LEIBNIZ VERSUS ISHIGURO: CLOSING A QUARTER CENTURY OF SYNCATEGOREMANIA

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Did Leibniz exploit infinitesimals and infinities à la rigueur or only as shorthand for quantified propositions that refer to ordinary Archimedean magnitudes? Hidé Ishiguro defends the latter position, which she reformulates in terms of Russellian logical fictions. Ishiguro does not explain how to reconcile this interpretation with Leibniz's repeated assertions that infinitesimals violate the Archimedean property (i.e., Euclid's *Elements*, V.4). We present textual evidence from Leibniz, as well as historical evidence from the early decades of the calculus, to undermine Ishiguro's interpretation. Leibniz frequently writes that his infinitesimals are useful fictions, and we agree, but we show that it is best not to understand them as logical fictions; instead, they are better understood as pure fictions.

1. Logical Fictions

If a publisher were to announce to the public that in addition to its fiction titles, it offers a variety of cookbooks, no one would interpret this as meaning that the fiction titles turn out to be cookbooks in disguise when their content is properly clarified and made explicit. Yet when Leibniz announced that "il ne faut pas s'imaginer que la science de l'infini est . . . reduite à des fictions; car il reste tousjours un infini syncategorematique," Hidé Ishiguro proposed just this

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type of interpretation of Leibnizian fictions in terms of a Weierstrassian cookbook (Leibniz 1702, 93).[1]

Twenty-five years ago Ishiguro presented her interpretation of Leibnizian infinitesimals as logical fictions (Ishiguro 1990, chap. 5). Ishiguro's interpretive strategy employs what Russell called logical or symbolic fictions (Russell 1919, 45 and 184).

Ishiguro's analysis has not been seriously challenged. The situation has reached a point where the literature contains statements of Ishiguro's hypothesis as fact, without any further attribution, as follows: "Robinson's infinitesimal is a static quantity, whereas Leibniz's infinitesimals are 'syncategorematic,' i.e., they are as small as is necessary, such that there is always a quantity that is smaller than the smallest given quantity" (Duffy 2013, 15). Such a syncategorematic reading of Leibnizian infinitesimals is endorsed by Leibniz scholars Arthur, Goldenbaum, Knobloch, Levey, Nachtomy, and others, as detailed in section 2.2 below. Recent advances in Leibniz scholarship suggest the time has come to reevaluate Ishiguro's interpretation. This text presents a number of difficulties for the thesis that Leibnizian infinitesimals are logical fictions.

1.1. Defending Leibniz's Honor

The context for Ishiguro's analysis was the general sense that no appeal to infinitesimals *à la rigueur* could stand philosophical scrutiny. More specifically, her reading is based on the premise that prior interpretations of Leibnizian infinitesimals, in the spirit of the infinitesimals of Bernoulli and Euler, must surely involve confusion or even logical inconsistency. This premise is spelled out in the title of her text "La notion dite confuse de l'infinitésimal chez Leibniz," an early version of her chapter 5 (Ishiguro 1986). As she writes there, "This is because the concept of infinitesimal was seen as being *confused*" (Ishiguro 1990, 79; emphasis added). Furthermore, "the second kind of critic acknowledges that Leibniz was interested in foundational issues, but after examination sees basic *inconsistencies* in his views" (80; emphasis added).

Thus, Ishiguro purports to defend Leibniz's honor as an unconfused and consistent logician by means of her syncategorematic reading. Meanwhile, in the first edition of her book, Ishiguro wrote: "Leibniz's philosophy of logic and language makes far more sense in every aspect than has generally been thought,

1. This passage is discussed in more detail in sec. 7.1.

let alone that his thought is more coherent than Russell allowed" (1972, 16). We argue that such an appreciation of Leibniz applies equally well to his infinitesimal calculus in the spirit of Bernoulli and Euler. Ishiguro goes on to write: "In many respects, it is much less dated than the theories of Locke and Berkeley, and even of Kant" (16).

According to Ishiguro, the superiority of Leibniz's thought over that of Locke and Berkeley manifests itself also in Leibniz's rejection of empiricism.[2] We similarly believe that Leibniz was not confused and likewise intend to defend his honor, in this case against Ishiguro's reading. We will see that, at a few key junctures, Ishiguro is forced to defend her reading by attributing confusion to Leibniz (see secs. 6.1 and 6.3). On at least one occasion, Ishiguro misrepresents what Leibniz wrote so as to buttress her position (see sec. 6.2). We argue that the appeal to logical fictions is neither necessary to defend Leibniz's honor nor warranted in view of the actual content of Leibniz's mathematics and philosophy.

1.2. Categorematic versus Syncategorematic

According to Ishiguro, expressions like *dy*/*dx*, which appear to refer to infinitesimals, are not in fact referring, denoting, or categorematic, expressions. Rather, they are *syncategorematic* expressions, namely, expressions that disappear when the logical content of the propositions in which they occur is properly clarified and made explicit. Writes Ishiguro: "The word 'infinitesimal' does not designate a special kind of magnitude. In fact, it does not designate at all" (Ishiguro 1990, 83).[3] A few pages later, she clarifies the nature of her nondesignating claim in the following terms: "we can paraphrase the proposition with a universal proposition with an embedded existential claim" (87).

2. Thus, Leibniz's disagreements with empiricism are mentioned in the final paragraph of Ishiguro (1972, 45): "[Leibniz's] disagreement with many of the views of the empiricists, as with those of the Cartesians, sprang from his belief that their theories failed to account for the complex facts which fascinated him, whether these were about the language we have or about the concepts we use." These comments on empiricism and Cartesianism appeared at the end of sec. 6, titled "Concepts Resolvable at Infinity," in the final chap. 7, titled "Necessity and Contingency." Ishiguro's sec. 6 is still present in the second edition (1990), although "Necessity and Contingency" is now chap. 9 rather than 7 (this is due in part to the addition of chap. 5, seeking to reduce infinitesimals to quantified propositions). The comment on empiricism and Cartesianism disappeared from the second edition, but here Ishiguro writes that Leibniz's reasoning "is not of an empiricist kind like that of Berkeley" (85).

3. Ishiguro uses *designate* as an intransitive verb, and similarly for the verbs *denote* and *refer*. A term is said not to refer when the term does not actually refer to anything but rather is awaiting a clarification of the logical content of the sentence it occurs in, which would make the term disappear. An example is provided by Weierstrass's use of the term *infinitesimal* as discussed in sec. 2.1.

In conclusion, "Fictions [such as Leibnizian infinitesimals] are *not entities* to which we refer. . . . They are correlates of ways of speaking which can be reduced to talk about more standard kinds of entities" (Ishiguro 1990, 100; emphasis added). Such fictions (which are not entities) are exemplified by Leibnizian infinitesimals, in Ishiguro's view. Her contention is that, when Leibniz talked about infinitesimals, what he really meant was a certain quantified proposition, or more precisely a quantifier-equipped proposition. In short, Leibniz was talking about ordinary numbers. For the seventeenth-century context, see Alexander (2014). Ishiguro does mention "Leibniz's followers like Johann Bernoulli, de l'Hospital, or Euler, who were all brilliant mathematicians rather than philosophers" (1990, 79–80) but then goes on to yank Leibniz right out of his historical context by claiming that their modus operandi "is prima facie a strange thing to ascribe to someone who, like Leibniz, was obsessed with general methodological issues, and with the logical analysis of all statements and the well-foundedness of all explanations" (80).

Having thus abstracted Leibniz from his late seventeenth-century context, Ishiguro proceeds to insert him in a late nineteenth-century Weierstrassian cookbook. Such an approach to a historical figure would apparently not escape Unguru's censure: "It is . . . a historically unforgiveable sin … to assume wrongly that mathematical equivalence is tantamount to historical equivalence" (1976, 783). Ishiguro seems to have been aware of the problem, and at the end of chapter 5 she tries again to explain—"why I believe that Leibniz's views on the contextual definition of infinitesimals is [*sic*] different from those of other mathematicians of his own time who sought for operationist definitions for certain mathematical notions"—but with limited success (1990, 99).

2. Testing the Limits of Syncategorematics

We take it that 'infinitesimal' expressions do designate insofar as our symbolism allows us to think about infinitesimals. It should be emphasized that our contention that a Leibnizian infinitesimal does designate does not imply that it designates entities on a par with monads, material objects, or ideal entities. While *infinitesimal* is a designating expression for Leibniz, it designates a fictional entity. Likewise, for Leibniz, *imaginary quantity* designates a fictional entity (see sec. 7.5). The literature contains a considerable amount of confusion on this subject, as in the following quotation: "The use of fictitious quantities could lead to the erroneous idea of objects whose existence is assured by their very definition and therefore to ascribing a modern conception to Leibniz. In reality, what finds its foundation in Nature cannot be created by the human mind by means of a definition" (Ferraro 2008, 36). Now the matter of creating by definition is a tricky one. Leibniz certainly denies that definitions carry existential commitments. In Leibniz, infinitesimals are created at the syntactic level by postulation, which has a subtle relation to existence. This must be so, since the difficulty Ferraro perceives arises equally for real numbers. The article (Leibniz 1695a, 322) introduces infinitesimals specifically by invoking a definition, namely, Euclid V.4, and postulating that infinitesimals are entities that fail to satisfy the latter. In order to test the range of applicability of Ishiguro's syncategorematic reading, we consider the following two extreme cases.

2.1. Weierstrass on Infinitesimals

On the one hand, there does exist a context in which Ishiguro's logical fiction hypothesis may be on solid ground. On occasion, Weierstrass mentions an infinitesimal definition of continuity. This is Cauchy's (1821, 34) original definition of continuity of a function $y = f(x)$: "infinitesimal x-increment always produces an infinitesimal change in y." Thus, Weierstrass wrote: "Finally, once the concept of the infinitely small has been grasped correctly, one can define the concept of the continuity of a function in the vicinity of a as follows: that infinitely small changes in the arguments correspond to infinitely small changes in the value of the function in the vicinity of a" (1886/1989, 74).[4] It may be reasonable to conjecture that when Weierstrass refers to an infinitesimal, he always means (unlike Leibniz, on our reading) a kind of logical fiction. Here an infinitesimal is shorthand for a longer paraphrase expressed by a proposition whose quantifiers range over ordinary real numbers, namely, the sort of proposition that typifies Weierstrass's contribution to the foundations of analysis.

On the other hand (and at the other extreme), an infinitesimal is not meant to be a shorthand for a quantified paraphrase in the context of modern infinitesimal frameworks such as those of Robinson (1961), Bell (2006), or Kock (2006). Note that Robinson as a formalist distanced himself from Platonist and foundationalist views: "mathematical theories which, allegedly, deal with infinite totalities do not have any detailed . . . reference" (1975, 42).

Ishiguro's argument is based on first philosophical principles (rather than on historical analysis or careful textual study) that are so general that, while it

4. "Endlich kann man, den Begriff des unendlich Kleinen richtig gefaßt, den Begriff der Stetigkeit einer Funktion in der Nähe von a auch dadurch definieren, daß unendlich kleinen Änderungen der Argumente unendlich kleiner Änderungen des Funktionswertes in der Nähe von a entsprechen sollen."

might apply to Weierstrass, it is difficult to see what would prevent her from applying it to Robinson, as well. Yet scholars agree that Robinson's infinitesimals are not logical fictions, nor is his continuum Archimedean.

2.2. Syncategorematic versus Fictionalist

The syncategorematic interpretation of Leibnizian infinitesimals is the starting point of much recent Leibniz scholarship. Leading Leibniz scholar E. Knobloch writes: "To my knowledge most of the historians of mathematics are convinced that Leibniz used an Archimedean continuum: Leibniz himself referred to the Greek authority in order to justify his procedure" (private communication with Knobloch, December 29, 2014). Writes Goldenbaum: "That Leibniz as a mature mathematician and philosopher did not take infinitesimals to be real entities, but rather as finite quantities, was clarified as early as 1972 by Hidé Ishiguru [*sic*]" (2008, 76 n. 59).

Rabouin (2015, n. 25) similarly endorses Ishiguro's (1990) chapter 5. Both Levey (2008) and Arthur (2008, 20; 2013, 554) take Ishiguro's interpretation as settled and have concentrated, instead, on demonstrating that Leibniz embraced the syncategorematic interpretation of infinitesimals as early as 1676. The following commentary, from Levey, is typical: "By April of 1676, with his early masterwork on the calculus, *De Quadratura Arithmetica*, nearly complete, Leibniz has *abandoned an ontology of actual infinitesimals* and adopted the syncategorematic view of both the infinite and the infinitely small as a philosophy of mathematics and, correspondingly, he has arrived at the official view of infinitesimals as fictions in his calculus" (2008, 133; emphasis added). Nachtomy chimes in: "Richard Arthur makes a very convincing argument that Leibniz's syncategorematic view of infinitesimals was developed in the very early 1670s and matured in 1676" (2009).

We do not intend to disagree with Levey and others that Leibniz may have "abandoned an ontology of actual infinitesimals" early on. However, we object to the conflation of the views of the syncategorematicist and the fictionalist. The syncategorematic interpretation is a fictionalist interpretation, to be sure, but the converse is not the case.

In what follows, we demonstrate that Leibniz understood this point and had good reason to embrace a different variety of fictionalism, which we call *pure* fictionalism.[5] Modern exponents of this variety of fictionalism include Hilbert and Robinson (see Katz and Sherry 2013).

5. Perhaps we may be allowed to quote Leibniz's own description of his method: "My arithmetic of infinites is pure, Wallis' is figurate" (Arithmetica infinitorum mea est pura, Wallisii figurata; Leibniz

That Leibniz considered an alternative version of fictionalism will come as a surprise mainly to scholars whose outlook presumes that the epsilon-delta style of analysis, promoted by the "triumvirate" of Cantor, Dedekind, and Weierstrass (see Boyer 1949, 298), is the embodiment of inevitable progress climaxing in the establishment of the foundations of real analysis purged of infinitesimals. Related issues are explored in Kanovei et al. (2015) and Katz and Kutateladze (2015).

2.3. Summary of Ishiguro's Hypothesis

According to Ishiguro, Leibniz's conception of continuity (i.e., the continuum) is Archimedean. On the syncategorematic reading, talk about infinitesimals involves only expressions that do not denote anything. The position as expressed in Ishiguro (1990, chap. 5) can therefore be summarized in terms of the following three contentions.

1. Taking Leibnizian infinitesimals at face value requires one to see Leibniz as confused (1990, 79) or inconsistent (80).
2. A term that seems to express a Leibnizian infinitesimal does not actually designate, denote, or refer and is a logical fiction.
3. Leibniz's continuum is Archimedean.

None of these can be sustained in light of Leibniz's philosophical and mathematical texts.

3. Analysis of Ishiguro's Contentions

Let us analyze Ishiguro's hypothesis as summarized in section 2.3. Ishiguro's contention 1 concerns scholars like Boyer and Earman (see Ishiguro 1990, 80). However, a perusal of their work reveals that the ultimate source of the inconsistency claim is Berkeley's *departed quantities*. Thus, Ishiguro's contention 1 echoes Berkeley's claim that inconsistent properties have been attributed to dx (i.e., $(dx \neq 0) \wedge (dx = 0)$).[6] However, Berkeley's claim ignores Leibniz's generalized relation of equality (see sec. 3.2).

1672, 102), as translated in Beeley (2008, 46). Note that the text in question, *De progressionibus et de arithmetica infinitorum*, predates the *Arithmetic Quadrature*. What Leibniz meant by *figurata* is not entirely clear, nor does the context offer an indication. The most likely interpretation seems to be that Wallis relied on induction from geometric figures, which Leibniz's method did not require.

6. Berkeley's critique was dissected into its logical and metaphysical components in Sherry (1987). The logical criticism concerns the alleged inconsistency expressed by the conjunction

3.1. Of *dx* and (*d*)*x*

Ishiguro does not appear to be an attentive reader of Bos, and in fact she misrepresents his position (1990, 81). She quotes Bos to the effect that Leibniz eventually introduced the finite (assignable) differentials. For these he used the notation $(d)x$ in place of dx. These satisfy the equality on the nose: $(d)y = L(d)x$, where L is what we would call today the derivative. Bos does say that. However, Ishiguro further implies that, according to Bos, these $(d)x$'s completely replaced the earlier dx's. She then goes on to disagree with her straw man Bos by claiming that Leibniz never changed his mind about infinitesimals (i.e., that they were always logical fictions). Ishiguro writes: "Bos talks (perhaps naturally as a post-Robinsonian) as if it is quite clear what it means for a magnitude to be infinitely small, and that Leibniz first assumed the existence of such things" (1990, 81). Bos may well be surprised to find himself described as a post-Robinsonian, especially given what he wrote about Robinson (Bos 1974, app. 2).

Contrary to Ishiguro, Bos never asserted that the dx's disappeared with the introduction of $(d)x$'s. Bos merely reports that Leibniz introduced the additional concepts $(d)x$, not that they completely replaced the dx's, which they certainly never did. Thus, the late piece *Cum Prodiisset* (Leibniz 1701a/1846) features both the $(d)x$'s and the dx's, as well as the crucial distinction between *assignable* and *inassignable*: "although we may be content with the assignable quantities $(d)y$, $(d)v$, $(d)z$, $(d)x$, etc., . . . yet it is plain from what I have said that, at least in our minds, the unassignables [*inassignables* in the original Latin] dx and dy may be substituted for them by a method of supposition even in the case when they are evanescent" (as translated in Child 1920/2005, 153).

3.2. Law of Homogeneity

Bos notes that Leibniz already mentioned his law of homogeneity in *Nova Methodus* (Leibniz 1684). Leibniz explained the law in a 1699 letter to Wallis and gave the most detailed presentation in his 1710 piece mentioning the transcendental law of homogeneity (TLH) in the title (Leibniz 1699, 63; 1710a). The law involves, roughly, discarding higher-order terms.

Leibniz was using the relation of equality in a generalized sense of *equality up to*, as mentioned in his *Responsio* (see, e.g., sec. 6.2, sentences labeled 1 and 2). This means that dx does not turn out to be zero at the end of the calculation

$(dx \neq 0) \wedge (dx = 0)$, while the metaphysical criticism is fueled by Berkeley's empiricist doubts about entities that are below any finite perceptual threshold; see sec. 1.1 and n. 2.

but, rather, is discarded in an application of TLH. *Equality up to* undermines the claim of logical inconsistency (alleged by Berkeley) without a need to dip into a Weierstrassian cookbook with hidden quantifiers à la Frege.

An antecedent to the Leibnizian generalized equality is found in Fermat's relation of *adequality* (see Cifoletti 1990; Bair et al. 2013; Katz et al. 2013; Bascelli et al. 2014). Leibniz in fact mentions Fermat's method in the context of a discussion of the generalized notion of equality. Here Leibniz is objecting to Nieuwentijt's postulation that the square of an infinitesimal term should be exactly nothing:

> Quod autem in aequationibus Fermatianis abjiciuntur termini, quos ingrediuntur talia quadrata vel rectangula, non vero illi quos ingrediuntur simplices lineae infinitesimae, ejus ratio non est, quod hae sint aliquid, illae vero sint nibil, sed quod termini ordinarii per se destruuntur, hinc restant tum termini, quos ingrediuntur Iineae simplices infinite parvae, tum quos ingrediuntur harum quadrata vel rectangula: cum vero hi termini sint illis incomparabiliter minores, abjiciuntur. Quod si termini ordinarii non evanuissent, etiam termini infinitesimarum linearum non minus, quam ab his quadratorum abjici debuissent. (Leibniz 1695a, 323)

We translate this as follows:

> But the reason that in Fermat's equations, terms incorporating squares or similar products are discarded, but not those containing simple infinitesimal lines [i.e., segments], is not that the latter are something, whereas the former are, on the contrary, nothing; but rather that the ordinary terms cancel each other out, whence there then remain terms containing infinitely small simple lines, and also those containing their squares or products: but since the latter terms are incomparably smaller than the former, they are discarded. Because if the ordinary terms did not disappear, then the terms of the infinitesimal lines would have to be discarded no less than their squares or products.[7]

Here Leibniz describes Fermat's method in a way similar to Leibniz's own.

Ishiguro's contention 2 is based on the fictional status of Leibnizian infinitesimals. To be sure, Leibniz frequently describes his infinitesimals as "useful fictions." However, their fictional nature could merely mean to Leibniz that they

7. A French translation is in Leibniz (1989), 329.

lack reference to either a material object or an ideal entity, as Leibniz often writes, not necessarily that they are logical fictions as Ishiguro claims.

Ishiguro's contentions 2 and 3 amount to a claim of proto-Weierstrassian hidden quantifier ranging over ordinary Archimedean quantities. One of the passages claimed to support such a reading of Leibniz is a letter to Pinson dated August 29, 1701, where Leibniz writes: "in lieu of the infinite or infinitely small, we take quantities as great or as small as it is required so that the error would be less than the given error such that we do not differ from the style of Archimedes except in the expressions" (as translated in Tho 2012, 71; we have retained Tho's precise punctuation, which turns out to be significant; see below). This passage is an optimistic expression of, in Jesseph's phrase, a *grand programmatic statement* (see sec. 4). We will analyze this passage further in section 5.

3.3. Ishiguro, Bos, Robinson

Given Ishiguro's post-Robinsonian description of Bos (see sec. 3.1), it will prove instructive to examine the matter in more detail. On the one hand, Robinson famously argued for continuity between the Leibnizian framework and his own. On the other, Bos rejected such claims of continuity in his appendix 2: "the most essential part of non-standard analysis, namely the proof of the existence of the entities it deals with, was entirely absent in the Leibnizian infinitesimal analysis, and this constitutes, in my view, so fundamental a difference between the theories that the Leibnizian analysis cannot be called an early form, or a precursor, of non-standard analysis" (1974, 83). Bos's comment is not sufficiently sensitive to the dichotomy of practice (or procedures) versus ontology (or foundational account for the entities such as numbers). While it is true that Leibniz's calculus contains nothing like a set-theoretical existence proof, nonetheless there do exist Leibnizian procedures exploiting infinitesimals that find suitable proxies in the procedures in the hyperreal framework. In other words, there are close formal analogies between inference procedures in the Leibnizian calculus and the Robinsonian calculus (see Reeder [2013] for a related discussion in the context of Euler). The relevance of such hyperreal proxies is in no way diminished by the fact that set-theoretic foundations of the latter ("proof of the existence of the entities," as Bos put it) were obviously as unavailable in the seventeenth century as set-theoretic foundations of the real numbers.

In the context of his discussion of "present-day standards of mathematical rigor," Bos writes: "it is understandable that for mathematicians who believe that these present-day standards are final, nonstandard analysis answers positively the question whether, after all, *Leibniz was right*" (1974, 82, item 7.3; emphasis added). The context of the discussion makes it clear that Bos's criticism targets Robinson. If so, Bos's criticism suffers from a straw man fallacy, for Robinson specifically wrote that he did not consider set theory to be the foundation of mathematics, and being a formalist, he did not subscribe to the view attributed to him by Bos that "present-day standards are final." Robinson expressed his position on the status of set theory as follows: "an infinitary framework such as set theory . . . cannot be regarded as the ultimate foundation for mathematics" (1969, 45; see also Robinson 1966, 281).

Furthermore, contrary to Bos's claim, Robinson's goal was not to show that "Leibniz was right." Rather, Robinson sought to provide hyperreal proxies for the inferential procedures commonly found in Leibniz as well as Euler and Cauchy (for the latter, see, e.g., Borovik and Katz 2012). Leibniz's procedures, involving as they do infinitesimals and infinite numbers, seem far less puzzling in light of their B-track hyperreal proxies than from the viewpoint of the received A-track frameworks (see sec. 4).

Some decades later, Bos has distanced himself from his appendix 2 in the following terms (in response to a question from one of the authors of the current text): "An interesting question, what made me reject a claim some 35 years ago? I reread the appendix [i.e., app. 2] and was surprised about the self assurance of my younger self. I'm less definite in my opinions today—or so I think" (private communication with Bos, November 2, 2010). And he continues: "You're right that the appendix was not sympathetic to Robinson's view. Am I now more sympathetic? If you talk about 'historical continuity' I have little problem to agree with you, given the fact that one can interpret continuity in historical developments in many ways; even revolutions can come to be seen as continuous developments." While Bos acknowledges that his appendix 2 was "unsympathetic to Robinson's view," we must also point out that his opinions as expressed in appendix 2 were based on mathematical misunderstandings (particularly in connection with the transfer principle, as discussed in Katz and Sherry [2013], sec. 11.3), marring an otherwise excellent study of Leibnizian methodology to which we now turn (Bos 1974).

4. Grand Programmatic Statements

In his seminal study, Bos argued that Leibniz exploited two competing methods in his work, one Archimedean and the other involving the law of continuity and infinitesimals (see, e.g., Bos 1974, 57). In asserting that Leibniz exploited distinct methods in developing the calculus, we mean that he employed distinct conceptualizations of continua; that is, Leibniz employed different techniques for representing relations among continuously changing magnitudes. At a minimum, the techniques differed in the inferences they sanctioned and in the objects, whether ideal or fictional, which individual symbols in the technique purported to represent.

Such a dichotomy can be reformulated in the terminology of dual methodology as follows (see Katz and Sherry 2013). One finds both A-track (i.e., Archimedean) and B-track (Bernoullian, i.e., involving infinitesimals) methodologies in Leibniz.[8] In addition, Leibniz on occasion speculates as to how one might seek to reformulate B-track techniques in an A-track fashion.

Now there is no argument that such a pair of distinct methodologies, A and B, is present in Leibniz at the syntactic level. Ishiguro does not disagree with the apparent surface difference between them. What she argues, however, is that the syntactic difference is merely skin deep, so that once one clarifies the precise content of the sentences one arrives at the conclusion that at that deeper level, talk about infinitesimals (B-track) is merely shorthand for a quantified statement (A-track), a position we denote $B = A$ as shorthand for Ishiguro's contention that Leibnizian infinitesimals are logical fictions.

We argue that the syntactic difference in fact corresponds to a semantic difference. Each methodology has its respective ontology. The B method involves a richer numerical structure than the A method. Note that the structures have different ontological status. The B numerical structure involves pure fictions, while the A structure involves ideal entities. On this view, the A and B methods are truly distinct; that is, the Leibnizian infinitesimals are pure fictions, even though Leibniz occasionally argues that B should be paraphrasable in terms of A, given enough effort. This *hopefully paraphrasable* view can be denoted by the formula $B > A$, suggesting that what is involved in the B method is an extended number system including infinitesimals *à la rigueur* (as Leibniz put it with respect to "des infinis" in Leibniz 1702, 92), namely, what we refer to as a Bernoullian continuum. Jesseph expressed this aspect of Leibniz's position in the following terms: "Leibniz often makes *grand programmatic statements* to the effect that derivations which presuppose infinitesimals can always be re-cast as exhaustion proofs in the style of Archimedes. But Leibniz never, so far as I know, attempted anything like a general proof of the eliminability of the infinitesimal, or ordered anything approaching a universal scheme for re-writing

8. Scholars attribute the first systematic use of infinitesimals as a foundational concept to Johann Bernoulli. While Leibniz exploited both infinitesimal methods and "exhaustion" methods (usually interpreted in the context of an Archimedean continuum, but see n. 10), Bernoulli never wavered from the infinitesimal methodology. To note the fact of such systematic use by Bernoulli is not to say that Bernoulli's foundation is adequate or that it could distinguish between manipulations with infinitesimals that produce only true results and those manipulations that can yield false results.

the procedures of the calculus in terms of exhaustion proofs" (2008, 233; emphasis added). The most basic difference between the positions represented respectively by $B = A$ and $B > A$ is that the former implies that the background continuum of both the A method and the B method is Archimedean, whereas the latter recognizes a genuinely enriched (Bernoullian) continuum in the B method.

We grant that for any given Leibnizian passage discussing the relation between A method and B method, a plausible case can be made for either $B = A$ or $B > A$, given sufficient ingenuity. How can a scholar determine which interpretation is truer to Leibniz's intentions? In the next few sections, we present context-specific clues in Leibniz that would allow one to choose between the two interpretations.

5. Truncation Manipulations

Returning to the passage from the letter to Pinson quoted in section 3, one discovers that Tho truncated the passage to make it fit Ishiguro's analysis of Leibnizian infinitesimals. The full passage does not fit so well: "Car au lieu de l'infini ou de l'infiniment petit, on prend des quantités aussi grandes et aussi petites qu'il faut pour que l'erreur soit moindre que l'erreur donnée, de sorte que l'on ne differe du style d'Archimede que dans les expressions qui sont plus directes dans Nostre methode, et plus conforme à l'art d'inventer" (Leibniz 1701b, 96).[9] The conclusion of the passage, namely, the clause concerning the expressions "qui sont plus directes dans Nostre methode, et plus conforme à l'art d'inventer" was omitted from Tho's translation cited in section 3. This conclusion clearly indicates that Leibniz's (B-track) method, where the expressions are "plus directes," is distinct from the (A-track) "moindre que l'erreur donnée" paraphrase thereof. Leibniz's expression "plus directes" suggests a distinct method rather than merely a shorthand. Thus, Leibniz is following a distinct strategy that employs an enriched continuum.[10] The specific clues

9. We have retained Leibniz's spelling, which differs slightly from modern French spelling. We provide an English translation as found in Jesseph (2008, 229): "For in place of the infinite or the infinitely small we can take quantities as great or as small as is necessary in order that the error will be less than any given error. In this way we only differ from the style of Archimedes in the expressions, which are more direct in our method and better adapted to the art of discovery."

10. In the context of Leibniz's reference to Archimedes, it should be noted that there are other possible interpretations of the exhaustion method of Archimedes. The received interpretation, developed in Dijksterhuis (1956/1987), is in terms of the limit concept of real analysis. However, in the seventeenth century, Wallis developed a different interpretation in terms of approximation by infinite-sided polygons (1685, 280–90). The ancient exhaustion method has two components: (1) *geometric construction*, consisting of approximation by some simple figure, e.g., a polygon or a line built of

contained in this particular passage from Leibniz favor the $B > A$ reading over the $B = A$ reading (see sec. 4).

6. Incomparables

Let us examine Ishiguro's interpretation of Leibniz's notion of the *incomparably small*.

6.1. Misleading and Unfortunate

Ishiguro claims "the incomparable magnitude is not an infinitesimal magnitude" and continues (1990, 87): "It is misleading for Leibniz to call these magnitudes incomparably small. What his explanation gives us is rather that a certain truth about the existence of *comparably* smaller magnitudes gives rise to the notion of incomparable magnitudes, not incomparably smaller magnitudes. If magnitudes are incomparable, they can be neither bigger nor smaller" (87–88). She reiterates this claim in another sentence, regretting Leibniz's choice of unfortunate terminology: "As we have already mentioned, the unfortunate thing about Leibniz's vocabulary here is that he moves from incomparable to incomparably small or incomparably smaller (*incomparabilitier parva* or *incomparabilitier minor*), when smaller is already a notion involving comparison" (88). For all her professed good intentions of defending Leibniz's honor (see sec. 1), Ishiguro ends up being forced to defend her interpretation by tarnishing that honor. She reproaches him for employing purportedly "misleading" and "unfortunate" terminology in the context of incomparables. In fact, Leibniz's terminology for incomparables appears felicitous when the latter are interpreted as pure rather than logical fictions.

Ishiguro appears to claim that talk about incomparability excludes the relation of being smaller. Note, however, that the term *incomparable* can be used in two distinct senses:

1. it can refer to things that cannot be compared because they are of a different nature, for example, a line and a surface;
2. it can refer to things of the same nature but not comparable because they are of a different order of magnitude, for example, an ordinary nonzero real number and an infinitesimal.

segments, and (2) *justification* carried out in the theory of proportion developed in *Elements* Book V. In the seventeenth century, mathematicians adopted the first component and developed alternative justifications.

Ishiguro seems to assume meaning 1 without any historical evidence. Seeking the meaning of incomparability, she speculates further: "The fact that we cannot add or subtract the quantities in question to make one quantity constitutes, it seems, the very criterion of their nonhomogeneity and hence of their incomparability" (1990, 88). Thus, on Ishiguro's reading, "*x* is incomparably smaller than *y*" means that *x* and *y* are not comparable at all, while their incomparability means that *x* cannot be added to *y*.

6.2. Evidence from *Responsio*

To buttress her interpretation, Ishiguro cites a few passages from Leibniz. The case of the 1695 *Responsio* to Nieuwentijt is particularly instructive here. Ishiguro writes: "Leibniz, in the reply to Nieuwentijt of 1695 cited earlier, also asserts that the magnitude of a line and a point of another line cannot be added, nor can a line be added to a surface, and he says that they are incomparable since only homogeneous quantities are comparable. (Leibniz writes that all homogeneous quantities are comparable in the Archimedean sense.)" (1990, 88). However, in the very same *Responsio* we find an account of incomparability rather different from Ishiguro's. Leibniz writes:

> [1] Caeterum aequalia esse puto, non tantum quorum differentia est omnino nulla, sed et quorum differentia est incomparabiliter parva; [2] et licet ea Nihil omnino dici non debeat, non tamen est quantitas comparabilis cum ipsis, quorum est differentia. [3] Quemadmodum si lineae punctum alterius lineae addas, vel superficiei lineam, quantitatem non auges. [4] Idem est, si lineam quidem lineae addas, sed incomparabiliter minorem. [5] Nec ulla constructione tale augmentum exhiberi potest. [6] Scilicet eas tantum homogeneas quantitates comparabiles esse, cum Euclide lib. 5 defin. 5 censeo, quarum una numero, sed finito multiplicata,[11] alteram superare potest. [7] Et quae tali quantitate non differunt, aequalia esse statuo, quod etiam Archimedes sumsit, aliique post ipsum omnes. (Leibniz 1695a, 322; numerals 1–7 added)

11. When Leibniz mentions a *number*, he is paraphrasing Euclid's definition. Now, Euclid does not mention that he is speaking of the line being added to itself finitely many times. But this point is essential for Leibniz, as it is not obvious that an infinitesimal added to itself infinitely many times might not get to be larger than a given finite magnitude. Therefore, Leibniz inserts the term that Euclid does not use and writes *sed finito*, literally "of course finite." Note that there is no number in Euclid's V.4 but rather a multitude. Thus, Leibniz reads Euclid's *multitude* as *number*, and to be precise modifies it by *finite*.

We translate this passage as follows:

1. Furthermore I think that not only those things are equal whose difference is absolutely zero, but also whose difference is incomparably small.
2. And although this [difference] need not absolutely be called Nothing, neither is it a quantity comparable to those whose difference it is.
3. It is so when you add a point of a line to another line or a line to a surface, then you do not increase the quantity.
4. The same is when you add to a line a certain line that is incomparably smaller.
5. Such a construction entails no increase.
6. Now I think, in accordance with Euclid Book V def. 5, that only those homogeneous quantities one of which, being multiplied by a finite number, can exceed the other, are comparable.
7. And those that do not differ by such a quantity are equal, which was accepted by Archimedes and his followers.

Here Leibniz employs the term *line* in the sense of what we would today call a *segment*. In sentence 3, Leibniz exploits the classical example with indivisibles (adding a point to a line does not change its length), so as to motivate a similar phenomenon for infinitesimals in sentence 4 (adding an infinitesimal line to a finite line does not increase its quantity), namely, his law of homogeneity (TLH) explained in more detail elsewhere (see sec. 3).

Referring to the passage we quoted, Ishiguro claims that "Leibniz, in the reply to Nieuwentijt . . . asserts that the magnitude of a line and a point of another line cannot be added, nor can a line be added to a surface" (1990, 88). On the contrary, Leibniz wrote in sentence 3 that they can indeed be so added, although the addition of a point to a line does not increase the line. Thus, addition is possible according to Leibniz, contrary to what Ishiguro claims Leibniz asserts. It is just that according to Leibniz such an addition does not result in an increase. What Leibniz actually wrote undermines Ishiguro's claim about incomparables, rather than supporting it.

Leibniz goes on to give a parallel example with infinitesimals in sentence 4. Here addition is again possible, whereas its result is unchanged in accordance with TLH (see sec. 3). Ishiguro somehow fails to mention the fact that Leibniz goes on to give an example with infinitesimals. This is not merely an instance of truncation (see sec. 5). Rather, it constitutes a misrepresentation of Leibniz's position.

6.3. Indivisibles, Infinitesimals, and Dimension

Leibniz clearly understood the difference between infinitesimals (of the same dimension as the quantities they modify) and indivisibles (of positive codimension), contrary to Ishiguro's suggestion that "the homogeneity of quantities in Leibniz . . . seems not to depend on a prior notion of a common dimension as in earlier mathematicians, since Leibniz wanted to free mathematics from geometrical intuitions" (1990, 88). The notion of "common dimension" is what distinguishes infinitesimals from indivisibles. Ishiguro's suggestion that Leibniz did not distinguish between indivisibles and infinitesimals by means of the notion of common dimension does no honor to Leibniz (see sec. 1).

The passage cited above clearly indicates what Leibniz means by comparable quantities (Leibniz 1695a, 322). Namely, x and y are comparable when the following condition is satisfied: $(\exists n \in \mathrm{N})(nx > y)$; that is, x, y do obey Euclid's definition V.4 as cited in Leibniz's sentence 6.[12] Leibniz defines incomparably small in terms of a violation of V.4. Thus, even though Leibniz eschews geometrical intuition, he is still able to distinguish indivisibles from infinitesimals.

6.4. Theory of Magnitudes

Since Leibniz explicitly refers to Euclid's definition V.4 in the *Responsio* (see sec. 6.2, sentence 6), let us turn to the theory of magnitudes as developed in Book V of the *Elements*. Euclid's magnitudes of the same kind (homogeneous quantities in Leibniz's terminology) can be formalized as an ordered additive semigroup with a total order, $\mathrm{M} = (M, +, <)$, characterized by the five axioms given below.

Beckmann (1967–68) and Błaszczyk and Mrówka (2013, 101–22) provide detailed sources for the axioms below in the primary source (Euclid; see also Mueller [1981], 118–48, which mostly follows Beckmann's development). Axiom E1 below interprets Euclid V.4:

E1. $(\forall x, y \in M)(\exists n \in \mathrm{N})(nx > y)$
E2. $(\forall x, y \in M)(\exists z \in M)(x < y \Rightarrow x + z = y)$
E3. $(\forall x, y, z \in M)(x < y \Rightarrow x + z < y + z)$
E4. $(\forall x \in M)(\forall n \in \mathrm{N})(\exists y \in M)(x = ny)$
E5. $(\forall x, y, z \in M)(\exists v \in M)(x : y :: z : v)$

12. Leibniz lists V.5 for Euclid's definition instead of V.4. In some editions of the *Elements* this definition does appear as V.5. Thus, Euclid (1660) as translated by Barrow in 1660 provides the following definition in V.V (the notation "V.V" is from Barrow's translation): "Those numbers are said to have a ratio betwixt them, which being multiplied may exceed one the other."

Comparable quantities can both be added to one another, and they are also subject to the relations *greater than* and *less than*. It follows from these axioms that for any $x, y \in M$ the following inequality holds: $y < y + x$. In the realm of incomparable quantities this inequality does not hold, even though incomparables can be added. Leibniz's claim can be formalized as the relation characterizing incomparable quantities.

6.5. *Elements Book VI on Horn Angles*

We turn next to Ishiguro's claim that incomparable quantities cannot be compared at all by means of the relations *greater than* or *less than*. In Book VI of the *Elements*, one finds that line segments form a semigroup of magnitudes of the same kind (M1), triangles form another (M2), and rectilinear angles form yet another (M3); there are other kinds of magnitudes in addition to the ones just mentioned (see e.g., Euclid 2007, VI.1, 2, 33). Euclid deals with two kinds of angles in the *Elements*: the first kind consists of rectilinear angles, while the second kind consists of angles cut out/formed by a line and an arc of a circle.[13] These two kinds of angles are compared in proposition III.16. Its thesis reads as follows: "A (straight-line) drawn at right-angles to the diameter of a circle, from its end, will fall outside the circle. And another straight-line cannot be inserted into the space between the (aforementioned) straight-line and the circumference. And the angle of the semi-circle is greater than any acute rectilinear angle whatsoever, and the remaining (angle is) less (than any acute rectilinear angle)" (translated by Fitzpatrick in Euclid 2007; see fig. 1, the accompanying diagram). Here "the remaining" angle, that is, the one formed by the arc *CA* and the tangent line *EA*, does not belong to the kind (i.e., species) of rectilinear angles M3. Euclid proves it to be less than any acute rectilinear angle.[14]

From the point of view of Greek mathematics, one can construct incomparable quantities x, y, meaning that they are not of the same kind and do not obey the Archimedean axiom, while at the same time the relation $x < y$ obtains. Here "x is incomparably smaller than y" means x is smaller than y and x, y are incomparable, which can be formalized as follows: $x < y$ and $y \nless y + x$. Thus, incomparable quantities can be compared by inequalities both according to Euclid and according to Leibniz.

13. In the seventeenth century such angles were called *horn angles*.

14. It is worth noting that there are angles cut out by a line and a curve in Leibniz's papers, and the phrase 'infinitely small angle' occurs many times (see, e.g., Child 1920/2005).

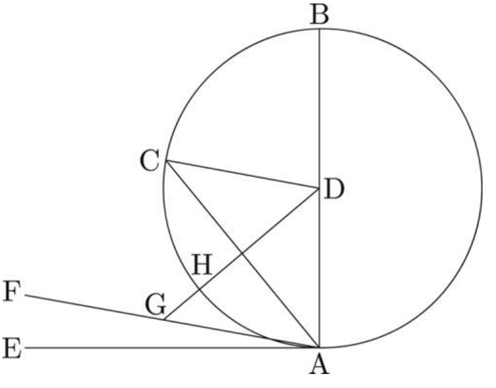

Figure 1. Source: Euclid (2007). *Elements*, III.16

7. Textual Evidence

In this section we will examine the textual evidence Ishiguro presents to support her claim that infinitesimals are logical fictions.

7.1. Letter to Varignon

Ishiguro's first piece of textual evidence in favor of her logical fiction hypothesis is a letter to Varignon dated February 2, 1702 (Leibniz 1702). Ishiguro does not provide a direct quotation but refers to Gerhardt (1850–63, 5:93), which contains what seems to be one of two occurrences in Leibniz of the term "syncategorematic infinite" (Ishiguro 1990, 82).[15]

The February 2, 1702, letter exploits the term *syncategorematic*. However, it is not obvious that Leibniz uses it in the same technical sense as Ishiguro. Leibniz discusses a number of examples, including imaginary numbers, dimensions beyond 3, and exponents that are not ordinary numbers, and then comments as follows: "Cependant il ne faut pas s'imaginer que la science de l'infini est degradée par cette explication et reduite à des fictions; car il reste tousjours un infini *syncategorematique*, comme parle l'ecole" (1702, 93; emphasis added). Leibniz then goes on to discuss the summation of a geometric series and points out that no infinitesimals need appear here. He is discussing a way of accounting for B methodology in terms of A methodology. The plain

15. The other one is in Leibniz's correspondence with des Bosses (Gerhardt 1875/1965, 2:314–15): "Datur infinitum syncategorematicum," etc.

meaning of the text, as mentioned in section 1, is that there is a pair of distinct methodologies, and if the fictions of the B method were found lacking, one could, at least in principle (recall Jesseph's remark concerning grand programmatic statements), fall back on an A-type syncategorematic paraphrase.

In analyzing this occurrence of the adjective *syncategorematic*, we again have the problem of investigating which of the two interpretations, $B = A$ or $B > A$, is more faithful to Leibniz's general philosophical outlook (see sec. 4). We will therefore look for additional clues in the letter that may favor one of the interpretations.

It is significant that the letter also contains a discussion of the law of continuity. Here Leibniz writes that the rules of the finite succeed in the infinite and vice versa: "il se trouve que *les règles du fini réussissent dans l'infini* comme s'il y avait des atomes (c'est à dire des éléments assignables de la nature) quoiqu'il n'y en ait point la matière étant actuellement sousdivisée sans fin; *et que vice versa les règles de l'infini réussissent dans le fini*, comme s'il y'avait des infiniment petits métaphysiques, quoiqu'on n'en n'ait point besoin" (1702, 93–94; emphasis added).

Leibniz goes on to mention the *souverain principe*: "et que la division de la matière ne parvienne jamais à des parcelles infiniment petites: c'est parce que tout se gouverne par raison, et qu'autrement il n'aurait point de science ni règle, ce qui ne serait point conforme avec la nature du *souverain principe*" (1702, 94; emphasis added).

A number of scholars, including Laugwitz (1992, 145) as well as Knobloch (2002, 67), identify the passage on 93–94 as an alternative formulation of the law of continuity; that is, "the rules of the finite succeed in the infinite, and conversely." Thus, recent scholarship has interpreted this passage as Leibniz's endorsement of the possibility of transferring properties from finite numbers to infinite (and infinitesimal) numbers and vice versa. For example, the usual rules governing the arithmetic operations and elementary functions should be obeyed by infinitesimals, as well.

Now if infinitesimal expressions were merely shorthand for talk about ordinary finite numbers or a sequence thereof, Leibniz's law of continuity would amount to the assertion that "each element in a sequence of ordinary numbers obeys the same rules as ordinary numbers."

But this seems anticlimactic and, moreover, too tautological to have been termed a law or a *souverain principe* by Leibniz. Leibniz writes further: "Et c'est pour cet effect que j'ay donné un jour des lemmes des incomparables dans les Actes de Leipzic, qu'on peut entendre comme on vent [*sic*], soit des infinis *à la rigueur*, soit des grandeurs seulement, qui n'entrent point en ligne de compte les unes au prix des autres" (1702, 92; emphasis added). Leibniz's pair of "soit"s in this remark indicates that there is a pair of distinct methodologies involved, as we elaborated in section 4. Note that Ishiguro quotes both a passage on page 91 preceding this remark and a passage on page 92 following the remark but fails to quote this crucial remark itself (1990, 86, 87; see our sec. 5 on truncation manipulations).

Thus, the letter offers support for the thesis that Leibniz thought infinitesimals (and infinite numbers) could stand on their own (*à la rigueur*), without paraphrase in terms of finite quantities. The letter fits well with the $B > A$ idea that on occasion Leibniz tried to argue optimistically that B-track techniques should be paraphrasable in terms of A-track ones (see sec. 4 on grand programmatic statements).

7.2. *Theodicy*

Ishiguro's second piece of textual evidence is from Leibniz's *Theodicy*: "every number is finite and assignable, every line is also finite and assignable. Infinites and infinitely small only signify magnitudes that one can take as big or as small as one wishes, in order to show that the error is smaller than the one that has been assigned" (*Theodicy*, sec. 70; Ishiguro 1990, 83). We have retained Ishiguro's precise punctuation, including the quotation marks. Note that her quotation marks close the citation without any punctuation mark at the end of the citation. There is no indication in Ishiguro that Leibniz's sentence does not end there, but rather continues. It is instructive to examine Leibniz's sentence in full: "Every number is finite and specific; every line is so likewise, and the infinite or infinitely small signify only magnitudes that one may take as great or as small as one wishes, to show that an error is smaller than that which has been specified, that is to say, that there is no error; or else by the infinitely small is meant the state of a magnitude at its vanishing point or its beginning, conceived after the pattern of magnitudes already actualized" (Leibniz 1710b; trans. Gutenberg Project). The closing phrase, "or else by the infinitely small is meant the state of a magnitude at its vanishing point or its beginning, conceived after the pattern of magnitudes already actualized," was truncated by Ishiguro. In this omission she is not without coconspirators: the same truncated passage appears in Ferraro (2008, 29) and Goldenbaum (2008, 76).

Leibniz's conclusion in section 70 suggests that there does exist a way of working with infinitesimals *à la rigueur*. This would presumably involve an enriched system of magnitudes, whose additional elements shared properties with the (already actualized) elements in the original system. Leibniz is being rather vague here, and it is hard to know what he means exactly by magnitudes being "conceived after the pattern of magnitudes already actualized," especially since section 70 is preceded by section 69 on free will and followed by section 71 on the Gospels, making it difficult to rely on the context for a clarification. However, our impression is that Ishiguro is not telling the full story here, for she observes: "As the *Theodicy* is a very late book (1710), it may be *thought* that this expresses a later-year shift to finitism brought about by *senility*. In order to see that this is not the case, let us trace some of the things Leibniz wrote on infinitesimals from his early years" (1990, 83; emphasis added)

To be sure, Ishiguro soon enough rejects her senility hypothesis. However, even a hypothesis that is ultimately rejected must have a grain of plausibility to it. Otherwise why would one want to consider it in the first place? Note that the Leibniz-Clarke correspondence is well regarded, and it comes at the very end of Leibniz's life. Leibniz died in 1716 before he had a chance to respond to Clarke's fifth letter.

Ishiguro's thought here seems at odds with her stated goal of defending Leibniz's honor (see sec. 1). Her thought seems to imply that Leibniz shifted to an infinitesimal-barring finitism in 1710. The truth is that, on the contrary, Leibniz was at that time at the height of his intellectual powers and was as committed as ever to developing the B methodology, including its foundations, as is evidenced by his extremely lucid 1710 text on the TLH analyzed in section 3 (Leibniz 1710a).

7.3. *Nova Methodus*

Ishiguro's third piece of textual evidence is drawn from *Nova Methodus* (Leibniz 1684, a text she misdates at 1685. She makes several dubious claims related to this work.

First, she alleges that in this text, differentials are "defined through the proportion of finite line segments" (Ishiguro 1990, 83). What Leibniz actually writes is as follows: "Now some right line taken arbitrarily may be called dx, and the right line which shall be to dx, as v (or w, y, z, resp.) is to VB (or WC, YD, ZE, respect.) may be called dv (or dw, dy, dz, resp.), or the differentials" (Leibniz 1684, 467). This passage in a notoriously (and deliberately) obscure work cannot qualify as a definition of *differential* and certainly offers no support for Ishiguro's claim that infinitesimal expressions are nonreferring. Leibniz scholars have argued that he had to conceal the use of infinitesimals in this publication to avoid the wrath of opponents: "The structure of the text [i.e., *Nova Methodus*], which was much more concise and complex than the primitive

Parisian manuscript essays, was complicated by the *need to conceal the use of infinitesimals*. Leibniz was well aware of the possible objections he would receive from mathematicians linked to classic tradition who would have stated that the infinitely small quantities were not rigorously defined, that there was not yet a theory capable of proving their existence and their operations, and hence they were not quite acceptable in mathematics" (Roero 2005, 49; emphasis added). This would account for the obscurities of Leibniz's discussion of differentials here, which offers no support at all for a syncategorematic reading of Leibnizian infinitesimals.

Furthermore, Ishiguro goes on to provide a syncategorematic interpretation of Leibniz's construction of a line through a pair of infinitely close points: "Leibniz writes that a tangent is found to be a straight line drawn between two points on a curve of infinitely small distance, or a side of a polygon of infinite angles. However, . . . infinitely small distances can be thought of as distances that can be taken smaller than any distances that are given" (1990, 84). This passage furnishes no explanation for the asymmetry of the two points involved in the received definition of the tangent line via secant lines (as discussed in sec. 8).

As we already mentioned in section 3, *Nova Methodus* contains the first mention of Leibniz's law of homogeneity, evidence in favor of infinitesimals *à la rigueur*. Thus, the clues contained in *Nova Methodus* support the $B > A$ reading.

7.4. *Responsio a Nieuwentijt*

Ishiguro's fourth piece of textual evidence is the 1695 response to Nieuwentijt published in *Acta Eruditorum* (Leibniz 1695a). She writes: "Leibniz explains that although he treats (*assumo*) infinitely small lines *dx* and *dy* as true quantities sui generis, this is just because he found them useful for reasoning and discovery. I take it that he is treating them as convenient *theoretical fictions* because using signs which looks [*sic*] as if they stand for quantities sui generis is useful" (Ishiguro 1990, 84; emphasis added).

In point of fact, theoretical fictions are on a par with what we refer to as pure fictions. What Ishiguro writes here undermines her syncategorematic interpretation and supports ours. The fact that this is what the passage means is demonstrated by the comparison with imaginaries, for which Leibniz has no syncategorematic account. The passage Ishiguro is referring to reads as follows: "Itaque non tantum lineas infinite parvas, ut *dx*, *dy*, pro quantitatibus veris in suo genere assumo, sed et earum quadrata vel rectangula *dxdx*, *dydy*, *dxdy*, idemque de cubis aliisque altioribus sentio, praesertim cum eas ad ratiocinandum inveniendumque utiles reperiam" (Leibniz 1695a, 322).[16]

According to Ishiguro, Leibniz says that he treats *dx* and *dy* as true quantities sui generis just because he found them useful. But is that really what he is saying? Leibniz refers to infinitesimals by the adjective *veris*, meaning "true" or "real." If his infinitesimals were logical fictions (i.e., merely shorthand for sequences of real values), what novelty would there be in emphasizing, as he does, that he includes infinitesimals (as well as those of higher order) among what he describes as true or real quantities? Why emphasize this point if infinitesimals were merely shorthand for sequences of what are already ordinary values drawn from an Archimedean system? Furthermore, why would he seek to buttress such a straightforward point by underscoring the usefulness of infinitesimals in reasoning and discovery?

A few lines earlier on page 322, Leibniz cites Euclid V.5 in a way similar to the 1695 letter to l'Hospital, indicating a violation of the Archimedean property (see sec. 8).[17] Remarkably, Leibniz uses the term *numerus infinitus*, meaning infinite number—rather than infinite quantity—here, blocking the option of interpreting it as a variable quantity increasing without bound.[18]

Leibniz not only speaks of two distinct methods but gives them names that suggest what his personal preferences are. Namely, Leibniz describes what we refer to as the A method as *reducendi via* (the way of reducing) and the infinitesimal method as *methodus directa* (the direct method).[19] It is instructive to analyze the relevant passage in detail. Leibniz writes in this *Responsio*: "Quoniam tamen methodus directa brevior est ad intelligendum et utilior ad inveniendum, sufficit cognita semel reducendi via postea methodum adhiberi, in qua incomparabiliter minora negliguntur, quae sane et ipsa secum fert demonstrationem suam secundum lemmata a me Febr. 1689 communicata" (Gerhardt 1850–63, 5:322). We translate this passage as follows: "But since the direct method [*methodus directa*] is shorter to understand and a more useful way of finding [i.e., discovering], it suffices, once the way of reducing [*reducendi via*] is known, to

16. This is translated as follows by Parmentier: "Ainsi au nombre des grandeurs réelles en leur genre, je ne compte pas seulement les lignes infiniment petites *dx*, *dy*, mais aussi leurs carrés ou leurs produits *dxdx*, *dxdy*, *dydy*, il en va de même d'après moi de leurs cubes et de leurs puissances supérieures, compte tenu notamment de la fécondité que j'y ai découverte dans les raisonnements et les inventions" (Leibniz 1989, 328).

17. This corresponds to V.4 in modern editions; see n. 12.

18. Parmentier in his 1989 French translation devotes a lengthy n. 30 on p. 325 to Leibniz's usage of the term *numerus* here (Leibniz 1989).

19. "The way of reducing" was rendered "cette démonstration régressive" in Parmentier's translation (Leibniz 1989, 327).

apply afterward the method in which quantities that are incomparably smaller are neglected, which in fact carries its own demonstration according to the lemmas that I communicated in February, 1689."

What emerges from this sentence is that there are two distinct methods: (A) "via reduction" and (B) a "direct method" using infinitesimals. The infinitesimal method is riskier but more powerful, and what Leibniz is pointing out is that having gained some experience with the traditional method so that one already knows what kind of results to expect, one can safely use the infinitesimal method that yields the same results but more efficiently. Leibniz points out that once the reductive method A has been used and understood, from that point onward one can systematically use the direct method B (which involves discarding infinitesimals), since it is quicker and more useful. These clues furnish further evidence in favor of the $B > A$ reading over the $B = A$ reading (see sec. 4).

7.5. The June 7, 1698, Letter to Bernoulli

Ishiguro's fifth piece of textual evidence is the June 7, 1698, letter to Bernoulli (Leibniz 1698).[20] She writes that Leibniz likens the status of infinitesimals to that of imaginary numbers in this letter (Ishiguro 1990, 84).

Since Ishiguro does not elaborate any further, it is difficult to see how this could be interpreted as a piece of evidence in favor of her logical fiction hypothesis, since in point of fact complex numbers could not (in Leibniz's day) be replaced by quantified paraphrases ranging over ordinary numbers, so complex numbers (or imaginary quantities, as Leibniz called them) are pure fictions par excellence. Leibniz repeatedly insisted (not merely in this letter to Bernoulli) on the analogy between the fictional status of infinitesimals and complex numbers. Meanwhile, Leibniz described imaginaries as having their *fundamentum in re* (basis in fact; Leibniz 1695b, 93). The comparison to complex numbers tends to undermine the logical fiction hypothesis concerning Leibnizian infinitesimals. This theme was explored more fully in Sherry and Katz (2012).

8. Euclid V.4, Apollonius, and Tangent Line

According to the letter to l'Hospital, Leibniz's infinitesimals violate Euclid V.4: "J'appelle *grandeurs incomparables* dont l'une multipliée par quelque nombre fini que ce soit, ne sçauroit exceder l'autre, de la même façon qu'Euclide la pris

20. Ishiguro gives the page range as 499–500. Actually the letter occupies four pages, 497–500.

dans sa cinquieme definition du cinquieme livre" (Leibniz 1695b, 288).[21] Note Leibniz's use of the term *grandeur* (i.e., magnitude) rather than the more ambiguous term *quantity*. A magnitude (e.g., 5 feet) is a level of a quantity (length). Here the option of interpreting this as shorthand for a variable quantity is not available, barring also a logical fiction reading. The definition Leibniz refers to is a variant of what is known today as the Archimedean property of continua. This indicates that Leibniz embraces what we refer to as a Bernoullian continuum (although certainly not a non-Archimedean continuum in a modern set-theoretic sense), contrary to Ishiguro's (1990) chapter 5.

Jesseph shows that strategies [Leibniz] employed in the attempt to show that such fictions are acceptable because the use of infinitesimals can ultimately be eliminated have to presume the correctness of an infinitesimal inference (i.e., inference-exploiting infinitesimals), namely, identifying the tangent line to a curve as part of the construction (2015). In the case of conic sections, this strategy succeeds because the tangents are already known from Apollonius. But for general curves (including transcendental ones treated by Leibniz), infinitesimals *à la rigueur* remain an irreducible part of the Leibnizian framework, contrary to Ishiguro's (1990) chapter 5.

In 1684, Leibniz wrote concerning the tangent line that to find a tangent is to draw a straight line, which joins two points of the curve having an infinitely small difference (1684). The definition of a tangent line as the line through a pair of infinitely close points on the curve poses a challenge to a proto-Weierstrassian reading. Such a reading involves having to fix one of the points and to vary the other and construct a sequence of secant lines producing the tangent line in the limit. In such a reading, one of Leibniz's points would be a genuine mathematical concept (the future point of tangency), while the other, merely a syncategorematic device or a shorthand for a sequence of ordinary values.

However, nothing whatsoever about Leibniz's wording would indicate that there is such an asymmetry between the two points, and on the contrary it implies a symmetry between them: either both denote, or neither denotes. Leibniz's definition of the tangent line is at odds with Ishiguro's (1990) chapter 5.

The most devastating blow to Ishiguro's (1990) chapter 5 is the hierarchical structure on the Leibnizian dx's, dx^2's, ddx's, and so on, ubiquitous in Leibniz's texts. One can replace dx by a sequence of finite values ε_n and furnish a concealed quantifier incorporated into a hidden proto-Weierstrassian limit notion so as to interpret dx as shorthand for the sequence $(\varepsilon_n : n \in \mathbb{N})$.

21. Leibniz actually refers to V.5; see n. 12.

However, one notices that $\lim_{n\to\infty}\varepsilon_n = 0$, as well as $\lim_{n\to\infty}\varepsilon_n^2 = 0$, and also unsurprisingly $\lim_{n\to\infty}(\varepsilon_n + \varepsilon_n^2) = 0$. Thus, the Leibnizian substitution $dx + dx^2 = dx$ in accordance with the TLH becomes a meaningless tautology $0 + 0 = 0$. To interpret it in a meaningful fashion, Ishiguro would have to introduce additional ad hoc proto-Weierstrassian devices with no shadow of a hint in the original Leibniz.[22]

9. Conclusion

Leibniz on occasion writes that arguments using infinitesimals (B-track terminology) could be paraphrased in terms of ordinary numbers drawn from an Archimedean number system (A-track terminology). The question we have investigated is what exactly is involved in such a paraphrase. Ishiguro argued that Leibnizian infinitesimals do not designate, so that when one clarifies the logical content of his propositions mentioning infinitesimals, the infinitesimals disappear and one is left with a suitable quantified proposition. Ishiguro's claim is that Leibnizian infinitesimals are logical fictions. We have argued that Leibnizian infinitesimals are pure fictions not eliminable by paraphrase.

This does not mean that Leibniz's infinitesimals are Robinson's infinitesimals; far from it. The well-known differences between them (Leibniz's continuum being nonpunctiform, whereas Robinson's is punctiform) should be approached from the viewpoint of the distinction between mathematical practice and the ontology of mathematical entities developed in Benacerraf (1965) and Quine (1968). What emerges from our analysis is that modern infinitesimal frameworks provide better proxies for understanding Leibnizian procedures and actual mathematical practice than the Weierstrassian framework (similarly punctiform, like Robinson's) Ishiguro seeks to read into Leibniz.

Ishiguro's syncategorematic reading is contrary to explicit Leibnizian texts, such as his 1695 *Responsio* to Nieuwentijt and letter to l'Hospital in which he writes that his differentials violate Euclid V.4, closely related to the Archimedean property of continua. Leibniz describes B-track methods as being direct and A-track methods as involving (indirect) reductio arguments, implying distinct methodologies. Leibniz repeatedly likens infinitesimals to imaginaries and at least once described the latter as having their *fundamentum in re* (basis in fact), suggesting that both are entities. In some cases, Ishiguro resorts to misrepresentation of what Leibniz wrote so as to buttress her position (see sec. 6.2 on the possibility of addition of incomparables).

22. For example, for every positive ε there exists a positive δ such that whenever dx is less than δ, the difference $|[(dx + dx^2)/dx]-1|$ is less than ε.

In view of all the difficulties with Ishiguro's reading, we can only conclude that the legitimate grounds for a "rehabilitation" (if any is needed) of Leibniz's infinitesimal calculus are to be found in the Leibnizian theory itself (including his TLH), rather than in Fregean quantifiers, Weierstrassian epsilontics, or Russellian logical fictions.[23]

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23. "If the Leibnizian calculus needs a rehabilitation because of too severe treatment by historians in the past half century, as Robinson suggests (1966, 250), I feel that the legitimate grounds for such a rehabilitation are to be found in the Leibnizian theory itself" (Bos 1974, 82–83).

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