsymbol{x}_\xi[\mathbb{G}]$ is not $(\{\mathbb{G} \restriction C_{\neq\xi}\} \cup \mathbf{Ord})$-definable in $\mathbf{L}[\mathbb{G}]$, in particular, $\boldsymbol{x}_\xi[\mathbb{G}] \notin \mathbf{L}[\mathbb{G} \restriction C_{\neq\xi}]$.*

*Proof.* See the proof of Lemma 14.5 in [21], based on the product forcing theorem and the $\mathsf{E}_0$-invariance of each component $\mathbb{P}(\xi)$ in the sense of 6.1(B). ∎

The next theorem belongs to the type of "continuous reading of names" theorems in the theory of forcing extensions. It involves the coding of continuous maps by Definition 12.1, and asserts that reals $x \in 2^\omega$ in $\Pi$-generic extensions are obtained by applications of continuous maps coded in $\mathbf{L}$ to suitable sequences of generic reals. To render the notation less cumbersome, if $\mathbf{c} \in \mathbf{L}$ and $\mathbf{c} \in \mathrm{CCF}$ in $\mathbf{L}$, and $\mathbb{G} \subseteq \Pi$ is generic over $\mathbf{L}$, then we put $f^{\mathbf{c}}[\mathbb{G}] := f^{\mathbf{c}}(\boldsymbol{x}[G] \restriction B)$, where $B = |\mathbf{c}|$.

LEMMA 17.4. *If $C \in \mathbf{L}$, $C \subseteq \omega_1^{\mathbf{L}}$, $\mathbb{G} \subseteq \mathbb{\Pi}$ is generic $\mathbf{L}$, and $x \in 2^\omega \cap \mathbf{L}[\mathbb{G} \restriction C]$, then there is a code $\mathbf{c} \in \mathrm{CCF} \cap \mathbf{L}$ such that $|\mathbf{c}| \subseteq C$ and $x = f^{\mathbf{c}}[\mathbb{G}]$.*

*Proof.* Let $\dot{\boldsymbol{x}}$ be a name for $x$ in the forcing language related to the forcing notion $\mathbb{\Pi}$. Thus the indexed family of sets

$$A_{ki} = \big\{ \mathbf{T} \in \mathbb{\Pi} : \mathbf{T} \text{ forces that } \dot{\boldsymbol{x}}(k) = i \big\}, \quad k < \omega, \; i = 0, 1,$$

belongs to $\mathbf{L}$ and we have (A) $x(k) = i \Leftrightarrow \mathbb{G} \cap A_{ki} \neq \varnothing$, (B) $A_{k0} \cap A_{k1} = \varnothing$, and (C) each set $A_k = A_{k0} \cup A_{k1}$ is open dense in $\mathbb{\Pi}$. We can assume that $\dot{\boldsymbol{x}}$ contains an explicit effective construction of $x$ from $\mathbb{G} \restriction C$, and then (*) if $\mathbf{S} \in A_{ki}$ then $\mathbf{S} \restriction (C \cap |\mathbf{S}|) \in A_{ki}$ as well.

The set $D = \{ \mathbf{T} \in \mathbb{\Pi} : \forall k \, (\mathbf{T} \subseteq^{\mathtt{fd}} \bigvee A_k) \}$ also is dense in $\mathbb{\Pi}$ by Lemma 16.3. Therefore, by genericity, there is a multitree $\mathbf{T}' \in \mathbb{G}$ such that $\mathbf{T}' \subseteq^{\mathtt{fd}} \bigvee A_k$ for all $k$. In addition, (*) implies that the multitree $\mathbf{T} = \mathbf{T}' \restriction (C \cap |\mathbf{T}|) \in \mathbb{G}$ also satisfies $\mathbf{T} \subseteq^{\mathtt{fd}} \bigvee A_k$ for all $k$, but now $|\mathbf{T}| \subseteq C$.

This means (Definition 7.2) that, in $\mathbf{L}$, there exists a sequence of finite sets $F_k \subseteq A_k$ which ensure $\mathbf{T} \subseteq^{\mathtt{fd}} \bigvee A_k$ in the sense that: (1) $|\mathbf{U}| \subseteq B = |\mathbf{T}|$ for all $\mathbf{U} \in F_k$, (2) $[\mathbf{T}] \subseteq \bigcup_{\mathbf{U} \in F_k} [\mathbf{U} \uparrow B]$, and (3) $[\mathbf{U} \uparrow B] \cap [\mathbf{V} \uparrow B] = \varnothing$ for all $\mathbf{V} \neq \mathbf{U}$ in $F_k$. We put $F_{ki} = F_k \cap A_{ki}$, $i = 0, 1$.

Now arguing in $\mathbf{L}$ we define a continuous $f : [\mathbf{T}] \to 2^\omega$ as follows: $f(y)(k) = i$ if there is a multitree $\mathbf{S} \in F_{ki}$ with $y \restriction |\mathbf{S}| \in [\mathbf{S}]$. Then $f = f^{\mathbf{c}} \restriction [\mathbf{S}]$ by Lemma 12.2, where $\mathbf{c}$ is a suitable code $\mathrm{CCF}_B \cap \mathbf{L}$. One easily verifies that $x = f^{\mathbf{c}}[\mathbb{G}]$. ∎

By the next theorem, the relation $y \notin \mathbf{L}[x]$ between reals $x, y \in 2^\omega$ in $\mathbf{L}[\mathbb{G}]$ is fully determined by a generic real $\boldsymbol{x}_\xi[\mathbb{G}]$, so that $\boldsymbol{x}_\xi[\mathbb{G}]$ belongs to $\mathbf{L}[y]$ but $x$ belongs to $\mathbf{L}[\boldsymbol{x}[\mathbb{G}] \restriction C_{\neq \xi}]$, while definitely $\boldsymbol{x}_\xi[\mathbb{G}] \notin \mathbf{L}[\boldsymbol{x}[\mathbb{G}] \restriction C_{\neq \xi}]$ by Proposition 17.3.

THEOREM 17.5. *If a set $\mathbb{G} \subseteq \mathbb{\Pi}$ is generic over $\mathbf{L}$, $x, y \in 2^\omega \cap \mathbf{L}[\mathbb{G}]$, and $y \notin \mathbf{L}[x]$, then there is an ordinal $\xi < \omega_1^{\mathbf{L}}$ such that $x \in \mathbf{L}[\boldsymbol{x}[\mathbb{G}] \restriction C_{\neq \xi}]$ but $\boldsymbol{x}_\xi[\mathbb{G}] \in \mathbf{L}[y]$, and in addition $\boldsymbol{x}_\xi[\mathbb{G}] = g(y)$, where $g : 2^\omega \to 2^\omega$ is a continuous map coded in $\mathbf{L}$.*

*Proof.* By Lemma 17.4, there exist codes $\mathbf{c}, \mathbf{d} \in \mathrm{CCF} \cap \mathbf{L}$ such that $x = f^{\mathbf{c}}[\mathbb{G}]$ and $y = f^{\mathbf{d}}[\mathbb{G}]$. Let $B = |\mathbf{c}| \cup |\mathbf{d}|$; we can assume that $|\mathbf{c}| = |\mathbf{d}| = B$.

*We argue in $\mathbf{L}$.* The set $\mathbf{D}$ of all multitrees $\mathbf{S} \in \mathbf{MT}_B$ such that either (i) $f^{\mathbf{d}}$ is reduced to $f^{\mathbf{c}}$ on $[\mathbf{S}]$, or (ii) $f^{\mathbf{d}}$ captures some ordinal $\xi \in B$ on $[\mathbf{S}]$ and $f^{\mathbf{c}}$ is reduced to the set $B \smallsetminus \{\xi\}$ on $[\mathbf{S}]$, is dense in $\mathbf{MT}_B$ by Theorem 8.2. It follows from Lemma 16.4 that the set $\mathbf{D}' = \mathbf{D} \cap \mathbb{\Pi}_B$ is dense in $\mathbb{\Pi}_B = \{ \mathbf{R} \in \mathbb{\Pi} : |\mathbf{R}| = B \}$.

*We argue in $\mathbf{L}[\mathbb{G}]$.* We have $\mathbb{G} \cap \mathbf{D} \neq \varnothing$ by Corollary 17.2. Let $\mathbf{S} \in \mathbb{G} \cap \mathbf{D}$; then $\boldsymbol{x}[\mathbb{G}] \restriction B \in [\mathbf{S}]$. Note that (i) fails for this $\mathbf{S}$, since (i) implies $f^{\mathbf{d}}(z) = g(f^{\mathbf{c}}(z))$ for all $z \in [\mathbf{S}]$, where $g : 2^\omega \to 2^\omega$ is a continuous map coded in $\mathbf{L}$, thus (with $z = \boldsymbol{x}[\mathbb{G}] \restriction B$) we get $y = g(x)$, and further $y \in \mathbf{L}[x]$ (as $g$ is coded

in **L**), a contradiction to the assumption of the theorem. Thus (ii) holds, i.e., *still in* **L**, $f^{\mathbf{d}}$ captures an ordinal $\xi \in B$ on $[\mathbf{S}]$, while $f^{\mathbf{c}}$ is reduced to $B \smallsetminus \{\xi\}$ on $[\mathbf{S}]$.

By the compactness of the spaces considered, this implies the existence of continuous maps $f : (2^\omega)^{B \smallsetminus \{\xi\}} \to 2^\omega$ and $g : 2^\omega \to 2^\omega$, both coded in **L** and satisfying $f^{\mathbf{c}}(z) = f(z{\restriction}(B \smallsetminus \{\xi\}))$ and $z(\xi) = g(f^{\mathbf{d}}(z))$ for all $z \in \mathbf{S}$. In particular, for $z = \boldsymbol{x}[\mathbb{G}]{\restriction}B$, we have $x = f^{\mathbf{c}}(\boldsymbol{x}[\mathbb{G}]{\restriction}(B \smallsetminus \{\xi\}))$, hence $x \in \mathbf{L}[\boldsymbol{x}[\mathbb{G}]{\restriction}C_{\neq\xi}]$, and $\boldsymbol{x}_\xi[\mathbb{G}] = g(y)$, hence $\boldsymbol{x}_\xi[\mathbb{G}] \in \mathbf{L}[y]$. ∎

**18. Definability of generic reals.** We continue to argue in terms of Definitions 15.5 and 17.1. Now the main goal will be to study $\mathbb{P}(\xi)$-generic reals $x \in 2^\omega$ in $\mathbb{\Pi}$-extensions of **L**.

THEOREM 18.1. *In any* $\mathbb{\Pi}$*-generic extension* $\mathbf{L}[\mathbb{G}]$ *of* **L**, *it is true that*: *if* $\xi < \omega_1^{\mathbf{L}}$ *then the set* $X_\xi = [\boldsymbol{x}_\xi[\mathbb{G}]]_{\mathsf{E}_0} = \{\sigma\cdot\boldsymbol{x}_\xi[\mathbb{G}] : \sigma \in 2^{<\omega}\}$ *is equal to the set* $Y_\xi = \bigcap_{\xi \leq \alpha < \omega_1^{\mathbf{L}}} \bigcup_{U \in \mathbb{P}_\alpha(\xi)}[U]$.

*Proof.* The real $x = \boldsymbol{x}_\xi[\mathbb{G}] \in 2^\omega$ is $\mathbb{P}(\xi)$-generic, while every set of the form $\mathbb{P}_\alpha(\xi)$ is pre-dense in $\mathbb{P}(\xi)$ by Lemma 16.2. Therefore $x \in Y_\xi$. Moreover all sets $\mathbb{P}_\alpha(\xi)$ are **LT**-forcings by construction, hence they are $\mathsf{E}_0$-invariant in the sense of 6.1(B). It follows that $X_\xi \subseteq Y_\xi$.

To establish the converse, assume that $y_0 \in Y_\xi$ in $\mathbf{L}[\mathbb{G}]$. By Lemma 17.4, there is a code $\mathbf{c} \in \mathrm{CCF} \cap \mathbf{L}$ such that $y_0 = f^{\mathbf{c}}[\mathbb{G}] = f^{\mathbf{c}}(\boldsymbol{x}[\mathbb{G}]{\restriction}B)$, where $B = |\mathbf{c}|$. Consider the set $\mathbf{D}$ of all multitrees $\mathbf{S} \in \mathbb{\Pi}_B$ such that either (i) there is a string $\sigma \in 2^{<\omega}$ such that $f^{\mathbf{c}}(x) = \sigma\cdot x(\xi)$ for all $x \in [\mathbf{S}]$, or (ii) there exists an ordinal $\alpha$, $\xi \leq \alpha < \omega_1$, such that $f^{\mathbf{c}}(x) \notin \bigcup_{U \in \mathbb{P}_\alpha(\xi)}[U]$ holds for all $x \in [\mathbf{S}]$.

LEMMA 18.2. *The set* $\mathbf{D}$ *is dense in* $\mathbb{\Pi}_B$.

*Proof of Lemma.* Let $\mathbf{T} \in \mathbb{\Pi}_B$; then $|\mathbf{T}| = B$. There exists an ordinal $\alpha < \omega_1^{\mathbf{L}}$ such that (1) $B \subseteq \alpha$, hence $\xi < \alpha$, (2) $\mathbf{T} \in \mathfrak{S}_\alpha = \mathfrak{S}(\mathbb{P}_{<\alpha})$, and (3) $\mathbf{c} \in \mathfrak{M}_\alpha$. Note that $\mathbb{P}_{<\alpha} \sqsubset^+_{\mathfrak{M}_\alpha} \mathbb{P}_\alpha$ holds by 15.5($*$). Therefore by Definition 12.3(D) there is a multitree $\mathbf{S} \in \mathbf{MT}(\mathbb{P}_\alpha)$ such that $|\mathbf{S}| = |\mathbf{T}| = B$, $\mathbf{S} \leqslant \mathbf{T}$, and either (i) there is a string $\sigma \in 2^{<\omega}$ satisfying $f^{\mathbf{c}}(x) = \sigma\cdot x(\xi)$ for all $x \in [\mathbf{S}]$, or (ii) $f^{\mathbf{c}}(x) \notin \bigcup_{U \in \mathbb{P}_\alpha(\xi)}[U]$ for all $x \in [\mathbf{S}]$. Thus we have $\mathbf{S} \in \mathbf{D}$, getting the density. ∎ Lemma

We return to the theorem. Corollary 17.2 implies $\mathbb{G} \cap \mathbf{D} \neq \varnothing$ by the lemma. Let $\mathbf{S} \in \mathbb{G} \cap \mathbf{D}$. In particular $x_0 = \boldsymbol{x}[\mathbb{G}]{\restriction}B \in [\mathbf{S}]$. It follows that $\mathbf{S}$ does not satisfy (ii) of the definition of $\mathbf{D}$, since $y_0 = f^{\mathbf{c}}(x_0) \in Y_\xi$. Therefore $\mathbf{S}$ satisfies (i) of the definition of $\mathbf{D}$ with some $\sigma \in 2^{<\omega}$. Then $y_0 = f^{\mathbf{c}}(x_0) = \sigma\cdot x_0(\xi) = \sigma\cdot\boldsymbol{x}[\mathbb{G}](\xi) = \sigma\cdot\boldsymbol{x}_\xi[\mathbb{G}]$, that is, $y_0 \in X$, as required. ∎

One easily proves that, under the assumptions of the theorem, the set $X_\xi = Y_\xi$ is equal to the set of all $\mathbb{P}(\xi)$-generic reals $y \in 2^\omega$ (see [18]).

**19. Non-uniformizable set.** Here we prove claim (i) of Theorem 15.6. To begin, we define a non-uniformizable set in the "rectangle" $\omega_1^{\mathbf{L}} \times 2^\omega$.

LEMMA 19.1. *Under the assumptions of Theorem 15.6, the set $K = \{\langle \xi, x \rangle : \xi < \omega_1^{\mathbf{L}} \wedge x \in [\boldsymbol{x}_\xi[\mathbb{G}]]_{\mathsf{E}_0} \}$ belongs to $\mathbf{L}[\mathbb{G}]$ and has the following properties in $\mathbf{L}[\mathbb{G}]$ :*

(i) *$K$ belongs to the definability class $\Pi_{\mathbbm{n}-1}^{\mathrm{HC}}$;*
(ii) *if $\xi < \omega_1$ then the cross-section $K_\xi = \{x : \langle \xi, x \rangle \in K\}$ is an $\mathsf{E}_0$ -class;*
(iii) *the set $K$ is not ROD-uniformizable.*

*Proof.* (ii) is quite obvious: $K_\xi = [x_\xi[\mathbb{G}]]_{\mathsf{E}_0}$. To prove (i) we note that Lemma 16.3 implies $\omega_1 = \omega_1^{\mathbf{L}}$ in $\mathbf{L}[\mathbb{G}]$. Therefore by Theorem 18.1, the sentence $\langle \xi, x \rangle \in K$ is equivalent to

$$\xi < \omega_1 \ \wedge \ \forall \alpha \left( \xi \leq \alpha < \omega_1 \Longrightarrow \exists T \in \mathbb{P}_\alpha(\xi) \, (x \in [T]) \right).$$

Yet the formula in the outer brackets here expresses a $\Pi_{\mathbbm{n}-1}^{\mathrm{HC}}$ relation by condition (i) of Theorem 15.4. (The quantifier $\exists T \in \mathbb{P}_\alpha(\xi)$ is bounded, hence it does not affect the definability estimation.)

To prove (iii) suppose towards the contrary that it is true in $\mathbf{L}[\mathbb{G}]$ that $R \subseteq K$ is a uniformizing ROD set. Let $r \in 2^\omega \cap \mathbf{L}[\mathbb{G}]$ be a real such that $R$ is $\{r\} \cup \mathbf{Ord}$-definable in $\mathbf{L}[\mathbb{G}]$. Lemma 16.3 (preservation of $\omega_1^{\mathbf{L}}$) implies the existence of an ordinal $\xi < \omega_1^{\mathbf{L}}$ such that $r \in \mathbf{L}[\mathbb{G}\!\restriction\!\{\eta : \eta < \xi\}]$, hence $r \in \mathbf{L}[\mathbb{G}\!\restriction\! C_{\neq\xi}]$, where $C_{\neq\xi} = \omega_1^{\mathbf{L}} \smallsetminus \{\xi\}$. Therefore the unique real $x \in 2^\omega$, satisfying $\langle \xi, x \rangle \in R$, is $(\{\mathbb{G}\!\restriction\! C_{\neq\xi}\} \cup \mathbf{Ord})$-definable in $\mathbf{L}[\mathbb{G}]$. However, $R \subseteq K$, thus $x \, \mathsf{E}_0 \, \boldsymbol{x}_\xi[\mathbb{G}]$. It follows that the generic real $\boldsymbol{x}_\xi[\mathbb{G}]$ itself is $(\{\mathbb{G}\!\restriction\! C_{\neq\xi}\} \cup \mathbf{Ord})$-definable in the model $\mathbf{L}[\mathbb{G}]$. But this contradicts Proposition 17.3. ∎

To convert the set $K = K[\mathbb{G}]$ into a similar non-uniformizable set in the plane $2^\omega \times 2^\omega$, we make use of the following elementary transformation.

Let $\mathbb{Q} = \{q_n : n < \omega\}$ be a recursive enumeration of the rationals. If $z \in 2^\omega$ then let $Q_z = \{q_n : z(n) = 1\} \subseteq \mathbb{Q}$, let $Q_z' \subseteq Q_z$ be the largest (perhaps, empty) well-ordered initial segment of $Q_z$, and let $|z| < \omega_1$ be the ordinal number of $Q_z'$; thus obviously $\{|z| : z \in 2^\omega\} = \omega_1$.

LEMMA 19.2. *Under the assumptions of Theorem 15.6, the set*

$$W = \{\langle z, x \rangle \in 2^\omega \times 2^\omega : \langle |z|, x \rangle \in K\}$$

*belongs to $\mathbf{L}[\mathbb{G}]$ and has the following properties in $\mathbf{L}[\mathbb{G}]$:*

(i) *$W$ belongs to the definability class $\Pi_{\mathbbm{n}}^1$;*
(ii) *if $z \in 2^\omega$ then the cross-section $W_z = \{x : \langle z, x \rangle \in W\}$ is an $\mathsf{E}_0$-class;*
(iii) *the set $W$ is not ROD-uniformizable.*

*Proof.* The set $W$ belongs to $\Pi^{\mathrm{HC}}_{\mathfrak{n}-1}$ since so does $K$; indeed, the map $z \mapsto |z|$ is $\Delta^{\mathrm{HC}}_1$. Thus by the transfer theorem (see e.g. [14, 9.1]), $W$ is a $\Pi^1_{\mathfrak{n}}$ set.

Further, each cross-section $W_z$ coincides with the corresponding cross-section $K_\xi$ of $K$, where $\xi = |z|$, thus $W_z$ is an $\mathsf{E}_0$-class.

To prove (iii), suppose to the contrary that $W$ is uniformized by a ROD set $S \subseteq W$. As $\omega^{\mathbf{L}}_1 = \omega_1$ holds, for every ordinal $\xi < \omega_1$ there is a real $z \in 2^\omega \cap \mathbf{L}$ satisfying $|z| = \xi$. Let $z(\xi)$ be the $\leqslant_{\mathbf{L}}$-least of such reals. Then

$$R = \{\langle \xi, x\rangle \in K : \langle z(\xi), x\rangle \in S\}$$

is a ROD subset of $K$ which uniformizes the set $K$, contrary to Lemma 19.1. Thus $W$ satisfies (i)–(iii). ∎

*Proof of Theorem 15.6(i).* Obvious by Lemma 19.2. ∎

**20. Auxiliary forcing relation.** Here we define a key instrumentarium for the proof of (ii) of Theorem 15.6. This is a forcing-type relation `forc`. It is not directly connected with the forcing notion $\mathbb{P}$, but rather related to the countable-support product $\mathbf{LT}^{\omega_1}$. But it happens to be compatible with the $\mathbb{\Pi}$-forcing relation for formulas of certain quantifier complexity (Lemma 21.2). An important property of `forc` will be its permutation-invariance (Lemma 21.3), a property which the $\mathbb{P}$-forcing relation definitely lacks. This will be the key argument in the proof of Theorem 22.1.

*We argue in* $\mathbf{L}$. Let $\mathscr{L}$ be a *language* containing variables $i, j, k, \ldots$ of type 0 with the domain $\omega$, and variables $x, y, z, \ldots$ of type 1 with the domain $2^\omega$. Let *terms* be variables of type 0 and expressions of the form $x(k)$. Atomic formulas are those of the form $R(t_1, \ldots, t_n)$, where $R \subseteq \omega^n$ is any $n$-ary relation on $\omega$ in $\mathbf{L}$. Formulas containing no quantifiers over type 1 variables are *arithmetic*. Formulas of the form

$$\exists\, x_1 \,\forall\, x_2 \,\exists\, x_3 \,\ldots\, \exists\,(\forall)\, x_n\, \Psi \quad \text{and} \quad \forall\, x_1 \,\exists\, x_2 \,\forall\, x_3 \,\ldots\, \forall\,(\exists)\, x_n\, \Psi,$$

where $\Psi$ is arithmetic, belong to types $\mathscr{L}\Sigma^1_n$ and $\mathscr{L}\Pi^1_n$ respectively.

Additionally, we allow codes $\mathbf{c} \in \mathrm{CCF}$ to substitute free variables of type 1. We let $|\varphi| = \bigcup_{\mathbf{c} \in \varphi} |\mathbf{c}|$ for any $\mathscr{L}$-formula, where $\mathbf{c} \in \varphi$ means that a code $\mathbf{c}$ occurs in $\varphi$. The semantics is as follows. Let $\varphi := \varphi(\mathbf{c}_1, \ldots, \mathbf{c}_k)$ be an $\mathscr{L}$-formula, and suppose all codes in CCF occurring in $\varphi$ are explicitly indicated, and $|\varphi| \subseteq B \subseteq \omega_1$. If $x \in (2^\omega)^B$ then let $\varphi[x]$ denote the formula $\varphi(f^{\mathbf{c}_1}(x{\restriction}|f^{\mathbf{c}_1}|), \ldots, f^{\mathbf{c}_k}(x{\restriction}|f^{\mathbf{c}_k}|))$; all elements $f^{\mathbf{c}_i}(x{\restriction}|f^{\mathbf{c}_i}|)$ are reals in $2^\omega$.

Arithmetic formulas and those in $\mathscr{L}\Sigma^1_n \cup \mathscr{L}\Pi^1_n$, $n \geq 1$, will be called *normal*. If $\varphi$ is a formula in $\mathscr{L}\Sigma^1_n$ or $\mathscr{L}\Pi^1_n$ then $\varphi^-$ is the result of canonical transformation of $\neg\, \varphi$ to resp. $\mathscr{L}\Pi^1_n$, $\mathscr{L}\Sigma^1_n$ form. We let $\varphi^- := \neg\, \varphi$ for arithmetic formulas.

DEFINITION 20.1 (in **L**). We define a relation **T** forc $\varphi$ between multi-trees $\mathbf{T} \in \mathbf{MT}$ and closed normal $\mathscr{L}$-formulas:

(I) if $\varphi$ is a closed $\mathscr{L}$-formula, arithmetic or in $\mathscr{L}\Sigma_1^1 \cup \mathscr{L}\Pi_1^1$, and $|\varphi| \subseteq B = |\mathbf{T}|$, then $\mathbf{T}$ forc $\varphi$ whenever $\varphi[x]$ holds for all $x \in [\mathbf{T}]$;

(II) if $\varphi := \exists x\, \psi(x)$ is a closed $\mathscr{L}\Sigma_{n+1}^1$-formula, $n \geq 1$ ($\psi$ belongs to $\mathscr{L}\Pi_n^1$), then $\mathbf{T}$ forc $\varphi$ whenever there is a code $\mathbf{c} \in \mathrm{CCF}$ such that $\mathbf{T}$ forc $\psi(\mathbf{c})$;

(III) if $\varphi$ is a closed $\mathscr{L}\Pi_n^1$-formula, $n \geq 2$, then $\mathbf{T}$ forc $\varphi$ whenever there is no multitree $\mathbf{S} \in \mathbf{MT}$ satisfying $\mathbf{S} \leqslant \mathbf{T}$ and $\mathbf{S}$ forc $\varphi^-$.

Let $\mathbf{Forc}(\varphi) = \{T \in \mathbf{ST} : T \text{ forc } \varphi\}$, $\mathbf{Des}(\varphi) = \mathbf{Forc}(\varphi) \cup \mathbf{Forc}(\varphi^-)$. □

LEMMA 20.2 (in **L**). *If $m \geq 2$ and $\varphi$ is a closed formula in $\mathscr{L}\Sigma_m^1$, resp., $\mathscr{L}\Pi_m^1$, then $\mathbf{Forc}(\varphi)$ belongs to $\Sigma_{m-1}^{\mathrm{HC}}(\mathrm{HC})$, resp. $\Pi_{m-1}^{\mathrm{HC}}(\mathrm{HC})$.*

*Proof.* If $\varphi$ is a $\mathscr{L}\Pi_1^1$ formula then $\mathbf{Forc}(\varphi) \in \mathbf{\Pi}_1^1$ by Definition 20.1(I), and hence $\mathbf{Forc}(\varphi)$ belongs to $\Delta_1^{\mathrm{HC}}(\mathrm{HC})$. Then argue by induction using 20.1(II, III). ∎

**21. Auxiliary forcing relation: two lemmas.** We here prove two key properties of the relation forc. They will be used in the proof of Theorem 15.6(ii). One of them (Lemma 21.2) says that forc is connected with the truth in $\mathbb{P}$-generic extensions similarly to the ordinary $\mathbb{P}$-forcing, for formulas of certain complexity. The other one (Lemma 21.3) claims the invariance of forc relative to the permutations of $\omega_1$.

Recall that a number $\mathbb{n} \geq 3$ is fixed by Definition 15.5.

LEMMA 21.1 (in **L**). *Let $\varphi$ be a closed normal $\mathscr{L}$-formula. Then the set $\mathbf{Des}(\varphi)$ is dense in $\mathbf{MT}$. If $\varphi$ is of type $\mathscr{L}\Sigma_m^1$, $m < \mathbb{n}$, then $\mathbf{Des}(\varphi) \cap \mathbb{P}$ is dense in $\mathbb{P}$.*

*Proof.* It suffices to prove the density of $\mathbf{Des}(\varphi)$ for formulas $\varphi$ as in 20.1(I). If $\varphi$ is such and $\mathbf{T} \in \mathbf{MT}$, $|\varphi| \subseteq B = |\mathbf{T}|$, then the set $X(\varphi) = \{x \in [\mathbf{T}] : \varphi[x]\}$ in $(2^\omega)^B$ belongs to $\mathbf{\Sigma}_1^1 \cup \mathbf{\Pi}_1^1$ and hence has the Baire property inside the closed set $[\mathbf{T}] \subseteq (2^\omega)^B$. It remains to refer to Lemma 10.1. The second claim follows by Lemmas 20.2 and 16.4. ∎

LEMMA 21.2. *Assume that $1 \leq n < \mathbb{n}$, $\varphi \in \mathbf{L}$ is a closed formula in $\mathscr{L}\Pi_n^1 \cup \mathscr{L}\Sigma_{n+1}^1$, and a set $\mathbb{G} \subseteq \mathbb{P}$ is generic over $\mathbf{L}$. Then $\varphi[\boldsymbol{x}[\mathbb{G}]]$ holds in $\mathbf{L}[\mathbb{G}]$ if and only if $\exists\, \mathbf{T} \in \mathbb{G}\, (\mathbf{T} \text{ forc } \varphi)$.*

*Proof. Base case*: $\varphi$ is arithmetic or $\mathscr{L}\Sigma_1^1 \cup \mathscr{L}\Pi_1^1$, as in 20.1(I). If $\mathbf{T} \in \mathbb{G}$ and $\mathbf{T}$ forc $\varphi$ then $\varphi[\boldsymbol{x}[\mathbb{G}]]$ holds in $\mathbf{L}[\mathbb{G}]$ by the Shoenfield absoluteness theorem, since $\boldsymbol{x}[\mathbb{G}]{\restriction}|\mathbf{T}| \in [\mathbf{T}]$. In the opposite direction apply Lemma 21.1.

*Step $\mathscr{L}\Pi_n^1 \Rightarrow \mathscr{L}\Sigma_{n+1}^1$*: $\varphi$ is $\exists x\, \psi(x)$, where $\psi$ belongs to $\mathscr{L}\Pi_n^1$. Let $\mathbf{T} \in \mathbb{G}$ and $\mathbf{T}$ forc $\varphi$. Then by Definition 20.1(II) there exists a code $\mathbf{c} \in \mathrm{CCF} \cap \mathbf{L}$

such that $\mathbf{T}\,\texttt{forc}\,\psi(\mathbf{c})$. By the inductive hypothesis, the formula $\psi(\mathbf{c})[\boldsymbol{x}[\mathbb{G}]]$, i.e., $\psi[\boldsymbol{x}[\mathbb{G}]](f^{\mathbf{c}}(\boldsymbol{x}[\mathbb{G}]\!\restriction\!B))$, where $B=|\mathbf{c}|$, holds in $\mathbf{L}[\mathbb{G}]$. Then $\varphi[\boldsymbol{x}[\mathbb{G}]]$ is true as well.

Conversely, assume that $\varphi[\boldsymbol{x}[\mathbb{G}]]$ holds in $\mathbf{L}[\mathbb{G}]$. There is a real $y\in\mathbf{L}[\mathbb{G}]\cap2^{\omega}$ such that $\psi[\boldsymbol{x}[\mathbb{G}]](y)$ holds. By Lemma 17.4, $y=f^{\mathbf{c}}[\mathbb{G}]=f^{\mathbf{c}}(\boldsymbol{x}[\mathbb{G}]\!\restriction\!B)$, where $\mathbf{c}\in\mathrm{CCF}\cap\mathbf{L}$ and $B=|\mathbf{c}|$. Then $\psi(\mathbf{c})[\boldsymbol{x}[\mathbb{G}]]$ holds in $\mathbf{L}[\mathbb{G}]$. By the inductive hypothesis, there is $\mathbf{T}\in\mathbb{G}$ such that $\mathbf{T}\,\texttt{forc}\,\psi(\mathbf{c})$, hence $\mathbf{T}\,\texttt{forc}\,\varphi$.

*Step* $\mathscr{L}\Sigma_n^1\Rightarrow\mathscr{L}\Pi_n^1$: $\varphi$ is a $\mathscr{L}\Pi_n^1$ formula, $n\geq2$. Lemma 21.1 yields a multitree $\mathbf{T}\in\mathbb{G}$ such that either $\mathbf{T}\,\texttt{forc}\,\varphi$ or $\mathbf{T}\,\texttt{forc}\,\varphi^-$. If $\mathbf{T}\,\texttt{forc}\,\varphi^-$ then $\varphi^-[\boldsymbol{x}[\mathbb{G}]]$ holds in $\mathbf{L}[\mathbb{G}]$ by the inductive hypothesis, hence $\varphi[\boldsymbol{x}[\mathbb{G}]]$ fails. Now assume that $\mathbf{T}\,\texttt{forc}\,\varphi$. We have to prove $\varphi[\boldsymbol{x}[\mathbb{G}]]$ in $\mathbf{L}[\mathbb{G}]$. Suppose to the contrary that $\varphi^-[\boldsymbol{x}[\mathbb{G}]]$ holds. By the inductive hypothesis, there exists a multitree $\mathbf{S}\in\mathbb{G}$ such that $\mathbf{S}\,\texttt{forc}\,\varphi^-$. But the multitrees $\mathbf{S},\mathbf{T}$ belong to the generic set $\mathbb{G}$, hence they are compatible, which contradicts the assumption $\mathbf{T}\,\texttt{forc}\,\varphi$. $\blacksquare$

**Invariance.** The relation $\texttt{forc}$ turns out to be *invariant* with respect to the natural action of the group $H$ of all self-inverse (i.e., $h=h^{-1}$) *permutations* of the set $\omega_1^{\mathbf{L}}$ in $\mathbf{L}$. Thus $h\in H$ iff $h\in\mathbf{L}$, $h:\omega_1^{\mathbf{L}}\xrightarrow{\mathrm{onto}}\omega_1^{\mathbf{L}}$ is a bijection, and $h=h^{-1}$.

*We argue in* $\mathbf{L}$. Let $h\in H$. If $B\subseteq\omega_1$ and $F$ is a function defined on $B$ then a function $hF=h\!\cdot\!F$ is defined on $h"B=\{h(\xi):\xi\in B\}$ so that $(hF)(h(\xi))=F(\xi)$ for all $\xi\in B$. Thus $hF$ is equal to the superposition $F\circ h^{-1}$, and even $hF=F\circ h$ by self-invertibility.

In particular, if $x\in(2^{\omega})^B$ then $hx\in(2^{\omega})^{h"B}$, and if $\mathbf{T}\in\mathbf{MT}_B$ then $h\mathbf{T}=h\!\cdot\!\mathbf{T}$ is a multitree in $\mathbf{MT}_{h"B}$. Further, if $\mathbf{c}\in\mathrm{CCF}_B$ then a code $h\mathbf{c}=h\!\cdot\!\mathbf{c}\in\mathrm{CCF}_{h"B}$ can be canonically defined so that $f^{h\mathbf{c}}(hx)=f^{\mathbf{c}}(x)$ for all $\xi\in B$. Finally if $\varphi:=\varphi(\mathbf{c}_1,\ldots,\mathbf{c}_k)$ is an $\mathscr{L}$-formula then $h\varphi$ or $h\!\cdot\!\varphi$ denotes the formula $\varphi(h\mathbf{c}_1,\ldots,h\mathbf{c}_k)$. Then $(h\varphi)[hx]$ coincides with $\varphi[x]$.

LEMMA 21.3 (in $\mathbf{L}$). *Let* $h\in H$, $\mathbf{T}\in\mathbf{MT}$, *and* $\varphi$ *is a closed normal* $\mathscr{L}$-*formula. Then* $\mathbf{T}\,\texttt{forc}\,\varphi$ *if and only if* $h\mathbf{T}\,\texttt{forc}\,h\varphi$.

*Proof.* If $\varphi$ is a formula of type 20.1(I) then $[h\mathbf{T}]=\{hx:x\in[\mathbf{T}]\}$, and on the other hand, if $x\in[\mathbf{T}]$ then $\varphi[x]$ coincides with $(h\varphi)[hx]$. We skip further routine inductive steps on the base of Definition 20.1(II, III). $\blacksquare$

**22. Proof of the uniformization claim.** To prove claim (i) of Theorem 15.6 in the end of this section, we establish Theorem 22.1 saying that in $\mathbb{P}$-generic extensions any element of a countable $\mathbf{\Sigma}_n^1$ set $X$ is constructible relative to the parameter of a $\mathbf{\Sigma}_n^1$ definition of $X$. The relation $\texttt{forc}$ and Lemma 21.2 will play the key role.

THEOREM 22.1. *If a set* $\mathbb{G} \subseteq \mathbb{P}$ *is* $\mathbb{P}$*-generic over* **L** *and* $p \in \mathbf{L}[\mathbb{G}] \cap 2^\omega$, *then it is true in* $\mathbf{L}[\mathbb{G}]$ *that any countable* $\Sigma_n^1(p)$ *set* $Y \subseteq 2^\omega$ *satisfies* $Y \subseteq \mathbf{L}[p]$.

In fact a stronger claim, $Y \in \mathbf{L}[p]$, holds. However, it requires more complex transformations beyond $H$, so we skip this issue whatsoever.

*Proof.* We argue in terms of Definition 15.5. Suppose to the contrary that $Y \not\subseteq \mathbf{L}[p]$. Then $Y = \{y \in 2^\omega : \varphi(p, y)\}$, where $\varphi(p, y) := \exists z \, \psi(p, y, z)$ is a $\Sigma_n^1$ formula with $p$ as the only parameter, and there is a real $y_0 \in Y$, $y_0 \notin \mathbf{L}[p]$. By Theorem 17.5, there is an ordinal $\eta < \omega_1^{\mathbf{L}}$ such that $p \in \mathbf{L}[\boldsymbol{x}[\mathbb{G}] \restriction C_{\neq \eta}]$ and $\boldsymbol{x}_\eta[\mathbb{G}] \in \mathbf{L}[y_0]$, and moreover $\boldsymbol{x}_\eta[\mathbb{G}] = g(y_0)$, where $g : 2^\omega \to 2^\omega$ is a continuous map coded in **L**. By Lemma 17.4, there exist codes $\mathbf{c}, \mathbf{d} \in \mathrm{CCF}$ satisfying $p = f^{\mathbf{d}}[\mathbb{G}] = f^{\mathbf{d}}(\boldsymbol{x}[\mathbb{G}] \restriction B)$ and $y_0 = f^{\mathbf{c}}[\mathbb{G}] = f^{\mathbf{c}}(\boldsymbol{x}[\mathbb{G}] \restriction B')$, where $B = |\mathbf{d}| \subseteq C_{\neq \eta}$ and $B' = |\mathbf{c}|$. We can assume that $B \subseteq B'$ and $\eta \in B'$. Note that definitely $\eta \notin B$.

The goal is *to get a contradiction.*

Consider the $\mathscr{L}\Sigma_n^1$ formula $\varphi(\mathbf{d}, \mathbf{c})$. Then $\varphi(\mathbf{d}, \mathbf{c})[\boldsymbol{x}[\mathbb{G}]]$ coincides with $\varphi(f^{\mathbf{d}}[\mathbb{G}], f^{\mathbf{c}}[\mathbb{G}])$ by the choice of the codes, therefore $\varphi(\mathbf{d}, \mathbf{c})[\boldsymbol{x}[\mathbb{G}]]$ holds in $\mathbf{L}[\mathbb{G}]$. By Lemma 21.2, there is a multitree $\mathbf{S} \in \mathbb{G}$ satisfying $\mathbf{S} \, \mathtt{forc} \, \varphi(\mathbf{d}, \mathbf{c})$.

Further, the equality $\boldsymbol{x}_\eta[\mathbb{G}] = g(y_0)$ (see above) can be rewritten as $f^{\mathbf{e}}(\boldsymbol{x}[\underline{G}] \restriction B') = g(f^{\mathbf{c}}(\boldsymbol{x}[\mathbb{G}] \restriction B'))$, where $\mathbf{e} \in \mathrm{CCF}_{B'} \cap \mathbf{L}$ is a canonical code of the map $f^{\mathbf{e}}(x) = x(\eta)$. We render this formula as

$$\exists z \, \big( z = f^{\mathbf{c}}(\boldsymbol{x}[\mathbb{G}] \restriction B') \wedge f^{\mathbf{e}}(\boldsymbol{x}[\underline{G}] \restriction B') = g(z) \big).$$

As above, Lemma 21.2 implies the existence of a multitree $\mathbf{S}' \in \mathbb{G}$ satisfying $\mathbf{S}' \, \mathtt{forc} \, \exists z \, (z = \mathbf{c} \wedge \mathbf{e} = g(z))$. We can assume that $\mathbf{S}' = \mathbf{S}$. (Otherwise replace both multitrees by a stronger multitree in $\mathbb{G}$). Thus we have

(∗) $\mathbf{S} \, \mathtt{forc} \, \varphi(\mathbf{d}, \mathbf{c})$ and $\mathbf{S} \, \mathtt{forc} \, \exists z \, (z = \mathbf{c} \wedge \mathbf{e} = g(z))$.

We can assume that $|\mathbf{S}| = B'$, as otherwise we just replace $B'$ by $B' \cup |\mathbf{S}|$ and $\mathbf{S}$ by $\mathbf{S} \!\uparrow\! (B' \cup |\mathbf{S}|)$.

If $\vartheta < \omega_1^{\mathbf{L}}$ then let $H_\vartheta$ denote the set of all permutations $h \in H$ such that $h(\xi) = \xi$ for all $\xi \in B$ and $h(\xi) > \vartheta$ for all $\xi \in B' \smallsetminus B$.

LEMMA 22.2. *If* $\vartheta < \omega_1^{\mathbf{L}}$ *then there is a permutation* $h \in H_\vartheta$ *and a multitree* $\mathbf{S}' \in \mathbb{G}$ *such that* $\mathbf{S}' \leqslant h \!\cdot\! \mathbf{S}$. *(It is not assumed that* $h \!\cdot\! \mathbf{S} \in \mathbb{\Pi}$.)

*Proof of Lemma.* Arguing in **L**, consider the set $\mathbf{D}_\vartheta$ of all multitrees $\mathbf{S}' \in \mathbf{MT}$ such that $\mathbf{S}' \leqslant \mathbf{S}$ and there exists a permutation $h \in H_\vartheta$ such that the multitree $h \!\cdot\! \mathbf{S}$ satisfies $\mathbf{S}' \leqslant h \!\cdot\! \mathbf{S}$. A routine estimation shows that $\mathbf{D}$ is a $\Sigma_1^{\mathrm{HC}}(\mathbf{S}, \vartheta)$ set. Therefore by Lemma 16.4 there is a multitree $\mathbf{S}' \in \mathbb{G}$ such that either (1) $\mathbf{S}' \in \mathbf{D}_\vartheta$, or (2) there is no multitree $\mathbf{R} \in \mathbf{D}_\vartheta$ satisfying $\mathbf{R} \leqslant \mathbf{S}'$. And as $\mathbf{S}$ also belongs to $\mathbb{G}$, we can assume that $\mathbf{S}' \leqslant \mathbf{S}$.

We claim that (2) is impossible. Indeed, let $\gamma < \omega_1^{\mathbf{L}}$ satisfy $|\mathbf{S}'| \subseteq \gamma$ and $\gamma \geq \vartheta$. Define a permutation $h$ by $h(\xi) = \xi$ for $\xi \in B$, $h(\xi) = h^{-1}(\xi) = \gamma + \xi$ for $\xi < \gamma$, $\xi \notin B$, and still $h(\xi) = \xi$ for all other $\xi < \omega_1^{\mathbf{L}}$. The multitrees $\mathbf{S}'$ and $\mathbf{U} = h \cdot \mathbf{S}'$ coincide on the common domain $|\mathbf{S}'| \cap |\mathbf{U}| = B$, hence are compatible. It follows that the union $\mathbf{R} = \mathbf{S}' \cup \mathbf{U}$ belongs to $\mathbf{MT}$ and $\mathbf{R} \leqslant \mathbf{S}', \mathbf{U}$. And further we have $\mathbf{R} \leqslant \mathbf{U} = h \cdot \mathbf{S}' \leqslant h \cdot \mathbf{S}$ by construction, hence $\mathbf{R} \in \mathbf{D}$, as required. Thus (2) fails. Therefore (1) holds, that is, $\mathbf{S}' \in \mathbf{D}_\vartheta$, as required. $\blacksquare_{\text{Lemma}}$

Coming back to Theorem 22.1, recall that $\omega_1^{\mathbf{L}}$ remains a cardinal in $\Pi$-generic extensions by Lemma 16.3. Therefore Lemma 22.2 allows one to define by induction an increasing sequence $\langle \vartheta_\nu \rangle_{\nu < \omega_1^{\mathbf{L}}}$ of ordinals $\vartheta_\nu < \omega_1^{\mathbf{L}}$ and a sequence of multitrees $\mathbf{S}_\nu \in \mathbb{G}$ and a sequence of permutations $h_\nu \in H_{\vartheta_\nu}$ satisfying $B' \subseteq \vartheta_0$ and $\mathbf{S}_\nu \leqslant h_\nu \cdot \mathbf{S}$ for all $\nu$, and $|\mathbf{S}_\mu| \subseteq \vartheta_\nu$ for $\mu < \nu$.

Let $\mathbf{T}_\nu = h_\nu \cdot \mathbf{S}$, $\mathbf{c}_\nu = h_\nu \cdot \mathbf{c}$, $\mathbf{d}_\nu = h_\nu \cdot \mathbf{d}$, $\mathbf{e}_\nu = h_\nu \cdot \mathbf{e}$ for all $\nu$. Then we have $\mathbf{T}_\nu \,\texttt{forc}\, \varphi(\mathbf{d}_\nu, \mathbf{c}_\nu)$ and $\mathbf{T}_\nu \,\texttt{forc}\, \exists z \, (z = \mathbf{c}_\nu \wedge \mathbf{e}_\nu = g(z))$ by $(*)$ and Lemma 21.3. It follows that

(†)    $\mathbf{S}_\nu \,\texttt{forc}\, \varphi(\mathbf{d}, \mathbf{c}_\nu)$  and  $\mathbf{S}_\nu \,\texttt{forc}\, \exists z \, (z = \mathbf{c}_\nu$ and $\mathbf{e}_\nu = g(z))$,

since $\mathbf{S}_\nu \leqslant \mathbf{T}_\nu$, and, with respect to the code $\mathbf{d}$: $\mathbf{d}_\nu = h_\nu \cdot \mathbf{d} = \mathbf{d}$. (Indeed, $h_\nu(\xi) = \xi$ whenever $\xi \in B = |\mathbf{d}|$.)

Recall that $f^{\mathbf{d}}(\boldsymbol{x}[\mathbb{G}] \restriction B) = p$. Let $B'_\nu = h'' B'$, $z_\nu = f^{\mathbf{c}_\nu}(\boldsymbol{x}[\mathbb{G}] \restriction B'_\nu)$, and $q_\nu = f^{\mathbf{e}_\nu}(\boldsymbol{x}[\mathbb{G}] \restriction B'_\nu)$. If $\nu < \omega_1^{\mathbf{L}}$ then, by (†) and Lemma 21.2, $\varphi(p, z_\nu)$ holds in $\mathbf{L}[\mathbb{G}]$—hence $z_\nu \in Y$, and we have $q_\nu = g(z_\nu)$ as well. Further,

$$q_\nu = f^{\mathbf{e}_\nu}(\boldsymbol{x}[\mathbb{G}] \restriction B'_\nu) = (h_\nu \cdot f^{\mathbf{e}})(\boldsymbol{x}[\mathbb{G}] \restriction B'_\nu) = f^{\mathbf{e}}(h_\nu^{-1}(\boldsymbol{x}[\mathbb{G}] \restriction B'_\nu))$$
$$= f^{\mathbf{e}}((h_\nu^{-1}(\boldsymbol{x}[\mathbb{G}]) \restriction B') = (h_\nu^{-1}(\boldsymbol{x}[\mathbb{G}]))(\eta) = (\boldsymbol{x}[\mathbb{G}])(\eta_\nu) = \boldsymbol{x}_{\eta_\nu}[\mathbb{G}],$$

where $\eta_\nu = h_\nu(\eta)$. Thus an uncountable sequence of the reals $z_\nu \in Y$ in $\mathbf{L}[\mathbb{G}]$ ($\nu < \omega_1^{\mathbf{L}}$) is defined, and it satisfies $g(z_\nu) = \boldsymbol{x}_{\eta_\nu}[\mathbb{G}]$ for all $\nu$. The ordinals $\eta_\nu = h_\nu(\eta)$ satisfy $\eta_\nu \geq \vartheta_\nu$ by the choice of $h_\nu$, since $\eta \in B' \setminus B$. Therefore there exist uncountably many pairwise different ordinals $\eta_\nu$ in $\mathbf{L}[\mathbb{G}]$. It follows that there exist uncountably many pairwise different generic reals $\boldsymbol{x}_{\eta_\nu}[\mathbb{G}]$. On the other hand, all reals $z_\nu$ belong to the countable set $Y$, and $\boldsymbol{x}_{\eta_\nu}[\mathbb{G}] = g(z_\nu)$, where $g$ does not depend on $\nu$. This is a *contradiction* required, and the theorem is proved. $\blacksquare$

*Proof of Theorem 15.6(ii).* Arguing under the requirements of Theorem 15.6, assume that, in $\mathbf{L}[\mathbb{G}]$, $p \in 2^\omega$ and $W \subseteq 2^\omega \times 2^\omega$ is a $\Sigma_\mathbb{n}^1(p)$ set whose cross-sections $W_x = \{y : \langle x, y \rangle \in W\}$ are at most countable. Every set $W_x$ is $\Sigma_\mathbb{n}^1(p, x)$, whence $W_x \subseteq \mathbf{L}[p, x]$ by Theorem 22.1. If $W_x \neq \varnothing$ then let $q_x$ be the $<_{px}$-least real in $W_x$, where $<_{px}$ is the Gödel well-ordering of $\mathbf{L}[p, x]$. The set $Q = \{\langle x, q_x \rangle : x \in 2^\omega \wedge W_x \neq \varnothing\}$ then uniformizes $W$. Moreover

$$\langle x, y \rangle \in Q \iff \langle x, y \rangle \in W \wedge \forall z \, (z <_{px} y \implies \langle x, y \rangle \notin W).$$

Therefore the set $Q$ belongs to $\Delta^1_{n+1}(p)$, or more exactly is the intersection of a $\Sigma^1_n(p)$ set and a $\Pi^1_n(p)$ set, because the Gödel well-orderings $<_{px}$ are well-known to be $\Sigma^1_2(p,x)$-definable uniformly in $p,x$. ∎ ∎ Theorems 2.2 and 2.1

**23. Open questions.** Surely there are open questions, concerning other problems in descriptive set theory, where one wants to control the least projective level of a counterexample. Of those, we mention the following two problems.

PROBLEM 1. Prove results similar to our Theorems 2.1 and 2.2, but without the restriction to sets with countable cross-sections, in items (ii) of both theorems.

PROBLEM 2. Uri Abraham [1] defined a generic extension $\mathbf{L}[a]$ of $\mathbf{L}$ by a real $a \in 2^\omega$ such that $a$ codes a collapse of $\aleph^{\mathbf{L}}_1$ and $a$ is a $\Pi^1_2$ singleton in $\mathbf{L}[a]$. Given $n \geq 3$, one may look for a model $\mathbf{L}[a]$, $a \in 2^\omega$, such that it holds in $\mathbf{L}[a]$ that $a$ is a $\Pi^1_n$ singleton that still collapses $\aleph^{\mathbf{L}}_1$, but no $\Sigma^1_n$ real collapses $\aleph^{\mathbf{L}}_1$. We solved this in the positive (to appear elsewhere), but a similar problem for Namba-collapse functions (yielding the cofinality of $\aleph^{\mathbf{L}}_2$ to $\omega$ without collapsing $\aleph_1$) remains open.

In this context, we mention a similar result, recently obtained in [22]. There is a generic extension $\mathbf{L}[a]$ of $\mathbf{L}$, by a real $a \in 2^\omega$, such that the following is true in $\mathbf{L}[a]$: the $\mathsf{E}_0$-equivalence class of $a$ is a (countable) lightface $\Pi^1_n$ set containing no ordinal-definable elements, but every countable $\Sigma^1_n$ set belongs to $\mathbf{L}$, hence consists of ordinal-definable elements.

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Vladimir Kanovei, Vassily Lyubetsky
Institute for Information Transmission Problems, IITP RAS
Bolshoy Karetny per. 19, build. 1
Moscow 127051, Russia
E-mail: kanovei@googlemail.com
        lyubetsk@iitp.ru