Izvestiya: Mathematics **85**:6 1181–1219 *Izvestiya RAN: Ser. Mat.* **85**:6 164–204
DOI: https://doi.org/10.1070/IM8937

Models of set theory in which the separation theorem fails

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Abstract. We use a finite-support product of Jensen-minimal forcings to define a model of set theory in which the separation theorem fails for the projective classes $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$, for a given $n \geqslant 3$.

Keywords: separability, models, Jensen forcing, iteration.

§ 1. Introduction

The separation problem was introduced in descriptive set theory by Luzin [1]. In particular, he asked whether (in the modern notation for projective classes)

(I) any pair of disjoint $\mathbf{\Sigma}^1_n$-sets of reals can be separated by a $\mathbf{\Delta}^1_n$-set,

(II) the remainders of two $\mathbf{\Sigma}^1_n$-sets after removing their intersection can be separated by disjoint $\mathbf{\Pi}^1_n$-sets, and

(III) there are two disjoint $\mathbf{\Pi}^1_n$-sets not separable by a $\mathbf{\Delta}^1_n$-set.

He stressed the importance and difficulty of these problems.[1] Novikov ([2], Russian p. 279) characterized the separation problem as one of the three main problems of descriptive set theory, along with the measurability problem for $\mathbf{\Sigma}^1_2$-sets and the cardinality problem for $\mathbf{\Pi}^1_1$-sets. (See, for example, [3] for the last two problems.)

The problem is well known in descriptive set theory. In modern terms (see Moschovakis [4], Kechris [5]), the (*first*) *separation theorem* for a class Γ of pointsets (sets in Polish spaces) is the claim that any two disjoint sets in Γ (in the same space) can be separated by a set in $\Gamma \cap \Gamma^{\complement}$, where $\Gamma^{\complement}$ is the class of complements of Γ-sets. The *second separation theorem* for Γ claims that if X, Y are sets in Γ (in the same space), then the sets $X' = X \setminus Y$ and $Y' = Y \setminus X$ are separable by two disjoint sets in $\Gamma^{\complement}$. Thus the content of the problems (I), (II), (III) is as follows:

– does the (first) separation theorem hold for $\mathbf{\Sigma}^1_n$?

[1] 'L'un des problémes les plus importants de la théorie des ensembles projectifs et qui attend encore sa solution, est celui de leur *séparabilité*. On sait que deux ensembles analytiques quelconques sans point commun sont toujours séparables B. Il serait trés important de démontrer que deux ensembles (A_n) quelconques sans point commun sont séparables (B_n). De même, nous savons que si l'on supprime la partie commune á deux ensembles analytiques, les parties restantes sont séparables au moyen de deux complémentairs analytiques. La question se pose naturellement de savoir si ce principe subsiste quand on remplace les ensembles analytiques par (A_n) et les complémentaires analytiques par (CA_n). C'est un probĺeme que mérite d'attirer l'attention des analystes malgré sa difficulté. D'ailleurs, il importe de savoir s'il existe deux ensembles (CA_n) qui ne soient pas séparables (B_n).' Luzin [1], p. 289.

This paper was written with the support of RFBR (grant no. 20-01-00670).

AMS 2020 *Mathematics Subject Classification*. 03E15, 03E35.

– does the second separation theorem hold for $\mathbf{\Sigma}^1_n$?
– does the (first) separation theorem fail for $\mathbf{\Pi}^1_n$?
Both separation theorems hold for $\mathbf{\Sigma}^1_1$ according to Luzin [6], [1], but fail for $\mathbf{\Pi}^1_1$ according to Novikov [7], and these results were known before the publication of (the French original of) [1] in 1930. Somewhat later, Novikov [8] established that the picture changes at the second projective level: both separation theorems hold for $\mathbf{\Pi}^1_2$ but fail for $\mathbf{\Sigma}^1_2$.

At the same time Kuratowski [9] proved the *reduction theorem* for $\mathbf{\Sigma}^1_2$, that is, if X, Y are sets in $\mathbf{\Sigma}^1_2$, then there are disjoint sets $X' \subseteq X$ and $Y' \subseteq Y$ in the same class $\mathbf{\Sigma}^1_2$ with the same union $X' \cup Y' = X \cup Y$. Kuratowski also observed that Luzin's arguments in the proof of the separation theorem for $\mathbf{\Sigma}^1_1$ yield the reduction theorem for $\mathbf{\Pi}^1_1$. Generally, if the reduction theorem holds for a projective class Γ, then both separation theorems hold for the dual class $\Gamma^{\complement}$.

Thus the classical studies showed that the reduction theorem holds for the projective classes $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$ and fails for $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_2$ while the separation theorems hold for $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_2$ and fail for $\mathbf{\Pi}^1_1$, $\mathbf{\Sigma}^1_2$. Note the inversion between the first and second levels of the hierarchy.

Concerning the higher levels of the projective hierarchy, all attempts to solve the separation/reduction problems above the second level by the methods of classical descriptive set theory were fruitless until some extra set-theoretic axioms were added. In particular, according to Novikov [2] (see also Addison [10]), Gödel's *axiom of constructibility* $\mathbf{V} = \mathbf{L}$ implies that, for every $n \geqslant 3$, the reduction theorem holds for $\mathbf{\Sigma}^1_n$ and fails for $\mathbf{\Pi}^1_n$ while the separation theorems hold for $\mathbf{\Pi}^1_n$ and fail for $\mathbf{\Sigma}^1_n$, like at the second level. On the other hand, according to Addison and Moschovakis [11] and Martin [12], the *axiom of projective determinacy* **PD** implies that, for every $m \geqslant 1$, the reduction theorem holds for the projective classes $\mathbf{\Pi}^1_{2m+1}$, $\mathbf{\Sigma}^1_{2m+2}$ and fails for $\mathbf{\Sigma}^1_{2m+1}$, $\mathbf{\Pi}^1_{2m+2}$ while the separation theorems hold for $\mathbf{\Sigma}^1_{2m+1}$, $\mathbf{\Pi}^1_{2m+2}$ and fail for $\mathbf{\Pi}^1_{2m+1}$, $\mathbf{\Sigma}^1_{2m+2}$, like what happens at the first and second levels ($n = 0$ in this scheme). Moreover, according to Steel [13], it is true under the full *axiom of determinacy* **AD** that if the class Γ of pointsets is closed under some simple operation and is not self-dual (that is, $\Gamma \neq \Gamma^{\complement}$), then the reduction theorem holds for exactly one of the classes Γ, $\Gamma^{\complement}$ and the separation theorems hold for the other. Conversely, Steel [14] proved that a more special form of separability for $\mathbf{\Pi}^1_3$ implies some otherwise impossible connections between certain determinacy hypotheses. See also [15] for other relevant results.

These achievements still leave open a number of important questions about the status of the separation theorems for higher projective classes. For example, consider the following problem.

Problem 1.1 (Mathias [16] for $n = 3$). Given a number $n \geqslant 3$, is it consistent with **ZFC** that the (first) separation theorem fails for $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$?

Harrington solved this problem in the affirmative using a generic extension of $\mathbf{L}$ in which the (first) separation theorem fails for $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$. The solution was obtained by the technique of almost-disjoint forcing [17] and was sketched in unpublished handwritten notes [18], Part 2. This result was mentioned by Moschovakis [4], 5B.3, with reference to Harrington. Moreover, Harrington [18] suggested substantial changes in the construction of the generic extension, which supposedly lead to the failure of separation for the classes $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ with a given $n > 3$, or even for all n, but this generalization has never been published in detail.

Our goal here is to prove the following theorem, which indeed solves Problem 1.1 in the affirmative for any given $n > 3$, albeit by a method different from that used in [18].

Theorem 1.2. *Let* $\mathfrak{n} \geqslant 3$. *It is true, in a suitable generic extension of* $\mathbf{L}$, *that*

(i) *there is a pair of disjoint* $\mathit{\Pi}^1_{\mathfrak{n}}$*-sets* $X, Y \subseteq 2^\omega$ *not separable by disjoint* $\mathbf{\Sigma}^1_{\mathfrak{n}}$*-sets, whence the separation theorem fails for* $\mathbf{\Pi}^1_{\mathfrak{n}}$ *and* $\mathit{\Pi}^1_{\mathfrak{n}}$;

(ii) *there is a pair of disjoint* $\mathit{\Sigma}^1_{\mathfrak{n}}$*-sets* $X, Y \subseteq 2^\omega$ *not separable by disjoint* $\mathbf{\Pi}^1_{\mathfrak{n}}$*-sets, whence the separation theorem fails for* $\mathbf{\Sigma}^1_{\mathfrak{n}}$ *and* $\mathit{\Sigma}^1_{\mathfrak{n}}$.

§ 2. Outline of the proof

Given an $\mathfrak{n} \geqslant 3$, we define a sequence of forcing notions $\mathbb{P}_\xi$, $\xi < \omega_1$, in $\mathbf{L}$ whose finite-support product $\mathbb{P} = \prod_\xi \mathbb{P}_\xi$ satisfies the countable antichain condition CCC (abbreviation of Countable Chain Condition) and adjoins a sequence of generic reals $x_\xi \in 2^\omega$ that are independent of each other in the sense that

(I) if $\eta < \omega_1$, then (a) the submodel $\mathbf{L}[\langle x_\xi \rangle_{\xi \neq \eta}]$ contains no reals $\mathbb{P}_\eta$-generic over $\mathbf{L}$ and, moreover, (b) x_η is the only real in $\mathbf{L}[\langle x_\xi \rangle_{\xi < \omega_1}]$, $\mathbb{P}_\eta$-generic over $\mathbf{L}$,

and the following definability property holds:

(II) the relation "$x \in 2^\omega$ is a real $\mathbb{P}_\xi$-generic over $\mathbf{L}$" (with arguments x, ξ) is of class $\mathit{\Pi}^1_{\mathfrak{n}-1}$ in the whole extension and any of its submodels.

Then, to construct an example for Theorem 1.2, (i), we can generically split ω_1 into three unbounded sets[2] $\omega_1 = \Omega_1 \cup \Omega_2 \cup \Omega_3$, put $\Delta = \{2\nu \colon \nu \in \Omega_1 \cup \Omega_3\} \cup \{2\nu+1 \colon \nu \in \Omega_2 \cup \Omega_3\}$ and prove that Ω_1 and Ω_2 (more precisely, the sets of codes in 2^ω for the ordinals in Ω_1 and Ω_2) are disjoint $\mathit{\Pi}^1_{\mathfrak{n}}$-sets non-separable by disjoint $\mathbf{\Sigma}^1_{\mathfrak{n}}$-sets in the model $M = \mathbf{L}[\langle x_\xi \rangle_{\xi \in \Delta}]$. Indeed, by (I) we have

$$\Omega_1 = \{\nu < \omega_1 \colon \neg \exists x \ (x \text{ is } \mathbb{P}_{2\nu+1}\text{-generic over } \mathbf{L})\}$$

in M, whence Ω_1 is $\mathit{\Pi}^1_{\mathfrak{n}}$ in M by (II), and similarly for Ω_2. The non-separability claim uses the following crucial property of $\mathbb{P}$-generic extensions:

(III) if a set $X \in \mathbf{L}$, $X \subseteq \omega_1$, is unbounded in ω_1 and a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$, then $\mathbf{L}[\langle x_\xi \rangle_{\xi \in X}]$ is an elementary submodel of $\mathbf{L}[G]$ with respect to all $\mathit{\Sigma}^1_{\mathfrak{n}-1}$-formulae.

Each factor forcing $\mathbb{P}_\xi$ in this scheme is a clone of Jensen minimal forcing defined in [19] (and henceforth referred to as *Jensen forcing* for brevity; see also [20], 28A, on this forcing). In particular, $\mathbb{P}_\xi$ consists of *perfect trees* in $2^{<\omega}$. The idea of using finite-support products of Jensen forcings to obtain models with various definability effects belongs to Enayat [21]. It was exploited to obtain generic models containing countable non-empty $\mathit{\Pi}^1_2$-sets (even E_o-classes) without OD-elements [22], [23], a countable $\mathit{\Pi}^1_2$ Groszek–Laver pair [24], OD-non-uniformizable planar $\mathit{\Pi}^1_2$-sets with countable cross-sections [25], [26], counterexamples to the separation theorem for both $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$ [25], counterexamples to the axiom

[2] In fact this partition is more complicated because we simultaneously construct an example for Theorem 1.2, (ii); see § 19.

of choice [27], as well as an ordinal-definable partition of the real line into two non-empty ordinal-undefinable sets [28].

The result in [25] corresponds to the case $\mathtt{n} = 3$ of Theorem 1.2, when (III) is immediately true according to Schoenfield. On the other hand, conditions similar to (I), (II) for $\mathtt{n} = 3$ occur in the forcing constructions in [22]–[26] as well as in [19] itself, where a CCC forcing $\mathbb{J} \in \mathbf{L}$ adding a real $a \in 2^\omega$ was constructed in such a way that a is the only $\mathbb{J}$-generic real in $\mathbf{L}[a]$ and the property 'to be a $\mathbb{J}$-generic real' is a $\mathit{\Pi}^1_2$-property. These properties are guaranteed by a special construction of $\mathbb{J} = \bigcup_{\alpha<\omega_1} \mathbb{J}_\alpha$ in $\mathbf{L}$ from countable sets $\mathbb{J}_\alpha$ of perfect trees. This construction may be regarded as a maximal branch of some mega-tree $\mathscr{P}$ whose nodes are countable sets of perfect trees and each $\mathbb{J}_\alpha$ is chosen as the $\leqslant_{\mathbf{L}}$-smallest appropriate extension. The complexity of this construction is $\mathit{\Delta}^1_2$ in the codes. This gives rise to the $\mathit{\Pi}^1_2$-definability of the property of being generic, while a special type of extension in the mega-tree enables one to 'kill' all possible competitors of a to be $\mathbb{J}$-generic.

Pretty similar ideas and constructions work in the papers mentioned above, including [25], where a model was constructed in which $\mathbf{\Pi}^1_3$-separation fails.

The method of transferring generic counterexamples (which were originally defined at the second and third projective levels) to higher projective levels $\mathtt{n}$ was introduced by Harrington [18] on the basis of an almost-disjoint forcing [17] and independently in [29] on the base of Jensen forcing [19]. In the above terms, this method consists in defining a maximal branch in $\mathscr{P}$ that intersects all dense sets in $\mathscr{P}$ of descriptive complexity $\mathtt{n}$ (or $\mathtt{n} + c$, where c is a small entire constant depending on the nature of the problem). It was recently used to construct models in which, for a given $\mathtt{n} \geqslant 2$, there are

(a) a $\mathit{\Pi}^1_{\mathtt{n}}$ E_o-equivalence class containing no OD elements while every countable $\mathit{\Sigma}^1_{\mathtt{n}}$-set of reals contains only OD reals [30],

(b) a $\mathit{\Pi}^1_{\mathtt{n}}$-singleton $\{a\}$ such that $a \in 2^\omega$ encodes a cofinal map $f\colon \omega \to \omega_1^{\mathbf{L}}$ minimal over $\mathbf{L}$ while every $\mathit{\Sigma}^1_{\mathtt{n}}$-set $X \subseteq \omega$ is constructive [31],

(c) a non-ROD-uniformizable $\mathit{\Pi}^1_{\mathtt{n}}$-set with countable cross-sections while all $\mathbf{\Sigma}^1_{\mathtt{n}}$-sets with countable cross-sections are $\mathbf{\Delta}^1_{\mathtt{n}+1}$-uniformizable [32],

and a model in which the family $\mathscr{P}(\omega) \cap \mathbf{L}$ of all constructive sets $x \subseteq \omega$ is equal to the family of all $\mathit{\Delta}^1_{\mathtt{n}}$-sets $x \subseteq \omega$; see the recent paper [33]. Here we use this method to prove Theorem 1.2.

Sections 3–7: perfect trees in $2^{<\omega}$, perfect tree forcing notions, multitrees (finite products of trees), multiforcings (countable products of forcings), splittings, refinements, generic refinements by Jensen splitting construction.

Sections 8–13: properties of generic refinements, sealing of dense sets, sealing of real names, applications to generic extensions.

Sections 14–16: we define the set $\overrightarrow{\mathbf{MF}}$ of all countable sequences $\vec{\boldsymbol{\pi}}$ of small multiforcings increasing in the sense of the refinement relation. Arguing in $\mathbf{L}$, we define a maximal branch $\vec{\Pi}$ in $\overrightarrow{\mathbf{MF}}$ of class $\mathit{\Delta}^1_{\mathtt{n}-1}$ (in codes) that blocks all the $\mathit{\Sigma}^1_{\mathtt{n}-2}$-sets in $\overrightarrow{\mathbf{MF}}$, where $\mathtt{n}$ is the number in Theorem 1.2. Here we say that $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ *blocks* a set $W \subseteq \overrightarrow{\mathbf{MF}}$ if either $\vec{\boldsymbol{\pi}} \in W$ or no extension of $\vec{\boldsymbol{\pi}}$ in $\overrightarrow{\mathbf{MF}}$ belongs to W. The forcing notion $\mathbb{P}$ for Theorem 1.2 is a derivate of $\vec{\Pi}$.

Sections 17–20: we show that $\mathbb{P}$ satisfies (I) and (II).

Sections 22–26: to achieve (III), we develop an auxiliary forcing relation forc which approximates the truth in $\mathbb{P}$-generic extensions for $\Sigma^1_{\mathfrak{n}-1}$-formulae and below. The restriction of the relation forc to Σ^1_m, Π^1_m, $m \geqslant 2$, is of type Σ^1_m, Π^1_m respectively. Using the invariance of the relation forc under certain transformations (while the forcing notion $\mathbb{P}$ is not invariant!), we complete the proof of (III) and Theorem 1.2.

§ 3. Trees and perfect-tree forcing notions

We write $2^{<\omega}$ for the set of all strings (finite sequences) of the numbers 0, 1. If $t \in 2^{<\omega}$ and $i = 0, 1$, then $t^\frown i$ stands for the extension of t by i as the rightmost term. If $s, t \in 2^{<\omega}$, then $s \subseteq t$ means that t extends s, while $s \subset t$ means a proper extension. $\operatorname{lh}(t)$ is the length of $t \in 2^{<\omega}$, and $2^n = \{s \in 2^{<\omega} : \operatorname{lh}(s) = n\}$ (strings of length n).

A set $T \subseteq 2^{<\omega}$ is a *tree* if, given any $s \subset t$ in $2^{<\omega}$, the inclusion $t \in T$ implies that $s \in T$. Thus every non-empty tree $T \subseteq 2^{<\omega}$ contains the *empty string* Λ.

When $T \subseteq 2^{<\omega}$ is a tree and $s \in T$, we put $T{\restriction_s} = \{t \in T : s \subseteq t \text{ or } t \subseteq s\}$.

Definition 3.1. **PT** is the set of all *perfect* trees $\varnothing \neq T \subseteq 2^{<\omega}$. A tree T belongs to **PT** if it has no endpoints and no isolated branches. Given any $T \in \mathbf{PT}$, we define a perfect set

$$[T] = \{a \in 2^\omega : \forall\, n\, (a \restriction n \in T)\} \subseteq 2^\omega.$$

Trees $T, S \in \mathbf{PT}$ are said to be *almost disjoint*, or *ad* for brevity, if the intersection $S \cap T$ is finite or, equivalently, $[S] \cap [T] = \varnothing$. A set $\mathbb{A} \subseteq \mathbf{PT}$ is an *antichain* if any two trees $T \neq S$ in $\mathbb{A}$ are ad.

Consider pairs of the form $\langle n, T\rangle$, where $n < \omega$ and $T \in \mathbf{PT}$. Following [34], we order the set $\omega \times \mathbf{PT}$ of all such pairs by a special relation $\preccurlyeq$ in such a way[3] that $\langle n, T\rangle \preccurlyeq \langle m, S\rangle$ (reads: $\langle n, T\rangle$ *extends* $\langle m, S\rangle$) if $m \leqslant n$, $T \subseteq S$ and $T \cap 2^m = S \cap 2^m$. The role of the number m in a pair $\langle m, S\rangle$ is to preserve the value $S \cap 2^m$ under $\preccurlyeq$-extensions.

While the implication $m > n \Longrightarrow \langle m, T\rangle \preccurlyeq \langle n, T\rangle$ (with the same T!) always holds, it is not always true that $S \subseteq T \Longrightarrow \langle n, S\rangle \preccurlyeq \langle n, T\rangle$: we need the equality $T \cap 2^n = S \cap 2^n$.

Lemma 3.2 (see [34]). *Let* $\cdots \preccurlyeq \langle n_2, T_2\rangle \preccurlyeq \langle n_1, T_1\rangle \preccurlyeq \langle n_0, T_0\rangle$ *be a decreasing sequence in* $\omega \times \mathbf{PT}$ *with* $n_0 < n_1 < n_2 < \cdots$ *which is minimally generic in the sense that it has a non-empty intersection with every set of the form*

$$D_t = \{\langle n, T\rangle \in \omega \times \mathbf{PT} : t \notin T \text{ or } \exists\, s \in T\, (t \subseteq s \wedge s^\frown 0,\ s^\frown 1 \in T)\}, \qquad t \in 2^{<\omega}.$$

Then $T = \bigcap_n T_n \in \mathbf{PT}$, *and if* $i < \omega$, *then* $\langle n_i, T\rangle \preccurlyeq \langle n_i, T_i\rangle$.

[3] This definition does not explicitly contain any splitting condition. This is why the genericity assumption is needed in Lemma 3.2. An earlier definition in [35] required that for every $s \in S \cap 2^m$ there are two strings $s' \neq s''$ in $T \cap 2^n$ such that $s \subset s'$ and $s \subset s''$. For this ordering, Lemma 3.2 holds without the genericity assumption.

Definition 3.3. An *arboreal forcing* is any set $\mathbb{P} \subseteq \mathbf{PT}$ such that if $u \in T \in \mathbb{P}$, then $T{\restriction}_u \in \mathbb{P}$. We write $\mathbf{AF}$ for the set of all arboreal forcings $\mathbb{P}$. A forcing $\mathbb{P} \in \mathbf{AF}$ is said to be

– *regular* if, for any $S, T \in \mathbb{P}$, the intersection $[S] \cap [T]$ is closed and open in $[S]$ or in $[T]$ (or in $[S]$ and $[T]$ simultaneously),

– *special* if there is a finite or countable antichain $\mathbb{A} \subseteq \mathbb{P}$ such that $\mathbb{P} = \{T{\restriction}_s : s \in T \in \mathbb{A}\}$ (then the antichain $\mathbb{A}$ is unique and $\mathbb{P}$ is clearly regular).

Example 3.4. If $s \in 2^{<\omega}$, then the tree $T[s] = \{t \in 2^{<\omega} : s \subseteq t \text{ or } t \subseteq s\}$ belongs to $\mathbf{PT}$ and $T[s] = (2^{<\omega}){\restriction}_s\ \forall\, s$. The set $\mathbb{P}_{\mathrm{coh}} = \{T[s] : s \in 2^{<\omega}\}$ (the Cohen forcing) is a regular and special arboreal forcing notion.

Any set $\mathbb{P} \in \mathbf{AF}$ may be regarded as a forcing notion (if $T \subseteq T'$, then T is a stronger 'condition'). Such a forcing $\mathbb{P}$ clearly adds a real in 2^ω.

To carry out the splitting constructions (as in Lemma 3.2) over a forcing $\mathbb{P} \in \mathbf{AF}$, we use the bigger forcing $\bigcup^{\mathrm{fin}} \mathbb{P} \in \mathbf{AF}$ which consists of all finite unions of trees in $\mathbb{P}$. Then the set $\mathbb{P}$ is dense in $\bigcup^{\mathrm{fin}} \mathbb{P}$, so that the forcing properties of both sets coincide. Yet $\bigcup^{\mathrm{fin}} \mathbb{P}$ is more flexible with respect to tree constructions.

The following lemma implies that the compatibility of 'conditions' in a *regular* forcing is absolute.

Lemma 3.5. *Assume that $\mathbb{P} \in \mathbf{AF}$ is regular and the trees $S, T \in \mathbb{P}$ are not ad. Then $S \cap T \in \bigcup^{\mathrm{fin}} \mathbb{P}$, whence S, T are compatible in $\mathbb{P}$.*

Proof. By regularity, the intersection $[S] \cap [T]$ is closed and open, say, in $[S]$. Then there is a finite set $U \subseteq S$ such that $[S] \cap [T] = \bigcup_{u \in U} [S{\restriction}_u]$. But every $S{\restriction}_u$ belongs to $\mathbb{P}$ because the latter is an arboreal forcing. □

Lemma 3.6. *Suppose that $\mathbb{P} \in \mathbf{AF}$ and $S, T \in \bigcup^{\mathrm{fin}} \mathbb{P}$, $u \in S$, $n = \mathrm{lh}(u)$, $T \subseteq S{\restriction}_u$. Then the tree $S' = T \cup \bigcup_{v \in S \cap 2^n,\, v \neq u} S{\restriction}_v$ belongs to $\bigcup^{\mathrm{fin}} \mathbb{P}$, $\langle n, S' \rangle \preccurlyeq \langle n, S \rangle$, $S'{\restriction}_u = T$, and $S'{\restriction}_v = S{\restriction}_v$ for all strings $v \in S$ with $\mathrm{lh}(v) = n$ and $v \neq u$.*

Corollary 3.7. *Suppose that $\mathbb{P}, \mathbb{P}' \in \mathbf{AF}$. Then the following assertions hold.*

(i) *If $n < \omega$ and $T \in \bigcup^{\mathrm{fin}} \mathbb{P}$, then there is a tree $S \in \bigcup^{\mathrm{fin}} \mathbb{P}$ such that $\langle n, S \rangle \preccurlyeq \langle n, T \rangle$ and $S{\restriction}_t \in \mathbb{P}$* (not only $\in \bigcup^{\mathrm{fin}} \mathbb{P}$!) *for all $t \in 2^n \cap S$.*

(ii) *If $T \in \mathbb{P}$ and $T' \in \mathbb{P}'$, then there are trees $S \in \mathbb{P}$, $S' \in \mathbb{P}'$ such that $S \subseteq T$, $S' \subseteq T'$ and $[S] \cap [S'] = \varnothing$.*

(iii) *If $n < \omega$, $T \in \bigcup^{\mathrm{fin}} \mathbb{P}$ and $T' \in \bigcup^{\mathrm{fin}} \mathbb{P}'$, then there are trees $S \in \bigcup^{\mathrm{fin}} \mathbb{P}$, $S' \in \bigcup^{\mathrm{fin}} \mathbb{P}'$ such that $\langle n, S \rangle \preccurlyeq \langle n, T \rangle$, $\langle n, S' \rangle \preccurlyeq \langle n, T' \rangle$, $[S] \cap [S'] = \varnothing$.*

Proof. (ii) If $T = T'$, then pick a pair of strings $u \neq v$ in $T = T'$ with $\mathrm{lh}(u) = \mathrm{lh}(v)$ and put $S = T'{\restriction}_u$, $S' = T'{\restriction}_v$. If, for example, $T \not\subseteq T'$, then take any $u \in T \setminus T'$ and put $S = T{\restriction}_u$ and simply $S' = T'$. To prove (iii), iterate (ii) and use Lemma 3.6. □

§ 4. Multiforcings and multitrees

A *multiforcing* is any map $\boldsymbol{\pi} : |\boldsymbol{\pi}| \to \mathbf{AF}$, where $|\boldsymbol{\pi}| = \operatorname{dom} \boldsymbol{\pi} \subseteq \omega_1$. Let $\mathbf{MF}$ be the family of all multiforcings. We typically represent multiforcings $\boldsymbol{\pi} \in \mathbf{MF}$ as indexed sets $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$, where $\mathbb{P}_\xi \in \mathbf{AF}$ for all $\xi \in |\boldsymbol{\pi}|$, so that each set $\mathbb{P}_\xi = \mathbb{P}^{\boldsymbol{\pi}}_\xi = \boldsymbol{\pi}(\xi)$, $\xi \in |\boldsymbol{\pi}|$, is an arboreal forcing. A multiforcing $\boldsymbol{\pi}$ is said to be

– *small* if the set $|\boldsymbol{\pi}|$ and each forcing $\mathbb{P}^{\boldsymbol{\pi}}_\xi$, $\xi \in |\boldsymbol{\pi}|$, is countable,

– *special* if each forcing $\mathbb{P}^{\boldsymbol{\pi}}_\xi$ is special in the sense of Definition 3.3,
– *regular* if each forcing $\mathbb{P}^{\boldsymbol{\pi}}_\xi$ is regular in the sense of Definition 3.3.

A *multitree* is any function $\boldsymbol{p}\colon |\boldsymbol{p}| \to \mathbf{PT}$ whose *support* $|\boldsymbol{p}| = \operatorname{dom} \boldsymbol{p}$ is finite. Let $\mathbf{MT}$ be the family of all multitrees. We typically represent multitrees $\boldsymbol{p} \in \mathbf{MT}$ as indexed sets $\boldsymbol{p} = \langle T^{\boldsymbol{p}}_\xi \rangle_{\xi \in |\boldsymbol{p}|}$, where $T^{\boldsymbol{p}}_\xi = \boldsymbol{p}(\xi) \in \mathbf{PT}$ for all $\xi \in |\boldsymbol{p}|$.

Let $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ be a multiforcing. In this case, a $\boldsymbol{\pi}$-*multitree* is any multitree $\boldsymbol{p} \in \mathbf{MT}$ such that $|\boldsymbol{p}| \subseteq |\boldsymbol{\pi}|$ and if $\xi \in |\boldsymbol{p}|$, then the tree $\boldsymbol{p}(\xi) = T^{\boldsymbol{p}}_\xi$ belongs to $\mathbb{P}_\xi$. If $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, then the set

$$[\boldsymbol{p}] = \{x \in (2^\omega)^{|\boldsymbol{\pi}|} \colon \forall\, \xi \in |\boldsymbol{p}|\, (x(\xi) \in [T^{\boldsymbol{p}}_\xi])\}$$

is a cofinite-dimensional perfect cube in $(2^\omega)^{|\boldsymbol{\pi}|}$. We order $\mathbf{MT}$ and each $\mathbf{MT}(\boldsymbol{\pi})$ componentwise: $\boldsymbol{q} \leqslant \boldsymbol{p}$ ($\boldsymbol{q}$ is stronger than $\boldsymbol{p}$) if $|\boldsymbol{p}| \subseteq |\boldsymbol{q}|$ and $T^{\boldsymbol{q}}_\xi \subseteq T^{\boldsymbol{p}}_\xi$ for all $\xi \in |\boldsymbol{p}|$ or, equivalently, $[\boldsymbol{q}] \subseteq [\boldsymbol{p}]$. The empty multitree $\boldsymbol{\Lambda}$ is defined by putting $|\boldsymbol{\Lambda}| = \varnothing$. It belongs to $\mathbf{MT}(\boldsymbol{\pi})$ and is the weakest.

Remark 4.1. If $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ is a multiforcing, then the set $\mathbf{MT}(\boldsymbol{\pi})$ of all $\boldsymbol{\pi}$-multitrees can be identified with the finite-support product $\prod_{\xi \in |\boldsymbol{\pi}|} \mathbb{P}_\xi$ of the arboreal forcings $\mathbb{P}_\xi$.

Definition 4.2. Two multitrees $\boldsymbol{p}, \boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ are *somewhere almost disjoint* (sad) if there is a $\xi \in |\boldsymbol{p}| \cap |\boldsymbol{q}|$ such that $T^{\boldsymbol{p}}_\xi$ and $T^{\boldsymbol{q}}_\xi$ are ad. The property of being sad is equivalent to the equality $[\boldsymbol{p}] \cap [\boldsymbol{q}] = \varnothing$ and, in the case of regular multiforcings $\boldsymbol{\pi}$, to *incompatibility* in $\mathbf{MT}(\boldsymbol{\pi})$ by the following result.

Corollary 4.3 (of Lemma 3.5). *Assume that $\boldsymbol{\pi}$ is a regular multiforcing and $\boldsymbol{p}, \boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ are not sad. Then there is a finite set $R \subseteq \mathbf{MT}(\boldsymbol{\pi})$ such that $[\boldsymbol{p}] \cap [\boldsymbol{q}] = \bigcup_{\boldsymbol{r} \in R} [\boldsymbol{r}]$. Therefore $\boldsymbol{p}$, $\boldsymbol{q}$ are compatible in $\mathbf{MT}(\boldsymbol{\pi})$, that is, there is a multitree $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying $\boldsymbol{r} \leqslant \boldsymbol{p}$ and $\boldsymbol{r} \leqslant \boldsymbol{q}$.*

Definition 4.4. The *componentwise union* of two multiforcings $\boldsymbol{\pi}$, $\boldsymbol{\varrho}$ is the multiforcing $\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varrho}$ such that $|(\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varrho})| = |\boldsymbol{\pi}| \cup |\boldsymbol{\varrho}|$ and

$$(\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varrho})(\xi) = \boldsymbol{\pi}(\xi), \quad \boldsymbol{\varrho}(\xi), \quad \boldsymbol{\pi}(\xi) \cup \boldsymbol{\varrho}(\xi)$$

when $\xi \in |\boldsymbol{\pi}| \setminus |\boldsymbol{\varrho}|$, $\xi \in |\boldsymbol{\varrho}| \setminus |\boldsymbol{\pi}|$, $\xi \in |\boldsymbol{\varrho}| \cap |\boldsymbol{\pi}|$ respectively.

Given any sequence $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_\alpha \rangle_{\alpha < \lambda}$ of forcings in $\mathbf{MF}$, we define $\boldsymbol{\pi} = \bigcup^{\mathrm{cw}} \vec{\boldsymbol{\pi}} = \bigcup^{\mathrm{cw}}_{\alpha < \lambda} \boldsymbol{\pi}_\alpha \in \mathbf{MF}$ in such a way that $|\boldsymbol{\pi}| = \bigcup_{\alpha < \lambda} |\boldsymbol{\pi}_\alpha|$ and $\boldsymbol{\pi}(\xi) = \bigcup_{\alpha < \lambda,\, \xi \in |\boldsymbol{\pi}_\alpha|} \boldsymbol{\pi}_\alpha(\xi)$ for every $\xi \in |\boldsymbol{\pi}|$.

Remark 4.5. Any forcing of the form $\mathbf{MT}(\boldsymbol{\pi})$, where $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|} \in \mathbf{MF}$, adjoins a generic sequence $\langle x_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$, where each $x_\xi = x_\xi[G] \in 2^\omega$ is a $\mathbb{P}_\xi$-generic real. Reals of the form $x_\xi[G]$ will be called *principal generic reals* in $\mathbf{V}[G]$.

§ 5. Refining arboreal forcings

Given any $T \in \mathbf{PT}$ (a tree) and $D \subseteq \mathbf{PT}$, we write $X \subseteq^{\mathrm{fin}} \bigcup D$ if there is a finite set $D' \subseteq D$ such that $T \subseteq \bigcup D'$ or, equivalently, $[T] \subseteq \bigcup_{S \in D'} [S]$.

Definition 5.1. Let $\mathbb{P}, \mathbb{Q} \in \mathbf{AF}$ be arboreal forcings. We say that $\mathbb{Q}$ is a *refinement* of $\mathbb{P}$ (and write $\mathbb{P} \sqsubset \mathbb{Q}$) if the following conditions hold.

(1) The set $\mathbb{Q}$ is dense[4] in $\mathbb{P} \cup \mathbb{Q}$: if $T \in \mathbb{P}$, then $\exists Q \in \mathbb{Q}\,(Q \subseteq T)$.

(2) If $Q \in \mathbb{Q}$, then $Q \subseteq^{\text{fin}} \bigcup \mathbb{P}$.

(3) If $Q \in \mathbb{Q}$ and $T \in \mathbb{P}$, then $[Q] \cap [T]$ is closed and open in $[Q]$ and $T \not\subseteq Q$.

Lemma 5.2. (i) *If* $\mathbb{P} \sqsubset \mathbb{Q}$ *and* $S \in \mathbb{P}$, $T \in \mathbb{Q}$, *then* $[S] \cap [T]$ *is meagre in* $[S]$ *and, therefore,* $\mathbb{P} \cap \mathbb{Q} = \varnothing$ *and* $\mathbb{Q}$ *is open-dense in* $\mathbb{P} \cup \mathbb{Q}$.

(ii) *If* $\mathbb{P} \sqsubset \mathbb{Q} \sqsubset \mathbb{R}$, *then* $\mathbb{P} \sqsubset \mathbb{R}$. *Thus* $\sqsubset$ *is a strict partial order.*

(iii) *If* $\langle \mathbb{P}_{\alpha\,\alpha<\lambda} \rangle$ *is a* $\sqsubset$*-increasing sequence in* **AF** *and* $0 < \mu < \lambda$, *then* $\mathbb{P} = \bigcup_{\alpha<\mu} \mathbb{P}_\alpha \sqsubset \mathbb{Q} = \bigcup_{\mu \leqslant \alpha < \lambda} \mathbb{P}_\alpha$.

(iv) *If* $\langle \mathbb{P}_{\alpha\,\alpha<\lambda} \rangle$ *is a* $\sqsubset$*-increasing sequence in* **AF** *and each* $\mathbb{P}_\alpha$ *is special, then* $\mathbb{P} = \bigcup_{\alpha<\lambda} \mathbb{P}_\alpha \in$ **AF** *is a regular forcing and each of the* $\mathbb{P}_\gamma$ *is pre-dense in* $\mathbb{P}$.

Proof. (i) Otherwise there is a string $u \in S$ such that $S{\restriction}_u \subseteq [T] \cap [S]$. But $S{\restriction}_u \in \mathbb{P}$, which contradicts 5.1, (3).

(ii), (iii) We use part (i) to establish 5.1, (3).

(iv) To check the regularity, suppose that $S \in \mathbb{P}_\alpha$, $T \in \mathbb{P}_\beta$, $\alpha \leqslant \beta$. If $\alpha = \beta$, then in view of the speciality of $\mathbb{P}_\alpha$ Lemma 3.5 yields that the trees S, T are either ad or $\subseteq$-comparable. If $\alpha < \beta$, then $[S] \cap [T]$ is open and closed in $[T]$ by 5.1, (3).

To check the pre-density, suppose that $S \in \mathbb{P}_\alpha$, $\alpha \neq \gamma$. If $\alpha < \gamma$, then by 5.1, (1) there is a tree $T \in \mathbb{P}_\gamma$, $T \subseteq S$. Now let $\gamma < \alpha$. Then $S \subseteq^{\text{fin}} \bigcup \mathbb{P}_\gamma$ by 5.1, (2). Hence there is a tree $T \in \mathbb{P}_\gamma$ such that $[S] \cap [T] \neq \varnothing$. However, $[S] \cap [T]$ is closed and open in $[S]$ by 5.1, (3). Therefore, $S{\restriction}_u \subseteq T$ for an appropriate $u \in S$. Finally, $S{\restriction}_u \in \mathbb{P}_\alpha$ since $\mathbb{P}_\alpha \in$ **AF**. □

Note that if $\mathbb{P}, \mathbb{Q} \in$ **AF** and $\mathbb{P} \sqsubset \mathbb{Q}$, then a dense set $D \subseteq \mathbb{P}$ need not be dense or even pre-dense in $\mathbb{P} \cup \mathbb{Q}$. However, there is a special type of refinement which preserves at least pre-density.

Definition 5.3. Suppose that $\mathbb{P}, \mathbb{Q} \in$ **AF** and $D \subseteq \mathbb{P}$. We say that $\mathbb{Q}$ *seals* D *over* $\mathbb{P}$ (and write $\mathbb{P} \sqsubset_D \mathbb{Q}$) if $\mathbb{P} \sqsubset \mathbb{Q}$ and every tree $S \in \mathbb{Q}$ satisfies $S \subseteq^{\text{fin}} \bigcup D$. Then the "simple" $\mathbb{P} \sqsubset \mathbb{Q}$ is equivalent to $\mathbb{P} \sqsubset_{\mathbb{P}} \mathbb{Q}$.

We shall see that a sealed set has to be pre-dense before and after the refinement. Sealing refinements are important because the property of being sealed is preserved under further simple refinements, that is, $\sqsubset_D$ is transitive when combined with $\sqsubset$ in the sense of part (ii) of the following lemma.

Lemma 5.4. (i) *If* $\mathbb{P} \sqsubset_D \mathbb{Q}$, *then* D *is pre-dense in* $\mathbb{P} \cup \mathbb{Q}$. *If in addition* $\mathbb{P}$ *is regular, then* D *is also pre-dense in* $\mathbb{P}$.

(ii) *If* $\mathbb{P} \sqsubset_D \mathbb{Q} \sqsubset \mathbb{R}$ (note: the second $\sqsubset$ is not $\sqsubset_D$!), *then* $\mathbb{P} \sqsubset_D \mathbb{R}$.

(iii) *If* $\langle \mathbb{P}_{\alpha\,\alpha<\lambda} \rangle$ *is a* $\sqsubset$*-increasing sequence in* **AF**, $0 < \mu < \lambda$, *and*

$$\mathbb{P} = \bigcup_{\alpha<\mu} \mathbb{P}_\alpha \sqsubset_D \mathbb{P}_\mu,$$

then $\mathbb{P} \sqsubset_D \mathbb{Q} = \bigcup_{\mu \leqslant \alpha < \lambda} \mathbb{P}_\alpha$.

[4] Given any $\mathbb{P} \subseteq \mathbb{R} \subseteq$ **PT**, we say that the set $\mathbb{P}$ is 1) *dense* in $\mathbb{R}$ if $\forall T \in \mathbb{R}\, \exists S \in \mathbb{P}\, (S \subseteq T)$; 2) *open-dense* in $\mathbb{R}$ if, moreover, $\forall T \in \mathbb{R}\, \forall S \in \mathbb{P}\, (T \subseteq S \Longrightarrow T \in \mathbb{P})$; and 3) *pre-dense* in $\mathbb{R}$ if the derived set $\mathbb{P}' = \{T \in \mathbb{R} \colon \exists S \in \mathbb{P}\, (T \subseteq S)\}$ is dense in $\mathbb{R}$.

Proof. (i) To see that D is pre-dense in $\mathbb{P}\cup\mathbb{Q}$, suppose that $T_0\in\mathbb{P}\cup\mathbb{Q}$. By 5.1, (1) there is a tree $T\in\mathbb{Q}$ with $T\subseteq T_0$. Then $T\subseteq^{\mathrm{fin}}\bigcup D$ and, in particular, there is a tree $S\in D$ with $X=[S]\cap[T]\neq\varnothing$. However, X is closed and open in $[T]$ by 5.1, (3). Therefore there is a tree $T'\in\mathbb{Q}$ with $[T']\subseteq X$. Thus, $T'\subseteq S\in D$ and $T'\subseteq T\subseteq T_0$. We conclude that T_0 is compatible with $S\in D$ in $\mathbb{P}\cup\mathbb{Q}$.

To see that D is pre-dense in $\mathbb{P}$ (in the case when $\mathbb{P}$ is regular), suppose that $S_0\in\mathbb{P}$. We have already seen that then S_0 is compatible with some $S\in D$. Hence S and S_0 are not ad. It remains to use Lemma 3.5.

To prove (ii), in addition to Lemma 5.2, (ii), suppose that $R\in\mathbb{R}$. Then $R\subseteq^{\mathrm{fin}}\bigcup\mathbb{Q}$, but every $T\in\mathbb{Q}$ satisfies $T\subseteq^{\mathrm{fin}}\bigcup D$. The same goes for (iii). □

§ 6. Refining multiforcings

Let $\boldsymbol{\pi}$, $\boldsymbol{\varphi}$ be multiforcings. We say that $\boldsymbol{\varphi}$ is a *refinement* of $\boldsymbol{\pi}$ (and write $\boldsymbol{\pi}\sqsubset\boldsymbol{\varphi}$) if $|\boldsymbol{\pi}|\subseteq|\boldsymbol{\varphi}|$ and $\boldsymbol{\pi}(\xi)\sqsubset\boldsymbol{\varphi}(\xi)$ whenever $\xi\in|\boldsymbol{\pi}|$.

Corollary 6.1 (of Lemma 5.2). *If $\boldsymbol{\pi}\sqsubset\boldsymbol{\varphi}\sqsubset\boldsymbol{\rho}$, then $\boldsymbol{\pi}\sqsubset\boldsymbol{\rho}$.*

If $\boldsymbol{\pi}\sqsubset\boldsymbol{\varphi}$, then the set $\mathbf{MT}(\boldsymbol{\varphi})$ is open-dense[5] *in $\mathbf{MT}(\boldsymbol{\pi}\cup^{\mathrm{cw}}\boldsymbol{\varphi})$.*

Our next goal is to adjust Definition 5.3 for multiforcings in such a way that an analogue of Lemma 5.4 holds.

We first adjust the definition of the relation $\subseteq^{\mathrm{fin}}$ in § 5 for multitrees. Namely, given a multitree $\boldsymbol{u}$ and a set $\boldsymbol{D}$ of multitrees, we shall write $\boldsymbol{u}\subseteq^{\mathrm{fin}}\bigvee\boldsymbol{D}$ if there is a finite set $\boldsymbol{D}'\subseteq\boldsymbol{D}$ such that 1) $|\boldsymbol{v}|=|\boldsymbol{u}|$ for all $\boldsymbol{v}\in\boldsymbol{D}'$, and 2) $[\boldsymbol{u}]\subseteq\bigcup_{\boldsymbol{v}\in\boldsymbol{D}'}[\boldsymbol{v}]$.

Definition 6.2. Suppose that $\boldsymbol{\pi}$, $\boldsymbol{\varphi}$ are multiforcings and $\boldsymbol{\pi}\sqsubset\boldsymbol{\varphi}$. We say that $\boldsymbol{\varphi}$ *seals a set $\boldsymbol{D}\subseteq\mathbf{MT}(\boldsymbol{\pi})$ over $\boldsymbol{\pi}$* (and write $\boldsymbol{\pi}\sqsubset_{\boldsymbol{D}}\boldsymbol{\varphi}$) if the following condition holds.

(∗) If $\boldsymbol{p}\in\mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u}\in\mathbf{MT}(\boldsymbol{\varphi})$, $|\boldsymbol{u}|\subseteq|\boldsymbol{\pi}|$ and $|\boldsymbol{u}|\cap|\boldsymbol{p}|=\varnothing$, then there is a $\boldsymbol{q}\in\mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q}\leqslant\boldsymbol{p}$, $|\boldsymbol{q}|\cap|\boldsymbol{u}|=\varnothing$ and $\boldsymbol{u}\subseteq^{\mathrm{fin}}\bigvee\boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}$, where

$$\boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}=\{\boldsymbol{u}'\in\mathbf{MT}(\boldsymbol{\pi})\colon|\boldsymbol{u}'|=|\boldsymbol{u}|\text{ and }\boldsymbol{u}'\cup\boldsymbol{q}\in\boldsymbol{D}\}.$$

Note that if $\boldsymbol{p},\boldsymbol{u},\boldsymbol{D},\boldsymbol{q}$ are as indicated, then $\boldsymbol{u}\cup\boldsymbol{q}\subseteq^{\mathrm{fin}}\bigvee\boldsymbol{D}'$, where $\boldsymbol{D}'=\{\boldsymbol{u}'\cup\boldsymbol{q}\colon\boldsymbol{u}'\in\boldsymbol{D}_{\boldsymbol{q}}^{|\boldsymbol{u}|}\}\subseteq\boldsymbol{D}$. The definition of $\sqsubset_{\boldsymbol{D}}$ in 6.2 looks somewhat different from and more complicated than that of $\sqsubset_D$ in 5.3. This reflects the fact that finite-support products of forcing notions in **AF** behave differently and in a more complicated way than single arboreal forcings. The following lemma, which is similar to Lemma 5.4, is accordingly somewhat less obvious.

Lemma 6.3. *Suppose that $\boldsymbol{\pi}$, $\boldsymbol{\varphi}$, $\boldsymbol{\sigma}$ are multiforcings and $\boldsymbol{D}\subseteq\mathbf{MT}(\boldsymbol{\pi})$. Then the following assertions hold.*

(i) *If $\boldsymbol{\pi}\sqsubset_{\boldsymbol{D}}\boldsymbol{\varphi}$, then $\boldsymbol{D}$ is dense in $\mathbf{MT}(\boldsymbol{\pi})$ and pre-dense in $\mathbf{MT}(\boldsymbol{\pi}\cup^{\mathrm{cw}}\boldsymbol{\varphi})$.*

(ii) *If $\boldsymbol{\pi}$ is regular, $\boldsymbol{\pi}\sqsubset_{\boldsymbol{D}_i}\boldsymbol{\varphi}$ for $i=1,\dots,n$, all the sets $\boldsymbol{D}_i\subseteq\mathbf{MT}(\boldsymbol{\pi})$ are open-dense in $\mathbf{MT}(\boldsymbol{\pi})$, and $\boldsymbol{D}=\bigcap_i\boldsymbol{D}_i$, then $\boldsymbol{\pi}\sqsubset_{\boldsymbol{D}}\boldsymbol{\varphi}$.*

(iii) *If $\boldsymbol{D}$ is open-dense in $\mathbf{MT}(\boldsymbol{\pi})$ and $\boldsymbol{\pi}\sqsubset_{\boldsymbol{D}}\boldsymbol{\varphi}\sqsubset\boldsymbol{\sigma}$, then $\boldsymbol{\pi}\sqsubset_{\boldsymbol{D}}\boldsymbol{\sigma}$.*

[5] Given any $\boldsymbol{P}\subseteq\boldsymbol{R}\subseteq\mathbf{MT}$, we say as in footnote 4 that $\boldsymbol{P}$ is 1) *dense* in $\boldsymbol{R}$ if $\forall\,\boldsymbol{r}\in\boldsymbol{R}\ \exists\,\boldsymbol{p}\in\boldsymbol{P}\,(\boldsymbol{p}\leqslant\boldsymbol{r})$, 2) *open-dense* in $\boldsymbol{R}$ if in addition $\forall\,\boldsymbol{r}\in\boldsymbol{R}\ \forall\,\boldsymbol{p}\in\boldsymbol{P}\,(\boldsymbol{p}\leqslant\boldsymbol{r}\implies\boldsymbol{p}\in\boldsymbol{R})$, and 3) *pre-dense* in $\boldsymbol{R}$ if the set $\boldsymbol{P}'=\{\boldsymbol{r}\in\boldsymbol{R}\colon\exists\,\boldsymbol{p}\in\boldsymbol{P}\,(\boldsymbol{r}\leqslant\boldsymbol{p})\}$ is dense in $\boldsymbol{R}$.

(iv) *If* $\langle \boldsymbol{\pi}_{\alpha\alpha<\lambda}\rangle$ *is a* $\sqsubset$*-increasing sequence in* $\mathbf{MF}$, $0<\mu<\lambda$, $\boldsymbol{\pi}=\bigcup^{\mathrm{cw}}_{\alpha<\mu}\boldsymbol{\pi}_\alpha$, *the set* $\boldsymbol{D}$ *is open-dense in* $\mathbf{MT}(\boldsymbol{\pi})$, *and* $\boldsymbol{\pi}\sqsubset_{\boldsymbol{D}}\boldsymbol{\pi}_\mu$, *then* $\boldsymbol{\pi}\sqsubset_{\boldsymbol{D}}\boldsymbol{\varphi}=\bigcup^{\mathrm{cw}}_{\mu\leqslant\alpha<\lambda}\boldsymbol{\pi}_\alpha$.

Proof. (i) To check that $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\boldsymbol{\pi}\cup^{\mathrm{cw}}\boldsymbol{\varphi})$, suppose that $\boldsymbol{r}\in\mathbf{MT}(\boldsymbol{\pi}\cup^{\mathrm{cw}}\boldsymbol{\varphi})$. Since $\mathbf{MT}(\boldsymbol{\pi}\cup^{\mathrm{cw}}\boldsymbol{\varphi})$ is a product, we can assume that $|\boldsymbol{r}|\subseteq|\boldsymbol{\pi}|$. Put

$$X=\{\xi\in|\boldsymbol{r}|\colon T^{\boldsymbol{r}}_\xi\in\mathbf{MT}(\boldsymbol{\varphi})\},\qquad Y=\{\xi\in|\boldsymbol{r}|\colon T^{\boldsymbol{r}}_\xi\in\mathbf{MT}(\boldsymbol{\pi})\}.$$

Then $\boldsymbol{r}=\boldsymbol{u}\cup\boldsymbol{p}$, where $\boldsymbol{u}=\boldsymbol{r}\restriction X\in\mathbf{MT}(\boldsymbol{\varphi})$, $\boldsymbol{p}=\boldsymbol{r}\restriction Y\in\mathbf{MT}(\boldsymbol{\pi})$. Since $\boldsymbol{\varphi}$ seals $\boldsymbol{D}$, there is a multitree $\boldsymbol{q}\in\mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q}\leqslant\boldsymbol{p}$, $|\boldsymbol{q}|\cap|\boldsymbol{u}|=\varnothing$ and $\boldsymbol{u}\subseteq^{\mathrm{fin}}\bigvee\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. It is easy to see that there is a multitree $\boldsymbol{u}'\in\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ compatible with $\boldsymbol{u}$ in $\mathbf{MT}(\boldsymbol{\varphi})$. Suppose that $\boldsymbol{w}\in\mathbf{MT}(\boldsymbol{\varphi})$, $\boldsymbol{w}\leqslant\boldsymbol{u}$, $\boldsymbol{w}\leqslant\boldsymbol{u}'$, $|\boldsymbol{w}|=|\boldsymbol{u}'|=|\boldsymbol{u}|$. Then the multitree $\boldsymbol{r}'=\boldsymbol{w}\cup\boldsymbol{q}\in\mathbf{MT}(\boldsymbol{\pi}\vee\boldsymbol{\varphi})$ satisfies $\boldsymbol{r}'\leqslant\boldsymbol{r}$ and $\boldsymbol{r}'\leqslant\boldsymbol{u}'\cup\boldsymbol{q}\in\boldsymbol{D}$.

To check that $\boldsymbol{D}$ is dense in $\mathbf{MT}(\boldsymbol{\pi})$, suppose that $\boldsymbol{p}\in\mathbf{MT}(\boldsymbol{\pi})$. Taking $\boldsymbol{u}=\boldsymbol{\Lambda}$ (the empty multitree) in the condition $(*)$ of Definition 6.2, we see that $|\boldsymbol{u}|=\varnothing$ and $\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}=\boldsymbol{D}$.

(ii) Suppose that $\boldsymbol{p}\in\mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u}\in\mathbf{MT}(\boldsymbol{\varphi})$, $|\boldsymbol{u}|\subseteq|\boldsymbol{\pi}|$, $|\boldsymbol{u}|\cap|\boldsymbol{p}|=\varnothing$. Iterating $(*)$ for the sets $\boldsymbol{D}_i$, $i=1,\dots,n$, we find a multitree $\boldsymbol{q}\in\mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q}\leqslant\boldsymbol{p}$, $|\boldsymbol{q}|\cap|\boldsymbol{u}|=\varnothing$ and $\boldsymbol{u}\subseteq^{\mathrm{fin}}\bigvee(\boldsymbol{D}_i)^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ for all i, where

$$(\boldsymbol{D}_i)^{|\boldsymbol{u}|}_{\boldsymbol{q}}=\{\boldsymbol{u}'\in\mathbf{MT}(\boldsymbol{\pi})\colon|\boldsymbol{u}'|=|\boldsymbol{u}|\text{ and }\boldsymbol{u}'\cup\boldsymbol{q}\in\boldsymbol{D}_i\}.$$

Thus there are finite sets $U_i\subseteq(\boldsymbol{D}_i)^{|\boldsymbol{u}|}_{\boldsymbol{q}}$ such that $[\boldsymbol{u}]\subseteq\bigcup_{\boldsymbol{v}\in U_i}[\boldsymbol{v}]$ for all i. Using the regularity assumption and Corollary 4.3, we obtain a finite set $W\subseteq\mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{w}|=|\boldsymbol{u}|$ for all $\boldsymbol{w}\in W$, $\bigcap_i\bigcup_{\boldsymbol{v}\in U_i}[\boldsymbol{v}]=\bigcup_{\boldsymbol{w}\in W}[\boldsymbol{w}]$ and if $i=1,\dots,n$ and $\boldsymbol{w}\in W$, then $[\boldsymbol{w}]\subseteq[\boldsymbol{v}]$ for some $\boldsymbol{v}\in U_i$, whence $\boldsymbol{w}\cup\boldsymbol{q}\in\boldsymbol{D}_i$. We conclude that if $\boldsymbol{w}\in W$, then $\boldsymbol{w}\cup\boldsymbol{q}\in\boldsymbol{D}$ and, therefore, $\boldsymbol{w}\in\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. Thus $W\subseteq\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$. However, $[\boldsymbol{u}]\subseteq\bigcup_{\boldsymbol{w}\in W}[\boldsymbol{w}]$ by the choice of W. Thus $\boldsymbol{u}\subseteq^{\mathrm{fin}}\bigvee\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$.

(iii) We have $\boldsymbol{\pi}\sqsubset\boldsymbol{\sigma}$ by Corollary 6.1. To check that $\boldsymbol{\sigma}$ seals $\boldsymbol{D}$ over $\boldsymbol{\pi}$, suppose that $\boldsymbol{u}\in\mathbf{MT}(\boldsymbol{\sigma})$, $|\boldsymbol{u}|\subseteq|\boldsymbol{\pi}|$, $\boldsymbol{p}\in\mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{u}|\cap|\boldsymbol{p}|=\varnothing$. Since $\boldsymbol{\varphi}\sqsubset\boldsymbol{\sigma}$, there is a finite set $U\subseteq\mathbf{MT}(\boldsymbol{\varphi})$ such that $|\boldsymbol{v}|=|\boldsymbol{u}|$ for all $\boldsymbol{v}\in U$, and $[\boldsymbol{u}]\subseteq\bigcup_{\boldsymbol{v}\in U}[\boldsymbol{v}]$. Since $\boldsymbol{\pi}\sqsubset_{\boldsymbol{D}}\boldsymbol{\varphi}$, an iterated application of Definition 6.2, $(*)$ yields a multitree $\boldsymbol{q}\in\mathbf{MT}(\boldsymbol{\pi})$ such that $\boldsymbol{q}\leqslant\boldsymbol{p}$, $|\boldsymbol{q}|\cap|\boldsymbol{u}|=\varnothing$ and if $\boldsymbol{v}\in U$, then $\boldsymbol{v}\subseteq^{\mathrm{fin}}\bigvee\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$, where

$$\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}=\{\boldsymbol{v}'\in\mathbf{MT}(\boldsymbol{\pi})\colon|\boldsymbol{v}'|=|\boldsymbol{v}|=|\boldsymbol{u}|\wedge\boldsymbol{v}'\cup\boldsymbol{q}\in\boldsymbol{D}\}.$$

We finally note that $\boldsymbol{u}\subseteq^{\mathrm{fin}}\bigvee U$ by construction. Hence we also have $\boldsymbol{u}\subseteq^{\mathrm{fin}}\bigvee\boldsymbol{D}^{|\boldsymbol{u}|}_{\boldsymbol{q}}$.

(iv) We have to check that $\boldsymbol{\varphi}$ seals $\boldsymbol{D}$ over $\boldsymbol{\pi}$. Suppose that $\boldsymbol{u}\in\mathbf{MT}(\boldsymbol{\varphi})$, $|\boldsymbol{u}|\subseteq|\boldsymbol{\pi}|$, $\boldsymbol{p}\in\mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{u}|\cap|\boldsymbol{p}|=\varnothing$. There is a finite set $U\subseteq\mathbf{MT}(\boldsymbol{\pi}_\mu)$ such that $|\boldsymbol{v}|=|\boldsymbol{u}|$ for all $\boldsymbol{v}\in U$ and $[\boldsymbol{u}]\subseteq\bigcup_{\boldsymbol{v}\in U}[\boldsymbol{v}]$. We proceed as in the proof of (iii). □

§ 7. Generic refinement of a multiforcing according to Jensen

Here we define a splitting/fusion construction of refinements. The construction was originally invented as a method for obtaining perfect sets in Polish spaces. Jensen modified it in [19] in order to obtain refinements of certain countable subforcings of the Saks forcing. Our next definition introduces what is essentially a product version of Jensen refiniements which is applicable to arboreal forcings and multiforcings. Since we are dealing with *finite-support* products (see Remark 4.1), the standard technique in the theory of countable-support Saks products (as, for example, in [36] or [37]–[39]) is not fully applicable. The notion of a *system* in our next definition contains appropriate changes of instrumentation related to the splitting/fusion construction. It was established in [22], [23] that infinite finite-support products of Jensen-type forcing notions are CCC, preserve cardinals (in contrast to finite-support Saks products) and admit a suitable version of the splitting/fusion technique.

Definition 7.1. Suppose that $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ is a small multiforcing.

(0) A $\boldsymbol{\pi}$-*system* is any indexed set of the form

$$\varphi = \langle T^{\varphi}_{\xi k} \rangle_{\langle \xi, k\rangle \in |\varphi|},$$

where $|\varphi| \subseteq |\boldsymbol{\pi}| \times \omega$ is finite and $T^{\varphi}_{\xi k} = \varphi(\xi,k) \in \bigcup^{\mathrm{fin}} \mathbb{P}_\xi$ for all ξ, k. (We recall that $\bigcup^{\mathrm{fin}} \mathbb{P}_\xi$ consists of all finite unions of trees in $\mathbb{P}_\xi$.) We order the set $\mathbf{Sys}(\boldsymbol{\pi})$ of all $\boldsymbol{\pi}$-systems componentwise: $\varphi \leqslant \psi$ (φ extends ψ) if $|\psi| \subseteq |\varphi|$ and $T^{\varphi}_{\xi k} \subseteq T^{\psi}_{\xi k}$ for all $\langle \xi, k\rangle \in |\psi|$. Accordingly, the set $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$ is ordered in such a way that $\langle n, \varphi\rangle \preccurlyeq \langle m, \psi\rangle$ if and only if $|\psi| \subseteq |\varphi|$ and $\langle n, T^{\varphi}_{\xi k}\rangle \preccurlyeq \langle m, T^{\psi}_{\xi k}\rangle$ in $\omega \times \mathbf{PT}$ (§ 3) for all ξ, k. This implies that $m \leqslant n$.

(1) Let $\mathfrak{M} \in \mathrm{HC}$ be any set.[6] The set $\mathfrak{M}^+$ of all sets $X \in \mathrm{HC}$ that are $\in$-separable in HC by formulae with sets in $\mathfrak{M}$ as parameters is still countable. Hence there is a $\preccurlyeq$-decreasing sequence $\Phi = \langle \langle n_j, \varphi_j\rangle\rangle_{j<\omega}$ of pairs $\langle n_j, \varphi_j\rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ which is $\mathfrak{M}^+$-*generic* in the sense that it intersects all the sets $D \in \mathfrak{M}$ with $D \subseteq \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ that are open-dense[7] in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$. We fix such an $\mathfrak{M}^+$-generic sequence Φ.

By definition, each φ_j is of the form $\varphi_j = \langle T^{\varphi_j}_{\xi k} \rangle_{\langle \xi, k\rangle \in |\varphi_j|}$, where $|\varphi_j| \subseteq |\boldsymbol{\pi}| \times \omega$ is finite and each tree $T^{\varphi_j}_{\xi k}$ belongs to $\bigcup^{\mathrm{fin}} \mathbb{P}_\xi$. Since $n_j \to \infty$ by genericity, we can assume without loss of generality that $n_0 < n_1 < n_2 < \cdots$ strictly.

(2) Suppose that $\xi \in |\boldsymbol{\pi}|$, $k < \omega$. By the genericity assumption, there is a number $j(\xi,k)$ such that if $j \geqslant j(\xi,k)$, then $\langle \xi, k\rangle \in |\varphi_j|$. Hence the tree $\varphi_j(\xi,k) = T^{\varphi_j}_{\xi k} \in \bigcup^{\mathrm{fin}} \mathbb{P}_\xi$ is defined and we have

$$\cdots \preccurlyeq \langle n_{j(\xi,k)+2}, T^{\varphi_{j(\xi,k)+2}}_{\xi k}\rangle \preccurlyeq \langle n_{j(\xi,k)+1}, T^{\varphi_{j(\xi,k)+1}}_{\xi k}\rangle \preccurlyeq \langle n_{j(\xi,k)}, T^{\varphi_{j(\xi,k)}}_{\xi k}\rangle,$$

with $n_{j(\xi,k)} < n_{j(\xi,k)+1} < n_{j(\xi,k)+2} < \cdots$ strictly; see (1) above.

(3) It then follows by Lemma 3.2 that each intersection $\boldsymbol{Q}^{\Phi}_{\xi k} = \bigcap_{j \geqslant j(\xi,k)} T^{\varphi_j}_{\xi k}$ is a tree in $\mathbf{PT}$ (not necessarily in $\mathbb{P}_\xi$) and the relation $\langle n_j, \boldsymbol{Q}^{\Phi}_{\xi k}\rangle \preccurlyeq \langle n_j, T^{\varphi_j}_{\xi k}\rangle$ holds for all $j \geqslant j(\xi,k)$. We define $\mathbb{Q}^{\Phi}_\xi = \{\boldsymbol{Q}^{\Phi}_{\xi k} \restriction_s : k < \omega \wedge s \in \boldsymbol{Q}^{\Phi}_{\xi k}\}$.

(4) We finally put $\boldsymbol{\varphi} = \langle \mathbb{Q}^{\Phi}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ and $\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varphi} = \langle \mathbb{P}_\xi \cup \mathbb{Q}^{\Phi}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$.

(5) Let $\boldsymbol{\varphi} = \boldsymbol{\varphi}[\Phi]$ be the multiforcing obtained in this way from an $\mathfrak{M}^+$-generic sequence Φ. Then $\boldsymbol{\varphi}$ is called an $\mathfrak{M}$-*generic refinement* of $\boldsymbol{\pi}$.

[6] We recall that HC = all *hereditarily countable* sets, that is, those having at most countable transitive closures.

[7] The density means that for any $\langle m, \psi\rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ there is an $\langle n, \varphi\rangle \in D$ with $\langle n, \varphi\rangle \preccurlyeq \langle m, \psi\rangle$. The openness means that if $\langle m, \psi\rangle \in D$ and $\langle n, \varphi\rangle \preccurlyeq \langle m, \psi\rangle$, then $\langle n, \varphi\rangle \in D$.

Lemma 7.2 (by the countability of $\mathfrak{M}^+$). *If* $\mathfrak{M} \in \mathrm{HC}$, *then for every small multiforcing* $\boldsymbol{\pi}$ *there is an* $\mathfrak{M}$*-generic refinement* $\boldsymbol{\varphi}$.

The following theorem is formulated under the hypotheses and notation of Definition 7.1. Its goal is to demonstrate that the construction in Definition 7.1 results in refinements of types $\sqsubset_D$ and $\sqsubset_{\boldsymbol{D}}$.

Theorem 7.3. *Suppose that* $\mathfrak{M} \in \mathrm{HC}$ *is transitive and* $\boldsymbol{\varphi} = \boldsymbol{\varphi}[\Phi] = \langle \mathbb{Q}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|}$ *is an* $\mathfrak{M}$*-generic refinement of a small multiforcing* $\boldsymbol{\pi} = \langle \mathbb{P}_\xi \rangle_{\xi \in |\boldsymbol{\pi}|} \in \mathfrak{M}$. *Then the following assertions hold.*

(i) $\boldsymbol{\varphi}$ *is a small special multiforcing*, $|\boldsymbol{\varphi}| = |\boldsymbol{\pi}|$ *and* $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$.

(ii) *If pairs* $\langle \xi, k \rangle \neq \langle \eta, \ell \rangle$ *belong to* $|\boldsymbol{\pi}| = |\boldsymbol{\varphi}|$, *then* $[\boldsymbol{Q}^\Phi_{\xi k}] \cap [\boldsymbol{Q}^\Phi_{\eta \ell}] = \varnothing$.

(iii) *If* $\xi \in |\boldsymbol{\pi}|$, $S \in \mathbb{Q}_\xi$ *and* $T \in \mathbb{P}_\xi$, *then* $[S] \cap [T]$ *is closed and open in* $[S]$ *and* $T \not\subseteq S$. *Hence* $\mathbb{Q}_\xi \cap \mathbb{P}_\xi = \varnothing$.

(iv) *If* $\xi \in |\boldsymbol{\pi}|$, *then the set* $\mathbb{Q}_\xi$ *is open-dense in* $\mathbb{Q}_\xi \cup \mathbb{P}_\xi$.

(v) *If* $\xi \in |\boldsymbol{\pi}|$ *and* $D \in \mathfrak{M}$ *is such that* $D \subseteq \mathbb{P}_\xi$ *is pre-dense in* $\mathbb{P}_\xi$, *then* $\mathbb{P}_\xi \sqsubset_D \mathbb{Q}_\xi$.

(vi) *If in addition* $\boldsymbol{\pi} = \bigcup^{\mathrm{cw}}_{\alpha<\lambda} \boldsymbol{\pi}_\alpha$, *where* $\lambda < \omega_1$ *and* $\langle \boldsymbol{\pi}_\alpha \rangle_{\alpha<\lambda} \in \mathfrak{M}$ *is a* $\sqsubset$*-increasing sequence of small special multiforcings, then* $\boldsymbol{\pi}_\alpha \sqsubset \boldsymbol{\varphi}$ *for all* $\alpha < \lambda$.

Proof. We argue using the notation in Definition 7.1.

(ii) By Corollary 3.7, (iii), the set D of all pairs $\langle n, \varphi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, where φ is a pairwise ad system and $|\varphi|$ contains $\langle \xi, k \rangle$ and $\langle \eta, \ell \rangle$, is dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$ and, obviously, $D \in \mathfrak{M}^+$. Thus $\langle n_j, \varphi_j \rangle \in D$ for some $j < \omega$. Then $T^{\varphi_j}_{\xi k} \cap T^{\varphi_j}_{\eta \ell} = \varnothing$ since φ_j is ad. But $\boldsymbol{Q}^\Phi_{\xi k} \subseteq T^{\varphi_j}_{\xi k}$ and $\boldsymbol{Q}^\Phi_{\eta \ell} \subseteq T^{\varphi_j}_{\eta \ell}$ by construction.

(iii) Let $S = \boldsymbol{Q}^\Phi_{\xi k}$. To prove the closedness and openness, we consider the set $D(T)$ of all pairs $\langle n, \varphi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ such that $\langle \xi, k \rangle \in |\varphi|$ and if $s \in 2^n$, then $T^\varphi_{\xi k} \restriction_s \subseteq T$ or $[T^\varphi_{\xi k}] \cap [T] = \varnothing$, and note that this set is dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$. To prove that $T \not\subseteq S$, we similarly consider the set $D'(T)$ of all pairs $\langle n, \varphi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ such that $\langle \xi, k \rangle \in |\varphi|$ and $T \not\subseteq T^\varphi_{\xi k}$. This set is dense. We note that $D(T), D'(T) \in \mathfrak{M}^+$ and argue as above.

(iv) The openness follows easily from (iii). To prove the density, suppose that $T \in \mathbb{P}_\xi$. Let $\Delta(T)$ be the set of all pairs $\langle n, \varphi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ such that $\langle \xi, k \rangle \in |\varphi|$ and $T^\varphi_{\xi k} = T$ for some k. Then this set belongs to $\mathfrak{M}^+$ and is dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$.

(i) By construction, the sets $\boldsymbol{\varphi}(\xi) = \mathbb{Q}^\Phi_\xi$ are special arboreal forcings. Hence $\boldsymbol{\varphi}$ is a small special multiforcing and $|\boldsymbol{\varphi}| = |\boldsymbol{\pi}|$. To establish that $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$, suppose that $\xi \in |\boldsymbol{\pi}|$. We have to prove that $\mathbb{P}_\xi \sqsubset \mathbb{Q}_\xi$. Condition (1) of Definition 5.1 follows from (iv), condition (3) from (iii), and (2) holds because $\boldsymbol{Q}^\Phi_{\xi k} \subseteq T^{\varphi_j}_{\xi k} \in \bigcup^{\mathrm{fin}} \mathbb{P}_\xi$ for some j.

(v) Assume that $\xi \in |\boldsymbol{\pi}|$, $k < \omega$ and $D \in \mathfrak{M}^+$ is pre-dense in $\mathbb{P}_\xi$. Then the set $D' = \{T \in \mathbb{P}_\xi \colon \exists S \in D (T \subseteq S)\}$ is open-dense in $\mathbb{P}_\xi$ and, therefore, Δ is dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$ by Lemma 3.6, where $\Delta \in \mathfrak{M}^+$ is the set of all pairs $\langle n, \varphi \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ such that $\langle \xi, k \rangle \in |\varphi|$ and $T^\varphi_{\xi k} \restriction_s \in D'$ for all $s \in 2^n \cap T^\varphi_{\xi k}$. Thus $\langle n_j, \varphi_j \rangle \in \Delta$ for some j. It follows that $\boldsymbol{Q}^\Phi_{\xi k} \subseteq T^{\varphi_j}_{\xi k} \subseteq^{\mathrm{fin}} \bigcup D$.

(vi) We need to prove that $\boldsymbol{\pi}_\alpha(\xi) \sqsubset \boldsymbol{\varphi}(\xi)$ when $\xi \in |\boldsymbol{\pi}_\alpha|$. Since the relation $\boldsymbol{\pi}(\xi) \sqsubset \boldsymbol{\varphi}(\xi)$ has already been checked, it suffices to prove that $\boldsymbol{Q}^\Phi_{\xi k} \subseteq^{\mathrm{fin}} \bigcup \boldsymbol{\pi}_\alpha(\xi)$. However, $D = \boldsymbol{\pi}_\alpha(\xi)$ is pre-dense in $\boldsymbol{\pi}(\xi) = \mathbb{P}_\xi$ by Lemma 5.2, (iv) and we still have $D \in \mathfrak{M}^+$. Hence (v) can be applied. $\square$

Corollary 7.4. *Under the hypotheses of Lemma 7.2, if $|\boldsymbol{\pi}| \subseteq Z \subseteq \omega_1$ and Z is countable, then there is a small special multiforcing $\boldsymbol{\varphi}$ such that $|\boldsymbol{\varphi}| = Z$ and $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$.*

Proof. If $|\boldsymbol{\pi}| = Z$, then let $\mathfrak{M} \in \mathrm{HC}$ be any countable set containing $\boldsymbol{\pi}$, pick $\boldsymbol{\varphi}$ by Lemma 7.2 and apply Theorem 7.3. But if $|\boldsymbol{\pi}| \subsetneqq Z$, then we trivially extend the construction by $\boldsymbol{\varphi}(\xi) = \mathbb{P}_{\mathrm{coh}}$ (Example 3.4) for all $\xi \in Z \setminus |\boldsymbol{\pi}|$. □

§ 8. Generic refinement: sealing dense sets

Here we prove an important corollary of the $\mathfrak{M}^+$-genericity of multiforcing refinements: the relation $\sqsubset_{\boldsymbol{D}}$ (see Definition 6.2) between a multiforcing and a refinement of it, where $\boldsymbol{D}$ is a dense set.

Theorem 8.1. *Under the hypotheses of Theorem 7.3 suppose that $\boldsymbol{D} \in \mathfrak{M}^+$, $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$ and $\boldsymbol{D}$ is open-dense in $\mathbf{MT}(\boldsymbol{\pi})$. Then $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi}$.*

Proof. By hypothesis, $\boldsymbol{\varphi} = \boldsymbol{\varphi}[\Phi]$ is obtained from a decreasing $\mathfrak{M}^+$-generic sequence Φ of pairs $\langle n_j, \varphi_j\rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ as in Definition 7.1, (1). We work in the notation of Definition 7.1. Suppose that $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varphi})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$ as in the condition $(*)$ of Definition 6.2. The extra condition $|\boldsymbol{u}| \subseteq |\boldsymbol{\pi}|$ holds automatically since $|\boldsymbol{\varphi}| = |\boldsymbol{\pi}|$. We have to find a multitree $\boldsymbol{q}$ which gives 6.2, $(*)$ for $\boldsymbol{u}$.

Each term $T^{\boldsymbol{u}}_\xi$ of $\boldsymbol{u}$ ($\xi \in |\boldsymbol{u}|$) is equal to some $\boldsymbol{Q}^{\Phi}_{\xi,k_\xi}\restriction_{t_\xi}$, where $t_\xi \in \boldsymbol{Q}^{\Phi}_{\xi,k_\xi}$. There is no loss of generality in assuming that $t_\xi = \Lambda$ and, therefore, $T^{\boldsymbol{u}}_\xi = \boldsymbol{Q}^{\Phi}_{\xi,k_\xi}$ $\forall\,\xi$.

Definition 8.2. If $n < \omega$, then $\mathbf{Sys}_n(\boldsymbol{\pi})$ contains all systems $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ such that $\langle \xi, k_\xi\rangle \in |\varphi|$ for all $\xi \in |\boldsymbol{u}|$ and $T^{\varphi}_{\xi k}\restriction_t \in \mathbb{P}_\xi = \boldsymbol{\pi}(\xi)$ (not merely $\in \bigcup^{\mathrm{fin}} \mathbb{P}_\xi$!) for all $\langle \xi, k\rangle \in |\varphi|$ and $t \in 2^n \cap T^{\varphi}_{\xi k}$.

If $\varphi \in \mathbf{Sys}_n(\boldsymbol{\pi})$, then $\mathbf{S}^n_\varphi$ contains all multistrings $\mathbf{s} = \langle s_\xi\rangle_{\xi\in|\boldsymbol{u}|}$ such that $s_\xi \in 2^n \cap T^{\varphi}_{\xi,k_\xi}$ for all $\forall\,\xi \in |\boldsymbol{u}|$. If $\mathbf{s} = \langle s_\xi\rangle_{\xi\in|\boldsymbol{u}|} \in \mathbf{S}^n_\varphi$, then we define $\boldsymbol{v}^{\mathbf{s}}_\varphi \in \mathbf{MT}(\boldsymbol{\pi})$ by $|\boldsymbol{v}^{\mathbf{s}}_\varphi| = |\boldsymbol{u}|$ and $T^{\boldsymbol{v}^{\mathbf{s}}_\varphi}_\xi = T^{\varphi}_{\xi,k_\xi}\restriction_{s_\xi}$ for all $\xi \in |\boldsymbol{u}|$.

Lemma 8.3. *Suppose that $n < \omega$ and $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$. Then there is a system $\psi \in \mathbf{Sys}_n(\boldsymbol{\pi})$ such that $\langle n, \psi\rangle \preccurlyeq \langle n, \varphi\rangle$.*

Proof. Adjoin every absent pair $\langle \xi, k_\xi\rangle \notin |\varphi|$ to $|\psi|$ and define $T^{\psi}_{\xi,k_\xi} \in \mathbb{P}_\xi$ arbitrarily. If $\langle \xi, k\rangle \in |\psi|$ and $t \in 2^n \cap T^{\psi}_{\xi k}$, but $T^{\psi}_{\xi k}\restriction_t \in \bigcup^{\mathrm{fin}} \mathbb{P}_\xi \setminus \mathbb{P}_\xi$, then shrink $T^{\psi}_{\xi k}$ to a tree in $\mathbb{P}_\xi$ by Lemma 3.7, (i) and do this for all such triples ξ, k, t. □

Lemma 8.4. *Suppose that $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$ and $|\boldsymbol{r}| \cap |\boldsymbol{u}| = \varnothing$. Let $\Delta_{\boldsymbol{r}} \in \mathfrak{M}$ be the set of all pairs $\langle n, \varphi\rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ such that $\varphi \in \mathbf{Sys}_n(\boldsymbol{\pi})$ and there is a $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying $\boldsymbol{q} \leqslant \boldsymbol{r}$, $|\boldsymbol{u}| \cap |\boldsymbol{q}| = \varnothing$ and (1) if $\mathbf{s} \in \mathbf{S}^n_\varphi$, then $\boldsymbol{v}^{\mathbf{s}}_\varphi \cup \boldsymbol{q} \in \boldsymbol{D}$. Then $\Delta_{\boldsymbol{r}}$ is dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$.*

Proof. Given any $\langle n, \psi\rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$, we shall find a pair $\langle n, \varphi\rangle \in \Delta_{\boldsymbol{r}}$ (same n!) such that $\langle n, \varphi\rangle \preccurlyeq \langle n, \psi\rangle$. By Lemma 8.3, there is no loss of generality in assuming that $\psi \in \mathbf{Sys}_n(\boldsymbol{\pi})$.

Let $\mathbf{s} = \langle s_\xi\rangle_{\xi\in|\boldsymbol{u}|} \in \mathbf{S}^n_\psi$. Consider the multitree $\boldsymbol{v}^{\mathbf{s}}_\psi \in \mathbf{MT}(\boldsymbol{\pi})$. Since $\boldsymbol{D}$ is dense, there are multitrees $\boldsymbol{r}', \boldsymbol{v} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{v}| = |\boldsymbol{u}|$, $\boldsymbol{v} \leqslant \boldsymbol{v}^{\mathbf{s}}_\psi$, $|\boldsymbol{r}'| \cap |\boldsymbol{u}| = \varnothing$, $\boldsymbol{r}' \leqslant \boldsymbol{r}$ and $\boldsymbol{v} \cup \boldsymbol{r}' \in \boldsymbol{D}$. Define a system $\psi' \in \mathbf{Sys}(\boldsymbol{\pi})$ with $|\psi'| = |\psi|$ that extends ψ

by shrinking each tree $T^{\psi}_{\xi,k_\xi}{\restriction}_{s_\xi}$ to $T^{\boldsymbol v}_{\xi}$, so that $T^{\psi'}_{\xi,k_\xi}{\restriction}_{s_\xi} = T^{\boldsymbol v}_{\xi}$ but $T^{\psi'}_{\xi,k_\xi}{\restriction}_t = T^{\psi}_{\xi,k_\xi}{\restriction}_t$ for all $t \in 2^n \cap T^{\psi}_{\xi,k_\xi}$, $t \ne s_\xi$, and $T^{\psi'}_{\eta k} = T^{\psi}_{\eta k}$ whenever $\langle \eta, k\rangle \in |\psi|$ is not of the form $\langle \xi, k_\xi\rangle$, where $\xi \in |\boldsymbol u|$. We have $\langle n, \psi'\rangle \preccurlyeq \langle n, \psi\rangle$ by construction. Therefore $\mathbf{S}^n_{\psi'} = \mathbf{S}^n_{\psi}$.

This construction can be iterated, taking all the strings $\mathbf{s} \in \mathbf{S}^n_{\psi}$ one by one. This results in a system $\varphi \in \mathbf{Sys}(\boldsymbol\pi)$ such that $|\varphi| = |\psi|$ and $\langle n, \varphi\rangle \preccurlyeq \langle n, \psi\rangle$. Then $\mathbf{S}^n_{\varphi} = \mathbf{S}^n_{\psi}$ and there is a multitree $\boldsymbol q \in \mathbf{MT}(\boldsymbol\pi)$ with $\boldsymbol q \leqslant \boldsymbol r$ and $|\boldsymbol q| \cap |\boldsymbol u| = \varnothing$ such that if $\mathbf{s} \in \mathbf{S}^n_{\psi}$, then the multitree $\boldsymbol v^{\mathbf s}_{\psi}$ satisfies $\boldsymbol v^{\mathbf s}_{\psi} \cup \boldsymbol q \in \boldsymbol D$. This $\boldsymbol q$ shows that $\langle n, \varphi\rangle \in \Delta_r$. $\square$

By the lemma, we have $\langle n_j, \varphi_j\rangle \in \Delta_{\boldsymbol p}$ for some j. Suppose that this is shown by a multitree $\boldsymbol q \in \mathbf{MT}(\boldsymbol\pi)$, so that $\boldsymbol q \leqslant \boldsymbol p$, $|\boldsymbol u| \cap |\boldsymbol q| = \varnothing$ and condition (1) of Lemma 8.4 holds with $n = n_j$, $\varphi = \varphi_j$. We easily conclude that $[\boldsymbol u] \subseteq \bigcup_{\mathbf s \in \mathbf S^n_{\varphi_j}} [\boldsymbol v^{\mathbf s}_{\varphi_j}]$. However, $\boldsymbol v^{\mathbf s}_{\varphi_j} \in \boldsymbol D^{|\boldsymbol u|}_{\boldsymbol q}$ $\forall \mathbf s$ by (1). $\square$

Corollary 8.5. *Under the hypotheses of Theorem 7.3 assume that $\boldsymbol D \in \mathfrak M$ and $\boldsymbol D \subseteq \mathbf{MT}(\boldsymbol\pi)$ is pre-dense in $\mathbf{MT}(\boldsymbol\pi)$. Then it is also pre-dense in $\mathbf{MT}(\boldsymbol\pi \cup^{\mathrm{cw}} \boldsymbol\varrho)$.*

Proof. There is no loss of generality in assuming that $\boldsymbol D$ is open-dense in $\mathbf{MT}(\boldsymbol\pi)$. (Otherwise consider $\boldsymbol D' = \{\boldsymbol p \in \mathbf{MT}(\boldsymbol\pi) \colon \exists\, \boldsymbol q \in \boldsymbol D\, (\boldsymbol p \leqslant \boldsymbol q)\}$.) Note that $\boldsymbol\pi \sqsubset_{\boldsymbol D} \boldsymbol\varrho$ by Theorem 8.1 and use Lemma 6.3, (i). $\square$

§ 9. Real names and direct forcing

Our next goal is to introduce suitable notation related to the names of reals in 2^ω in the context of forcing notions of the form $\mathbf{MT}(\boldsymbol\pi)$.

Definition 9.1. A *real name* is any set $\mathbf c \subseteq \mathbf{MT} \times (\omega \times 2)$ such that the sets $K^{\mathbf c}_{ni} = \{\boldsymbol p \in \mathbf{MT} \colon \langle \boldsymbol p, n, i\rangle \in \mathbf c\}$ satisfy the following condition. If $n < \omega$ and $\boldsymbol p \in K^{\mathbf c}_{n0}$, $\boldsymbol q \in K^{\mathbf c}_{n1}$, then $\boldsymbol p, \boldsymbol q$ are sad.[8] We put $K^{\mathbf c}_n = K^{\mathbf c}_{n0} \cup K^{\mathbf c}_{n1}$.

A real name $\mathbf c$ is *small* if each $K^{\mathbf c}_n$ is at most countable. Then the set $|\mathbf c| = \bigcup_n \bigcup_{\boldsymbol p \in K^{\mathbf c}_n} |\boldsymbol p|$ and $\mathbf c$ itself are also countable.

Let $\boldsymbol\pi$ be a multiforcing. A real name $\mathbf c$ is $\boldsymbol\pi$-*complete* if every set $K^{\mathbf c}_n \uparrow \boldsymbol\pi = \{\boldsymbol p \in \mathbf{MT}(\boldsymbol\pi) \colon \exists\, \boldsymbol q \in K^{\mathbf c}_n\, (\boldsymbol p \leqslant \boldsymbol q)\}$ (the $\boldsymbol\pi$-cone of $K^{\mathbf c}_n$) is pre-dense in $\mathbf{MT}(\boldsymbol\pi)$. In this case, if a set (a filter) $G \subseteq \mathbf{MT}(\boldsymbol\pi)$ is $\mathbf{MT}(\boldsymbol\pi)$-generic over the family of all sets $K^{\mathbf c}_n$, then we define a real $\mathbf c[G] \in 2^\omega$ in such a way that $\mathbf c[G](n) = i$ if and only if $G \cap C_{ni} \ne \varnothing$.

We do not require in this case that $\mathbf c \subseteq \mathbf{MT}(\boldsymbol\pi) \times (\omega \times 2)$ or, equivalently, $K^{\mathbf c}_n \subseteq \mathbf{MT}(\boldsymbol\pi)$ for all n. When this inclusion does hold, this will be mentioned explicitly.

Let $\mathbf c$ be a real name in the sense of 9.1. We say that the multitree $\boldsymbol p$

– *directly forces* $\mathbf c(n) = i$ (where $n < \omega$ and $i = 0, 1$) if there is a multitree $\boldsymbol q \in K^{\mathbf c}_{ni}$ such that $\boldsymbol p \leqslant \boldsymbol q$;

– *directly forces* $s \subset \mathbf c$ (where $s \in 2^{<\omega}$) if, for every $n < \mathrm{lh}(s)$, $\boldsymbol p$ directly forces $\mathbf c(n) = i$, where $i = s(n)$;

[8] We recall that the condition sad of somewhere almost disjointness (Definition 4.2) is equivalent to the incompatibility of $\boldsymbol p, \boldsymbol q$ in $\mathbf{MT}$ and in any set of the form $\mathbf{MT}(\boldsymbol\pi)$, where $\boldsymbol\pi$ is a regular multiforcing, by Corollary 4.3.

– *directly forces* $\mathbf{c} \notin [T]$ (where $T \in \mathbf{PT}$) when there is a string $s \in 2^{<\omega} \setminus T$ such that $\boldsymbol{p}$ directly forces $s \subset \mathbf{c}$.

The definition of direct forcing is not explicitly associated with any concrete forcing notion, but it is in fact compatible with any multiforcing.

Lemma 9.2. *Let $\boldsymbol{\pi}$ be a multiforcing, $\mathbf{c}$ a $\boldsymbol{\pi}$-complete real name and $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$. When $n < \omega$, we can find an $i = 0, 1$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ with $\boldsymbol{q} \leqslant \boldsymbol{p}$ that directly forces $\mathbf{c}(n) = i$. When $T \in \mathbf{PT}$, we can find an $s \in T$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ with $\boldsymbol{q} \leqslant \boldsymbol{p}$ that directly forces $\mathbf{c} \notin [T{\restriction_s}]$.*

Proof. To prove the first claim, we use the density of the sets $K_n^{\mathbf{c}}$ in the light of Definition 9.1. To prove the second claim, pick an n such that $T \cap 2^n$ contains at least two strings. By the first claim, one can find a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ with $\boldsymbol{q} \leqslant \boldsymbol{p}$ and a string $t \in T \cap 2^n$ such that $\boldsymbol{q}$ directly forces $t \subset \mathbf{c}$. Now take any $s \in T \cap 2^n$, $s \neq t$. □

§ 10. Sealing real names and avoiding refinements

The following definition extends Definition 6.2 to real names.

Definition 10.1. Assume that $\boldsymbol{\pi}$, $\boldsymbol{\varrho}$ are multiforcings, $\mathbf{c}$ is a $\boldsymbol{\pi}$-complete real name and $\boldsymbol{\pi} \sqsubset \boldsymbol{\varrho}$. We say that $\boldsymbol{\varrho}$ *seals* $\mathbf{c}$ *over* $\boldsymbol{\pi}$ (and write $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho}$) if $\boldsymbol{\varrho}$ seals every set $K_n^{\mathbf{c}} \uparrow \boldsymbol{\pi} = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) \colon \exists\, \boldsymbol{q} \in K_n^{\mathbf{c}}\, (\boldsymbol{p} \leqslant \boldsymbol{q})\}$ over $\boldsymbol{\pi}$ in the sense of Definition 6.2.

Corollary 10.2. *Under the hypotheses of Theorem 7.3, if $\mathbf{c} \in \mathfrak{M}^+$ and $\mathbf{c}$ is a $\boldsymbol{\pi}$-complete real name, then $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho}$.*

Proof. Each set $K_n^{\mathbf{c}} \uparrow \boldsymbol{\pi}$ belongs to $\mathfrak{M}^+$ (along with $\boldsymbol{\pi}$ and $\mathbf{c}$) and is open-dense in $\mathbf{MT}(\boldsymbol{\pi})$. It remains to apply Theorem 8.1. □

Lemma 10.3. *Let $\boldsymbol{\pi}, \boldsymbol{\varrho}, \boldsymbol{\sigma}$ be multiforcings and let $\mathbf{c}$ be a real name. Then the following assertions hold.*

(i) *If $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho}$, then $\mathbf{c}$ is a $\boldsymbol{\pi}$-complete and $(\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varrho})$-complete real name.*

(ii) *If $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho} \sqsubset \boldsymbol{\sigma}$, then $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\sigma}$.*

(iii) *If $\langle \boldsymbol{\pi}_{\alpha\,\alpha<\lambda} \rangle$ is a $\sqsubset$-increasing sequence in $\mathbf{MF}$, $0 < \mu < \lambda$, $\boldsymbol{\pi} = \bigcup^{\mathrm{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha$, and $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\pi}_\mu$, then $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varrho} = \bigcup^{\mathrm{cw}}_{\mu \leqslant \alpha < \lambda} \boldsymbol{\pi}_\alpha$.*

Proof. (i) By definition, we have $\boldsymbol{\pi} \sqsubset_{K_n^{\mathbf{c}} \uparrow \boldsymbol{\pi}} \boldsymbol{\varrho}$ for each n. Therefore $K_n^{\mathbf{c}} \uparrow \boldsymbol{\pi}$ is dense in $\mathbf{MT}(\boldsymbol{\pi})$ (and then clearly open-dense) and pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varrho})$ by Lemma 6.3, (i). Hence $K_n^{\mathbf{c}} \uparrow (\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varrho})$ is dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varrho})$.

To prove (ii) and (iii), use parts (iii) and (iv) of Lemma 6.3. □

If $\boldsymbol{\pi}$ is a multiforcing, then $\mathbf{MT}(\boldsymbol{\pi})$ adjoins a family of *principal generic reals* $x_\xi = x_\xi[G] \in 2^\omega$, $\xi \in |\boldsymbol{\pi}|$, where each x_ξ is $\boldsymbol{\pi}(\xi)$-generic over the ground-set universe; see Remark 4.5. Clearly, many more reals are adjoined, and given a $\boldsymbol{\pi}$-complete real name $\mathbf{c}$, we can establish various requirements for a condition $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$ to force that $\mathbf{c}$ is the name of a real of the form $x_{\xi k}$, or to force the opposite. The following definition provides such a condition related to the 'opposite' direction.

Definition 10.4. Let $\boldsymbol{\pi}$ be a multiforcing, $\xi \in |\boldsymbol{\pi}|$. A real name $\mathbf{c}$ is said to be *non-principal over $\boldsymbol{\pi}$ at ξ* if the following set is open-dense in $\mathbf{MT}(\boldsymbol{\pi})$:

$$\boldsymbol{D}^{\xi}_{\mathbf{c}}(\boldsymbol{\pi}) = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) \colon \xi \in |\boldsymbol{p}| \wedge \boldsymbol{p} \text{ directly forces } \mathbf{c} \notin [T^{\boldsymbol{p}}_{\xi}]\}.$$

We shall prove below (Theorem 12.2, (i)) that non-principality implies that $\mathbf{c}$ is not a name of the real $x_\xi[\underline{G}]$. We shall further show that the avoidance condition in the next definition implies that $\mathbf{c}$ is a name of a non-generic real.

Definition 10.5. Suppose that $\boldsymbol{\pi}$, $\boldsymbol{\varphi}$ are multiforcings, $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$, $\xi \in |\boldsymbol{\pi}|$. We say that $\boldsymbol{\varphi}$ *avoids a real name* $\mathbf{c}$ *over* $\boldsymbol{\pi}$ *at* ξ (and write $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\varphi}$) if, for every $Q \in \boldsymbol{\varphi}(\xi)$, $\boldsymbol{\varphi}$ seals the set

$$\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi}) = \{\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi}) \colon \xi \in |\boldsymbol{r}| \wedge \boldsymbol{r} \text{ directly forces } \mathbf{c} \notin [Q]\}$$

over $\boldsymbol{\pi}$ in the sense of Definition 6.2, that is, formally $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},Q,\boldsymbol{\pi})} \boldsymbol{\varphi}$.

Lemma 10.6. *Assume that* $\boldsymbol{\pi}$, $\boldsymbol{\varphi}$, $\boldsymbol{\sigma}$ *are multiforcings,* $\xi \in |\boldsymbol{\pi}|$ *and* $\mathbf{c}$ *is a* $\boldsymbol{\pi}$*-complete real name. Then the following assertions hold.*

(i) *If* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\varphi}$ *and* $Q \in \boldsymbol{\varphi}(\xi)$, *then the set* $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ *is open-dense in* $\mathbf{MT}(\boldsymbol{\pi})$ *and pre-dense in* $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varphi})$.

(ii) *If* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\varphi} \sqsubset \boldsymbol{\sigma}$, *then* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\sigma}$.

(iii) *If* $\langle \boldsymbol{\pi}_{\alpha\alpha<\lambda} \rangle$ *is a* $\sqsubset$*-increasing sequence in* $\mathbf{MF}$, $0 < \mu < \lambda$, $\boldsymbol{\pi} = \bigcup^{\mathrm{cw}}_{\alpha<\mu} \boldsymbol{\pi}_\alpha$ *and* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\pi}_\mu$, *then* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\varphi} = \bigcup^{\mathrm{cw}}_{\mu\leqslant\alpha<\lambda} \boldsymbol{\pi}_\alpha$.

Proof. (i) Apply Lemma 6.3, (i). To prove part (ii), suppose that $S \in \boldsymbol{\sigma}(\xi)$. Since $\boldsymbol{\varphi} \sqsubset \boldsymbol{\sigma}$, there is a finite set $\{Q_1, \dots, Q_m\} \subseteq \boldsymbol{\varphi}(\xi)$ such that $S \subseteq Q_1 \cup \dots \cup Q_m$. We have $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},Q_i,\boldsymbol{\pi})} \boldsymbol{\varphi}$ for all i because $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\varphi}$. Hence $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},Q_i,\boldsymbol{\pi})} \boldsymbol{\sigma}$ $\forall\, i$, by Lemma 6.3, (iii). Note that $\bigcap_i \boldsymbol{D}(\mathbf{c}, Q_i, \boldsymbol{\pi}) \subseteq \boldsymbol{D}(\mathbf{c}, S, \boldsymbol{\pi})$ since $S \subseteq \bigcup_i Q_i$. We conclude that $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}(\mathbf{c},S,\boldsymbol{\pi})} \boldsymbol{\sigma}$ by Lemma 6.3, (ii). Therefore $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\sigma}$, as required.

To prove (iii), we use Lemma 6.3, (iv) in the same way. □

§ 11. Generic refinement avoids non-principal names

The following theorem says that generic refinements as in § 7 avoid non-principal names. This resembles Theorem 8.1 to some extent, but the latter is not directly applicable here because the multitree Q and the set $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ depend on $\boldsymbol{\varphi}$ and, therefore, the sets $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ need not belong to $\mathfrak{M}^+$. However, the proof will be based on rather similar arguments.

Theorem 11.1. *Under the hypotheses of Theorem* 7.3, *if* $\eta \in |\boldsymbol{\pi}| \subseteq \mathfrak{M}$ *and* $\mathbf{c} \in \mathfrak{M}$ *is a* $\boldsymbol{\pi}$*-complete real name non-principal over* $\boldsymbol{\pi}$ *at* η, *then* $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\eta} \boldsymbol{\varphi}$.

Proof. By hypothesis, $\boldsymbol{\varphi} = \boldsymbol{\varphi}[\Phi]$ is obtained from a decreasing $\mathfrak{M}^+$-generic sequence Φ of pairs $\langle n_j, \varphi_j \rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ as in Definition 7.1, (1). We work in the notation of Definition 7.1.

Suppose that $Q \in \boldsymbol{\varphi}(\eta) = \mathbb{P}^{\boldsymbol{\varphi}}_{\eta}$. We need to prove that $\boldsymbol{\varphi}$ seals the set $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ over $\boldsymbol{\pi}$. By construction, $Q = \boldsymbol{Q}^{\Phi}_{\eta K}{\restriction}_s$ for some $K < \omega$ and $s \in \boldsymbol{Q}^{\Phi}_{\eta K}$. It can be assumed that simply $Q = \boldsymbol{Q}^{\Phi}_{\eta K}$. Following the proof of Theorem 8.1, we suppose that $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{u} \in \mathbf{MT}(\boldsymbol{\varphi})$, $|\boldsymbol{u}| \cap |\boldsymbol{p}| = \varnothing$ and $T^{\boldsymbol{u}}_{\xi} = \boldsymbol{Q}^{\Phi}_{\xi,k_\xi}$ for all $\xi \in |\boldsymbol{u}|$. We have to find a multitree $\boldsymbol{q}$ which gives 6.2, $(*)$ for $\boldsymbol{u}$, $\boldsymbol{p}$, $\boldsymbol{D} = \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$. Note that η may or may not belong to the set $|\boldsymbol{u}|$ and, even when $\eta \in |\boldsymbol{u}|$ and, therefore, k_η is defined, K may or may not be equal to k_η. In the remainder of the proof, we use the notation in Definition 8.2, in particular, $\mathbf{Sys}_n(\boldsymbol{\pi})$, $\mathbf{S}^n_\varphi$, $\boldsymbol{v}^{\mathbf{s}}_\varphi$.

Assume that $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$, $|\boldsymbol{r}| \cap |\boldsymbol{u}| = \varnothing$. Consider the set $\Delta_{\boldsymbol{r}} \in \mathfrak{M}$ of all pairs $\langle n, \varphi\rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$ such that $\varphi \in \mathbf{Sys}_n(\boldsymbol{\pi})$ (Definition 8.2), $\langle \eta, K\rangle \in |\varphi|$ and there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$ satisfying $\boldsymbol{q} \leqslant \boldsymbol{r}$, $|\boldsymbol{u}| \cap |\boldsymbol{q}| = \varnothing$ and

(1′) if $\mathbf{s} \in \mathbf{S}^n_\varphi$ and $t \in T^\varphi_{\eta K} \cap 2^n$, then $\boldsymbol{v}^{\mathbf{s}}_\varphi \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^\varphi_{\eta K}\restriction_t]$.

Condition (1′) is similar to condition (1) of Lemma 8.4, of course. Note that direct forcing of $\mathbf{c} \notin [Q]$ cannot be used in (1′) since Q need not belong to $\mathfrak{M}$. But $\mathbf{c} \notin [T^\varphi_{\eta K}]$ appears to be an effective replacement.

Lemma 11.2. *If $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$ and $|\boldsymbol{r}| \cap |\boldsymbol{u}| = \varnothing$, then $\Delta_{\boldsymbol{r}}$ is dense in $\omega \times \mathbf{Sys}(\boldsymbol{\pi})$.*

Proof. We follow the proof of Lemma 8.4. Suppose that $\langle n, \psi\rangle \in \omega \times \mathbf{Sys}(\boldsymbol{\pi})$. There is no loss of generality in assuming that $\psi \in \mathbf{Sys}_n(\boldsymbol{\pi})$ (see Lemma 8.4), whence $\langle \xi, k_\xi\rangle \in |\psi|$ for all $\xi \in |\boldsymbol{u}|$, $T^\psi_{\xi k}\restriction_t \in \mathbb{P}_\xi$ for all $\langle \xi, k\rangle \in |\psi|$ and $t \in 2^n \cap T^\psi_{\xi k}$, and $\langle \eta, K\rangle \in |\psi|$ as well.

We have to define a system $\varphi \in \mathbf{Sys}_n(\boldsymbol{\pi})$ such that $\langle n, \varphi\rangle \preccurlyeq \langle n, \pi\rangle$ and $\varphi \in \Delta_r$. As in the proof of Lemma 8.4, it suffices to fulfill (1′) for one particular pair of $\mathbf{s} = \langle s_\xi\rangle_{\xi \in |\boldsymbol{u}|} \in \mathbf{S}^n_\psi$ and $t \in T^\psi_{\eta K} \cap 2^n$; the final goal is then achieved by a simple iteration through all such pairs. We have two cases.

Case 1. $\eta \in |\boldsymbol{u}|$, $K = k_\eta$, $t = s_\eta$. Consider the multitree $\boldsymbol{v}^{\mathbf{s}}_\psi \in \mathbf{MT}(\boldsymbol{\pi})$. The set $\boldsymbol{D}^\eta_{\mathbf{c}}(\boldsymbol{\pi})$, as in Definition 10.4, is dense because $\mathbf{c}$ is non-principal. Hence there are multitrees $\boldsymbol{q}, \boldsymbol{v} \in \mathbf{MT}(\boldsymbol{\pi})$ such that $|\boldsymbol{v}| = |\boldsymbol{u}|$, $\boldsymbol{v} \leqslant \boldsymbol{v}^{\mathbf{s}}_\psi$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, $\boldsymbol{q} \leqslant \boldsymbol{r}$ and $\boldsymbol{v} \cup \boldsymbol{q} \in \boldsymbol{D}^\eta_{\mathbf{c}}(\boldsymbol{\pi})$. Therefore $\boldsymbol{v} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\boldsymbol{q}}_\eta]$. Using ψ, we define a system $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ with $|\varphi| = |\psi|$ by performing the following operations:

(a) shrinking each tree $T^\psi_{\xi,k_\xi}\restriction_{s_\xi}$ $(\xi \in |\boldsymbol{u}|)$ to $T^{\boldsymbol{v}}_\xi$, so that $T^\varphi_{\xi,k_\xi}\restriction_{s_\xi} = T^{\boldsymbol{v}}_\xi$,

(b) in particular, shrinking $T^\psi_{\eta K}\restriction_t$ to $T^{\boldsymbol{v}}_\eta$, so that $T^\varphi_{\eta K}\restriction_t = T^{\boldsymbol{v}}_\eta$,

and making no other changes. We have $\langle n, \varphi\rangle \preccurlyeq \langle n, \psi\rangle$, $\boldsymbol{v}^{\mathbf{s}}_\varphi = \boldsymbol{v}$ and $T^\varphi_{\eta K}\restriction_t = T^{\boldsymbol{v}}_\eta$ by construction. In particular, $\boldsymbol{v}^{\mathbf{s}}_\varphi \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^\varphi_{\eta K}\restriction_t]$. Thus (1′) holds.

Case 2: not Case 1. By Lemma 9.2 there are multitrees $\boldsymbol{q}, \boldsymbol{v} \in \mathbf{MT}(\boldsymbol{\pi})$ and a tree $T \in \mathbb{P}_\eta$ such that $T \subseteq T^\psi_{\eta K}\restriction_t$, $|\boldsymbol{v}| = |\boldsymbol{u}|$, $\boldsymbol{v} \leqslant \boldsymbol{v}^{\mathbf{s}}_\psi$, $|\boldsymbol{q}| \cap |\boldsymbol{u}| = \varnothing$, $\boldsymbol{q} \leqslant \boldsymbol{r}$ and $\boldsymbol{v} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T]$. We define a system $\varphi \in \mathbf{Sys}(\boldsymbol{\pi})$ with $|\varphi| = |\psi|$ which extends ψ by performing the operations (a) and

(c) shrinking $T^\psi_{\eta K}\restriction_t$ to T, so that $T^\varphi_{\eta K}\restriction_t = T$

and making no other changes. Note that (a) and (c) do not contradict each other since $\langle \eta, T, t\rangle \neq \langle \xi, k_\xi, s_\xi\rangle$ for all $\xi \in \boldsymbol{u}$ by the hypotheses of Case 2. By construction, we have $\langle n, \varphi\rangle \preccurlyeq \langle n, \psi\rangle$, $\boldsymbol{v}^{\mathbf{s}}_\varphi = \boldsymbol{v}$ and $T^\varphi_{\eta K}\restriction_t = T^{\boldsymbol{v}}_\eta$. In particular, $\boldsymbol{v}^{\mathbf{s}}_\varphi \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^\varphi_{\eta K}\restriction_t]$. Thus (1′) holds. □

We return to the theorem. Since $\Delta_{\boldsymbol{p}} \in \mathfrak{M}^+$, we have $\langle n_j, \varphi_j\rangle \in \Delta_{\boldsymbol{p}}$ for some j by the lemma. Let this be shown by a multitree $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$, so that $\boldsymbol{q} \leqslant \boldsymbol{p}$, $|\boldsymbol{u}| \cap |\boldsymbol{q}| = \varnothing$ and (1′) holds with $n = n_j$, $\varphi = \varphi_j$. In particular, since $T^{\varphi_j}_{\eta K} = \bigcup_{t \in T^{\varphi_j}_{\eta K} \cap 2^n} T^{\varphi_j}_{\eta K}\restriction_t$, the multitree $\boldsymbol{v}^{\mathbf{s}}_{\varphi_j} \cup \boldsymbol{q}$ directly forces $\mathbf{c} \notin [T^{\varphi_j}_{\eta K}]$ whenever $\mathbf{s} \in \mathbf{S}^n_{\varphi_j}$ and, therefore, directly forces $\mathbf{c} \notin [Q]$ because $Q = \boldsymbol{Q}^{\Phi}_{\eta K} \subseteq T^{\varphi_j}_{\eta K}$ by construction. Thus, if $\mathbf{s} \in \mathbf{S}^n_{\varphi_j}$, then $\boldsymbol{v}^{\mathbf{s}}_{\varphi_j} \cup \boldsymbol{q} \in \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ and, therefore, $\boldsymbol{v}^{\mathbf{s}}_{\varphi_j} \in \boldsymbol{D}(\mathbf{c}, Q, \varphi_j)^{|\boldsymbol{u}|}_{\boldsymbol{q}})$. On the other hand, $[\boldsymbol{u}] \subseteq \bigcup_{\mathbf{s} \in \mathbf{S}^n_{\varphi_j}} [\boldsymbol{v}^{\mathbf{s}}_{\varphi_j}]$, so that $\boldsymbol{u} \subseteq^{\text{fin}} \bigvee \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi}^{|\boldsymbol{u}|}_{\boldsymbol{q}})$, as required. □

§ 12. Consequences for generic extensions

We first prove a lemma on the adequate representation of reals in $\mathbf{MT}(\boldsymbol{\pi})$-generic extensions by real names. Then Theorem 12.2 will have corollaries for non-principal names.

Lemma 12.1. *Suppose that $\boldsymbol{\pi}$ is a regular multiforcing and $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is generic over the ground-set universe $\mathbf{V}$.*

If $x \in \mathbf{V}[G] \cap 2^\omega$, then there is a $\boldsymbol{\pi}$-complete real name $\mathbf{c} \in \mathbf{V}$, $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times \omega \times 2$, such that $x = \mathbf{c}[G]$.

If $\mathbf{MT}(\boldsymbol{\pi})$ is a CCC forcing[9] *in $\mathbf{V}$, and $\mathbf{c} \in \mathbf{V}$, $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times \omega \times 2$, is a $\boldsymbol{\pi}$-complete real name, then there is a small $\boldsymbol{\pi}$-complete real name $\mathbf{d} \in \mathbf{V}$, $\mathbf{d} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times \omega \times 2$, such that $\mathbf{MT}(\boldsymbol{\pi})$ forces $\mathbf{c}[\underline{G}] = \mathbf{d}[\underline{G}]$ over $\mathbf{V}$.*

Proof. The first part is a particular case of a general forcing theorem. To prove the second part, we extend each set $K_n^{\mathbf{c}} \subseteq \mathbf{MT}(\boldsymbol{\pi})$ to an open-dense set $K_n^{\mathbf{c}} \uparrow \boldsymbol{\pi} = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) \colon \exists \boldsymbol{q} \in K_n^{\mathbf{c}}\, (\boldsymbol{p} \leqslant \boldsymbol{q})\}$, choose maximal antichains $A_n \subseteq K_n^{\mathbf{c}} \uparrow \boldsymbol{\pi}$ (they are countable by the CCC) and then put $A_{ni} = \{\boldsymbol{p} \in A_n \colon \exists \boldsymbol{q} \in K_{ni}^{\mathbf{c}}(\boldsymbol{p} \leqslant \boldsymbol{q})\}$ and $\mathbf{d} = \{\langle \boldsymbol{p}, n, i\rangle \colon \boldsymbol{p} \in A_{ni}\}$. $\square$

Theorem 12.2. *Suppose that $\boldsymbol{\pi}$ is a regular multiforcing and $\xi \in |\boldsymbol{\pi}|$. Then the following assertions hold.*

(i) *If $\mathbf{MT}(\boldsymbol{\pi})$ is CCC, a set $G \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is generic over the ground-set universe $\mathbf{V}$ and $x \in \mathbf{V}[G] \cap 2^\omega$, then $x \neq x_\xi[G]$ if and only if there is a small $\boldsymbol{\pi}$-complete real name $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times (\omega \times 2)$ which is non-principal over $\boldsymbol{\pi}$ at ξ and satisfies $x = \mathbf{c}[G]$.*

(ii) *If $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times (\omega \times 2)$ is a $\boldsymbol{\pi}$-complete real name, $\boldsymbol{\varphi}$ is a multiforcing, $\boldsymbol{\pi} \sqsubset_\xi^{\mathbf{c}} \boldsymbol{\varphi}$ and $G \subseteq \mathbf{MT}(\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varphi})$ is generic over $\mathbf{V}$, then $\mathbf{c}[G] \notin \bigcup_{Q \in \boldsymbol{\varphi}(\xi)} [Q]$.*

Proof. (i) Suppose that $x \neq x_\xi[G]$. By a known forcing theorem, there is a $\boldsymbol{\pi}$-complete real name $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times (\omega \times 2)$ such that $x = \mathbf{c}[G]$ and $\mathbf{MT}(\boldsymbol{\pi})$ forces that $\mathbf{c} \neq x_\xi[\underline{G}]$. By Lemma 12.1, $\mathbf{c}$ is small since $\mathbf{MT}(\boldsymbol{\pi})$ is CCC. It remains to show that $\mathbf{c}$ is a non-principal name over $\boldsymbol{\pi}$ at ξ, that is, the set

$$\boldsymbol{D}_{\mathbf{c}}^{\xi}(\boldsymbol{\pi}) = \{\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) \colon \xi \in |\boldsymbol{p}| \wedge \boldsymbol{p} \text{ directly forces } \mathbf{c} \notin [T_\xi^{\boldsymbol{p}}]\}$$

is open-dense in $\mathbf{MT}(\boldsymbol{\pi})$. The openness is clear. To prove the density consider any $\boldsymbol{q} \in \mathbf{MT}(\boldsymbol{\pi})$. Then $\boldsymbol{q}$ $\mathbf{MT}(\boldsymbol{\pi})$-forces $\mathbf{c} \neq x_\xi[\underline{G}]$ by the choice of $\mathbf{c}$. Hence we can assume that there is an n such that $\boldsymbol{q}$ $\mathbf{MT}(\boldsymbol{\pi})$-forces $\mathbf{c}(n) \neq x_\xi[\underline{G}](n)$. Then, by Lemma 9.2, we can find a multitree $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi})$, $\boldsymbol{p} \leqslant \boldsymbol{q}$, and an $s \in \omega^{n+1}$ such that $\boldsymbol{p}$ directly forces $s \subseteq \mathbf{c}$. It suffices to show that $s \notin T_\xi^{\boldsymbol{p}}$. Assume the opposite: $s \in T_\xi^{\boldsymbol{p}}$. Then the tree $T = T_\xi^{\boldsymbol{p}} {\restriction}_s$ belongs to $\mathbf{MT}(\boldsymbol{\pi})$. Therefore, defining a multitree $\boldsymbol{r}$ by the formulae $T_\xi^{\boldsymbol{r}} = T$ and $T_{\xi'}^{\boldsymbol{r}} = T_{\xi'}^{\boldsymbol{p}}$ for every $\xi' \neq \xi$, we see that $\boldsymbol{r}$ belongs to $\mathbf{MT}(\boldsymbol{\pi})$ and $\boldsymbol{r} \leqslant \boldsymbol{p} \leqslant \boldsymbol{q}$. However, $\boldsymbol{r}$ directly forces $\mathbf{c}(n)$ and $x_\xi[\underline{G}](n)$ to be equal to the same quantity $\ell = s(n)$. This contradicts the choice of n.

To prove the converse, suppose that $\mathbf{c} \subseteq \mathbf{MT}(\boldsymbol{\pi}) \times (\omega \times 2)$ is a $\boldsymbol{\pi}$-complete real name, non-principal over $\boldsymbol{\pi}$ at ξ, and $x = \mathbf{c}[G]$. Assume the opposite: $x = x_\xi[G]$. There is a multitree $\boldsymbol{q} \in G$ which $\mathbf{MT}(\boldsymbol{\pi})$-forces $\mathbf{c} = x_\xi[\underline{G}]$. Since $\mathbf{c}$ is non-principal,

[9]The CCC property means that every antichain $A \subseteq \mathbf{MT}(\boldsymbol{\pi})$ is at most countable.

there is a multitree $\boldsymbol{p} \in G \cap \boldsymbol{D}^{\xi}_{\mathbf{c}}(\boldsymbol{\pi})$, $\boldsymbol{p} \leqslant \boldsymbol{q}$. Thus $\boldsymbol{p}$ directly forces $\mathbf{c} \notin [T^{\boldsymbol{p}}_{\xi}]$ and, therefore, $\mathbf{MT}(\boldsymbol{\pi})$-forces the same statement. However, $\boldsymbol{p}$ clearly $\mathbf{MT}(\boldsymbol{\pi})$-forces $x_{\xi}[\underline{G}] \in [T^{\boldsymbol{p}}_{\xi}]$. This is a contradiction.

(ii) Assume the opposite: $Q \in \boldsymbol{\varphi}(\xi)$ and $\mathbf{c}[G] \in [Q]$. By definition, $\boldsymbol{\varphi}$ seals the set

$$\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi}) = \{\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi}) \colon \xi \in |\boldsymbol{r}| \wedge \boldsymbol{r} \text{ directly forces } \mathbf{c} \notin [Q]\}$$

over $\boldsymbol{\pi}$. Hence $\boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi})$ is pre-dense in $\mathbf{MT}(\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varphi})$ by Lemma 6.3 and, therefore, $G \cap \boldsymbol{D}(\mathbf{c}, Q, \boldsymbol{\pi}) \neq \varnothing$. In other words, there is a multitree $\boldsymbol{r} \in \mathbf{MT}(\boldsymbol{\pi})$ which directly forces $\mathbf{c} \notin [Q]$. It follows easily that $\mathbf{c}[G] \notin [Q]$. This is a contradiction. □

§ 13. Combining types of refinement

Here we summarize the properties of generic refinements considered above. The following definition combines the types $\sqsubset_D$, $\sqsubset_{\boldsymbol{D}}$, $\sqsubset_{\mathbf{c}}$, $\sqsubset^{\mathbf{c}}_{\xi}$.

Definition 13.1. Suppose that $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi}$ are multiforcings, $\xi \in |\boldsymbol{\pi}|$ and $\mathfrak{M} \in \mathrm{HC}$ is any set. We write $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\varphi}$ if the following four requirements hold.

(1) If $D \in \mathfrak{M}$, $D \subseteq \boldsymbol{\pi}(\xi)$ and D is pre-dense in $\boldsymbol{\pi}(\xi)$, then $\boldsymbol{\pi}(\xi) \sqsubset_D \boldsymbol{\varphi}(\xi)$.

(2) If $\boldsymbol{D} \in \mathfrak{M}$, $\boldsymbol{D} \subseteq \mathbf{MT}(\boldsymbol{\pi})$ and $\boldsymbol{D}$ is open-dense in $\mathbf{MT}(\boldsymbol{\pi})$, then $\boldsymbol{\pi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\varphi}$.

(3) If $\mathbf{c} \in \mathfrak{M}$ is a $\boldsymbol{\pi}$-complete real name, then $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varphi}$.

(4) If $\mathbf{c} \in \mathfrak{M}$ is a $\boldsymbol{\pi}$-complete real name, *non-principal over* $\boldsymbol{\pi}$ *at* ξ, then $\boldsymbol{\pi} \sqsubset^{\mathbf{c}}_{\xi} \boldsymbol{\varphi}$.

Corollary 13.2 (of Lemmas 5.4, 6.3, 10.3, 10.6). *Let* $\boldsymbol{\pi}, \boldsymbol{\varphi}, \boldsymbol{\sigma}$ *be multiforcings and* $\mathfrak{M}$ *a countable set. Then the following assertions hold.*

(i) *If* $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\varphi} \sqsubset \boldsymbol{\sigma}$, *then* $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\sigma}$.

(ii) *If* $\langle \boldsymbol{\pi}_{\alpha} \rangle_{\alpha<\lambda}$ *is a* $\sqsubset$*-increasing sequence in* $\mathbf{MF}$, $0 < \mu < \lambda$, $\boldsymbol{\pi} = \bigcup^{\mathrm{cw}}_{\alpha<\mu} \boldsymbol{\pi}_{\alpha}$ *and* $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\pi}_{\mu}$, *then* $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\varphi} = \bigcup^{\mathrm{cw}}_{\mu\leqslant\alpha<\lambda} \boldsymbol{\pi}_{\alpha}$.

Corollary 13.3. *If* $\boldsymbol{\pi}$ *is a small multiforcing*, $\mathfrak{M} \in \mathrm{HC}$ *and* $\boldsymbol{\varphi}$ *is an* $\mathfrak{M}$*-generic refinement of* $\boldsymbol{\pi}$ (*which exists by Lemma* 7.2!), *then* $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\varphi}$.

Proof. We have $\boldsymbol{\pi} \mathrel{\sqsubset\!\!\!\sqsubset}_{\mathfrak{M}} \boldsymbol{\varphi}$ in view of 7.3, (v), 8.1, 8.5, 11.1. □

§ 14. Increasing sequences of multiforcings

Recall that $\mathbf{MF}$ is the family of all multiforcings (§ 4). Put

$$\mathbf{sMF} = \{\boldsymbol{\pi} \in \mathbf{MF} \colon \boldsymbol{\pi} \text{ is a } \textit{small} \text{ multiforcing}\};$$
$$\mathbf{spMF} = \{\boldsymbol{\pi} \in \mathbf{MF} \colon \boldsymbol{\pi} \text{is a small and } \textit{special} \text{ multiforcing}\}.$$

Thus a multiforcing $\boldsymbol{\pi} \in \mathbf{MF}$ belongs to $\mathbf{sMF}$ if $|\boldsymbol{\pi}| \subseteq \omega_1$ is (at most) countable and if $\xi \in |\boldsymbol{\pi}|$, then $\boldsymbol{\pi}(\xi)$ is a countable forcing in $\mathbf{AF}$. The inclusion $\boldsymbol{\pi} \in \mathbf{spMF}$ requires in addition that each $\boldsymbol{\pi}(\xi)$ is special (Definition 3.3).

Definition 14.1. If $\kappa \leqslant \omega_1$, then we write $\overrightarrow{\mathbf{MF}}_{\kappa}$ for the set of all $\sqsubset$-increasing sequences $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_{\alpha} \rangle_{\alpha<\kappa}$ of multiforcings $\boldsymbol{\pi}_{\alpha} \in \mathbf{spMF}$ which are *domain-continuous* in the sense that if $\lambda < \kappa$ is a limit ordinal, then $|\boldsymbol{\pi}_{\lambda}| = \bigcup_{\alpha<\lambda} |\boldsymbol{\pi}_{\alpha}|$. We put $\overrightarrow{\mathbf{MF}} = \bigcup_{\kappa<\omega_1} \overrightarrow{\mathbf{MF}}_{\kappa}$.

We order $\overrightarrow{\mathbf{MF}} \cup \overrightarrow{\mathbf{MF}}_{\omega_1}$ by the ordinary relations $\subseteq$ and $\subset$ of extension of sequences. Namely, $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\varphi}}$ if and only if $\kappa = \mathrm{dom}(\vec{\boldsymbol{\pi}}) < \lambda = \mathrm{dom}(\vec{\boldsymbol{\varphi}})$ and $\boldsymbol{\pi}_{\alpha} = \boldsymbol{\varphi}_{\alpha}$

for all $\alpha < \kappa$. In this case, if $\mathfrak{M}$ is any set and $\boldsymbol{\varphi}_\kappa$ (the first term of $\vec{\boldsymbol{\varphi}}$ which is absent from $\vec{\boldsymbol{\pi}}$) satisfies $\boldsymbol{\pi} \sqsubset_{\mathfrak{M}} \boldsymbol{\varphi}_\kappa$, where $\boldsymbol{\pi} = \bigcup^{\mathrm{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$, then we write $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$.

If $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}_\kappa$, then we put $\mathbf{MT}(\vec{\boldsymbol{\pi}}) = \mathbf{MT}(\boldsymbol{\pi})$, where $\boldsymbol{\pi} = \bigcup^{\mathrm{cw}} \vec{\boldsymbol{\pi}} = \bigcup^{\mathrm{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$ (the componentwise union). Accordingly, a $\vec{\boldsymbol{\pi}}$-*complete real name* means a $\boldsymbol{\pi}$-complete real name.

Lemma 14.2. *If $\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}}$, $\mathbf{c}$ is a $\vec{\boldsymbol{\pi}}$-complete real name and $\vec{\boldsymbol{\pi}} \subset_{\{\mathbf{c}\}} \vec{\boldsymbol{\varphi}}$, then $\mathbf{c}$ is a $\vec{\boldsymbol{\varphi}}$-complete real name.*

Proof. Suppose that $\kappa = \mathrm{dom}(\vec{\boldsymbol{\pi}}) < \lambda = \mathrm{dom}(\vec{\boldsymbol{\varphi}})$ and $\boldsymbol{\pi} = \bigcup^{\mathrm{cw}} \vec{\boldsymbol{\pi}} = \bigcup^{\mathrm{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$. Then, by definition, $\boldsymbol{\pi} \sqsubset_{\{\mathbf{c}\}} \boldsymbol{\varphi}_\kappa$. Hence $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varphi}_\kappa$ because $\mathbf{c}$ is a $\boldsymbol{\pi}$-complete real name. However, $\boldsymbol{\pi} \sqsubset_{\mathbf{c}} \boldsymbol{\varphi} = \bigcup^{\mathrm{cw}}_{\kappa\leqslant\alpha<\lambda} \boldsymbol{\varphi}_\alpha$ by Lemma 10.3, (iii). Therefore, $\mathbf{c}$ is a ($\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varphi}$)-complete name by Lemma 10.3, (i). However, $\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varphi} = \bigcup^{\mathrm{cw}}_{\alpha<\lambda} \boldsymbol{\varphi}_\alpha = \bigcup^{\mathrm{cw}} \vec{\boldsymbol{\varphi}}$. □

Definition 14.3. Let $\mathbf{ZFL}^-$ be the subtheory of $\mathbf{ZFC}$ including all axioms except the power-set axiom, plus the axiom of constructivity $\mathbf{V} = \mathbf{L}$, and plus the axiom saying that $\mathscr{P}(\omega)$ exists. (Then the sets ω_1, HC and, more generally, sets related to the continuum, like 2^ω and $\mathbf{PT}$, also exist.) The axiom of choice is included in $\mathbf{ZFL}^-$ in the form of the well-ordering principle.

If $x \in \mathrm{HC}$ (HC = hereditarily countable sets; see footnote 6), then we write $\mathfrak{L}(x)$ for the least *countable* transitive model (CTM) of the theory $\mathbf{ZFL}^-$ which contains x and satisfies $x \in (\mathrm{HC})^{\mathfrak{L}(x)}$. It is necessarily of the form $\mathfrak{L}(x) = \mathbf{L}_\mu$, where $\mu = \mu_x < \omega_1$.

An ordinal $\xi < \kappa$ is said to be *crucial* for a sequence $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_{\alpha \alpha<\kappa} \rangle \in \overrightarrow{\mathbf{MF}}_\kappa$ if we have $(\bigcup^{\mathrm{cw}}_{\alpha<\xi} \boldsymbol{\pi}_\alpha) \sqsubset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\xi)} \boldsymbol{\pi}_\xi$. This is equivalent to $\vec{\boldsymbol{\pi}} \restriction \xi \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\xi)} \vec{\boldsymbol{\pi}}$.

Lemma 14.4. *Suppose that $\kappa \leqslant \omega_1$ and $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_{\alpha \alpha<\kappa} \rangle \in \overrightarrow{\mathbf{MF}}_\kappa$. Then the following assertions hold.*

(i) *$\boldsymbol{\pi} = \bigcup^{\mathrm{cw}} \vec{\boldsymbol{\pi}} = \bigcup^{\mathrm{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$ is a regular multiforcing.*

(ii) *If $\kappa < \lambda \leqslant \omega_1$ and $\mathfrak{M} \in \mathrm{HC}$, then there is a sequence $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}}$ such that $\mathrm{dom}(\vec{\boldsymbol{\varphi}}) = \lambda$ and $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$.*

(iii) *If $\xi < \kappa$ is a crucial ordinal for $\vec{\boldsymbol{\pi}}$, $\boldsymbol{\pi}_{<\xi} = \bigcup^{\mathrm{cw}}_{\alpha<\xi} \boldsymbol{\pi}_\alpha$ and $\boldsymbol{\pi}_{\geqslant\xi} = \bigcup^{\mathrm{cw}}_{\xi\leqslant\beta<\kappa} \boldsymbol{\pi}_\beta$, then $\boldsymbol{\pi}_{<\xi} \sqsubset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\xi)} \boldsymbol{\pi}_{\geqslant\xi}$ and $\boldsymbol{\pi}_{<\xi} \sqsubset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\xi)} \boldsymbol{\pi}_\beta$ for $\xi \leqslant \beta < \kappa$, whence,*

(a) *$\mathbf{MT}(\boldsymbol{\pi}_{\geqslant\xi})$ is open-dense in $\mathbf{MT}(\vec{\boldsymbol{\pi}})$,*

(b) *if $\boldsymbol{D} \in \mathfrak{L}(\vec{\boldsymbol{\pi}} \restriction \xi)$, $\boldsymbol{D} \subseteq \mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \xi)$ and $\boldsymbol{D}$ is open-dense in $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \xi)$, then $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\boldsymbol{\pi}_{<\xi} \cup^{\mathrm{cw}} \boldsymbol{\pi}_{\geqslant\xi}) = \mathbf{MT}(\vec{\boldsymbol{\pi}})$.*

Proof. (i) Use Lemma 5.2, (iv).

(ii) We use induction to define the terms $\boldsymbol{\varphi}_\alpha$ of the required sequence $\vec{\boldsymbol{\varphi}}$.

We naturally put $\boldsymbol{\varphi}_\alpha = \boldsymbol{\pi}_\alpha$ for all $\alpha < \kappa$. To define the crucial term $\boldsymbol{\varphi}_\kappa$, we can assume without loss of generality that $\mathfrak{M}$ contains $\vec{\boldsymbol{\pi}}$ and satisfies $\kappa \subseteq \mathfrak{M}$ (otherwise take a bigger set). By Lemma 7.2, there is an $\mathfrak{M}$-generic refinement $\boldsymbol{\pi}'$ of $\boldsymbol{\pi} = \bigcup^{\mathrm{cw}}_{\alpha<\kappa} \boldsymbol{\pi}_\alpha$. By Theorem 7.3, $\boldsymbol{\pi}'$ is a small special multiforcing, $\boldsymbol{\pi} \sqsubset \boldsymbol{\pi}'$ and $\boldsymbol{\pi}_\alpha \sqsubset \boldsymbol{\pi}'$ for all $\alpha < \kappa$. In addition, $\boldsymbol{\pi} \sqsubset_{\mathfrak{M}} \boldsymbol{\pi}'$ by Corollary 13.3. We put $\boldsymbol{\varphi}_\kappa = \boldsymbol{\pi}'$. The extended sequence $\vec{\boldsymbol{\varphi}}_+ = \langle \boldsymbol{\varphi}_\alpha \rangle_{\alpha<\kappa+1}$ belongs to $\overrightarrow{\mathbf{MF}}_{\kappa+1}$ and satisfies $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}_+$.

The following steps are pretty similar, but we can take $\mathfrak{M} = \varnothing$.

To prove the main claim of (iii), we use Corollary 13.2.

To prove (iii), (a), use Corollary 6.1. We now prove (iii), (b). Since $\boldsymbol{\pi}_{<\xi} \sqsubset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\xi)} \boldsymbol{\pi}_{\geqslant\xi}$ and $\boldsymbol{D} \in \mathfrak{L}(\vec{\boldsymbol{\pi}} \restriction \xi)$, we have $\boldsymbol{\pi}_{<\xi} \sqsubset_{\boldsymbol{D}} \boldsymbol{\pi}_{\geqslant\xi}$. Therefore $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\vec{\boldsymbol{\pi}})$ by Lemma 6.3, (i). □

§ 15. The key sequence

In this section we define the forcing notion used to prove Theorem 1.2. It will be of the form $\mathbf{MT}(\mathbb{\Pi})$ for a certain multiforcing $\mathbb{\Pi}$ with $|\mathbb{\Pi}| = \omega_1$. The multiforcing $\mathbb{\Pi}$ will be equal to the componentwise union of the terms of the sequence $\vec{\mathbb{\Pi}} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$. The construction of this sequence in the constructible universe $\mathbf{L}$ will employ some ideas related to diamond-style constructions as well as to a special form of *definable genericity*. The following definition introduces an important notion used in this construction.

Definition 15.1. We say that a sequence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ *blocks* the set W if either $\vec{\boldsymbol{\pi}} \in W$ (a *positive* block), or there is no sequence $\vec{\boldsymbol{\varphi}} \in W$ extending $\vec{\boldsymbol{\pi}}$ (a *negative* block).

We recall that HC = all hereditarily countable sets; see footnote 6.

Definition 15.2. We use the standard notation $\mathit{\Sigma}_n^{\mathrm{HC}}$, $\mathit{\Pi}_n^{\mathrm{HC}}$, $\mathit{\Delta}_n^{\mathrm{HC}}$ (slanted $\mathit{\Sigma}$, $\mathit{\Pi}$, $\mathit{\Delta}$) for the classes of *lightface* definability in HC (no parameters allowed) and we write $\mathbf{\Sigma}_n(\mathrm{HC})$, $\mathbf{\Pi}_n(\mathrm{HC})$, $\mathbf{\Delta}_n(\mathrm{HC})$ for *boldface* definability in HC (parameters in HC allowed). It is well known that if $n \geqslant 1$ and $X \subseteq 2^\omega$, then

$$X \in \mathit{\Sigma}_n^{\mathrm{HC}} \iff X \in \mathit{\Sigma}_{n+1}^1, \qquad X \in \mathbf{\Sigma}_n(\mathrm{HC}) \iff X \in \mathbf{\Sigma}_{n+1}^1,$$

and the same for $\mathit{\Pi}$, $\mathbf{\Pi}$, $\mathit{\Delta}$, $\mathbf{\Delta}$.

Theorem 15.3 (in $\mathbf{L}$). *Suppose that $\mathfrak{n} \geqslant 3$. There is a sequence $\vec{\mathbb{\Pi}} = \langle \mathbb{\Pi}_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ satisfying the following requirements.*

(i) *$\vec{\mathbb{\Pi}}$ belongs to the definability class $\mathit{\Delta}_{\mathfrak{n}-2}^{\mathrm{HC}}$.*

(ii) $|\bigcup^{\mathrm{cw}} \vec{\mathbb{\Pi}}| = \omega_1$.

(iii) *If $\mathfrak{n} \geqslant 4$ and $W \subseteq \overrightarrow{\mathbf{MF}}$ is a boldface $\mathbf{\Sigma}_{\mathbf{n}-3}(\mathrm{HC})$-set, then there is an ordinal $\gamma < \omega_1$ such that the sequence $\vec{\mathbb{\Pi}} \restriction \gamma$ blocks W.*

(iv) *There is a closed unbounded set $\mathbb{C} \subseteq \omega_1$ such that every $\gamma \in \mathbb{C}$ is a crucial ordinal for $\vec{\mathbb{\Pi}}$ in the sense of Definition 14.3.*

Proof. We work under the hypothesis $\mathbf{V} = \mathbf{L}$. When $\mathfrak{n} \geqslant 4$, let $\mathbf{un}_{\mathfrak{n}}(p, x)$ be the canonical universal $\mathit{\Sigma}_{\mathfrak{n}-3}$-formula, so that the family of all boldface $\mathbf{\Sigma}_{\mathbf{n}-3}(\mathrm{HC})$-sets $X \subseteq \mathrm{HC}$ is equal to the family of all sets of the form $\Upsilon_{\mathfrak{n}}(p) = \{x \in \mathrm{HC} \colon \mathrm{HC} \models \mathbf{un}_{\mathfrak{n}}(p, x)\}$, $p \in \mathrm{HC}$.

Claim. *If $\mathfrak{n} \geqslant 4$, then the set $\{\langle \vec{\boldsymbol{\pi}}, p \rangle \colon \vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}} \wedge p \in \mathrm{HC} \wedge \vec{\boldsymbol{\pi}}$ blocks $\Upsilon_{\mathfrak{n}}(p)\}$ belongs to $\mathit{\Delta}_{\mathfrak{n}-2}^{\mathrm{HC}}$.*

Proof. We skip the routine verification that $\overrightarrow{\mathbf{MF}}$ is $\mathit{\Delta}_1^{\mathrm{HC}}$. Furthermore, if $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ and $p \in \mathrm{HC}$, then for $\vec{\boldsymbol{\pi}}$ to block $\Upsilon_{\mathfrak{n}}(p)$, it is necessary and sufficient that

$$\underbrace{\vec{\boldsymbol{\pi}} \in \Upsilon_{\mathfrak{n}}(p)}_{\mathit{\Sigma}_{\mathfrak{n}-3}^{\mathrm{HC}}} \vee \underbrace{\neg \exists \vec{\boldsymbol{\varphi}} \bigl(\underbrace{\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}} \wedge \vec{\boldsymbol{\varphi}} \text{ extends } \vec{\boldsymbol{\pi}}}_{\mathit{\Delta}_1^{\mathrm{HC}}} \wedge \underbrace{\vec{\boldsymbol{\varphi}} \in \Upsilon_{\mathfrak{n}}(p)}_{\mathit{\Sigma}_{\mathfrak{n}-3}^{\mathrm{HC}}}\bigr)}_{\mathit{\Pi}_{\mathfrak{n}-3}^{\mathrm{HC}}},$$

so this is a disjunction of formulae of types $\mathit{\Sigma}_{\mathfrak{n}-3}^{\mathrm{HC}}$ and $\mathit{\Pi}_{\mathfrak{n}-3}^{\mathrm{HC}}$, whence $\mathit{\Delta}_{\mathfrak{n}-2}^{\mathrm{HC}}$. □

For $\alpha < \omega_1$ we define a sequence $\vec{\pi}[\alpha] \in \overrightarrow{\mathbf{MF}}$ by induction as follows. Put $\vec{\pi}[0] = \varnothing$, the empty sequence.

Step $\alpha \to \alpha + 1$. Suppose that $\vec{\pi}[\alpha] \in \overrightarrow{\mathbf{MF}}$ is defined. $\kappa = \operatorname{dom} \vec{\pi}[\alpha]$, $\mathfrak{M} = \mathfrak{L}(\vec{\pi}[\alpha])$ and p_α is the α th element of the set $\mathrm{HC} = \mathbf{L}_{\omega_1}$ in the sense of the Gödel well-ordering $\leqslant_{\mathbf{L}}$. By Lemma 14.4, (ii), there is a sequence $\vec{\tau} \in \overrightarrow{\mathbf{MF}}_{\kappa+1}$ such that $\vec{\pi}[\alpha] \subset_{\mathfrak{M}} \vec{\tau}$. By Corollary 7.4, there is a sequence $\vec{\varphi} \in \overrightarrow{\mathbf{MF}}_{\kappa+2}$ such that $\vec{\tau} \subset \vec{\varphi}$ and $\alpha \in |\vec{\varphi}(\kappa+1)|$. Finally, if $\mathfrak{n} \geqslant 4$, then there is a sequence $\vec{\pi} \in \overrightarrow{\mathbf{MF}}$ satisfying $\vec{\varphi} \subset \vec{\pi}$ and blocking the set $\Upsilon_{\mathfrak{n}}(p_\alpha)$. But if $\mathfrak{n} = 3$, then we simply put $\vec{\pi} = \vec{\varphi}$. Thus we finally have

(∗) $\vec{\pi}[\alpha] \subset_{\mathfrak{M}} \vec{\pi}$, $\kappa + 1 < \operatorname{dom} \vec{\pi}$, $\alpha \in |\vec{\varphi}(\kappa+1)|$ and if $\mathfrak{n} \geqslant 4$, then $\vec{\pi}$ blocks the set $\Upsilon_{\mathfrak{n}}(p_\alpha)$.

Let $\vec{\pi}[\alpha+1]$ be the $\leqslant_{\mathbf{L}}$-least sequence among the sequences $\vec{\pi} \in \overrightarrow{\mathbf{MF}}$ satisfying (∗). Note that the axiom $\mathbf{V} = \mathbf{L}$ is a *sine qua non* of this construction since otherwise the $\leqslant_{\mathbf{L}}$-least choice of $\vec{\pi}[\alpha+1]$ would not necessarily be possible.

Limit step. If $\lambda < \omega_1$ is limit, then we naturally define $\vec{\pi}[\lambda] = \bigcup_{\alpha<\lambda} \vec{\pi}[\alpha]$.

We have $\alpha < \beta \Longrightarrow \vec{\pi}[\alpha] \subset \vec{\pi}[\beta]$ by construction. Hence $\vec{\sqcap} = \bigcup_\alpha \vec{\pi}[\alpha] \in \overrightarrow{\mathbf{MF}}_{\omega_1}$. To prove (i), note that the relation $R(\vec{\pi}, \vec{\varphi}, \mathfrak{M}) :=$ "$\vec{\pi} \subset_{\mathfrak{M}} \vec{\varphi}$" is absolute for all transitive models of $\mathbf{ZFL}^-$, whence R is Δ_1^{HC}. We easily see that the map $\vec{\pi} \mapsto \mathfrak{L}(\vec{\pi})$ is also Δ_1^{HC}. Finally, the relation 'to block $\Upsilon_{\mathfrak{n}}(p)$' is a $\Delta_{\mathfrak{n}-2}^{\mathrm{HC}}$-relation by the claim above. Using these facts, we routinely verify that (∗) is a $\Delta_{\mathfrak{n}-2}^{\mathrm{HC}}$-relation (in $\mathbf{L}$). On the other hand, it is known that, under $\mathbf{V} = \mathbf{L}$, choosing the $\leqslant_{\mathbf{L}}$-least element in every non-empty section of a Δ_k^{HC}-set, $k \geqslant 1$, results in a set (a transversal) of the same class Δ_k^{HC}. This completes the verification of (i).

To check (ii), note that $\alpha \in |\bigcup^{\mathrm{cw}} \vec{\pi}[\alpha+1]|$ by construction.

To check (iii) ($\mathfrak{n} \geqslant 4$), note that any boldface $\mathbf{\Sigma}_{\mathbf{n}-3}(\mathrm{HC})$-set $W \subseteq \overrightarrow{\mathbf{MF}}$ is equal to $\Upsilon_{\mathfrak{n}}(p_\alpha)$ for some $\alpha < \omega_1$. Hence we put $\gamma = \operatorname{dom} \vec{\pi}[\alpha+1]$.

(iv) The set $\mathbb{C} = \{\operatorname{dom} \vec{\pi}[\alpha] \colon \alpha < \omega_1\}$ is closed and unbounded by the limit step of the construction. Moreover, if $\gamma = \operatorname{dom} \vec{\pi}[\alpha] \in \mathbb{C}$, then $\vec{\sqcap} \restriction \gamma = \vec{\pi}[\alpha]$ and, therefore, γ is crucial for $\vec{\sqcap}$ by construction. □

Blanket Assumption 15.4 (in $\mathbf{L}$). From now on we fix a number $\mathfrak{n} \geqslant 3$ as in Theorem 1.2. We also fix a sequence $\vec{\sqcap} = \langle \sqcap \rangle_{\alpha\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ satisfying parts (i)–(iv) of Theorem 15.3 for this $\mathfrak{n}$. This fixed sequence $\vec{\sqcap}$ is referred to as the *key sequence.*

Lemma 15.5. *If* $\mathfrak{n} \geqslant 4$ *and* $W \subseteq \overrightarrow{\mathbf{MF}}$ *is a* $\mathbf{\Sigma}_{\mathbf{n}-3}(\mathrm{HC})$*-set dense in* $\overrightarrow{\mathbf{MF}}$, *then there is an ordinal* $\gamma < \omega_1$ *such that* $\vec{\sqcap} \restriction \gamma \in W$.

Proof. By 15.4, $\vec{\sqcap}$ satisfies part (iii) of Theorem 15.3. Hence there is an ordinal $\gamma < \omega_1$ such that $\vec{\sqcap} \restriction \gamma$ blocks W. The negative block is impossible because W is dense. Hence we have $\vec{\sqcap} \restriction \gamma \in W$. □

§ 16. The key forcing notion

We continue to argue in $\mathbf{L}$ and we now use the key sequence $\vec{\sqcap} = \langle \sqcap_{\alpha\alpha<\omega_1} \rangle$ introduced in 15.4.

Definition 16.1 (in $\mathbf{L}$). Define the multiforcings

$$\Pi = \bigcup^{\text{cw}}_{\alpha<\omega_1} \Pi_\alpha \in \mathbf{MF},$$

$$\Pi_{<\gamma} = \bigcup^{\text{cw}}_{\alpha<\gamma} \Pi_\alpha \in \mathbf{sMF} \quad \text{for every} \quad \gamma < \omega_1,$$

$$\Pi_{\geqslant\gamma} = \bigcup^{\text{cw}}_{\gamma\leqslant\alpha<\omega_1} \Pi_\alpha \in \mathbf{MF} \quad \text{for every} \quad \gamma < \omega_1.$$

We further define $\mathbb{P} = \mathbf{MT}(\Pi) = \mathbf{MT}(\vec{\Pi})$ and, for all $\gamma < \omega_1$,

$$\mathbb{P}_{<\gamma} = \mathbf{MT}(\Pi_{<\gamma}) = \mathbf{MT}(\vec{\Pi} \restriction \gamma), \qquad \mathbb{P}_{\geqslant\gamma} = \mathbf{MT}(\Pi_{\geqslant\gamma}) = \mathbf{MT}(\vec{\Pi} \restriction (\omega_1 \setminus \gamma)).$$

The set $\mathbb{P} = \mathbf{MT}(\Pi)$ will be our *key forcing notion*.

Corollary 16.2 (in $\mathbf{L}$, by 15.3, (ii)). *Π is a regular multiforcing and $|\Pi| = \omega_1$, whence $\mathbb{P} = \prod_{\xi<\omega_1} \Pi(\xi)$ (a finite-support product).*

If $\xi < \omega_1$, then (in accordance with the corollary) let $\alpha(\xi) < \omega_1$ be the least ordinal α such that $\xi \in |\Pi_\alpha|$. Thus the forcing $\Pi_\alpha(\xi) \in \mathbf{AF}$ is defined whenever α satisfies $\alpha(\xi) \leqslant \alpha < \omega_1$. Moreover, $\langle \Pi_\alpha(\xi)\rangle_{\alpha(\xi)\leqslant\alpha<\omega_1}$ is a $\sqsubset$-increasing sequence of special forcings in $\mathbf{AF}$. Note that $\Pi(\xi) = \bigcup_{\alpha(\xi)\leqslant\alpha<\omega_1} \Pi_\alpha(\xi)$ by construction.

Corollary 16.3 (in $\mathbf{L}$). *The sequence $\langle \alpha(\xi)\rangle_{\xi<\omega_1}$ of ordinals and the sequence $\langle \Pi_\alpha(\xi)\rangle_{\xi<\omega_1,\, \alpha(\xi)\leqslant\alpha<\omega_1}$ of forcings are $\varDelta^{\text{HC}}_{\mathfrak{n}-2}$.*

Proof. The following double equivalence holds by construction:

$$\begin{aligned} \alpha < \alpha(\xi) \quad &\Longleftrightarrow \quad \exists\, \boldsymbol{\pi}(\boldsymbol{\pi} = \Pi_\alpha \wedge \xi \in \operatorname{dom} \boldsymbol{\pi}) \\ &\Longleftrightarrow \quad \forall\, \boldsymbol{\pi}(\boldsymbol{\pi} = \Pi_\alpha \Longrightarrow \xi \in \operatorname{dom} \boldsymbol{\pi}). \end{aligned}$$

However $\boldsymbol{\pi} = \Pi_\alpha$ is a $\varDelta^{\text{HC}}_{\mathfrak{n}-2}$-relation by Theorem 15.3, (i). It follows that the sequence $\langle \alpha(\xi)\rangle_{\xi<\omega_1}$ is also $\varDelta^{\text{HC}}_{\mathfrak{n}-2}$. The second assertion is similar. □

Corollary 16.4 (in $\mathbf{L}$, of Lemma 5.2, (iv)). *If $\xi < \omega_1$ and $\alpha(\xi) \leqslant \alpha < \omega_1$, then the set $\Pi_\alpha(\xi)$ is pre-dense in $\Pi(\xi)$ and in Π.*

In spite of Corollary 16.2, the sets $|\Pi_{<\gamma}|$ can be quite arbitrary (countable) subsets of ω_1. However, we obtain the following corollary.

Corollary 16.5 (in $\mathbf{L}$, of Corollary 16.2). *The set $\mathbb{C}' = \{\gamma < \omega_1 \colon |\Pi_{<\gamma}| = \gamma\}$ is closed and unbounded in ω_1.*

To prove the CCC property, we need the following lemma.

Lemma 16.6 (in $\mathbf{L}$). *If $X \subseteq \text{HC} = \mathbf{L}_{\omega_1}$, then the set $\mathscr{O}_X$ of all ordinals $\gamma < \omega_1$ such that $\langle \mathbf{L}_\gamma; X\cap\mathbf{L}_\gamma\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; X\rangle$ and $X\cap\mathbf{L}_\gamma \in \mathfrak{L}(\vec{\Pi} \restriction \gamma)$ is stationary and, therefore, unbounded in ω_1.*

More generally, if $X_n \subseteq \text{HC}$ for all n, then the set $\mathscr{O}$ of all ordinals $\gamma < \omega_1$ such that $\langle \mathbf{L}_\gamma; \langle X_n \cap \mathbf{L}_\gamma\rangle_{n<\omega}\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; \langle X_n\rangle_{n<\omega}\rangle$ and $\langle X_n \cap \mathbf{L}_\gamma\rangle_{n<\omega} \in \mathfrak{L}(\vec{\Pi} \restriction \gamma)$ is stationary and, therefore, unbounded in ω_1.

Proof. Let $C \subseteq \omega_1$ be a closed unbounded set, M a countable elementary submodel of $\mathbf{L}_{\omega_2}$ containing C, ω_1, X, $\vec{\sqcap}$ and such that $M \cap \mathbf{L}_{\omega_1}$ is transitive, and $\phi\colon M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ the Mostowski collapse. Put $\gamma = \phi(\omega_1)$. Then

$$\gamma < \lambda < \omega_1, \qquad \phi(X) = X \cap \mathbf{L}_\gamma, \qquad \phi(C) = C \cap \gamma, \qquad \phi(\vec{\sqcap}) = \vec{\sqcap} \restriction \gamma$$

by the choice of M. It follows that $\langle \mathbf{L}_\gamma; X \cap \mathbf{L}_\gamma, C \cap \gamma, \vec{\sqcap} \restriction \gamma\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}; X, C, \vec{\sqcap}\rangle$, whence $\gamma \in \mathscr{O}_X$. Moreover, γ is uncountable in $\mathbf{L}_\lambda$. Hence $\mathbf{L}_\lambda \subseteq \mathfrak{L}(\vec{\sqcap} \restriction \gamma)$. (See Defintion 14.3 on the models $\mathfrak{L}(\boldsymbol{\vec{\pi}}) \models \mathbf{ZFL}^-$.) We conclude that $X \cap \mathbf{L}_\gamma \in \mathfrak{L}(\vec{\sqcap} \restriction \gamma)$ since $X \cap \mathbf{L}_\gamma \in \mathbf{L}_\lambda$ by construction. On the other hand, $C \cap \gamma$ is unbounded in γ by elementarity. Hence $\gamma \in C$, as required.

The second, more general, claim does not differ much. $\square$

Corollary 16.7 (in $\mathbf{L}$). *The forcing $\mathbb{P}$ satisfies CCC. Therefore $\mathbb{P}$-generic extensions of $\mathbf{L}$ preserve cardinals.*

Proof. Suppose that $A \subseteq \mathbb{P} = \mathbf{MT}(\vec{\sqcap})$ is a maximal antichain. By 15.4 and Theorem 15.3, (iv), there is a closed unbounded set $\mathbb{C} \subseteq \omega_1$ such that every $\gamma \in \mathbb{C}$ is a crucial ordinal for $\vec{\sqcap}$. By Lemma 16.6, there is an ordinal $\gamma \in \mathbb{C}$ such that $A' = A \cap \mathbb{P}_{<\gamma}$ is a maximal antichain in $\mathbb{P}_{<\gamma} = \mathbf{MT}(\vec{\sqcap} \restriction \gamma)$ and $A' \in \mathfrak{L}(\vec{\sqcap} \restriction \gamma)$. It follows that the set $D(A') = \{\boldsymbol{p} \in \mathbb{P}_{<\gamma} \colon \exists\, \boldsymbol{q} \in A\,(p \leqslant q)\} \in \mathfrak{L}(\vec{\sqcap} \restriction \gamma)$ is open-dense in $\mathbb{P}_{<\gamma}$.

However, γ is a crucial ordinal for $\vec{\sqcap}$. Therefore, by Lemma 14.4, (iii), (b), the sets $D(A')$ and A' remain pre-dense in the whole set $\mathbb{P} = \mathbf{MT}(\vec{\sqcap})$. We conclude that $A = A'$ is countable. $\square$

Corollary 16.8 (in $\mathbf{L}$). *If a set $D \subseteq \mathbb{P}$ is pre-dense in $\mathbb{P}$, then there is an ordinal $\gamma < \omega_1$ such that $D \cap \mathbb{P}_{<\gamma}$ is pre-dense in $\mathbb{P}$.*

Proof. We can even assume that D is dense. Let $A \subseteq D$ be a maximal antichain in D. Then A is a maximal antichain in $\mathbb{P}$ because D is dense. Hence $A \subseteq \mathbb{P}_{<\gamma}$ for some $\gamma < \omega_1$ by Corollary 16.7. But A is pre-dense in $\mathbb{P}$. $\square$

§ 17. The basic generic extension

We recall that the key sequence $\vec{\sqcap} = \langle \sqcap_\alpha \rangle_{\alpha<\omega_1}$ of small special multiforcings $\sqcap_\alpha$ is defined in $\mathbf{L}$ by 15.4. Moreover, the componentwise union $\sqcap = \bigcup^{\text{cw}}_{\alpha<\omega_1} \sqcap_\alpha$ is a multiforcing, $|\sqcap| = \omega_1$ in $\mathbf{L}$, and $\mathbb{P} = \mathbf{MT}(\vec{\sqcap}) = \mathbf{MT}(\sqcap) \in \mathbf{L}$ is our key forcing notion, equal to the finite-support product $\prod_{\xi<\omega_1} \sqcap(\xi)$ of arboreal forcings $\sqcap(\xi)$ in $\mathbf{L}$. Some properties of $\mathbb{P}$ were established in § 16, including the CCC and the definability of the factors $\sqcap(\xi)$ in $\mathbf{L}$. Our next goal is to show that certain submodels of $\mathbb{P}$-generic models prove Theorem 1.2.

Remark 17.1. From now on, we typically argue in $\mathbf{L}$ and in $\omega_1^{\mathbf{L}}$-preserving generic extensions of $\mathbf{L}$ (this includes, for example, $\mathbb{P}$-generic extensions by Corollary 16.7). Thus we always have $\omega_1^{\mathbf{L}} = \omega_1$. This enables us to assume that $|\sqcap| = \omega_1$ (rather than $\omega_1^{\mathbf{L}}$).

Definition 17.2. Let $G \subseteq \mathbb{P}$ be a generic set over the constructible set universe $\mathbf{L}$. If $\xi < \omega_1$, then following Remark 4.5, we

– define $G(\xi) = \{T^{\boldsymbol{p}}_\xi : \boldsymbol{p} \in G \wedge \xi \in |\boldsymbol{p}|\} \subseteq \sqcap(\xi)$;

– write $x_\xi = x_\xi[G] \in 2^\omega$ for the only real in $\bigcap_{T \in G(\xi)} [T]$;

– put $\boldsymbol{X} = \boldsymbol{X}[G] = \langle x_\xi[G] \rangle_{\xi < \omega_1} = \{\langle \xi, x_\xi[G] \rangle : \xi < \omega_1\}$.

Thus, $\mathbb{P}$ adjoins an array $\boldsymbol{X}[G]$ of reals to $\mathbf{L}$, where each $x_\xi[G] \in 2^\omega \cap \mathbf{L}[G]$ is a $\sqcap(\xi)$-generic real over $\mathbf{L}$ and we have $\mathbf{L}[G] = \mathbf{L}[\boldsymbol{X}[G]]$.

If $\Delta \subseteq \omega_1$, then we put $\mathbb{P} \restriction \Delta = \mathbf{MT}(\sqcap \restriction \Delta) = \{\boldsymbol{p} \in \mathbb{P} : |\boldsymbol{p}| \subseteq \Delta\}$.

The following lemma uses the product structure of $\mathbb{P}$.

Lemma 17.3. *Suppose that $\Delta \in \mathbf{L}$, $\Delta \subseteq \omega_1$. Then $\mathbb{P} = \mathbf{MT}(\sqcap)$ is equal to the product $(\mathbb{P} \restriction \Delta) \times (\mathbb{P} \restriction \Delta')$, where $\Delta' = \omega_1 \setminus \Delta$. If $G \subseteq \mathbb{P}$ is generic over $\mathbf{L}$, then the set $G \restriction \Delta = \{\boldsymbol{p} \in G : |\boldsymbol{p}| \subseteq \Delta\}$ is $(\mathbb{P} \restriction \Delta)$-generic over $\mathbf{L}$. If $\xi < \omega_1$ and $\xi \notin \Delta$, then $x_\xi[G] \notin \mathbf{L}[G \restriction \Delta]$.*

§ 18. Definability of generic reals

Recall that the factors $\sqcap(\xi)$ of the forcing notion $\mathbb{P} = \mathbf{MT}(\sqcap) = \prod_{\xi < \omega_1} \sqcap(\xi)$ are defined by putting $\sqcap(\xi) = \bigcup_{\alpha(\xi) \leqslant \alpha < \omega_1} \sqcap_\alpha(\xi)$, where $\alpha(\xi) < \omega_1$, the sets $\sqcap_\alpha(\xi)$ are countable sets of perfect trees, and their definability in $\mathbf{L}$ is determined by Corollary 16.3. We shall freely use the notation introduced in Definition 17.2.

Theorem 18.1. *Assume that $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic set over $\mathbf{L}$, $\xi < \omega_1$ and $x \in \mathbf{L}[G] \cap 2^\omega$. Then the following assertions are equivalent:*

(1) $x = x_\xi[G]$;

(2) *x is $\sqcap(\xi)$-generic over $\mathbf{L}$*;

(3) $x \in \bigcap_{\alpha(\xi) \leqslant \alpha < \omega_1} \bigcup_{T \in \sqcap_\alpha(\xi)} [T]$.

Proof. (1) $\Longrightarrow$ (2) is routine; see Remark 4.5. To check (2) $\Longrightarrow$ (3), we recall that each set $\sqcap_\alpha(\xi)$ is pre-dense in $\sqcap(\xi)$ by Lemma 5.2, (iv). We now prove that (3) $\Longrightarrow$ (1). Suppose that $x \in \mathbf{L}[G] \cap 2^\omega$ but (1) fails, that is, $x \neq x_\xi[G]$. By Theorem 12.2, (i) there is a $\sqcap$-complete small (since $\mathbb{P} = \mathbf{MT}(\sqcap)$ is CCC by 16.7) real name $\mathbf{c} \in \mathbf{L}$ such that $\mathbf{c} \subseteq \mathbb{P} \times \omega \times 2$, $x = \mathbf{c}[G]$ and $\mathbf{c}$ is non-principal over $\sqcap$ at ξ in the sense that the set

$$\boldsymbol{D}^\xi_{\mathbf{c}}(\sqcap) = \{\boldsymbol{p} \in \mathbb{P} = \mathbf{MT}(\sqcap) : \xi \in |\boldsymbol{p}| \wedge \boldsymbol{p} \text{ directly forces } \mathbf{c} \notin [T^{\boldsymbol{p}}_\xi]\}$$

is open-dense in $\mathbb{P} = \mathbf{MT}(\sqcap)$. Since $\mathbf{c}$ is small, there is an ordinal $\gamma < \omega_1$ such that $\mathbf{c}$ is a $\sqcap_{<\gamma}$-complete real name, and we can assume by Corollary 16.8 that the set $\boldsymbol{D}^\xi_{k\mathbf{c}}(\sqcap) \cap \mathbb{P}_{<\gamma}$ is pre-dense in $\mathbb{P}$ and, therefore, open-dense in $\mathbb{P}_{<\gamma}$. Then $\mathbf{c}$ is non-principal over $\sqcap_{<\gamma}$ at ξ. We can further assume that $\mathbf{c} \in \mathfrak{L}(\vec{\sqcap} \restriction \gamma)$. Finally, we can assume that γ belongs to the set $\mathbb{C}$ in Theorem 15.3, (iv). In other words, γ is a crucial ordinal for $\vec{\boldsymbol{\pi}}$, that is, $\sqcap_{<\gamma} \sqsubset_{\mathfrak{L}(\vec{\sqcap} \restriction \gamma)} \sqcap_\gamma$. It follows that $\sqcap_{<\gamma} \sqsubset_{\mathfrak{L}(\vec{\sqcap} \restriction \gamma)} \sqcap_{\geqslant \gamma}$ by Lemma 14.4, (iii). Then we also have $\sqcap_{<\gamma} \sqsubset^{\mathbf{c}}_\xi \sqcap_{\geqslant \gamma}$ by Lemma 13.1, (4) since $\mathbf{c} \in \mathfrak{L}(\vec{\sqcap} \restriction \gamma)$ and $\mathbf{c}$ is non-principal. Now Theorem 12.2, (ii) with $\boldsymbol{\pi} = \sqcap_{<\gamma}$ and $\boldsymbol{\varphi} = \sqcap_{\geqslant \gamma}$ (note that $\boldsymbol{\pi} \cup^{\mathrm{cw}} \boldsymbol{\varphi} = \sqcap$) implies that $x = \mathbf{c}[G] \notin \bigcup_{Q \in \sqcap_{\geqslant \gamma}(\xi)} [Q]$ and, in particular, $x \notin \bigcup_{Q \in \sqcap_\gamma(\xi)} [Q]$. In other words, (3) fails as well. $\square$

Corollary 18.2. *Assume that* $G \subseteq \mathbb{P}$ *is* $\mathbb{P}$*-generic over* $\mathbf{L}$*. Let* M *be a generic extension of* $\mathbf{L}$ *satisfying* $2^\omega \cap M \subseteq \mathbf{L}[G]$*. Then* $\boldsymbol{X}[G] \cap M$ *is a set of definability class* $\varPi^{\mathrm{HC}}_{\mathbb{n}-2}$ *in* M*.*

We recall that $\boldsymbol{X}[G]$ was introduced in Definition 17.2.

Proof. By the theorem, the relation $\langle \xi, x\rangle \in \boldsymbol{X}[G]$ is equivalent in M to the assertion

$$\forall\, \alpha < \omega_1\, \exists\, T \in \mathbb{\Pi}_\alpha(\xi)\big(\alpha(\xi) \leqslant \alpha \Longrightarrow x \in [T]\big),$$

which can be rewritten as

$$\forall\, \alpha < \omega_1\, \forall\, \mu < \omega_1\, \forall\, Y\ \exists\, T \in Y\big(\mu = \alpha(\xi) \wedge Y = \mathbb{\Pi}_\alpha(\xi) \wedge \mu \leqslant \alpha \Longrightarrow x \in [T]\big).$$

Here the equality $\mu = \alpha(\xi)$ is of type $\varDelta^{\mathrm{HC}}_{\mathbb{n}-2}$ by Corollary 16.3, and so is the equality $Y = \mathbb{\Pi}_\alpha(\xi)$, again by 16.3. It follows that the whole relation is of type $\varPi^{\mathrm{HC}}_{\mathbb{n}-2}$ since the quantifier $\exists\, T \in Y$ is bounded. □

Corollary 18.3. *If* $G \subseteq \mathbb{P}$ *is* $\mathbb{P}$*-generic over* $\mathbf{L}$*, then in* $\mathbf{L}[G]$ *there is a 'good'* $\varDelta^1_{\mathbb{n}}$*-well-ordering of* 2^ω *of length* ω_1*.*

Proof. If $\gamma < \omega_1$, then let $\boldsymbol{X}_\gamma = \langle x_\xi[G]\rangle_{\xi<\gamma}$. The equality $Y = \boldsymbol{X}_\gamma$ is a $\varPi^{\mathrm{HC}}_{\mathbb{n}-2}$-relation in $\mathbf{L}[G]$ (with arguments γ, Y) by Corollary 18.2. If $x \in 2^\omega \cap \mathbf{L}[G]$, then let $\gamma(x)$ be the least $\gamma < \omega_1$ such that $x \in \mathbf{L}[\boldsymbol{X}_\gamma]$, and let $\nu(x) < \omega_1$ be the index of x in the canonical well-ordering of 2^ω in $\mathbf{L}[\boldsymbol{X}_\gamma]$. We well-order $2^\omega \cap \mathbf{L}[G]$ lexicographically for the triples $\langle \max\{\gamma(x), \nu(x)\}, \gamma(x), \nu(x)\rangle$. This well-ordering is of class $\varDelta^{\mathrm{HC}}_{\mathbb{n}-1}$ by what was said above and, therefore, of class $\varDelta^1_{\mathbb{n}}$. One can also easily verify the 'goodness' (that is, the set of all coded proper initial segments has to be $\varSigma^1_{\mathbb{n}}$). □

§ 19. The non-separation model

The model for Theorem 1.2 will be defined on the basis of a $\mathbb{P}$-generic extension $\mathbf{L}[G]$ of $\mathbf{L}$. More precisely, it will be of the form $\mathbf{L}[G \restriction \Delta]$, where $\Delta \subseteq \omega_1^{\mathbf{L}}$ will itself be a generic set over $\mathbf{L}[G]$.

Put $\mathbb{Q} = \{1,2,3\}^{\omega_1^{\mathbf{L}}} \cap \mathbf{L}$ with *countable* support, that is, a typical element of $\mathbb{Q}$ is a partial function $q \in \mathbf{L}$ from $\omega_1^{\mathbf{L}}$ to the 3-element set $\{1,2,3\}$ such that the domain $\operatorname{dom} q \subseteq \omega_1^{\mathbf{L}}$ is countable in $\mathbf{L}$, that is, just bounded in $\omega_1^{\mathbf{L}}$. (The choice of the 3-element set $\{1,2,3\}$ will be explained below; see Definition 19.3.) We order $\mathbb{Q}$ oppositely to inclusion, that is, we put $q \leqslant q'$ (q is stronger) if and only if $q' \subseteq q$. Thus $\mathbb{Q} \in \mathbf{L}$ and, inside $\mathbf{L}$, $\mathbb{Q}$ is equal to the countable-support product $\{1,2,3\}^{\omega_1}$. Accordingly, a $\mathbb{Q}$-generic object is a $\mathbb{Q}$-generic map $H\colon \omega_1^{\mathbf{L}} \to \{1,2,3\}$.

Recall that $\mathbb{P}$ is a CCC forcing in $\mathbf{L}$ by Corollary 16.7.

Lemma 19.1. $\mathbb{P}$ *remains CCC in any* $\mathbb{Q}$*-generic extension* $\mathbf{L}[H]$ *of* $\mathbf{L}$*. Therefore* $\mathbb{P} \times \mathbb{Q}$ *preserves cardinals over* $\mathbf{L}$*.*

Proof. Assume the opposite: some $q' \in \mathbb{Q}$ forces C to be an uncountable antichain in $\mathbb{P}$, where C is a $\mathbb{Q}$-name. Note that $\mathbb{Q}$ is *countably complete* in $\mathbf{L}$. This means that if $q_0 \geqslant q_1 \geqslant q_2 \geqslant \cdots$ is a sequence in $\mathbb{Q}$, then there is a condition $q = \bigcup_k q_k \in \mathbb{Q}$; $q \leqslant q_k$, $\forall\, k$. Therefore, *arguing in* $\mathbf{L}$, we can define by induction a decreasing sequence $\langle q_{\xi_{\xi<\omega_1}}\rangle$ in $\mathbb{Q}$ and a sequence of pairwise-incompatible conditions $p_\xi \in \mathbb{P}$ such that $q_0 \leqslant q'$ and each q_ξ forces that $p_\xi \in C$. But then $A = \{p_\xi : \xi < \omega_1\} \in \mathbf{L}$ is an uncountable antichain in $\mathbb{P}$, a contradiction. $\square$

Lemma 19.2. *Suppose that the set $G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over $\mathbf{L}$. Then the following assertions hold.*

(i) $2^\omega \cap \mathbf{L}[G,H] \subseteq \mathbf{L}[G]$. *Therefore,* $\omega_1^{\mathbf{L}} = \omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}[G,H]}$.

(ii) *If* $\Delta \in \mathbf{L}$, $\Delta \subseteq \omega_1^{\mathbf{L}}$, *then* $\mathbf{L}[G \restriction \Delta, H] \cap 2^\omega \subseteq \mathbf{L}[G \restriction \Delta]$.

(iii) *If* $\Delta \in \mathbf{L}[H]$, $\Delta \subseteq \omega_1^{\mathbf{L}}$ *and* $\xi < \omega_1^{\mathbf{L}}$, *then* $x_\xi[G] \in \mathbf{L}[G \restriction \Delta]$ *is equivalent to* $\xi \in \Delta$.

Proof. Note that $\mathbb{Q}$ may not be countably complete in $\mathbf{L}[G]$. Hence the most elementary way to prove (ii) does not work. However, regard $\mathbf{L}[G,H]$ as a $\mathbb{P}$-generic extension $\mathbf{L}[H][G]$ of $\mathbf{L}[H]$. Suppose that $x \in 2^\omega \cap \mathbf{L}[H][G]$. Since $\mathbb{P} = \mathbf{MT}(\sqcap)$ is CCC in $\mathbf{L}[H]$ by Lemma 19.1, Lemma 12.1 yields a small $\sqcap$-complete real name $\mathbf{c} \in \mathbf{L}[H]$ such that $\mathbf{c} \subseteq \mathbb{P} \times \omega \times 2$ and $x = \mathbf{c}[G]$. In view of its smallness, $\mathbf{c}$ is effectively coded by a real. Hence we have $\mathbf{c} \in \mathbf{L}$ because $\mathbf{L}[H] \cap 2^\omega = \mathbf{L} \cap 2^\omega$. Thus $x = \mathbf{c}[G] \in \mathbf{L}[G]$.

The proof of (ii) is similar.

(iii) In the non-trivial direction, suppose that $\xi \notin \Delta$. Consider the set $\Delta' = \omega_1^{\mathbf{L}} \setminus \{\xi\} \in \mathbf{L}$. It is clear that $G \restriction \Delta \in \mathbf{L}[G \restriction \Delta', H]$ and, therefore, any real in $2^\omega \cap \mathbf{L}[G \restriction \Delta]$ belongs to $\mathbf{L}[G \restriction \Delta']$ by (ii). But $x_\xi[G] \notin \mathbf{L}[G \restriction \Delta']$ by Lemma 17.3. $\square$

Recall that if $\nu \in \mathbf{Ord}$, then the ordinal product 2ν is regarded as the ordered sum of ν copies of the ordinal $2 = \{0,1\}$. (In contrast to $\nu 2 = \nu + \nu$.) Thus if $\nu = \lambda + m$, where λ is a limit ordinal or 0 and $m < \omega$, then $2\nu = \lambda + 2m$ and $2\nu + 1 = \lambda + 2m + 1$ while $\langle \nu, i\rangle \mapsto 2\nu + i$ is a bijection of $\omega_1 \times 2$ onto ω_1.

Definition 19.3. If $H : \omega_1^{\mathbf{L}} \to \{1,2,3\}$, then we define the sets

$$\mathbb{1}_H = \{2\nu : H(2\nu) = 1\}, \qquad \mathbb{2}_H = \{2\nu : H(2\nu) = 2\}, \qquad \mathbb{3}_H = \{2\nu : H(2\nu) = 3\},$$
$$\mathbb{4}_H = \{2\nu + 1 : H(2\nu+1) = 1\}, \qquad \mathbb{5}_H = \{2\nu + 1 : H(2\nu+1) = 2\},$$

and $\mathbb{6}_H = \{2\nu+1 : H(2\nu+1) = 3\}$, and further

$$\begin{aligned}\Delta_H = \{4\nu : 2\nu \in \mathbb{1}_H \cup \mathbb{3}_H\} \cup \{4\nu + 1 : 2\nu \in \mathbb{2}_H \cup \mathbb{3}_H\} \\ \cup \{4\nu + 2 : 2\nu + 1 \in \mathbb{4}_H\} \cup \{4\nu + 3 : 2\nu + 1 \in \mathbb{5}_H\}.\end{aligned}$$

Note that the relation $\mathbf{L}[G \restriction \Delta_H] \subseteq \mathbf{L}[G]$ is not necessarily true since the set Δ_H does not necessarily belong to $\mathbf{L}[G]$, but we have $\mathbf{L}[G \restriction \Delta_H] \subseteq \mathbf{L}[G][H]$, of course.

§ 20. The non-separation theorem: the HC-version

We now prove the following version of Theorem 1.2 for HC-definability.

Theorem 20.1. *Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic set over $\mathbf{L}$ and let $H : \omega_1^{\mathbf{L}} \to \{1,2,3\}$ be a $\mathbb{Q}$-generic function over $\mathbf{L}[G]$. Then it is true in $\mathbf{L}[G \restriction \Delta_H]$ that*

(i) $\mathbb{1}_H$, $\mathbb{2}_H$ *are disjoint* $\varPi^{\mathrm{HC}}_{\mathbb{n}-1}$*-sets not separable by disjoint* $\mathbf{\Sigma}_{\mathbf{n}-1}(\mathrm{HC})$*-sets;*

(ii) $\mathbb{4}_H$, $\mathbb{5}_H$ *are disjoint* $\varSigma^{\mathrm{HC}}_{\mathbb{n}-1}$*-sets not separable by disjoint* $\mathbf{\Pi}_{\mathbf{n}-1}(\mathrm{HC})$*-sets.*

The proof of Theorem 20.1 in this section includes a reference to the following theorem, whose separate lengthy proof occupies the remainder of the paper.

Theorem 20.2 (to be proved in § 26). *Assume that* $X \in \mathbf{L}$, $X \subseteq \omega_1^{\mathbf{L}}$ *is unbounded in* $\omega_1^{\mathbf{L}}$, *and* $G \subseteq \mathbb{P}$ *is a* $\mathbb{P}$*-generic set over* $\mathbf{L}$. *Then* $\mathbf{L}[G \upharpoonright X] \cap 2^\omega$ *is an elementary submodel of* $\mathbf{L}[G] \cap 2^\omega$ *with respect to all* $\Sigma^1_{\mathfrak{n}-1}$*-formulae with parameters in* $2^\omega \cap \mathbf{L}[G \upharpoonright X]$.

Corollary 20.3. *Under the hypotheses of Theorem* 20.1, $\mathrm{HC}^{\mathbf{L}[G \upharpoonright \Delta_H]}$ *is an elementary submodel of* $\mathrm{HC}^{\mathbf{L}[G]}$ *with respect to all* $\Sigma_{\mathfrak{n}-2}$*-formulae.*

Note that $\mathrm{HC}^{\mathbf{L}[G \upharpoonright \Delta_H]} \subseteq \mathrm{HC}^{\mathbf{L}[G]}$ by Lemma 19.2 while $\mathbf{L}[G \upharpoonright \Delta_H] \not\subseteq \mathbf{L}[G]$.

Proof (of Corollary 20.3). We have $\omega_1^{\mathbf{L}} = \omega_1^{\mathbf{L}[G]} = \omega_1^{\mathbf{L}[G \upharpoonright \Delta_H]}$ and $\Delta_H \cap \lambda \in \mathbf{L}$ for all $\lambda < \omega_1^{\mathbf{L}}$ by Lemma 19.2. We now use Theorem 20.2 with $\Sigma^{\mathrm{HC}}_{\mathfrak{n}-2}$-definability instead of $\Sigma^1_{\mathfrak{n}-1}$-definability. □

Proof (of Theorem 20.1). (i) To check that $\mathbb{1}_H$ is a $\Pi^{\mathrm{HC}}_{\mathfrak{n}-1}$-set in $\mathbf{L}[G \upharpoonright \Delta_H]$, it suffices to prove the equality

$$\mathbb{1}_H = \{2\nu < \omega_1 \colon \neg \exists x \ (\langle 4\nu + 1, x \rangle \in \boldsymbol{X})\}$$

in $\mathbf{L}[G \upharpoonright \Delta_H]$, where $\boldsymbol{X} = \boldsymbol{X}[G] \cap \mathbf{L}[G \upharpoonright \Delta_H]$ is a $\Pi^{\mathrm{HC}}_{\mathfrak{n}-2}$-set in $\mathbf{L}[G \upharpoonright \Delta_H]$ by Corollary 18.2. (For $\mathbb{2}_H$ it would be $\langle 4\nu, x \rangle \in \boldsymbol{X}$ in the displayed formula.)

Assuming first that $\nu < \omega_1^{\mathbf{L}}$, $\xi = 4\nu + 1$, $x \in \mathbf{L}[G \upharpoonright \Delta_H] \cap 2^\omega$ and $\langle \xi, x \rangle \in \boldsymbol{X}$, we claim that $2\nu \notin \mathbb{1}_H$. Indeed, $x = x_\xi[G]$ by definition and $\xi \in \Delta_H$ by Lemma 19.2,(iii). But then $2\nu \in \mathbb{2}_H \cup \mathbb{3}_H$ and, therefore, $2\nu \notin \mathbb{1}_H$, as required.

To prove the converse, suppose that $2\nu \notin \mathbb{1}_H$, whence $2\nu \in \mathbb{2}_H \cup \mathbb{3}_H$. Then $\xi = 4\nu + 1 \in \Delta_H$ and, therefore, $x = x_\xi \in \mathbf{L}[G \upharpoonright \Delta_H]$ and $\langle \xi, x \rangle = \langle 4\nu + 1, x \rangle \in \boldsymbol{X}$, as required.

To prove the non-separability, assume the opposite: the sets $\mathbb{1}_H, \mathbb{2}_H$ are separated by disjoint $\mathbf{\Sigma}_{\mathbf{n}-1}(\mathrm{HC})$-sets $A, B \subseteq \omega_1 = \omega_1^{\mathbf{L}}$ in $\mathbf{L}[G \upharpoonright \Delta_H]$. The sets A, B are defined in $\mathrm{HC}^{\mathbf{L}[G \upharpoonright \Delta_H]}$ by the $\Sigma_{\mathfrak{n}-1}$-formulae $\varphi(a, \xi), \psi(a, \xi)$ respectively with a real parameter $a \in \mathbf{L}[G \upharpoonright \Delta_H] \cap 2^\omega$. Hence $a \in \mathbf{L}[G]$ by Lemma 19.2. Let $\lambda < \omega_1^{\mathbf{L}}$ be a limit ordinal such that $a \in \mathbf{L}[G \upharpoonright \Delta_{H\lambda}]$, where $\Delta_{H\lambda} = \Delta_H \cap \lambda \in \mathbf{L}$.

If $K \colon \omega_1^{\mathbf{L}} \to \{1, 2, 3\}$ (for example, $K = H$), then we put

$$A^*_K = \{\xi < \omega_1^{\mathbf{L}} \colon \varphi(a, \xi)^{\mathrm{HC}^{\mathbf{L}[G \upharpoonright \Delta_K]}}\}, \qquad B^*_K = \{\xi < \omega_1^{\mathbf{L}} \colon \psi(a, \xi)^{\mathrm{HC}^{\mathbf{L}[G \upharpoonright \Delta_K]}}\}. \quad (*)$$

By definition, $\mathbb{1}_H \subseteq A = A^*_H$, $\mathbb{2}_H \subseteq B = B^*_H$ and $A^*_H \cap B^*_H = \varnothing$. We fix a condition $q_0 \in \mathbb{Q}$ which is compatible with H (this simply means that $q_0 \subset H$) and forces the properties mentioned of the sets A, B, so that

(†) if $K \colon \omega_1^{\mathbf{L}} \to \{1, 2, 3\}$ is a $\mathbb{Q}$-generic map over $\mathbf{L}[G]$ compatible with q_0, then $\mathbb{1}K \subseteq A^*_K$, $\mathbb{2}K \subseteq B^*_K$ and $A^*_K \cap B^*_K = \varnothing$.

We can assume that $\operatorname{dom} q_0 \subseteq \lambda$, otherwise simply increase λ.

Take any ordinal ν_0, $\lambda \leqslant \nu_0 < \omega_1$. Consider maps $H_1, H_2, H_3 \colon \omega_1^{\mathbf{L}} \to \{1, 2, 3\}$ which are generic over $\mathbf{L}[G]$, compatible with q_0 and satisfy $H_i(2\nu_0) = i$, $i = 1, 2, 3$, and $H_1(\alpha) = H_2(\alpha) = H_3(\alpha)$ for all $\alpha \neq 2\nu_0$. Then $\Delta_{H_3} = \Delta_{H_1} \cup \{4\nu_0 + 1\}$ by Definition 19.3. Hence $\mathbf{L}[G \upharpoonright \Delta_{H_1}] \subseteq \mathbf{L}[G \upharpoonright \Delta_{H_3}]$. It follows that $A^*_{H_1} \subseteq A^*_{H_3}$

by Corollary 20.3. Therefore $\mathbb{1}H_1 \subseteq A^*_{H_1} \subseteq A^*_{H_3}$ by (†). We conclude that $2\nu_0 \in A^*_{H_3}$ because $2\nu_0 \in \mathbb{1}H_1$ by the choice of H_1.

A similar argument with H_2 yields that $2\nu_0 \in B^*_{H_3}$. Thus, $A^*_{H_3} \cap B^*_{H_3} \neq \varnothing$, contrary to (†). This contradiction completes the proof of (i).

The proof of (ii) is similar. □

§ 21. The main theorem modulo Theorem 20.2

Proof (of Theorem 1.2). (i) We work under the hypotheses of Theorem 20.1. To define a non-separable pair of $\varPi^1_{\mathtt{n}}$-sets in $\mathbf{L}[G \restriction \Delta_H]$, we consider the $\varPi^1_1$-set $\mathbf{WO} \subseteq 2^\omega$ of codes of countable ordinals and, for any $w \in \mathbf{WO}$, let $|w| < \omega_1$ be the ordinal encoded by w. Since $\omega_1^{\mathbf{L}} = \omega_1$ by Corollary 16.7, for every $\xi < \omega_1$ there is a code $w \in \mathbf{WO} \cap \mathbf{L}$ with $|w| = \xi$. We write w_ξ for the $\leqslant_{\mathbf{L}}$-least of them and put $X = \{w_\xi : \xi \in \mathbb{1}_H\}$ and $Y = \{w_\xi : \xi \in \mathbb{2}_H\}$.

The sets $X, Y \subseteq \mathbf{WO} \cap \mathbf{L}$ are $\varPi^{\mathrm{HC}}_{\mathtt{n}-1}$ in $\mathbf{L}[G \restriction \Delta_H]$ together with $\mathbb{1}_H$, $\mathbb{2}_H$ (hence they are $\varPi^1_{\mathtt{n}}$) and $X \cap Y = \varnothing$. Assume the opposite: in $\mathbf{L}[G \restriction \Delta_H]$, $X', Y' \subseteq 2^\omega$ are disjoint sets in $\mathbf{\Sigma}^1_{\mathtt{n}}$ (and hence in $\mathbf{\Sigma_{n-1}}(\mathrm{HC})$) such that $X \subseteq X'$ and $Y \subseteq Y'$. Then, in $\mathbf{L}[G \restriction \Delta_H]$,

$$A = \{\xi < \omega_1^{\mathbf{L}} : w_\xi \in X'\} \quad \text{and} \quad B = \{\xi < \omega_1^{\mathbf{L}} : w_\xi \in Y'\}$$

are disjoint $\mathbf{\Sigma_{n-1}}(\mathrm{HC})$-sets and we have $\mathbb{1}_H \subseteq A$ and $\mathbb{2}_H \subseteq B$ by construction, contrary to Theorem 20.1. This contradiction completes the proof of (i). The proof of (ii) is similar. □

§ 22. An auxiliary forcing relation

We now begin the lengthy proof of Theorem 20.2. It uses an auxiliary forcing relation which is not explicitly connected with any particular forcing notion. In particular, it is not connected with the key forcing $\mathbb{P}$.

Blanket Assumption 22.1. We assume that $\mathtt{n} \geqslant 4$ because if $\mathtt{n} = 3$, then Theorem 20.2 holds by Schoenfield absoluteness.

We work in $\mathbf{L}$. Consider the second-order arithmetic language with variables $k, l, m, n, \ldots$ of type 0 over ω and variables $a, b, x, y, \ldots$ of type 1 over 2^ω whose atomic formulae are of the form $x(n) = i$. Let $\mathscr{L}$ be the extension of this language allowing the substitution of variables of type 0 (resp. 1) by positive integers (resp. *small real names* $\mathbf{c} \in \mathbf{L}$; see Definition 9.1).

We define natural classes $\mathscr{L}\varSigma^1_n$, $\mathscr{L}\varPi^1_n$ ($n \geqslant 1$) of $\mathscr{L}$-formulae. Let $\mathscr{L}(\varSigma\varPi)^1_1$ be the closure of $\mathscr{L}\varSigma^1_1 \cup \mathscr{L}\varPi^1_1$ under $\neg$, $\wedge$, $\vee$ and quantifiers over ω. If φ is a formula in $\mathscr{L}\varSigma^1_n$ (resp. $\mathscr{L}\varPi^1_n$), then let φ^- be the result of the canonical transformation of $\neg\,\varphi$ to $\mathscr{L}\varPi^1_n$-form (resp. $\mathscr{L}\varSigma^1_n$-form).

We now define a relation $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ between multitrees $\boldsymbol{p} \in \mathbf{MT}$, sequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ and closed formulae φ in $\mathscr{L}(\varSigma\varPi)^1_1$ or $\mathscr{L}\varSigma^1_n \cup \mathscr{L}\varPi^1_n$, $n \geqslant 2$, which suitably approximates the true $\mathbb{P}$-forcing relation. The definition is by induction on the complexity of φ.

1°. Suppose that $\vec{\pi} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol{p} \in \mathbf{MT}$ (not necessarily $\boldsymbol{p} \in \mathbf{MT}(\vec{\pi})$) and φ is a closed $\mathscr{L}(\Sigma\Pi)^1_1$-formula. We define $\boldsymbol{p} \operatorname{forc}_{\vec{\pi}} \varphi$ if and only if there are a CTM $\mathfrak{M} \models \mathbf{ZFL}^-$ (see Definition 14.3 about $\mathbf{ZFL}^-$), an ordinal $\vartheta < \operatorname{dom}\vec{\pi}$ and a multitree $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\pi} \restriction \vartheta)$ such that

(1) $\boldsymbol{p} \leqslant \boldsymbol{p}_0$ (meaning: $\boldsymbol{p}_0$ is weaker),

(2) $\mathfrak{M}$ contains $\vec{\pi} \restriction \vartheta$ (then it also contains $\mathbf{MT}(\vec{\pi} \restriction \vartheta)$ and $\boldsymbol{p}_0$),

(3) every name $\mathbf{c}$ in φ belongs to $\mathfrak{M}$ and is $\vec{\pi} \restriction \vartheta$-complete,

(4) $\vec{\pi} \restriction \vartheta \subset_{\mathfrak{M}} \vec{\pi}$ and, therefore, $\vec{\pi} \restriction \vartheta \subset_{\{\mathbf{c}\}} \vec{\pi}$ for any name $\mathbf{c}$ in φ, and

(5) $\boldsymbol{p}_0$ $\mathbf{MT}(\vec{\pi} \restriction \vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$ in the ordinary sense.[10]

2°. If $\varphi(x)$ is an $\mathscr{L}\Pi^1_n$-formula, $n \geqslant 1$, then we define $\boldsymbol{p} \operatorname{forc}_{\vec{\pi}} \exists x\, \varphi(x)$ if and only if there is a small real name $\mathbf{c}$ such that $\boldsymbol{p} \operatorname{forc}_{\vec{\pi}} \varphi(\mathbf{c})$.

3°. If $\vec{\pi} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol{p} \in \mathbf{MT}$ and φ is a closed $\mathscr{L}\Pi^1_n$-formula, $n \geqslant 2$, then we define $\boldsymbol{p} \operatorname{forc}_{\vec{\pi}} \varphi$ if and only if we cannot find a sequence $\vec{\tau} \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{p}' \in \mathbf{MT}(\vec{\tau})$ such that $\vec{\pi} \subseteq \vec{\tau}$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$ and $\boldsymbol{p}' \operatorname{forc}_{\vec{\tau}} \varphi^-$.

Remark 22.2. The condition '$\boldsymbol{p}_0$ $\mathbf{MT}(\vec{\pi} \restriction \vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$' in 1° is independent of the choice of the CTM $\mathfrak{M}$ containing $\vec{\pi} \restriction \vartheta$ and φ. Indeed, if φ is an $\mathscr{L}(\Sigma\Pi)^1_1$-formula, then all transitive models agree on the formula $\varphi[G]$ by Mostowski's absoluteness theorem ([20], Theorem 25.4).

Lemma 22.3. *Assume that sequences $\vec{\pi} \subseteq \vec{\varrho}$ belong to $\overrightarrow{\mathbf{MF}}$, $\boldsymbol{q}, \boldsymbol{p} \in \mathbf{MT}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$ and φ is an $\mathscr{L}$-formula. Then $\boldsymbol{p} \operatorname{forc}_{\vec{\pi}} \varphi$ implies $\boldsymbol{q} \operatorname{forc}_{\vec{\varrho}} \varphi$.*

Proof. Suppose that φ is an $\mathscr{L}(\Sigma\Pi)^1_1$-formula and $\boldsymbol{p} \operatorname{forc}_{\vec{\pi}} \varphi$ is given by $\mathfrak{M}$, ϑ, $\boldsymbol{p}_0$ as in 1°. Then the same $\mathfrak{M}$, ϑ, $\boldsymbol{p}_0$ give $\boldsymbol{q} \operatorname{forc}_{\vec{\varrho}} \varphi$.

The induction step $\exists$, as in 2°, is elementary.

We now consider the induction step $\forall$ as in 3°. Suppose that φ is a closed $\mathscr{L}\Pi^1_n$-formula, $n \geqslant 2$, and $\boldsymbol{p} \operatorname{forc}_{\vec{\pi}} \varphi$. Assume that $\boldsymbol{q} \operatorname{forc}_{\vec{\varrho}} \varphi$ fails. Then, by 3°, we can find a sequence $\vec{\varrho}\,' \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{q}' \in \mathbf{MT}(\vec{\varrho}\,')$ such that $\vec{\varrho} \subseteq \vec{\varrho}\,'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$ and $\boldsymbol{q}' \operatorname{forc}_{\vec{\varrho}\,'} \varphi^-$. But then $\vec{\pi} \subseteq \vec{\varrho}\,'$ and $\boldsymbol{q}' \leqslant \boldsymbol{p}$. Hence $\boldsymbol{p} \operatorname{forc}_{\vec{\pi}} \varphi$ fails by 3°. □

Definition 22.4. Let K be one of the classes $\mathscr{L}(\Sigma\Pi)^1_1$, $\mathscr{L}\Sigma^1_n$, $\mathscr{L}\Pi^1_n$ $(n \geqslant 2)$. Then $\mathbf{FORC}[K]$ is the set of all triples $\langle \vec{\pi}, \boldsymbol{p}, \varphi \rangle$ such that $\boldsymbol{p} \operatorname{forc}_{\vec{\pi}} \varphi$.

Thus, $\mathbf{FORC}[K]$ is a subset of HC.

Lemma 22.5 (definability, in $\mathbf{L}$). $\mathbf{FORC}[\mathscr{L}(\Sigma\Pi)^1_1] \in \Delta^{\mathrm{HC}}_1$. *If $n \geqslant 2$, then $\mathbf{FORC}[\mathscr{L}\Sigma^1_n]$ belongs to $\Sigma^{\mathrm{HC}}_{n-1}$ and $\mathbf{FORC}[\mathscr{L}\Pi^1_n]$ belongs to Π^{HC}_{n-1}.*

Proof. Relations such as $\vec{\pi} \in \overrightarrow{\mathbf{MF}}$, 'to be a formula in $\mathscr{L}(\Sigma\Pi)^1_1$, $\mathscr{L}\Sigma^1_n$, $\mathscr{L}\Pi^1_n$', $\boldsymbol{p} \in \mathbf{MT}(\vec{\rho})$, forcing over a CTM, and so on are definable in HC by bounded formulae. Hence they are Δ^{HC}_1. Moreover, the model $\mathfrak{M}$ can be tied by both $\exists$ and $\forall$ in 1°; see Remark 22.2. This wraps up the Δ^{HC}_1-estimation for $\mathscr{L}(\Sigma\Pi)^1_1$.

The inductive step by 2° is quite simple.

We now consider the inductive step by 3°. Assume that $n \geqslant 2$ and it is already known that $\mathbf{FORC}[\mathscr{L}\Sigma^1_n] \in \Sigma^{\mathrm{HC}}_{n-1}$. However, $\langle \vec{\pi}, \boldsymbol{p}, \varphi \rangle \in \mathbf{FORC}[\mathscr{L}\Pi^1_n]$ if and only

[10] Item 1° not only requires $\varphi[\underline{G}]$ to be forced but also suitably seals this status by $\vec{\pi} \restriction \vartheta \subset_{\mathfrak{M}} \vec{\pi}$. This will help us to prove the consistency of forc in Lemma 22.7.

if $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol{p} \in \mathbf{MT}$, φ is a closed $\mathscr{L}\Pi^1_n$-formula and, by 3°, there is no triple $\langle \vec{\boldsymbol{\tau}}, \boldsymbol{p}', \psi\rangle \in \mathbf{FORC}[\mathscr{L}\Sigma^1_n]$ such that $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{MF}}$, $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\tau}}$, $\boldsymbol{p}' \in \mathbf{MT}(\vec{\boldsymbol{\tau}})$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$ and ψ is φ^-. We easily get the required Π^{HC}_{n-1}-estimate for $\mathbf{FORC}[\mathscr{L}\Pi^1_n]$. □

Lemma 22.6 (in $\mathbf{L}$). *Suppose that $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ and φ is an $\mathscr{L}(\Sigma\Pi)^1_1$-formula.*

(i) *If $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}} \cup \overrightarrow{\mathbf{MF}}_{\omega_1}$, $\mathfrak{N} \models \mathbf{ZFL}^-$ is a TM containing $\vec{\boldsymbol{\varphi}}$ and φ, and $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$, then $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$-forces $\varphi[\underline{G}]$ over $\mathfrak{N}$ in the ordinary sense.*

(ii) *If $\mathfrak{N} \models \mathbf{ZFL}^-$ is a TM containing $\vec{\boldsymbol{\pi}}$, each name $\mathbf{c}$ in φ belongs to $\mathfrak{N}$ and is $\vec{\boldsymbol{\pi}}$-complete and $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$-forces $\varphi[\underline{G}]$ over $\mathfrak{N}$, then there is a $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}}$ such that $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{N}} \vec{\boldsymbol{\varphi}}$ and $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}}\,\varphi$.*

Proof. (i) By definition, there are an ordinal $\vartheta < \operatorname{dom}\vec{\boldsymbol{\pi}}$, a multitree $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$ and a CTM $\mathfrak{M} \models \mathbf{ZFL}^-$ containing $\vec{\boldsymbol{\pi}} \restriction \vartheta$ such that $\boldsymbol{p} \leqslant \boldsymbol{p}_0$, each name $\mathbf{c}$ in φ belongs to $\mathfrak{M}$ and is $\vec{\boldsymbol{\pi}} \restriction \vartheta$-complete, $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\mathfrak{M}} \vec{\boldsymbol{\pi}}$ and $\boldsymbol{p}_0$ $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{M}$. There is no loss of generality in assuming that $\mathfrak{M} \subseteq \mathfrak{N}$. (Otherwise $\mathfrak{N} \subseteq \mathfrak{M}$ and we replace $\mathfrak{N}$ by $\mathfrak{M}$.)

We now suppose that $G \subseteq \mathbf{MT}(\vec{\boldsymbol{\varphi}})$ is an $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$-generic set over $\mathfrak{N}$ and $\boldsymbol{p} \in G$. Then also $\boldsymbol{p}_0 \in G$. We have to prove that $\varphi[G]$ is true in $\mathfrak{N}[G]$.

We claim that the set $G' = G \cap \mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$ is $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-generic over $\mathfrak{M}$. Indeed, suppose that $\boldsymbol{D} \in \mathfrak{M}$ and $\boldsymbol{D} \subseteq \mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$ is open-dense in $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$. Since $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\mathfrak{M}} \vec{\boldsymbol{\varphi}}$, $\boldsymbol{D}$ is pre-dense in $\mathbf{MT}(\vec{\boldsymbol{\varphi}})$ by Lemma 14.4, (iii), (b). Hence $G \cap \boldsymbol{D} \neq \varnothing$ by the choice of G. It follows that $G' \cap \boldsymbol{D} \neq \varnothing$.

Now if $\mathbf{c}$ is a name in φ, then $\mathbf{c} \in \mathfrak{M}$ and $\mathbf{c}$ is $\vec{\boldsymbol{\pi}} \restriction \vartheta$-complete. By the above, it follows that $\mathbf{c}[G'] \in 2^\omega$ is defined. Hence $\mathbf{c}[G] = \mathbf{c}[G']$ because $G' \subseteq G$. Thus $\varphi[G]$ coincides with $\varphi[G']$. We also note that $\boldsymbol{p}_0 \in G'$. We conclude that $\varphi[G']$ holds in $\mathfrak{M}[G']$ because $\boldsymbol{p}_0$ forces $\varphi[\underline{G}]$ over $\mathfrak{M}$. Then the same formula $\varphi[G]$ holds in $\mathfrak{N}[G]$ by Mostowski absoluteness.

(ii) Lemma 14.4, (ii) yields the existence of a $\vec{\boldsymbol{\varphi}} \in \overrightarrow{\mathbf{MF}}$ such that $\vec{\boldsymbol{\pi}} \subset_{\mathfrak{N}} \vec{\boldsymbol{\varphi}}$. □

Lemma 22.7 (in $\mathbf{L}$). *Suppose that $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ and φ is a formula in $\mathscr{L}(\Sigma\Pi)^1_1$ or $\mathscr{L}\Sigma^1_n$, $n \geqslant 2$. Then the relations $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ and $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi^-$ cannot hold simultaneously.*

Proof. Suppose that $\varphi \in \mathscr{L}(\Sigma\Pi)^1_1$. If we simultaneously have $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ and $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi^-$, then Lemma 22.6 yields that $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\pi}})$-forces both $\varphi[\underline{G}]$ and $\varphi^-[\underline{G}]$ over a sufficiently large CTM $\mathfrak{M}$, a contradiction. If $\varphi \in \mathscr{L}\Sigma^1_n, n \geqslant 2$, then the result follows from 3°. □

§ 23. Tail invariance

Invariance theorems are very typical for all kinds of forcing. We prove two major invariance theorems for the auxiliary forcing. The first shows the tail invariance. The second (§ 24) explores the permutation invariance.

Suppose that $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_{\alpha\alpha<\lambda}\rangle \in \overrightarrow{\mathbf{MF}}$ and $\gamma < \lambda = \operatorname{dom}\vec{\boldsymbol{\pi}}$. Then the γ-*tail* $\vec{\boldsymbol{\pi}}\restriction_{\geqslant\gamma}$ is defined as the restriction $\vec{\boldsymbol{\pi}} \restriction [\gamma, \lambda)$ to the half-open ordinal interval $[\gamma, \lambda) = \{\alpha \colon \gamma \leqslant \alpha < \lambda\}$. Then the set $\mathbf{MT}(\vec{\boldsymbol{\pi}}\restriction_{\geqslant\gamma}) = \bigcup^{\mathrm{cw}}_{\gamma\leqslant\alpha<\lambda} \vec{\boldsymbol{\pi}}(\alpha)$ is open-dense in $\mathbf{MT}(\vec{\boldsymbol{\pi}})$ by Lemma 14.4, (iii), (a). Therefore it can be expected that if $\vec{\boldsymbol{\varphi}}$ is another sequence of the same length $\lambda = \mathrm{dom}(\vec{\boldsymbol{\varphi}})$ and $\vec{\boldsymbol{\varphi}}\restriction_{\geqslant\gamma} = \vec{\boldsymbol{\pi}}\restriction_{\geqslant\gamma}$, then the relation $\mathrm{forc}_{\vec{\boldsymbol{\pi}}}$ coincides with $\mathrm{forc}_{\vec{\boldsymbol{\varphi}}}$. This turns out to be (almost) the case.

Theorem 23.1. *Assume that $\vec{\boldsymbol{\pi}}, \vec{\boldsymbol{\varphi}}$ are sequences in $\overrightarrow{\mathbf{MF}}$, $\gamma < \lambda = \mathrm{dom}\,\vec{\boldsymbol{\pi}} = \mathrm{dom}\,\vec{\boldsymbol{\varphi}}$, $\vec{\boldsymbol{\varphi}}\restriction_{\geqslant\gamma} = \vec{\boldsymbol{\pi}}\restriction_{\geqslant\gamma}$, $\boldsymbol{p} \in \mathbf{MT}$, $n \geqslant 2$ and φ is a formula in $\mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$. Then $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ if and only if $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}}\,\varphi$.*

Proof. Part 1. *The $\mathscr{L}\Pi^1_2$ Case.* Let $\psi(x)$ be an $\mathscr{L}\Sigma^1_1$-formula. Suppose that $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}}\,\forall\,x\,\psi(x)$ fails. Hence there are $\vec{\boldsymbol{\varphi}}' \in \overrightarrow{\mathbf{MF}}$ and $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ such that $\vec{\boldsymbol{\varphi}} \subseteq \vec{\boldsymbol{\varphi}}'$, $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\boldsymbol{q}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}'}\,\exists\,x\,\psi^-(x)$. We can assume that $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}'\restriction_{\geqslant\gamma})$. By definition 2° (§ 22), there is a small real name $\mathbf{c}$ such that $\boldsymbol{q}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}'}\,\psi^-(\mathbf{c})$. By definition 1°, there is an ordinal $\vartheta < \lambda' = \mathrm{dom}\,\vec{\boldsymbol{\varphi}}'$ such that every name $\mathbf{c}'$ in the formula $\psi^-(\mathbf{c})$, including $\mathbf{c}' = \mathbf{c}$, is $\vec{\boldsymbol{\varphi}}' \restriction \vartheta$-complete and, therefore, $\vec{\boldsymbol{\varphi}}'$-complete by Lemma 14.2.

We define a sequence $\vec{\boldsymbol{\pi}}'$ such that $\mathrm{dom}\,\vec{\boldsymbol{\pi}}' = \lambda'$, $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$ and $\vec{\boldsymbol{\pi}}'\restriction_{\geqslant\lambda} = \vec{\boldsymbol{\varphi}}'\restriction_{\geqslant\lambda}$. Then $\vec{\boldsymbol{\pi}}'\restriction_{\geqslant\gamma} = \vec{\boldsymbol{\varphi}}'\restriction_{\geqslant\gamma}$. Hence $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}}'\restriction_{\geqslant\gamma}) \subseteq \mathbf{MT}(\vec{\boldsymbol{\pi}}')$.

Consider any CTM $\mathfrak{N} \models \mathbf{ZFL}^-$ containing ψ, $\mathbf{c}$, $\vec{\boldsymbol{\pi}}'$, $\vec{\boldsymbol{\varphi}}'$. Then $\boldsymbol{q}$ $\mathbf{MT}(\vec{\boldsymbol{\varphi}}')$-forces $\psi^-(\mathbf{c})[\underline{G}]$ over $\mathfrak{N}$ by Lemma 22.6. However, the forcing notions $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$, $\mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ contain one and the same dense set $\mathbf{MT}(\vec{\boldsymbol{\pi}}'\restriction_{\geqslant\gamma}) = \mathbf{MT}(\vec{\boldsymbol{\varphi}}'\restriction_{\geqslant\gamma})$. Therefore, each name $\mathbf{c}'$ in the formula $\psi^-(\mathbf{c})$, including $\mathbf{c}' = \mathbf{c}$, is $\vec{\boldsymbol{\pi}}'$-complete since it is $\vec{\boldsymbol{\varphi}}'$-complete, and $\boldsymbol{q}$ also $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$-forces $\psi^-(\mathbf{c})[\underline{G}]$ over $\mathfrak{N}$. Hence, by Lemma 22.6, (ii) there is a sequence $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{MF}}$ such that $\vec{\boldsymbol{\pi}}' \subseteq \vec{\boldsymbol{\tau}}$ and $\boldsymbol{q}\,\mathrm{forc}_{\vec{\boldsymbol{\tau}}}\,\psi^-(\mathbf{c})$. But then $\boldsymbol{q}\,\mathrm{forc}_{\vec{\boldsymbol{\tau}}}\,\exists\,x\,\psi^-(x)$ and, therefore, $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\forall\,x\,\psi(x)$ fails, as required.

Part 2. *The step $\mathscr{L}\Pi^1_n \to \mathscr{L}\Sigma^1_{n+1}$, $n \geqslant 2$.* Let $\varphi(x)$ be an $\mathscr{L}\Pi^1_n$-formula. Assume that $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\exists\,x\,\varphi(x)$. By definition (see 2° in § 22) there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi(\mathbf{c})$. Then we have $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}}\,\varphi(\mathbf{c})$ by the inductive hypothesis. Thus $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}}\,\exists\,x\,\psi(x)$.

Part 3. *The step $\mathscr{L}\Sigma^1_n \to \mathscr{L}\Pi^1_n$, $n \geqslant 3$.* Suppose that φ is an $\mathscr{L}\Pi^1_n$-formula and $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}}\,\varphi$ fails. By definition 3° (§ 22) one can find a sequence $\vec{\boldsymbol{\varphi}}' \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{p}' \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ such that $\vec{\boldsymbol{\varphi}} \subseteq \vec{\boldsymbol{\varphi}}'$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$ and $\boldsymbol{p}'\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}'}\,\varphi^-$. By Lemma 14.4, (ii), (a), there is a multitree $\boldsymbol{r} \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}'\restriction_{\geqslant\gamma})$, $\boldsymbol{r} \leqslant \boldsymbol{p}'$. Then $\boldsymbol{r} \leqslant \boldsymbol{p}$ and $\boldsymbol{r}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}'}\,\varphi^-$. Define a sequence $\vec{\boldsymbol{\pi}}' \in \overrightarrow{\mathbf{MF}}$ by putting $\mathrm{dom}\,\vec{\boldsymbol{\pi}}' = \lambda' = \mathrm{dom}\,\vec{\boldsymbol{\varphi}}'$, $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$ and $\vec{\boldsymbol{\pi}}'\restriction_{\geqslant\lambda} = \vec{\boldsymbol{\varphi}}'\restriction_{\geqslant\lambda}$. Then $\boldsymbol{r} \in \mathbf{MT}(\vec{\boldsymbol{\pi}}'\restriction_{\geqslant\gamma})$, $\boldsymbol{r} \leqslant \boldsymbol{p}$ and also $\boldsymbol{r}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}'}\,\varphi^-$ by the inductive hypothesis. We conclude that $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ also fails. □

§ 24. Permutations

Continuing to work in $\mathbf{L}$, we write PERM for the set of all bijections $\boldsymbol{h}\colon \omega_1 \xrightarrow{\text{onto}} \omega_1$ such that $\boldsymbol{h} = \boldsymbol{h}^{-1}$ and the *non-identity domain* $\mathbf{NID}(\boldsymbol{h}) = \{\xi\colon \boldsymbol{h}(\xi) \neq \xi\}$ is at most countable. The elements of PERM are called *permutations*.

Suppose that $\boldsymbol{h} \in \mathrm{PERM}$. We extend the action of $\boldsymbol{h}$ as follows.

– If $\boldsymbol{p}$ is a multitree, then $\boldsymbol{hp}$ is a multitree, $|\boldsymbol{hp}| = \boldsymbol{h}''\boldsymbol{p} = \{\boldsymbol{h}(\xi)\colon \xi \in |\boldsymbol{p}|\}$ and $(\boldsymbol{hp})(\boldsymbol{h}(\xi)) = \boldsymbol{p}(\xi)$ whenever $\xi \in |\boldsymbol{p}|$. In other words, $\boldsymbol{hp}$ coincides with the superposition $\boldsymbol{p} \circ (\boldsymbol{h}^{-1})$.

– If $\boldsymbol{\pi} \in \mathbf{MT}$ is a multiforcing, then so is $\boldsymbol{h} \cdot \boldsymbol{\pi} = \boldsymbol{\pi} \circ (\boldsymbol{h}^{-1})$, $|\boldsymbol{h} \cdot \boldsymbol{\pi}| = \boldsymbol{h}''\boldsymbol{\pi}$ and $(\boldsymbol{h} \cdot \boldsymbol{\pi})(\boldsymbol{h}(\xi)) = \boldsymbol{\pi}(\xi)$ whenever $\xi \in |\boldsymbol{\pi}|$.

– If $\mathbf{c} \subseteq \mathbf{MT} \times (\omega \times \omega)$ is a real name, then we put $\boldsymbol{h}\mathbf{c} = \{\langle \boldsymbol{hp}, n, i\rangle : \langle \boldsymbol{p}, n, i\rangle \in \mathbf{c}\}$. It is clear that $\boldsymbol{h}\mathbf{c}$ is also a real name.

– If $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_{\alpha\alpha<\kappa}\rangle \in \overrightarrow{\mathbf{MF}}$, then $\boldsymbol{h}\vec{\boldsymbol{\pi}} = \langle \boldsymbol{h} \cdot \boldsymbol{\pi}_{\alpha\alpha<\kappa}\rangle$ is still a sequence in $\overrightarrow{\mathbf{MF}}$.

– If $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$ is an $\mathscr{L}$-formula (with all names explicitly indicated), then $\boldsymbol{h}\varphi$ is $\varphi(\boldsymbol{h}\mathbf{c}_1, \dots, \boldsymbol{h}\mathbf{c}_n)$.

Many of the notions and relations defined above are clearly PERM-invariant. For example, $\boldsymbol{p} \in \mathbf{MT}(\boldsymbol{\pi}) \iff \boldsymbol{hp} \in \mathbf{MT}(\boldsymbol{h} \cdot \boldsymbol{\pi})$, $\boldsymbol{\pi} \sqsubset \boldsymbol{\varphi} \iff \boldsymbol{h} \cdot \boldsymbol{\pi} \sqsubset \boldsymbol{h} \cdot \boldsymbol{\varphi}$, and so on. The invariance also holds for the relation forc.

Theorem 24.1. *Assume that* $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$, $\boldsymbol{p} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$, $\boldsymbol{h} \in \mathrm{PERM}$, $n \geqslant 2$ *and* φ *belongs to* $\mathscr{L}\Pi^1_n \cup \mathscr{L}\Sigma^1_{n+1}$. *Then* $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ *if and only if* $(\boldsymbol{hp})\,\mathrm{forc}_{\boldsymbol{h}\vec{\boldsymbol{\pi}}}(\boldsymbol{h}\varphi)$.

Proof. Put $\vec{\boldsymbol{\varphi}} = \boldsymbol{h}\vec{\boldsymbol{\pi}}$, $\boldsymbol{q} = \boldsymbol{hp}$.
Part 1. *The* $\mathscr{L}\Pi^1_2$ *Case.* Assume that $\varphi(x)$ is an $\mathscr{L}\Sigma^1_1$-formula, $\psi(x) := \boldsymbol{h}\varphi(x)$ and $\boldsymbol{q}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}}\,\forall x\,\psi(x)$ fails. By definition (1°, 2° in § 22), we can find a sequence $\vec{\boldsymbol{\varphi}}' \in \overrightarrow{\mathbf{MF}}$, a multitree $\boldsymbol{q}' \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ and a small real name $\mathbf{d}$ such that $\vec{\boldsymbol{\varphi}} \subset \vec{\boldsymbol{\varphi}}'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$ and $\boldsymbol{q}'\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}'}\,\psi^-(\mathbf{d})$. Then the sequence $\vec{\boldsymbol{\pi}}' = \boldsymbol{h}\vec{\boldsymbol{\varphi}}'$ satisfies $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\pi}}'$, the multitree $\boldsymbol{p}' = \boldsymbol{hq}'$ belongs to $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$, $\boldsymbol{p}' \leqslant \boldsymbol{p}$ and $\mathbf{c} = \boldsymbol{h}\mathbf{d}$ is a small real name. However, we cannot now claim that $\boldsymbol{p}'\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}'}\,\varphi^-(\mathbf{c})$ since the existence of $\mathfrak{M}$ and ϑ as in 1° of § 22 is not necessarily preserved by the action of $\boldsymbol{h}^{-1}$ or $\boldsymbol{h}$.

To circumvent this difficulty, suppose that $\mathfrak{M} \models \mathbf{ZFL}^-$ is a CTM containing $\vec{\boldsymbol{\pi}}'$, $\vec{\boldsymbol{\varphi}}'$, $\boldsymbol{h}$, $\mathbf{c}$, $\mathbf{d}$ and (all names in) φ, ψ. Then $\boldsymbol{q}'$ $\mathbf{MT}(\vec{\boldsymbol{\varphi}}')$-forces $\psi^-(\mathbf{d})[\underline{G}]$ over $\mathfrak{M}$ by Lemma 22.6, (i). It follows that $\boldsymbol{p}'$ $\mathbf{MT}(\vec{\boldsymbol{\pi}}')$-forces $\varphi^-(\mathbf{c})[\underline{G}]$ over $\mathfrak{M}$ by standard theorems of forcing. Lemma 22.6, (ii) yields a sequence $\vec{\boldsymbol{\tau}} \in \overrightarrow{\mathbf{MF}}$ with $\vec{\boldsymbol{\pi}}' \subset \vec{\boldsymbol{\tau}}$ such that $\boldsymbol{p}'\,\mathrm{forc}_{\vec{\boldsymbol{\tau}}}\,\varphi^-(\mathbf{c})$ and, therefore, $\boldsymbol{p}'\,\mathrm{forc}_{\vec{\boldsymbol{\tau}}}\,\exists x\,\varphi^-(x)$ by 2°. However, $\vec{\boldsymbol{\pi}} \subset \vec{\boldsymbol{\pi}}' \subset \vec{\boldsymbol{\tau}}$ and $\boldsymbol{p}' \leqslant \boldsymbol{p}$. Therefore $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\forall x\,\varphi(x)$ fails by 3°, as required.
Part 2. *The step* $\mathscr{L}\Pi^1_n \to \mathscr{L}\Sigma^1_{n+1}$, $n \geqslant 2$. Let $\varphi(x)$ be a formula in $\mathscr{L}\Pi^1_n$. We put $\psi(x) := \boldsymbol{h}\varphi(x)$ and assume that $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\exists x\,\varphi(x)$. By definition (see 2° in § 22), there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi(\mathbf{c})$. Then we have $\boldsymbol{q}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}}\,\psi(\mathbf{d})$ by the inductive hypothesis, where $\mathbf{d} = \boldsymbol{h}\mathbf{c}$ is a small real name. Thus $\boldsymbol{q}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}}\,\exists x\,\psi(x)$.
Part 3. *The step* $\mathscr{L}\Sigma^1_n \to \mathscr{L}\Pi^1_n$, $n \geqslant 3$. Let $\varphi(x)$ be a formula in $\mathscr{L}\Pi^1_n$. Assume that $\boldsymbol{q}\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}}\,\psi$ fails, where $\boldsymbol{q} = \boldsymbol{hp}$, $\vec{\boldsymbol{\varphi}} = \boldsymbol{h}\vec{\boldsymbol{\pi}}$ and ψ is $\boldsymbol{h}\varphi$ as above. By 3° there are a sequence $\vec{\boldsymbol{\varphi}}' \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{q}' \in \mathbf{MT}(\vec{\boldsymbol{\varphi}}')$ such that $\vec{\boldsymbol{\varphi}} \subseteq \vec{\boldsymbol{\varphi}}'$, $\boldsymbol{q}' \leqslant \boldsymbol{q}$ and $\boldsymbol{q}'\,\mathrm{forc}_{\vec{\boldsymbol{\varphi}}'}\,\psi^-$. We now put $\boldsymbol{p}' = \boldsymbol{h}^{-1}\boldsymbol{q}'$ and $\vec{\boldsymbol{\pi}}' = \boldsymbol{h}^{-1}\vec{\boldsymbol{\varphi}}'$, so that $\boldsymbol{p}' \leqslant \boldsymbol{p}$ and $\vec{\boldsymbol{\pi}} \subseteq \vec{\boldsymbol{\pi}}'$. We have $\boldsymbol{p}'\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}'}\,\varphi^-$ by the inductive hypothesis. We conclude that $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$ fails, as required. □

§ 25. Forcing inside the key sequence

Theorem 25.3 below shows that the forcing relation $\mathrm{forc}_{\vec{\boldsymbol{\pi}}}$, when considered with countable initial segments $\vec{\boldsymbol{\pi}} = \vec{\sqcap} \restriction \alpha$ of the key sequence $\vec{\sqcap}$, coincides with the true $\mathbb{P}$-forcing relation up to level $\mathfrak{n} - 1$.

We work in $\mathbf{L}$. Recall that the number $\mathfrak{n}$ and the key sequence $\vec{\sqcap} = \langle \sqcap_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ (which satisfies parts (i)–(iv) of Theorem 15.3) were introduced in 15.4, and $\mathbb{P} = \mathbf{MT}(\vec{\sqcap})$ is our forcing notion. In addition, $\mathfrak{n} \geqslant 4$ by 22.1.

Definition 25.1. For brevity, we write $\boldsymbol{p}\,\mathrm{forc}_{\alpha\varphi}$ instead of $\boldsymbol{p}\,\mathrm{forc}_{\vec{\sqcap}\restriction\alpha}\,\varphi$. Let $\boldsymbol{p}\,\mathrm{forc}\,\varphi$ mean that $\boldsymbol{p}\,\mathrm{forc}_{\alpha\varphi}$ for some $\alpha < \omega_1$.

Lemma 25.2 (in $\mathbf{L}$). *Assume that* $\boldsymbol{p} \in \mathbb{P}$, $\alpha < \omega_1$ *and* $\boldsymbol{p}\,\mathrm{forc}_{\alpha\varphi}$. *Then the following assertions hold.*

(i) *If* $\alpha \leqslant \beta < \omega_1$, $\boldsymbol{q} \in \mathbb{P}_{<\beta} = \mathbf{MT}(\vec{\sqcap} \restriction \beta)$ *and* $\boldsymbol{q} \leqslant \boldsymbol{p}$, *then* $\boldsymbol{q}\,\mathrm{forc}_{\beta\varphi}$.

(ii) *If* $\boldsymbol{q} \in \mathbb{P}$ *and* $\boldsymbol{q} \leqslant \boldsymbol{p}$, *then* $\boldsymbol{q}\,\mathrm{forc}_{\beta\varphi}$ *for some* β, $\alpha \leqslant \beta < \omega_1$.

(iii) *If* $\boldsymbol{q} \in \mathbb{P}$ *and* $\boldsymbol{q}\,\mathrm{forc}\,\varphi^-$, *then* $\boldsymbol{p}, \boldsymbol{q}$ *are sad.*

(iv) *Therefore, first, if* $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{P}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$ *and* $\boldsymbol{p}\,\mathrm{forc}\,\varphi$, *then* $\boldsymbol{q}\,\mathrm{forc}\,\varphi$ *and, second,* $\boldsymbol{p}\,\mathrm{forc}\,\varphi$ *and* $\boldsymbol{p}\,\mathrm{forc}\,\varphi^-$ *cannot hold simultaneously.*

Proof. To prove (i), apply Lemma 22.3. To prove (ii), pick a β with $\alpha < \beta < \omega_1$ and a $\boldsymbol{q} \in \mathbf{MT}(\vec{\sqcap} \restriction \beta)$ and apply (i). To prove (iii), note that $\boldsymbol{p}, \boldsymbol{q}$ are incompatible in $\mathbb{P}$ since otherwise (i) gives rise to a contradiction. But incompatibility in $\mathbb{P}$ implies being sad by Corollary 4.3. □

Theorem 25.3. *If* φ *is a closed* $\mathscr{L}$*-formula in*

$$\mathscr{L}(\Sigma\Pi)^1_1 \cup \mathscr{L}\Sigma^1_2 \cup \mathscr{L}\Pi^1_2 \cup \cdots \cup \mathscr{L}\Sigma^1_{\mathbf{n}-2} \cup \mathscr{L}\Pi^1_{\mathbf{n}-2} \cup \mathscr{L}\Sigma^1_{\mathbf{n}-1}$$

and $\boldsymbol{p} \in \mathbb{P}$, *then* $\boldsymbol{p}$ $\mathbb{P}$*-forces* $\varphi[\underline{G}]$ *over* $\mathbf{L}$ *in the ordinary sense if and only if* $\boldsymbol{p}\,\mathrm{forc}\,\varphi$.

Proof. Let $\Vdash$ denote the ordinary $\mathbb{P}$-forcing relation over $\mathbf{L}$.

Part 1. *φ is a formula in* $\mathscr{L}(\Sigma\Pi)^1_1$. If $\boldsymbol{p}\,\mathrm{forc}\,\varphi$, then $\boldsymbol{p}\,\mathrm{forc}_{\vec{\sqcap}\restriction\gamma}\,\varphi$ for some $\gamma < \omega_1$, and then $\boldsymbol{p} \Vdash \varphi[\underline{G}]$ by Lemma 22.6, (i) with $\vec{\boldsymbol{\varphi}} = \vec{\sqcap}$ and $\mathfrak{N} = \mathbf{L}$.

We now suppose that $\boldsymbol{p} \Vdash \varphi[\underline{G}]$. There is an ordinal $\gamma_0 < \omega_1$ such that $\boldsymbol{p} \in \mathbb{P}_{\gamma_0} = \mathbf{MT}(\vec{\sqcap} \restriction \gamma_0)$ and all the names in φ belong to $\mathfrak{L}(\vec{\sqcap} \restriction \gamma_0)$. (We recall Definition 14.3 on models $\mathfrak{L}(x) \models \mathbf{ZFL}^-$.) Let U be the set of all sequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ such that $\gamma_0 < \mathrm{dom}\,\vec{\boldsymbol{\pi}}$ and there is an ordinal ϑ, $\gamma_0 < \vartheta < \mathrm{dom}\,\vec{\boldsymbol{\pi}}$, such that $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)} \vec{\boldsymbol{\pi}}$. Then U is dense in $\overrightarrow{\mathbf{MF}}$ by Lemma 14.4, (ii) and belongs to $\mathbf{\Delta}_1(\mathrm{HC})$. Therefore by Corollary 15.5 there is an ordinal $\gamma < \omega_1$ such that $\vec{\boldsymbol{\pi}} = \vec{\sqcap} \restriction \gamma \in U$. Let ϑ be the ordinal that shows this, that is, $\gamma_0 < \vartheta < \gamma = \mathrm{dom}\,\vec{\boldsymbol{\pi}}$ and $\vec{\boldsymbol{\pi}} \restriction \vartheta \subset_{\mathfrak{L}(\vec{\boldsymbol{\pi}}\restriction\vartheta)} \vec{\boldsymbol{\pi}}$. We claim that $\boldsymbol{p}$ $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-forces $\varphi[\underline{G}]$ over $\mathfrak{L}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$ in the ordinary sense. Then, by definition, $\boldsymbol{p}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$, and we are done.

To prove the claim, assume the opposite. Then there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\sqcap}\restriction\vartheta)$ with $\boldsymbol{q} \leqslant \boldsymbol{p}$ that $\mathbf{MT}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$-forces $\neg\,\varphi[\underline{G}]$ over $\mathfrak{L}(\vec{\boldsymbol{\pi}} \restriction \vartheta)$. It follows by definition (see 1° in § 22) that $\boldsymbol{q}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\neg\,\varphi$, whence $\boldsymbol{q}\,\mathrm{forc}\,\neg\,\varphi$ and, therefore, $\boldsymbol{q} \Vdash \neg\,\varphi[\underline{G}]$ (see above). This contradicts $\boldsymbol{p} \Vdash \varphi[\underline{G}]$.

Part 2. *The step* $\mathscr{L}\Pi^1_n \to \mathscr{L}\Sigma^1_{n+1}$ $(n \geqslant 1)$. Consider an $\mathscr{L}\Pi^1_n$-formula $\varphi(x)$. Assume that $\boldsymbol{p}\,\mathrm{forc}\,\exists\,x\,\varphi(x)$. By definition, there is a small real name $\mathbf{c}$ such that $\boldsymbol{p}\,\mathrm{forc}\,\varphi(\mathbf{c})$. By the inductive hypothesis, $\boldsymbol{p} \Vdash \varphi(c)[\underline{G}]$, whence $\boldsymbol{p} \Vdash \exists\,x\,\varphi(x)[\underline{G}]$. Conversely, suppose that $\boldsymbol{p} \Vdash \exists\,x\,\varphi(x)[\underline{G}]$. Since $\mathbb{P}$ is a CCC forcing, there is a small real name $\mathbf{c}$ (in $\mathbf{L}$) such that $\boldsymbol{p} \Vdash \varphi(\mathbf{c})[\underline{G}]$. Then $\boldsymbol{p}\,\mathrm{forc}\,\varphi(\mathbf{c})$ by the inductive hypothesis. Hence $\boldsymbol{p}\,\mathrm{forc}\,\exists\,x\,\varphi(x)$.

Part 3. *The step* $\mathscr{L}\Sigma^1_n \to \mathscr{L}\Pi^1_n$ $(2 \leqslant n \leqslant \mathbf{n} - 2)$. Assume that φ is a closed $\mathscr{L}\Sigma^1_n$-formula and $\boldsymbol{p}\,\mathrm{forc}\,\varphi^-$. By Lemma 25.2, (iv) there is no multitree $\boldsymbol{q} \in \mathbb{P}$ with $\boldsymbol{q} \leqslant \boldsymbol{p}$ such that $\boldsymbol{q}\,\mathrm{forc}\,\varphi$. Hence $\boldsymbol{p} \Vdash \varphi^-$ by the inductive hypothesis.

Conversely, suppose that $\boldsymbol{p} \Vdash \varphi^-$. There is an ordinal $\gamma_0 < \omega_1$ such that $\boldsymbol{p} \in \mathbb{P}_{\gamma_0} = \mathbf{MT}(\vec{\sqcap} \restriction \gamma_0)$ and φ belongs to $\mathfrak{L}(\vec{\sqcap} \restriction \gamma_0)$. Consider the set U of all sequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ such that $\mathrm{dom}\,\vec{\boldsymbol{\pi}} > \gamma_0$ and there is a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ satisfying $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\boldsymbol{q}\,\mathrm{forc}_{\vec{\boldsymbol{\pi}}}\,\varphi$. We have $U \in \mathbf{\Sigma}_{n-1}(\mathrm{HC})$ (with parameters φ, $\boldsymbol{p}_0$) by Lemma 22.5.

Hence $U \in \mathbf{\Sigma}_{\mathbf{n}-3}(\mathrm{HC})$, where $\mathbb{n} \geqslant 4$ by 22.1. Therefore, by 15.4 (and part (iii) of Theorem 15.3), there is an ordinal $\gamma < \omega_1$ such that $\vec{\mathbb{\Pi}} \restriction \gamma$ blocks U.
Case 1. $\vec{\mathbb{\Pi}} \restriction \gamma \in U$. Let $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ be the multitree that shows this. In particular, $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\gamma > \gamma_0$. We have $\boldsymbol{q} \in \mathbf{MT}(\vec{\mathbb{\Pi}} \restriction \gamma)$, $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\boldsymbol{q}\,\mathrm{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\,\varphi$. Thus $\boldsymbol{q} \Vdash \varphi[\underline{G}]$ by the inductive hypothesis, contrary to the choice of $\boldsymbol{p}$. Therefore Case 1 does not occur.
Case 2. *No sequence in U extends $\vec{\mathbb{\Pi}} \restriction \gamma$.* We can assume that $\gamma > \gamma_0$. (If not, replace γ by $\gamma_0 + 1$.) Then we claim that $\boldsymbol{p}\,\mathrm{forc}_\gamma\,\varphi^-$. Indeed, otherwise by 3° we can find a sequence $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ and a multitree $\boldsymbol{q} \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ such that $\vec{\mathbb{\Pi}} \restriction \gamma \subseteq \vec{\boldsymbol{\pi}}$, $\boldsymbol{q} \leqslant \boldsymbol{p}$ and $\boldsymbol{q}\,\mathrm{forc}_{\vec{\boldsymbol{\varrho}}}\,\varphi$. But then $\vec{\boldsymbol{\pi}}$ belongs to U. On the other hand, $\vec{\mathbb{\Pi}} \restriction \gamma \subseteq \vec{\boldsymbol{\pi}}$ contrary to the assumption of Case 2. Thus, $\boldsymbol{p}\,\mathrm{forc}\,\varphi^-$, as required. $\square$

§ 26. A theorem on elementary equivalence

Proof (of Theorem 20.2). Assume the opposite. Then one can find a $\mathit{\Pi}^1_{\mathbb{n}-2}$-formula $\varphi(r, x)$ with just one parameter $r \in 2^\omega \cap \mathbf{L}[G \restriction X]$ and a real $x_0 \in 2^\omega \cap \mathbf{L}[G]$ such that $\varphi(r, x_0)$ is true in $\mathbf{L}[G]$ but there is no $x \in 2^\omega \cap \mathbf{L}[G \restriction X]$ such that $\varphi(r, x)$ is true in $\mathbf{L}[G]$. By an appropriate version of Lemma 12.1, we have $r = \mathbf{c}_0[G]$, where $\mathbf{c}_0 \subseteq \mathbf{MT}(\mathbb{\Pi} \restriction X) \times \omega \times 2$ is a small $(\mathbb{\Pi} \restriction X)$-complete real name. (See § 17 for notation.) Moreover, there is a small $\mathbb{\Pi}$-complete real name $\mathbf{c} \subseteq \mathbb{P} \times \omega \times 2$ such that $x_0 = \mathbf{c}[\underline{G}]$.

By Theorem 25.3 there is a multitree $\boldsymbol{p}_0 \in G$ such that

(1) $\boldsymbol{p}_0$ $\mathbb{P}$-forces '$\varphi(\mathbf{c}_0[\underline{G}], \mathbf{c}[\underline{G}]) \wedge \neg\exists x \in \mathbf{L}[\underline{G} \restriction X]\,\varphi(\mathbf{c}_0[\underline{G}], x)$' over $\mathbf{L}$;

(2) $\boldsymbol{p}_0\,\mathrm{forc}\,\varphi(\mathbf{c}_0, \mathbf{c})$, that is, $\boldsymbol{p}_0\,\mathrm{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma_0}\,\varphi(\mathbf{c}_0, \mathbf{c})$, where $\gamma_0 < \omega_1$, and we can assume that $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\mathbb{\Pi}} \restriction \gamma_0)$ as well.

Since $\mathbf{c}$, $\mathbf{c}_0$ are small names, there is an ordinal $\delta < \omega_1$ such that

(3) $|\mathbf{c}_0| \subseteq \delta \cap X$, $|\mathbf{c}| \subseteq \delta$ and $|\boldsymbol{p}_0| \subseteq \delta$.

Since $|\vec{\mathbb{\Pi}}| = \omega_1$ by Corollary 16.2, we can enlarge γ_0 (if necessary) to guarantee that

(4) $\delta \subseteq |\vec{\mathbb{\Pi}} \restriction \gamma_0|$, that is, if $\eta < \delta$, then $\eta \in |\mathbb{\Pi}_{\alpha'}|$ for some $\alpha' = \alpha'(\eta) < \gamma_0$.

Our aim is to deduce a contradiction. We put $D = \delta \setminus X$.

Let U be the set of all sequences $\vec{\boldsymbol{\pi}} \in \overrightarrow{\mathbf{MF}}$ such that $\vec{\mathbb{\Pi}} \restriction \gamma_0 \subset \vec{\boldsymbol{\pi}}$, whence $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\boldsymbol{\pi}})$ by (2), and there are $\zeta < \mathrm{dom}\,\vec{\boldsymbol{\pi}}$ and $\boldsymbol{h} \in \mathrm{PERM}$ such that

(A) $\mathbf{NID}(\boldsymbol{h}) \cap (\delta \cap X) = \varnothing$ and $\boldsymbol{h}$ maps D onto a set $R \subseteq X \setminus \delta$;

(B) $\gamma_0 \leqslant \zeta < \mathrm{dom}\,\vec{\boldsymbol{\pi}}$ and $(\boldsymbol{h}\vec{\boldsymbol{\pi}})\restriction_{\geqslant\zeta} = \vec{\boldsymbol{\pi}}\restriction_{\geqslant\zeta}$, that is, $\boldsymbol{h}(\vec{\boldsymbol{\pi}}(\alpha)) = \vec{\boldsymbol{\pi}}(\alpha)$ whenever $\zeta \leqslant \alpha < \mathrm{dom}\,\vec{\boldsymbol{\pi}}$.

Routine estimations show that U is a $\mathbf{\Sigma}_1(\mathrm{HC})$-set (with [parameters $\vec{\mathbb{\Pi}} \restriction \gamma_0$, δ). Hence it is a $\mathbf{\Sigma}_{\mathbf{n}-3}(\mathrm{HC})$-set because $\mathbb{n} \geqslant 4$ by 22.1. Therefore, by 15.4, there is an ordinal $\gamma < \omega_1$ such that $\vec{\mathbb{\Pi}} \restriction \gamma$ blocks U.
Case 1. $\vec{\mathbb{\Pi}} \restriction \gamma \in U$, *so that* (A), (B) *hold for* $\vec{\boldsymbol{\pi}} = \vec{\mathbb{\Pi}} \restriction \gamma$ *in terms of some* $\zeta \in [\gamma_0, \gamma)$ *and* $\boldsymbol{h} \in \mathrm{PERM}$. In particular, by (B), $\boldsymbol{h}(\mathbb{\Pi}_\alpha) = \mathbb{\Pi}_\alpha$ whenever $\zeta \leqslant \alpha < \gamma$. We have $\boldsymbol{p}_0\,\mathrm{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\,\varphi(\mathbf{c}_0, \mathbf{c})$ by (2) and Lemma 22.3. Put $\mathbf{c}' = \boldsymbol{h}\mathbf{c}$, $\boldsymbol{p}_0' = \boldsymbol{h}\boldsymbol{p}_0$. Note that $\boldsymbol{h}\mathbf{c}_0 = \mathbf{c}_0$ since $|\mathbf{c}_0| \cap \mathbf{NID}(\boldsymbol{h}) = \varnothing$ by (A). Theorem 24.1 now yields that $\boldsymbol{p}_0'\,\mathrm{forc}_{\boldsymbol{h}\cdot(\vec{\mathbb{\Pi}}\restriction\gamma)}\,\varphi(\mathbf{c}_0, \mathbf{c}')$. Hence we have $\boldsymbol{p}_0'\,\mathrm{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\,\varphi(\mathbf{c}_0, \mathbf{c}')$ by (B) and Theorem 23.1. But the common domain $|\boldsymbol{p}_0| \cap |\boldsymbol{p}_0'|$ is disjoint from $\mathbf{NID}(\boldsymbol{h})$ in view of (A) since $|\boldsymbol{p}_0| \subseteq \delta$. It follows that $\boldsymbol{p}_0$, $\boldsymbol{p}_0'$ are compatible, so $\boldsymbol{p} = \boldsymbol{p}_0 \cup \boldsymbol{p}_0' \in \mathbf{MT}$ (not necessarily $\in \mathbf{MT}(\vec{\mathbb{\Pi}} \restriction \gamma)$) and $\boldsymbol{p} \leqslant \boldsymbol{p}_0'$, whence $\boldsymbol{p}\,\mathrm{forc}_{\vec{\mathbb{\Pi}}\restriction\gamma}\,\varphi(\mathbf{c}_0, \mathbf{c}')$.

Unfortunately, Theorem 25.3 is not immediately applicable here to conclude that $\boldsymbol{p}$ $\mathbb{P}$-forces $\varphi(\mathbf{c}_0[\underline{G}], \mathbf{c}'[\underline{G}])$ over $\mathbf{L}$. This is because $\boldsymbol{p}$ may not belong to $\mathbb{P}$. We need a further argument. Recall that $\boldsymbol{p}_0 \in \mathbf{MT}(\vec{\mathbb{\Pi}} \restriction \gamma_0)$, whence $\boldsymbol{p}_0' \in \mathbf{MT}(\boldsymbol{h} \cdot (\vec{\mathbb{\Pi}} \restriction \gamma_0))$. Since $\zeta > \gamma_0$, there is a multitree $\boldsymbol{q}_0 \in \mathbf{MT}(\boldsymbol{h} \cdot \mathbb{\Pi}_\zeta)$ satisfying $|\boldsymbol{q}_0| = |\boldsymbol{p}_0'|$ and $\boldsymbol{q}_0 \leqslant \boldsymbol{p}_0'$. Then $\boldsymbol{q}_0 \operatorname{forc}_{\vec{\mathbb{\Pi}} \restriction \gamma} \varphi(\mathbf{c}_0, \mathbf{c}')$ (because $\boldsymbol{p}_0' \operatorname{forc}_{\vec{\mathbb{\Pi}} \restriction \gamma} \varphi(\mathbf{c}_0, \mathbf{c}')$) and $\boldsymbol{q}_0 \in \mathbf{MT}(\mathbb{\Pi}_\zeta)$ since $\boldsymbol{h} \cdot \mathbb{\Pi}_\zeta = \mathbb{\Pi}_\zeta$. Thus, $\boldsymbol{q}_0 \in \mathbf{MT}(\vec{\mathbb{\Pi}} \restriction \gamma)$. Moreover, $\boldsymbol{q}_0$ is compatible with $\boldsymbol{p}_0$ in $\mathbf{MT}(\vec{\mathbb{\Pi}} \restriction \gamma)$ because $|\boldsymbol{q}_0| = |\boldsymbol{p}_0'|$ and $\boldsymbol{q}_0 \leqslant \boldsymbol{p}_0'$, and $\boldsymbol{p}_0'$ coincides with $\boldsymbol{p}_0$ on the common domain $|\boldsymbol{p}_0| \cap |\boldsymbol{p}_0'| = \delta \cap X$. Thus there is a $\boldsymbol{q} \in \mathbf{MT}(\vec{\mathbb{\Pi}} \restriction \gamma)$ with $\boldsymbol{q} \leqslant \boldsymbol{p}_0$, $\boldsymbol{q} \leqslant \boldsymbol{q}_0$. Then $\boldsymbol{q} \operatorname{forc}_{\vec{\mathbb{\Pi}} \restriction \gamma} \varphi(\mathbf{c}_0, \mathbf{c}')$ and we conclude that

(5) $\boldsymbol{q}$ $\mathbb{P}$-forces $\varphi(\mathbf{c}_0[\underline{G}], \mathbf{c}'[\underline{G}])$ over $\mathbf{L}$

by Theorem 25.3. However, $|\mathbf{c}'| \subseteq (\delta \cap X) \cup R \subseteq X$ by construction, whence $\mathbf{c}'[\underline{G}] \in \mathbf{L}[\underline{G} \restriction X]$ is forced. Thus $\boldsymbol{q}$ $\mathbb{P}$-forces $\exists x \in \mathbf{L}[\underline{G} \restriction X]\, \varphi(\mathbf{c}_0[\underline{G}], x)$ over $\mathbf{L}$ by (5), contrary to (1). This contradiction settles Case 1.

Case 2. *No sequence in U extends $\vec{\mathbb{\Pi}} \restriction \gamma$.* We can assume that $\gamma > \gamma_0$. (Otherwise replace γ by $\gamma_0 + 1$.) Pick any set $R \subseteq X \setminus \delta$ satisfying

$$\operatorname{card} R = \operatorname{card} D \quad \text{and} \quad R \cap \left(\bigcup_{\alpha < \gamma} |\mathbb{\Pi}_\alpha| \right) = \varnothing.$$

Since $D \subseteq \delta$, we have $D \cap R = \varnothing$ and there is a permutation $\boldsymbol{h} \in \mathrm{PERM}$, $\boldsymbol{h} \colon D \xrightarrow{\text{onto}} R$, satisfying $\mathbf{NID}(\boldsymbol{h}) = D \cup R$. Hence (A) holds.

Pick any ordinal λ, $\gamma < \lambda < \omega_1$. Our plan is to modify $\vec{\boldsymbol{\pi}} = \vec{\mathbb{\Pi}} \restriction \lambda$ in order to fulfill (B) with $\zeta = \gamma$. The modification replaces the R-part of $\vec{\mathbb{\Pi}} \restriction \lambda$ above γ by an $\boldsymbol{h}$-copy of its D-part. To perform this in detail, we recall that $\vec{\mathbb{\Pi}} \restriction \lambda = \langle \mathbb{\Pi}_{\alpha} \rangle_{\alpha < \lambda}$, where each $\mathbb{\Pi}_\alpha$ is a small special multiforcing whose domain $d_\alpha = |\mathbb{\Pi}_\alpha| \subseteq \omega_1$ is countable. If $\alpha < \gamma$, then we put $\boldsymbol{\pi}_\alpha = \mathbb{\Pi}_\alpha$. Suppose that $\gamma \leqslant \alpha < \lambda$. Then $D \subseteq |\mathbb{\Pi}_\alpha|$ by (4). On the basis of $\mathbb{\Pi}_\alpha$, we define a new multiforcing $\boldsymbol{\pi}_\alpha$ with the following properties.

(a) $|\boldsymbol{\pi}_\alpha| = d_\alpha \cup R$. Note that $D \subseteq d_\alpha = |\mathbb{\Pi}_\alpha| \subseteq |\boldsymbol{\pi}_\alpha|$ in this case because $D \subseteq \delta \subseteq |\vec{\mathbb{\Pi}} \restriction \gamma|$ by (4) (since $\gamma_0 \leqslant \gamma$).

(b) If $\xi \in d_\alpha \setminus R$, then $\boldsymbol{\pi}_\alpha(\xi) = \mathbb{\Pi}_\alpha(\xi)$.

(c) If $\xi \in D$, that is, $\boldsymbol{h}(\xi) = \eta \in R$, then $\boldsymbol{\pi}_\alpha(\eta) = \mathbb{\Pi}_\alpha(\xi)$.

We claim that $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_{\alpha} \rangle_{\alpha < \lambda} \in \overrightarrow{\mathbf{MF}}$, that is, if $\alpha < \beta < \lambda$, then $\boldsymbol{\pi}_\alpha \sqsubset \boldsymbol{\pi}_\beta$. In other words, if $\eta \in |\boldsymbol{\pi}_\alpha|$, then $\boldsymbol{\pi}_\alpha(\eta) \sqsubset \boldsymbol{\pi}_\beta(\eta)$.

If $\eta \notin R$, then $\boldsymbol{\pi}_\alpha(\eta) = \mathbb{\Pi}_\alpha(\eta)$ by construction. It remains to check that $\boldsymbol{\pi}_\alpha(\eta) \sqsubset \boldsymbol{\pi}_\beta(\eta)$ whenever $\alpha < \beta < \lambda$, $\eta = \boldsymbol{h}(\xi) \in R \cap |\boldsymbol{\pi}_\alpha|$ and $\xi \in D$. If now $\alpha < \gamma$, then $R \cap |\boldsymbol{\pi}_\alpha| = \varnothing$ by the choice of R. Hence it remains to consider the case when $\gamma \leqslant \alpha$. Then $\xi, \eta \in |\boldsymbol{\pi}_\alpha|$ by construction and we have $\boldsymbol{\pi}_\alpha(\eta) = \mathbb{\Pi}_\alpha(\xi)$ and $\boldsymbol{\pi}_\beta(\eta) = \mathbb{\Pi}_\beta(\xi)$. Therefore $\boldsymbol{\pi}_\alpha(\xi) \sqsubset \boldsymbol{\pi}_\beta(\xi)$, as required.

We claim that the sequence $\vec{\boldsymbol{\pi}} = \langle \boldsymbol{\pi}_{\alpha} \rangle_{\alpha < \lambda}$ satisfies $\vec{\mathbb{\Pi}} \restriction \gamma \subseteq \vec{\boldsymbol{\pi}}$ and (A), (B). Indeed, $\vec{\mathbb{\Pi}} \restriction \gamma \subseteq \vec{\boldsymbol{\pi}}$ since $\gamma \geqslant \gamma_0$. (A) holds by construction. We now show that (B) holds with $\zeta = \gamma$, that is, if $\gamma \leqslant \alpha < \lambda$, then $\boldsymbol{h} \cdot \boldsymbol{\pi}_\alpha = \boldsymbol{\pi}_\alpha$. Indeed, $D \cup R \subseteq |\boldsymbol{\pi}_\alpha|$ by (a) and, therefore, $\boldsymbol{h} \cdot \boldsymbol{\pi}_\alpha = \boldsymbol{\pi}_\alpha$ by (b), (c).

Thus, $\vec{\boldsymbol{\pi}} \in U$ and $\vec{\mathbb{\Pi}} \restriction \gamma \subset \vec{\boldsymbol{\pi}}$. But this contradicts the assumptions of Case 2.

Either of the two cases leads to a contradiction. This completes the proof of Theorem 20.2. Thus Theorem 1.2 is also proved. $\square$

§ 27. Remarks and problems

One may ask what happens with the separation theorem at other projective levels $m \neq \mathfrak{n}$ in the model of § 19. For higher levels, it turns out that in the model $\mathbf{L}[G \restriction \Delta_H]$ of Theorem 20.1 there is a 'good' $\varDelta^1_{\mathfrak{n}+1}$-well-ordering of the reals, of length ω_1. (The gaps in Δ_H do not enable us to perform the well-ordering construction of Corollary 18.3 at level $\mathfrak{n}$!) It follows by a standard argument that the separation theorem holds for $\mathbf{\Pi}^1_m$ and fails for $\mathbf{\Sigma}^1_m$ for all $m > \mathfrak{n}$ in the model $\mathbf{L}[G \restriction \Delta_H]$. Concerning the levels $3 \leqslant m < \mathfrak{n}$, we conjecture that separation holds for $\mathbf{\Pi}^1_m$ and fails for $\mathbf{\Sigma}^1_m$ in $\mathbf{L}[G \restriction \Delta_H]$, but this problem is open.

Let $\mathbb{P}_{\mathfrak{n}}$ be the forcing notion $\mathbb{P}$ defined in § 16 for a given $\mathfrak{n} \geqslant 3$. Using a certain *amalgamation* of all $\mathbb{P}_{\mathfrak{n}}$, $\mathfrak{n} \geqslant 3$, which is defined by a rather sophisticated product-like construction originally used in [18], part 1, and [40], one can construct a generic extension of $\mathbf{L}$ in which the separation theorem fails simultaneously for all the classes $\mathbf{\Sigma}^1_{\mathfrak{n}}$, $\mathbf{\Pi}^1_{\mathfrak{n}}$, $\mathfrak{n} \geqslant 3$.

Finally, there is the interesting and perhaps very difficult problem of constructing a generic extension of $\mathbf{L}$ in which the separation theorem holds for a given class $\mathbf{\Sigma}^1_{\mathfrak{n}}$, $\mathfrak{n} \geqslant 3$, beginning with $\mathbf{\Sigma}^1_3$ for example. This problem has been open since the early years of forcing; see [16], Problem 3029. In this regard, we mention a recent preprint by Hoffelner [41] with interesting results.

The authors are grateful to the anonymous referee for valuable remarks and suggestions, which contributed significantly to improving the quality of the publication.

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Received 28/MAY/19
21/JUL/20
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