# THE AXIOM OF DETERMINACY AND THE MODERN DEVELOPMENT OF DESCRIPTIVE SET THEORY

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This article contains a survey of modern investigations in descriptive set theory connected with the axiom of determinacy.

## Introduction

Descriptive set theory, whose origins go back to the works of Borel, Baire, and Lebesgue at the turn of the century, developed into an independent subject during the twenties and thirties, occupying at that time a prominent place in mathematical research. Such world-famous scholars as P. S. Aleksandrov, L. V. Kantorovich, A. N. Kolmogorov, and M. A. Lavrent'ev spent time studying the descriptive theory (and achieved recognized results in it); and this field became one of the important areas in the mathematical activity of N. N. Luzin and P. S. Novikov.

It was largely through the efforts of Soviet mathematicians that such divisions of the descriptive theory as the theory of Borel sets, the theory of $A$-sets (also called *analytic*, or *Suslin*, sets), the theory of lattices, indices, and constituents, the theory of $CA$-sets and second-level projective sets, and the general theory of operations on sets (from which the theory of $R$-sets later evolved) were established and achieved, in the main, their finished form by the end of the 30's.

All this research, now unified under the general name of classical descriptive set theory, is characterized from the modern point of view by the traditional concept, inherited from the theory of functions, of mathematical proofs as activity directed toward establishing the properties of objects having in some sense a real existence. One consequence of such an approach was the intuitive conviction of the researchers that every statement (or at least every "meaningful" statement) about "real" sets is either true—in which case it should be possible to prove it by a sufficiently great effort—or false, in which case it should be possible to refute it. The principal task of mathematicians is to find new techniques and methods of proof.

Such conceptions, typical of the majority of fields of mathematics, also "worked" well for a time in descriptive set theory, as long as the theory limited itself to such relatively "simple" sets as Borel sets or $A$-sets. However the situation changed completely when specialists in the descriptive theory turned to the study of the projective sets discovered by Luzin. While they had succeeded in establishing a theory rich in results about the first (lowest) level of projective sets formed by the Borel sets, $A$-sets, and $CA$-sets, yet only isolated essential results were obtained for sets of the second projective level, and the higher projective levels remained, in general, terra incognita. Essentially all that was known about them was that in each level there appear sets not found on the preceding levels. Moreover the reason why a definitive study was impossible lay not at all in deficiencies of technique. After the investigations of Novikov, R. Solovay, and others it became known (and Luzin had been convinced of this in the mid-20's) that many important questions on projective sets of higher levels—and in some cases also second- and even first-level sets—do not in principle admit of a definite positive or negative answer on the basis of accepted mathematical methods and ways of reasoning.

Thus, Novikov [11] showed that no contradiction could be deduced from the assumption that there exists a Lebesgue nonmeasurable set of the second projective level. Later Solovay [69] established that it is also impossible to deduce a contradiction from the assumption that all projective sets of the real line are measurable. Thus the problem of the measurability of projective sets turns out to be undecidable. The same fate awaited the majority of the remaining open problems of the classical descriptive theory.

Naturally, such a situation led mathematicians working in the descriptive theory to seek new axioms

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not among the traditional postulates of classical mathematics, but admitting a more or less acceptable foundation and making it possible to obtain definite answers to the questions that are undecidable within the framework of the traditional approach. The first such supplementary axiom to be considered was Gödel's axiom of constructivity, whose principal applications to the problems of the descriptive theory were obtained by Novikov [11]. Definite interest was also aroused by the measurable cardinal axiom and the "sharps hypothesis" which is equivalent to it as far as applications to the descriptive theory are concerned.[1] But the greatest attention and recognition among specialists in the descriptive theory over the last 10 or 15 years has been given to two axioms connected with infinite games: the axiom of determinacy (AD), and the axiom of projective determinacy (PD). It is to these axioms that the present article is devoted.

The popularity of this topic in contemporary research is attested by the mere fact that, besides a mass of journal articles, four volumes of the series *Lecture Notes in Mathematics* were devoted totally or to a significant degree to applications of determinacy. In chronological order, they were volumes 38, 37, 35, and 36. However, research in determinacy is reflected very little in Soviet publications: only §6 of Chapter 8 of the translated handbook [3] and Chapter 2 of the pamphlet [5] can be mentioned. This circumstance exerted a decisive influence on the choice of style for the present article. The author preferred to devote more space to the more important results, presenting them with proofs, rather than striving for a maximal coverage of all areas. The same method of exposition, we note, is adopted in the handbook mentioned above.

Now a few words about the structure of this survey. In the first section we present some necessary definitions and facts relating to projective sets. The following section §2 is an introduction to game theory and determinate sets. Then in §§3–7 we study the main applications of the axioms AD and PD to problems of the theory of projective sets connected with regularity properties, separability, uniformization, single- and countably-valued sets, and Borel and Suslin representations of projective sets. In the last section (§8) we present some results that are based on Martin's theorem on the determinacy of the Borel sets.

We close this introduction by pointing out the works from the bibliography which might be considered as an introductory course in the theory of determinacy. Article [59] (in particular, its first part) contains a survey of "early" research in determinacy. In the book [38] another field of applications of the axiom of determinacy is expounded—infinitary combinatorics. The fundamental monograph [58] included practically all the significant results of the descriptive theory connected with determinacy and obtained by the end of the 70's. The same could also be said of the articles [25, 34], except that here a narrower circle of questions is considered on a more popular level. To these we add the already-mentioned works [3, Ch. 8, §6] and [5]. In the works just enumerated, as in [23], certain philosophico-mathematical questions are also touched upon concerning the place of the hypothesis of determinacy among the accepted set-theoretical axioms.

## §1. Projective Sets and the Projective Hierarchy

Descriptive set theory is concerned with sets located in certain definite spaces and consequently possessing an inherited external structure. Originally the research was limited, as a rule, to subsets of the real line $R$ and the Euclidean spaces $R^n$. However, by the beginning of the 30's it was realized that for a variety of reasons it was more convenient to take as the basic space not the real line but a Baire space, leading to essential simplifications in certain important calculations.

Striving for geometric clarity, Luzin, in his *Lectures* [48], used a realization of Baire space in the form of the set $J$ of irrational points of the line $R$. In modern works a different realization is used more often—the product of a countable number of copies of the set of natural numbers $\omega = \{0, 1, 2, \dots\}$—which facilitates the use of logical methods in the reasoning. Thus Baire space is taken to be the set $\mathcal{N} = \omega^\omega$ of all $\omega$-sequences of natural numbers endowed with the product topology. (The topology on $\omega$ is discrete.) Each point $\alpha \in \mathcal{N}$ can be represented in the form $\alpha = \langle a_0, a_1, a_2, \dots, a_k, \dots \rangle$, where $a_k = \alpha(k) \in \omega$ for all $k$.

Together with Baire space, certain spaces derived from it of the form $\omega^l \times \mathcal{N}^m$ are usually considered; here $l$ and $m$ are natural numbers, not simultaneously zero. We shall call such spaces point spaces or

[1]Here is a list of articles presenting rather completely the trends mentioned: [4,§2], [14], [26], [34, par. 5.4], [50], [52], [67].

product spaces; sets located in them will be called point sets, and various families consisting of such sets will be called point classes.

In the literature on the descriptive theory some useful conventions have been worked out in regard to the use of letters. Natural numbers are denoted by small Latin letters ($i$, $a$, $k$ and the like); points of the space $\mathcal{N}$ are denoted by the letters $\alpha$, $\beta$, $\gamma$, $\delta$, $\varepsilon$, and points of an arbitrary point space $\mathcal{X}$ by the letters $x$, $y$, $z$. It is customary to denote point sets by capital Latin letters and point classes by the letters $\Sigma$, $\Pi$, $\Delta$, $\Gamma$ (as a rule, with various indices). The letters $\xi$, $\eta$, $\zeta$, $\varkappa$, $\lambda$ are reserved to denote ordinal numbers (both finite and transfinite).

The simplest point class is formed by the open sets, with which the construction of the projective hierarchy begins. This hierarchy consists of the projective classes $\Sigma^1_n$, $\Pi^1_n$, and $\Delta^1_n$, where $n \in \omega$. The initial class $\Sigma^1_0$ is the class of open sets of point spaces. For any $n$ the class $\Pi^1_n$ includes the complements of $\Sigma^1_n$-sets, and the class $\Delta^1_n$—the intersection of the classes $\Sigma^1_n$ and $\Delta^1_n$—includes all those sets that belong simultaneously to the classes $\Sigma^1_n$ and $\Pi^1_n$. Finally, the projections of sets of $\Pi^1_n$ are assigned to the class $\Sigma^1_{n+1}$. Here we have in mind the special method of projection whereby a set $P \subseteq \mathcal{X} \times \mathcal{N}$ (where $\mathcal{X}$ is any point space) goes to its projection

$$\pi P = \{x \in \mathcal{X} : \text{There exists } \alpha \in \mathcal{N} \text{ such that } \langle x, \alpha \rangle \in P\},$$

i.e., projection along the very last (="right-hand") axis, when this axis is $\mathcal{N}$.

A point set is called projective when it belongs to one of the projective classes. Projective sets constitute the smallest class of point sets containing all open sets which is closed with respect to the operations of complementation and projection.

The projective hierarchy and the concept of a projective set were introduced by Luzin in [46]. Luzin denoted the projective classes by $A_n$, $CA_n$, and $B_n$, corresponding to $\Sigma^1_n$, $\Pi^1_n$, and $\Delta^1_n$. From the work of Suslin [71] it follows that the class $\Sigma^1_1$ coincides with the class of all point $A$-sets (discovered by Suslin and intensively studied even before the publication of note [46]), and the class $\Delta^1_1$ coincides with the class of all Borel point sets.[2] As it happens the Borel sets themselves form a hierarchy of classes indexed by the finite and countable ordinals (cf. §8 below).

In carrying out practically any reasoning involving projective sets it is necessary to "calculate" the class of a set obtained by some operation from the sets of known classes. Such calculations are easier, as a rule, to carry out not on the point sets themselves but on relations that are determined by these sets.

The use of relations instead of sets can be illustrated by the example of subsets of the space $\mathcal{X} = \omega^2 \times \mathcal{N}$. Suppose $X \subseteq \mathcal{X}$. Writing $X(i,j,\alpha)$ instead of $\langle i,j,\alpha \rangle \in X$, we identify the set $X$ with the relation corresponding to it, denoting the latter by the same letter: $X(i,j,\alpha)$. In this notation the letter $\alpha$—an argument of type $\mathcal{N}$—denotes an arbitrary (i.e., "variable") point of $\mathcal{N}$, and the letters $i$ and $j$—arguments of type $\omega$—denote arbitrary natural numbers.

Operations on relations such as conjuction $\wedge$, disjunction $\vee$, and negation $\neg$, the quantifiers $\exists$ and $\forall$, and also substitution have a logical rather than a geometrical character, but nevertheless also admit a natural geometric interpretation (negation = complement, $\exists$ = projection, and the like). Dealing with projective relations calls for developing definite practical skills in such interpretation, and, if you will, a certain psychological bent. However, all this justifies itself by making it possible to calculate quickly the class of projective relations and sets using a small number of quite simple rules. We now give these rules.

1. THE EXPANSION RULE: $\Sigma^1_n \cup \Pi^1_n \subseteq \Delta^1_{n+1}$ for any $n$. This rule reflects the fact that projective classes expand as the index $n$ increases. We note that the expansion here is strict: to be precise, $\Sigma^1_n \not\subseteq \Pi^1_n$, $\Pi^1_n \not\subseteq \Sigma^1_n$, $\Delta^1_{n+1} \not\subseteq \Sigma^1_n \cup \Pi^1_n$ for all $n$.

2. THE CONTINUOUS SUBSTITUTION RULE. A relation obtained from a relation on a given projective class $\Gamma$ (here and below $\Gamma = \Sigma^1_n$ or $\Pi^1_n$ or $\Delta^1_n$) by substituting continuous functions that are defined everywhere on the corresponding point spaces belongs to the same class $\Gamma$. For example, if the relation $P(k,\alpha)$ (i.e., the set $P \subseteq \omega \times \mathcal{N}$) has class $\Gamma$, and the functions $F : \mathcal{N}^2 \times \omega \to \omega$, $G : \omega \times \mathcal{N} \to \mathcal{N}$,

[2] For early work on the descriptive theory see the surveys [2, 8, 10, 13].

$H : \mathcal{N} \times \omega \to \mathcal{N}$ are continuous, then the relation $Q(i, j, \alpha, \beta) \rightleftharpoons P(F(G(i, \alpha), \beta, i), H(\alpha, j))$ also has class $\Gamma$—in other words, the subset

$$Q = \{\langle i, j, \alpha, \beta\rangle : \langle F(G(i, \alpha), \beta, i), H(\alpha, j)\rangle \in P\}$$

of the space $\omega^2 \times \mathcal{N}^2$ belongs to $\Gamma$.

Geometrically the fact that every projective class is closed with respect to the operation of taking the complete inverse image under continuous everywhere-defined functions corresponds to this rule.

3. THE SUBSTITUTION-OF-PARAMETERS RULE. If $P(\ldots, \alpha, \ldots)$ is a relation of the projective class $\Gamma$ with the displayed argument $\alpha$ of type $\mathcal{N}$ (i.e., denoting a "variable" point of $\mathcal{N}$; naturally, instead of $\alpha$ here, there might be any other of the letters reserved above to denote points of $\mathcal{N}$), and $\alpha_0 \in \mathcal{N}$ is a fixed point of $\mathcal{N}$ (i.e., a parameter), then the relation $P(\ldots, \alpha_0, \ldots)$ also belongs to $\Gamma$ and analogously for arguments of type $\omega$ (they are of course replaced by natural numbers).

Thus, variable arguments of relations can be replaced by concrete parameters (points of $\mathcal{N}$ or natural numbers) while preserving the projective class. Geometrically such a replacement corresponds to sectioning with a hyperplane.

We note that rule 3 is a particular case of rule 2: actually it is a matter of substituting constant functions, each of which, of course, is continuous.

4. THE NEGATION RULE. The negation $\neg P(\ldots)$ of a relation $P(\ldots)$ of class $\Sigma^1_n$ is a relation of class $\Pi^1_n$. Symbolically $\neg\Sigma^1_n = \Pi^1_n$. Analogously $\neg\Pi^1_n = \Sigma^1_n$ and $\neg\Delta^1_n = \Delta^1_n$. From the geometric point of view this means that the operation of complementation takes $\Sigma^1_n$ into $\Pi^1_n$ and vice versa, while the complements of $\Delta^1_n$-sets remain in $\Delta^1_n$.

5. THE CONJUNCTION AND DISJUNCTION RULE. Any relation obtained from relations of a given projective class $\Gamma$ using the conjunction sign $\wedge$ ("and") and the disjunction sign $\vee$ (inclusive "or") is a relation of the same class $\Gamma$: $\wedge\Gamma = \vee\Gamma = \Gamma$.

Geometrically this corresponds to the proposition that every projective class is closed with respect to finite unions and intersections. Still the rule for relations has a wider field of action, for example, allowing the conclusion that the conjunction $P(i, \alpha) \wedge Q(\alpha, \beta)$ of two $\Gamma$-relations $P$ and $Q$ has class $\Gamma$, even though the corresponding set

$$\{\langle i, \alpha, \beta\rangle : P(i, \alpha) \wedge Q(\alpha, \beta)\}$$

is, of course, not the intersection of the sets $P$ and $Q$.

Before explaining the "quantifier" rules we recall that the notations $\exists x \ldots$ and $\forall x \ldots$ denote respectively: "there exists $x$ such that $\ldots$," and "for every $x$ $\ldots$ holds." Moreover, in accordance with the convention adopted above on the use of definite letters, the notation, say, $\exists\alpha$ will be understood as meaning $\exists\alpha \in \mathcal{N}$, and $\forall k$ as $\forall k \in \omega$ and the like.

6. RULES FOR QUANTIFIERS OF TYPE $\mathcal{N}$. (a) If $P(\ldots, \alpha, \ldots)$ is a relation of class $\Sigma^1_n$ with displayed argument $\alpha$, then the relation $\exists\alpha P(\ldots, \alpha, \ldots)$ also belongs to $\Sigma^1_n$. Geometrically this corresponds to the fact that each of the classes $\Sigma^1_n$ is closed under the operation of projection.

The rule just stated can be written in symbolic form as the equality $\exists^{\mathcal{N}}\Sigma^1_n = \Sigma^1_n$, where $\exists^{\mathcal{N}}$ denotes the application of the quantifier to one of the arguments of type $\mathcal{N}$ (i.e., denoting an arbitrary "variable" point of $\mathcal{N}$). Using this notation, we state several more points of rule 6 and the following rule 7.

(b) $\forall^{\mathcal{N}}\Pi^1_n = \Pi^1_n$; (c) $\exists^{\mathcal{N}}\Pi^1_n = \Sigma^1_{n+1}$; (d) $\forall^{\mathcal{N}}\Sigma^1_n = \Pi^1_{n+1}$.

7. THE RULE FOR QUANTIFIERS OF TYPE $\omega$.

$$\text{(a)}\ \exists^{\omega}\Sigma^1_n = \Sigma^1_n; \qquad \text{(b)}\ \forall^{\omega}\Pi^1_n = \Pi^1_n;$$

$$\text{(c) if } n \geqslant 1, \text{ then } \quad \forall^{\omega}\Sigma^1_n = \Sigma^1_n \text{ and } \exists^{\omega}\Pi^1_n = \Pi^1_n.$$

The content of rules 6 and 7 can also be expressed as follows. Each of the classes $\Sigma^1_n$ is closed with respect to $\exists^{\mathcal{N}}$ and $\exists^{\omega}$, and also with respect to $\forall^{\omega}$ provided $n \geqslant 1$. Each class $\Pi^1_n$ is closed with respect to $\forall^{\mathcal{N}}$ and $\forall^{\omega}$, and also with respect to $\exists^{\omega}$ provided $n \geqslant 1$. Hence, incidentally, it follows that the classes $\Delta^1_n$ for $n \geqslant 1$ are closed with respect to $\exists^{\omega}$ and $\forall^{\omega}$.

We now give another, final rule, which, in contrast to the preceding ones, is more naturally stated for sets than for relations.

8. THE RULE FOR COUNTABLE UNIONS AND INTERSECTIONS. If $n \geqslant 1$, then the classes $\Sigma^1_n$, $\Pi^1_n$, and $\Delta^1_n$ are closed with respect to the operations of countable union and countable intersection (applied naturally to families of sets situated in some point space). In addition, the class $\Sigma^1_0$ is closed with respect to countable union, and the class $\Pi^1_0$ with respect to countable intersection.

We shall not dwell on the justifications of these rules (the corresponding proofs can be found in [3, Ch. 8, §§1–3], [4, Sec. 9], and [12, final two chapters]); rather we pass on to explain the basic concepts connected with determinacy. It will be possible to see how rules 1–8 "work" in the calculations of the following sections.

## §2. Introduction to the theory of determinacy

Assume that some set $A$ of the "Baire plane" $\mathcal{N}^2 = \mathcal{N} \times \mathcal{N}$ is fixed and is a game set, or GS for short. By means of this set a two-person game $G(A)$ is defined, with players denoted, as a rule, I and II. The game proceeds as follows:

player I writes a natural number $a_0$;

player II, knowing the "move" $a_0$, writes his own natural number $b_0$;

again player I, knowing $b_0$, writes a natural number $a_1$;

player II, knowing $a_1$, writes a natural number $b_1$;

and so on ad infinitum. At the end there is a pair of points

$$\alpha = \langle a_i : i \in \omega \rangle = \langle a_0, a_1, a_2, \dots \rangle$$
$$\beta = \langle b_i : i \in \omega \rangle = \langle b_0, b_1, b_2, \dots \rangle$$

of the space $\mathcal{N}$ called a match. If it turns out that $\langle \alpha, \beta \rangle \in A$, the match is considered to have been won by player I; the opposite case is defined as a win for player II.

The players may make their moves by following strategies chosen in advance. Any function defined on the set $FC = \omega^{<\omega}$ of all finite sequences of natural numbers (with the empty sequence $\Lambda$) and assuming values in the set of natural numbers may serve as a strategy in games of the type under consideration. If player I adheres to a strategy $\sigma : FC \to \omega$, then he must make each of his moves $a_i$ in accordance with the equation $a_i = \sigma(b_0, \dots, b_{i-1})$, or, more briefly, $a_i = \sigma(\beta \restriction i)$, where $\beta \restriction i = \langle b_0, \dots, b_{i-1} \rangle$ is the sequence of the first $i$ moves of player II. In particular, the initial move $a_0$ is given by the equation $a_0 = \sigma(\Lambda)$; further $a_1 = \sigma(b_0)$, $a_2 = \sigma(b_0, b_1)$, etc. Thus the strategy $\sigma$ completely determines the sequence $\alpha = \langle a_0, a_1, a_2, \dots \rangle$ of moves of player I from the sequence $\beta = \langle b_0, b_1, b_2, \dots \rangle$ of movers of player II. It is conventional to denote the sequence $\alpha$ so defined by $\sigma * \beta$.

In a completely analogous manner if player II follows a strategy $\tau$, then he makes each of his moves $b_i$ according to the equation

$$b_i = \tau(a_0, \dots, a_i) = \tau(\alpha \restriction i + 1),$$

where $\alpha \restriction i + 1 = \langle a_0, \dots, a_i \rangle$ is the sequence of the first $(i+1)$ moves of player I. The sequence $\beta$ given by these equations is denoted $\alpha * \tau$.

The strategy $\sigma$ is said to be a winning strategy (WS for short) for player I in the game $G(A)$ (i.e., in the game with game set $A$), if $\langle \sigma * \beta, \beta \rangle \in A$, for any point $\beta \in \mathcal{N}$. In other words, a winning strategy guarantees a win, no matter how the opponent plays.

Analogously $\tau$ is a WS for player II when $\langle \alpha, \alpha * \tau \rangle \notin A$ for any point $\alpha \in \mathcal{N}$.

The set $A$ and the game $G(A)$ are called deterministic if one of the players has a WS in the game $G(A)$ (obviously they cannot both have one).

Various principles or determinacy hypotheses are considered which assert the determinacy of sets of one class or another. The following are the most interesting from the point of view of applications to descriptive set theory:

the axiom of determinacy (AD), which postulates the determinacy of every set $A \subseteq \mathcal{N}^2$; and the axiom of projective determinacy (PD), which postulates the determinacy of all projective sets $A \subseteq \mathcal{N}^2$.

In general, for each (say, projective) class $\Gamma$ there is a principle $\Gamma$-Det asserting the determinacy of all sets $A \subseteq \mathcal{N}^2$ of the class $\Gamma$.

Now a few words about the relation between determinacy hypotheses and the Zermelo–Fraenkel axiom system accepted by the majority of specialists as the foundation of set-theoretic constructions. This system is denoted by the abbreviation ZFC or ZF, according as it includes the axiom of choice (AC) or not (thus ZFC = ZF + AC). The axiom AD contradicts the axiom of choice (among other reasons, because it implies that every subset of the real line is Lebesgue-measurable, see below). At present the question of the consistency of the systems ZF + AD and ZFC + PD remains open. The only argument in favor of consistency is the actual absence of contradictions in those very rich tableaux of consequences which have been obtained for both of these theories.

The incompatibility of the axiom AD with the axiom of choice, of course, compels a certain skepticism in dealing with its possibilities. Fortunately axiom AD is compatible with the principle of dependent choice[3] (DC), which is weaker than AC, but sufficient to prove such 'positive' consequences of AC as the theorem that a countable union of countable sets is countable or the countable additivity of Lebesgue measure. To be specific Kechris showed in [33] that if the system ZF + AD is consistent, then it remains so when DC is added to it. At the same time neither the principle DC [70] nor the axiom of choice $AC_\omega$ for countable families of nonempty sets [33] is a theorem of ZF + AD.

However it is not difficult to prove that AD (plus the axioms of ZF) implies the axiom of choice for countable families of sets of Baire space (and hence also of any other of the spaces $\omega^l \times \mathcal{N}^m$, as well as any Euclidean space $R^m$). We now give this simple reasoning. We need to construct a choice function, assuming AD, for a family of nonempty sets $X_0, X_1, X_2, \ldots, \subseteq \mathcal{N}$. To do this consider the game $G(A)$ defined by the set $A = \{\langle\alpha,\beta\rangle : \beta \notin A_{\alpha(0)}\}$. Player I cannot have a winning strategy in this game, since no matter what opening move $a_0$ he makes, player II guarantees himself a win by making his moves $b_i$ so that their sequence coincides with a point $\beta \in X_{a_0}$ chosen by him in advance (after the move $a_0$!). Thus player II has a WS $\tau$ in the game $G(A)$. This strategy provides the required choice function. Indeed, let $k \in \omega$. Consider a match in the game $G(A)$ in which player I makes all his moves equal to $k$ and player II responds with strategy $\tau$. The sequence $\beta = \langle b_i : i \in \omega\rangle$ of moves of player II in this match (which is completely determined once $\tau$ is fixed and $k$ is prescribed) will be denoted $f(k)$. Then $f(k) \in X_k$ for any $k$ by the definition of $A$ and the choice of $\tau$, i.e., $f$ is a choice function for the family of sets $X_k$.

If we now turn from the question of consistency to the question of truth (=provability in ZF or ZFC) of determinacy hypotheses, we may begin with the following theorem, which gives a simple result, yet one sufficient for many applications.

THEOREM. (Gale–Stewart) [24] *Every open set $A \subseteq \mathcal{N}^2$ is determinate, i.e., $\Sigma^1_0$-Det holds.*

The proof of this theorem involves some devices that are quite common in work with determinacy—in particular the concept of a game beginning at a certain position.

Suppose that, in addition to the set $A \subseteq \mathcal{N}^2$, two finite sequences $u, v \in FC$ are given. The game $G(u; v; A)$—the game $G(A)$ from the position $u; v$—differs from the game $G(A)$ only in that player I is required to make his first $i$ moves so that they constitute $u$ and player II is required to makes his first $j$ moves so as to constitute the sequence $v$. Here $i$ and $j$ are the lengths of $u$ and $v$ respectively.

For example, if the finite sequences $u = \langle a_0, a_1\rangle$ and $v = \langle b_0\rangle$ are given, then the game $G(a_0, a_1; b_0; A)$ (= the game $G(A)$ from the position $a_0, a_1; b_0$) presupposes that player I makes initial move $a_0$, then player II makes the move $b_0$, and then player I makes the move $a_1$—in these three moves the players have no choice, being obliged to take as these moves the corresponding terms of the finite sequences $u$ and $v$—and all subsequent moves $b_1, a_2, b_2, a_3, b_3, \ldots$ can be arbitrarily chosen. The result of such a game is defined as in the game $G(A)$, i.e., taking account of the first moves dictated by the finite sequences $u$ and $v$.

[3]The principle DC postulates the following. If a binary relation $E$ on a set $X$ is such that $\forall x \in X \exists y \in X(xEy)$, then there exists a sequence $x_0, x_1, x_2 \ldots$ of elements $x_i$ of the set $X$ such that $x_i E x_{i+1}$ for all $i$.

The concept of strategy and winning strategy in games beginning at a certain position is illustrated by the example of the same game $G(a_0, a_1; b_0; A)$. A strategy for player I in this game is any function $\sigma : FC \to \omega$ satisfying the conditions: $\sigma(\Lambda) = a_0$ and $\sigma(b_0) = a_1$. Such a strategy is a winning strategy for player I if $\langle \sigma * \beta, \beta \rangle \in A$ for any point $\beta \in \mathcal{N}$ satisfying $\beta(0) = b_0$ (in the general case satisfying $v \subseteq \beta$). The concepts of strategy and winning strategy for player II are introduced in exactly the same way: In the game under consideration it is required that $\tau(a_0) = b_0$ and in this case $\tau$ will be a WS for player II if $\langle \alpha, \alpha * \tau \rangle \notin A$ for any point $\alpha \in \mathcal{N}$ such that $\alpha(0) = a_0$ and $\alpha(1) = a_1$ ($u \subset \alpha$ in the general case).

Finally, the position $u; v$ is said to be winning for player I (or for player II) in the game $G(A)$ when player I (resp. player II) has a WS in the game $G(u; v; A)$.

Having stated these definitions, we now turn directly to the proof of the Gale–Stewart theorem. Consider an arbitrary open set $A \subseteq \mathcal{N}^2$. Assuming that player I has no WS in the game $G(A)$, we shall show how player II must proceed in order to win the game.

Suppose player I makes some opening move $a_0$. By hypothesis the initial position $\Lambda; \Lambda$ is not winning for I; consequently the position $a_0; \Lambda$ is also not winning for this player. Therefore player II can make a move $b_0$ in such a way that the position $a_0; b_0$ is again not winning for I. Player II will take such a move $b_0$ (for definiteness, say the smallest such move) as his response to the move $a_0$ of his opponent.

Next suppose player I carries out his next move $a_1$. Analogous reasoning shows that player II has a move $b_1$ such that the position $a_0, a_1; b_0, b_1$ is not winning for his opponent, and so forth.

From this description of the moves of player II it is not difficult to extract a strategy $\tau$ possessing the property that for any sequence $\alpha \in \mathcal{N}$ of moves of player I, if we define $\beta = \alpha * \tau$ (the sequence of responses of II according to the strategy $\tau$), then for any $m$ the position $\alpha \restriction m; \beta \restriction m$ is not winning for I in the game $G(A)$. In particular, for every $m$ there exists a pair $\langle \alpha_m, \beta_m \rangle \notin A$ such that $\alpha \restriction m = \alpha_m \restriction m$ and $\beta \restriction m = \beta_m \restriction m$ (otherwise the position $\alpha \restriction m; \beta \restriction m$ would already be won by player I independently of all succeeding choices of moves by both players). In other words, there exists a sequence of points $\langle \alpha_m, \beta_m \rangle$ of the closed complement of the set $A$ converging to $\langle \alpha, \beta \rangle$. Consequently $\langle \alpha, \beta \rangle \notin A$ and this happens whenever $\beta = \alpha * \tau$. Hence the strategy $\tau$ thus found is indeed a WS for player II in the game $G(A)$, which was to be proved. □

COROLLARY. *All closed sets are determinate, i.e.,* $\Pi^1_0$-Det *holds.*

PROOF: For each point $\alpha = \langle a_0, a_1, a_2, \dots \rangle \in \mathcal{N}$ we set $\alpha^- = \langle a_1, a_2, \dots \rangle$. Consider an arbitrary closed set $A \subseteq \mathcal{N}^2$. By the continuity of the mapping $\alpha \mapsto \alpha^-$ the set $B = \{\langle \alpha, \beta \rangle : \langle \beta, \alpha^- \rangle \notin A\}$ is open and hence determinate. If now player I has a winning strategy in the game $G(B)$, then the strategy $\tau = \sigma$ is a winning strategy for II in the game $G(A)$. And if player II has a winning strategy $\tau$ in the game $G(B)$, then a winning strategy $\sigma$ for player I in the game $G(A)$ can be defined by the equation $\sigma(u) = \tau(u^-)$ for all $u \in FC$ (the finite sequence $u^-$ is obtained by removing from $u$ the leftmost term—compare with the definition of $\alpha^-$). □

The result of the Gale–Stewart theorem was later strengthened several times (see [34, §3]), until Martin [53] proved the following theorem:

THEOREM OF BOREL DETERMINACY. *All Borel sets are determinate, i.e.,* $\Delta^1_1$-Det *holds.*

This result is the strongest possible in ZFC (actually the proof uses only axioms of ZF + DC). The point is that the hypothesis $\Sigma^1_1$-Det (and $\Pi^1_1$-Det, which is equivalent to it) is nondeducible even in ZFC (see the following section) although, as Martin showed [52], it can be deduced from the measurable cardinal axiom. The work of Harrington [26] and Steel [72] (see also [34]) showed that the hypothesis $\Sigma^1_1$-Det is equivalent to the assertion that any two nonborel $\Sigma^1_1$-sets are Borel isomorphic, and also to the "sharps hypothesis", which has been intensively studied in connection with measurable and Ramsey cardinals (see [34]; [58, Ch. 8]; [69]).

Now a brief historical sketch. Infinite games of a certain special type connected with the proof of the Baire property first appear in the work of Banach and Mazur at the end of the 20's and beginning of the 30's. The general concept of games of the type considered here was introduced by Gale and Stewart [24]. Nevertheless the first serious study in connection with determinacy should be attributed to the note of

Mycielski and Steinhaus [60] and the subsequent works [59], [61], [22], where it was shown that AD implies Lebesgue measurability, the Baire property, and the existence of a perfect subset under the hypothesis of uncountability—for all subsets of the real line.

The next crucial step in the study of determinacy occurs in the second half of the 60's. In the note of Blackwell [16] it was shown how it is possible, using the Gale–Stewart theorem, to prove certain classical theorems on sets of the first projective level, in particular the separability theorem for the class $\Sigma^1_1$. Immediately Moschovakis (see [15]) and Martin [51] discovered that by adopting the axiom of projective determinacy PD it was possible to ascertain the laws of separability and reduction on all levels of the projective hierarchy (by classical methods this could be done only for the zeroth, first, and second levels). In this way a period of intensive development of the applications of determinacy to descriptive set theory was opened, and has continued down to the present.

In the course of this study the view of the nature of the axioms AD and PD also underwent certain changes. Whereas in the early stages of work on determinacy (the first half of the 60's) specialists were inclined to consider these axioms simply as interesting mathematical hypotheses with unusual consequences (approximately on the same level that topologists assign to Martin's axiom), by the 70's they had begun to look at AD and PD as postulates claiming to be true in the "world of real sets", or at least in certain natural parts of that "world". More or less convincing grounds are presented in support of such a view (for this see [5, Ch. 2 and conclusion], [3, Ch. 8, §6], [23], [58, parts 7,8 and conclusion]). This approach was reflected in the terminology as well: Moschovakis [55] introduced the concept of a "playful universe," by which he understood a world of sets in which a definite determinacy hypothesis holds, preferring to talk about truth in this universe rather than deducibility from the corresponding hypothesis.

In what follows we shall talk about truth in a completely determinate universe, in a projectively determinate universe, or, in general in a $\Gamma$-determinate universe, by which we understand, strictly speaking, deducibility from AD, PD or the hypothesis $\Gamma$-Det, respectively. (The "usual" mathematical universe of sets is $\Sigma^1_0$-determinate by the Gale–Stewart theorem, and even $\Delta^1_1$-determinate by Martin's theorem.) As the underlying set theory, (to which one determinacy hypothesis or another will be adjoined) we shall use the theory ZF + DC; any application of the "full" axiom of choice AC will be explicitly stipulated (actually, this affects only one proposition of §7).

## §3. Regularity properties of point sets in determinate universes

Under the general heading of regularity properties we usually understand the following three properties of point sets:

1) MEASURABILITY. In Euclidean spaces this property can be associated with Lebesgue measure. Spaces of the form $\omega^l \times \mathcal{N}^m$ have no one measure that is in any way distinguished from all others, and therefore it is more natural in these cases to talk about the property of absolute measurability. A point set $X$ is absolutely measurable if it is measurable (i.e., has a definite—finite or $+\infty$—measure value) in the sense of any prescribed countably additive $\sigma$-finite Borel measure on the space under consideration. (The phrase "$\sigma$-finite" means that the whole space is a countable union of sets of finite measure, and a "Borel measure" is a measure such that every measurable set coincides up to a set of measure zero with a suitable Borel set.)

2) THE BAIRE PROPERTY. A set $X$ has this property when it coincides up to a set of first category with a suitable open set. In other words, there must exist an open set $U$ in the space under consideration such that the symmetric difference $X \triangle U = (X - U) \cup (U - X)$ is a set of first category. Sets of first category, in turn, are defined as countable unions of sets that are nowhere dense in the given space.

3) THE PERFECT KERNEL PROPERTY. This property amounts to the assertion that a given set must either be at most countable or contain a perfect subset. Perfect subsets of the spaces $\omega^l \times \mathcal{N}^m$ with $m \geqslant 1$, as is known, have cardinality of the continuum $\mathfrak{c}$, so that a point set with the perfect kernel property has cardinality either $\leqslant \aleph_0$ or equal to $\mathfrak{c}$, and in either case cannot serve as a counterexample to the continuum hypothesis.

We have already mentioned in the preceding section the achievements of the early work on determinacy:

AD implies absolute measurability, the Baire property, and the perfect kernel property for all point sets. Elementary analysis of the proofs of this proposition showed that projective determinacy suffices to prove all three regularity properties for any projective set; and, more precisely, in a $\Sigma^1_n$-determinate universe all $\Sigma^1_n$-sets possess these properties. Then an even stronger result was obtained:

THEOREM 1 ($\Sigma^1_n$-DET)[4]. *All sets of the class $\Sigma^1_{n+1}$ are absolutely measurable and possess the Baire property and the measurable kernel property.*

(In the book [58], where apparently the proof of this theorem was first presented, it is credited to unpublished work of Martin and Kechris in the early 70's. The latter authors, however, indicate in [34] that the main technical device was invented by Solovay.)

We note one specific point in the theorem just stated (characteristic, as will be shown below, of many other theorems of the same type). For $n = 0$ the hypothesis $\Sigma^1_0$-Det is a theorem in ZF—the Gale–Stewart theorem of the preceding section. Consequently the conclusion on measurability, the Baire property, and the perfect kernel property for all $\Sigma^1_1$-sets is also an ordinary mathematical fact, not depending on any determinacy hypotheses. However, this result, of course, is not new: even in the earliest work on the descriptive theory (Aleksandrov, Suslin—see [13, §3]; Luzin [45]) it was shown that each $\Sigma^1_1$-set (i.e., $A$-set, in the terminology of the time) has all three regularity properties. Indeed what was noteworthy was that theorem 1 generalizes very naturally the classical results mentioned above, becoming these results when $n = 0$.

Here we shall give a proof only of that part of theorem 1 that concerns the perfect kernel property—it is especially important in applications to the theory of single- and countably-valued sets (see §6). As for the Baire property, at the end of this section we shall sketch the proof of an even stronger result than is contained in theorem 1.

Thus, we shall prove that in a $\Sigma^1_n$-determinate universe each $\Sigma^1_{n+1}$-point set $X \subseteq \mathcal{N}$ has the perfect kernel property. The plan of the proof reduces to the following: for a specially constructed game $G$ it will be shown that if player I has a WS, then the set $X$ contains a perfect subset, and if player II has a WS, then $X$ is at most countable. Then we shall deduce the determinacy of the game from the hypothesis $\Sigma^1_n$-Det.

Without loss of generality we may assume that $X \subseteq \mathcal{D}$, where

$$\mathcal{D} = 2^\omega = \{\delta \in \mathcal{N} : \forall k(\delta(k) = 0 \text{ or } 1)\}$$

is the Cantor discontinuum. (In fact, it is easy to arrange a homeomorphism between $\mathcal{N}$ and a suitable co-countable set in $\mathcal{D}$, which allows us to carry out the reduction to the indicated special case.) There exists a $\Pi^1_n$-set $Q \subseteq \mathcal{D} \times \mathcal{N}$ such that

$$X = \pi Q = \{\delta : \exists \gamma Q(\delta, \gamma)\}.$$

Fix an enumeration $\langle l[b], v[b]\rangle$, $b \in \omega$, of all pairs $\langle l, v\rangle \in \omega \times FC_{01}$, where $FC_{01}$ is the set of all finite sequences of zeros and ones (including the empty sequence $\Lambda$). To each pair of points $\alpha = \langle a_0, a_1, a_2, \ldots\rangle \in \mathcal{N}$ and $\beta = \langle b_0, b_1, b_2, \ldots,\rangle \in \mathcal{N}$ we assign the points

$$D(\alpha, \beta) = v[b_0]^\frown\langle a_1^*\rangle^\frown v[b_1]^\frown\langle a_2^*\rangle^\frown \cdots \in \mathcal{D},$$

where $a_i^* = \min\{1, a_i\}$, $\langle a_i^*\rangle$ is the finite sequence with the single term $a_i^*$, the sign $^\frown$ denotes the operation of concatenating finite sequences, and

$$H(\beta) = \langle l[b_0], l[b_1], l[b_2], \ldots\rangle \in \mathcal{N}.$$

The functions $D$ and $H$, of course, are continuous, whence by rule 2 of §1 it is not difficult to deduce that the set

$$A = \{\langle \alpha, \beta\rangle \in \mathcal{N}^2 : \langle D(\alpha, \beta), H(\beta)\rangle \notin Q\}$$

belongs to $\Sigma^1_n$, so that the corresponding game $G(A)$ is determinate.

[4] The notation, e.g., $\Sigma^1_n$-Det after the word 'theorem' indicates that the theorem in question is proved using ZF + DC + ($\Sigma^1_n$-Det).

A SMALL EXPLANATION. The game $G(A)$ essentially reduces to the following: player I makes moves $a_i^* = 0$ or 1, while the moves of player II must be pairs $\langle l_i, v_i \rangle$, where $l_i \in \omega$, and $v_i$ is a finite sequence of zeros and ones. The point $\gamma = \langle l_0, l_1, \dots \rangle \in \mathcal{N}$ is constructed from the numbers $l_i$, and the point $\delta = v_0{}^\wedge\langle a_1^* \rangle^\wedge v_1{}^\wedge \langle a_2^* \rangle^\wedge \cdots \in \mathcal{D}$ is constructed from the finite sequences $v_i$ and the numbers $a_i^*$. After this player I wins if and only if $\langle \delta, \gamma \rangle \notin Q$. It is useful to keep in mind this actual content of the game $G(A)$ when studying the following computations.

We note also that the opening move $a_0$ of player I has absolutely no influence on the outcome of the match; essentially I "loses" a move, effectively offering his opponent the right to open the match. In works on the descriptive theory in determinate universes proofs of perfect kernel theorems are usually constructed by reversing the functions of the two players, i.e., the moves of player I are pairs $\langle l, v \rangle \in \omega \times FC_{01}$, and player II responds with zeros and ones; in that case there are no longer any "indifferent" moves. The variant adopted in our exposition was motivated by the need to prepare for the application of a construction explained in the proof so as to prove (in §6) the theorem on the partition of countably-valued sets: it is necessary that the countability of $X$ correspond to the existence of a WS for player I.

Due to the determinacy of the game $G(A)$ one of the players has a winning strategy.

CASE 1: Player I has a WS $\sigma$ in the game $G(A)$. We shall verify that then $X$ is at most countable. We may suppose that the strategy $\sigma$ has as values only the numbers 0 and 1 (otherwise we simply replace each value of $\sigma$ greater than 1 by 1).

We now make some definitions. We shall call a finite sequence $t = \langle a_0, b_0, \dots, a_{k-1}, b_{k-1}, a_k \rangle$ (of odd length) a *$\sigma$-compatible sequence* (a $\sigma$-CS for short) if $a_i = \sigma(b_0, \dots, b_{i-1})$ for all $i \leqslant k$. Further let $\delta \in \mathcal{D}$. We shall agree to call a pair consisting of a number $l \in \omega$ and a $\sigma$-CS $t = \langle a_0, b_0, \dots, a_k \rangle$ *$\delta$-maximal* if, first, the finite sequence

$$w(t) = v[b_0]^\wedge \langle a_1 \rangle^\wedge v[b_1]^\wedge \langle a_2 \rangle^\wedge \dots {}^\wedge v[b_{k-1}]^\wedge \langle a_k \rangle$$

is the beginning of $\delta$ (i.e., $w(t) \subset \delta$), and second, there are no numbers $b_k \in \omega$ and $a_{k+1} = 0$ or 1 such that $l = l[b_k]$, $a_{k+1} = \sigma(b_0, \dots, b_{k-1}, b_k)$ and $w(t)^\wedge v[b_k]^\wedge \langle a_{k+1} \rangle \subset \delta$.

We claim that for each point $\delta \in X$ there is a $\delta$-maximal pair. Indeed, since $\delta \in X$, it follows that $\langle \delta, \gamma \rangle \in Q$ for some point $\gamma = \langle l_0, l_1, l_2, \dots \rangle \in \mathcal{N}$. Set $a_0 = \sigma(\Lambda)$. Then $t_0 = \langle a_0 \rangle$ will be a $\sigma$-CS and in addition $w(t_0) = \Lambda \subset \delta$. If the pair $\langle l_0, t_0 \rangle$ is not $\delta$-maximal, then $t_0$ can be extended, yielding a $\sigma$-CS $t_1 = \langle a_0, b_0, a_1 \rangle$ such that $w(t_1) \subset \delta$ and $l[b_0] = l_0$. If again the pair $\langle l_1, t_1 \rangle$ is not $\delta$-maximal, there exists a still longer $\sigma$-CS $t_2 = \langle a_0, b_0, a_1, b_1, a_2 \rangle$ such that $w(t_2) \subset \delta$ and $l[b_1] = l_1$, etc.

But this process cannot continue indefinitely, for then we would obtain a match $\alpha = \langle a_0, a_1, \dots \rangle, \beta = \langle b_0, b_1, \dots \rangle$ in the game $G(A)$ corresponding to the strategy $\sigma$ ($\alpha = \sigma * \beta$), and such that $\langle D(\alpha, \beta), H(\beta) \rangle = \langle \delta, \gamma \rangle \in Q$, which is impossible, since $\sigma$ is a WS for player I. Hence the construction terminates, and at the corresponding step $k$ we arrive at a $\delta$-maximal pair $\langle l, t \rangle$.

Thus, in fact, for any point $\delta \in X$ there exists a $\delta$-maximal pair $\langle l, t \rangle$. Let us verify that in such a situation $\delta$ is uniquely determined in $l$ and $t$ via $\sigma$. This will suffice to prove that $X$ is countable, since the totality of all pairs $\langle l, t \rangle$ of the type under consideration is countable.

Let $t = \langle a_0, b_0, \dots a_{k-1}, b_{k-1}, a_k \rangle$. Given that $w(t) \subset \delta$, there exists $m$ such that $w(t) = \delta \upharpoonright m$. All the values of $\delta(j)$ with $j < m$ are determined by this equation. We shall show how to compute all the values $\delta(m+i)$ by induction on $i$.

Set $v^i = \langle \delta(m), \dots, \delta(m+i-1) \rangle$ for each $i$ (in particular $v^0 = \Lambda$). For every $i$ there exists a unique natural number $b^i$ satisfying $v[b^i] = v^i$ and $l[b^i] = l$. Let us also set $a^i = \sigma(b_0, \dots, b_{k-1}, b^i)$. Then $t^i = t^\wedge \langle b^i, a^i \rangle$ is a $\sigma$-CS and $l[b^i] = l$. Consequently, in view of the $\delta$-maximality of the pair $\langle l, t \rangle$, we obtain $w(t^i) \not\subset \delta$. However $w(t^i) = w(t)^\wedge v^{i\wedge} \langle a^i \rangle$ and it is clear that $w(t)^\wedge v^i \subset \delta$. Thus, $a^i \neq \delta(m+i)$.

In addition $\delta(m+i) = 0$ or 1 since $\delta \in X \subseteq \mathcal{D}$, and $a^i = 0$ or 1 by the assumption made about the values of the strategy $\sigma$. Thus for any $i \in \omega$ the equation

$$\delta(m+i) = 1 - \sigma(b_0, b_1, \dots, b_{k-1}, b^i) \tag{*}$$

holds, making it possible to find all the numbers $\delta(m+i)$ sequentially.

CASE 2: Player II has a WS $\tau$ in the game $G(A)$. We shall verify that in this case our set $X$ contains a perfect subset. The function $F(\alpha) = D(\alpha, \alpha * \tau)$ is continuous, and the image $C = \{F(\alpha) : \alpha \in \mathcal{D}\}$ of the discontinuum $\mathcal{D}$ is contained in $X$ by the choice of $\tau$ in accordance with the definition of $A$. It remains to verify that $F$ and $D$ are in one-to-one correspondence; once that is done we can conclude that $C$ is the desired perfect subset of the set $X$.

Consider a pair of distinct points $\alpha = \langle a_0, a_1, \dots \rangle$ and $\alpha' = \langle a'_0, a'_1, \dots \rangle$ of the discontinuum and denote by $m$ the smallest index such that $a'_m \neq a_m$. Set

$$b_i = \tau(a_0, \dots, a_i) \quad \text{and} \quad b'_i = \tau(a'_0, \dots a'_i)$$

for all $i$. Then

$$F(\alpha) = v[b_0]^\frown \langle a_1 \rangle^\frown v[b_1]^\frown \langle a_2 \rangle^\frown \cdots ,$$
$$F(\alpha') = v[b'_0]^\frown \langle a'_1 \rangle^\frown v[b'_1]^\frown \langle a'_2 \rangle^\frown \cdots .$$

By the choice of $m$ we shall obtain $a'_i = a_i$ and then also $b'_i = b_i$ for all $i < m$, but $a'_m \neq a_m$. Thus $F(\alpha) \neq F(\alpha')$, as required. □

We now turn to the proof of the Baire property. The result whose proof will be explained here is connected with the "game-operator" of Moschovakis [56]. Let $\mathcal{X}$ be one of the point spaces and $B \subseteq \mathcal{X} \times \mathcal{N}^2$. Each point $x \in \mathcal{X}$ determines a section $B/x = \{\langle \alpha, \beta \rangle \colon B(x, \alpha, \beta)\}$ and thereby defines a game $G(B/x)$ in which one of the players may have a winning strategy. Moschovakis proposed the following definition:

$$\mathfrak{D}B = \{x \in \mathcal{X} : \text{ player I has a WS in the game } G(B/x)\}$$

The action of the operator $\mathfrak{D}$ can be symbolically pictured as an infinite string of alternating quantifiers over the natural numbers:

$$x \in \mathfrak{D}B \leftrightarrow \exists \alpha(0) \forall \beta(0) \exists \alpha(1) \forall \beta(1) \cdots B(x, \alpha, \beta),$$

but any attempt to give a precise sense to an infinite quantifying prefix leads inevitably back to strategies.

For each class $\Gamma$ of point sets, it is customary to denote by $\mathfrak{D}\Gamma$ the collection of all sets of the form $\mathfrak{D}B$, where $B$ is a set from $\Gamma$ situated in a space of the form $\mathcal{X} \times \mathcal{N}^2$. In a projectively determinate universe the operator $\mathfrak{D}$ acts on projective classes in such a way that the equalities $\mathfrak{D}\Pi^1_n = \Sigma^1_{n+1}$, $\mathfrak{D}\Sigma^1_n = \Pi^1_{n+1}$ hold (actually only the assumption $\Sigma^1_n$-Det is required)—see the following section. Therefore the result of theorem 1 for the Baire property follows from the following theorem of Kechris [31]:

THEOREM 2. *Let $\Gamma$ be a projective class, and suppose the hypothesis* $\Gamma$-Det *holds. Then every set of* $\mathfrak{D}\Gamma$ *has the Baire property.*

We shall carry out the proof only for sets of the space $\mathcal{N}$ and we shall limit ourselves to the exposition of only the main points. It suffices to show that for any $X \subseteq \mathcal{N}$ of the class $\mathfrak{D}\Gamma$ either $X$ has first category, or there is a Baire interval $\mathcal{N}_w = \{\alpha \in \mathcal{N} : w \subset \alpha\}$ (where $w \in FC$) in $\mathcal{N}$ such that the difference $\mathcal{N}_w - X$ has first category. Thus, let $X = \mathfrak{D}B \subseteq \mathcal{N}$, where the set $B \subseteq \mathcal{N}^3$ belongs to $\Gamma$.

Consider the game $G$ in which both players I and II make every move by choosing a pair of the form $\langle a, u \rangle$, where $a \in \omega$ and $u \in FC$, $u \neq \Lambda$. Thus player I (who, as usual, opens) makes moves $\langle a_0, u_0 \rangle, \langle a_1, u_1 \rangle, \langle a_2, u_2 \rangle, \dots$ and player II makes moves $\langle b_0, v_0 \rangle, \langle b_1, v_1 \rangle, \langle b_1, v_2 \rangle, \dots$. To determine the result the points

$$\alpha = \langle a_0, a_1, a_2, \dots \rangle, \quad \beta = \langle b_0, b_1, b_2, \dots \rangle,$$
$$\gamma = u_0{}^\frown v_0{}^\frown u_1{}^\frown v_1{}^\frown u_2{}^\frown v_2{}^\frown \cdots$$

of the space $\mathcal{N}$ are formed. Player I wins in the case when $\langle \gamma, \alpha, \beta \rangle \in B$; otherwise II wins.

The game $G$ is determinate. To prove this proposition we must fix an enumeration of all pairs $\langle a, u \rangle$ of the indicated type by natural numbers (as was done above in the proof of part of theorem 1). After such a transformation we arrive at a game of the form $G(A)$, where $A \subseteq \mathcal{N}^2$ is a set of class $\Gamma$ (it is obtained from $B$ by a continuous substitution).

In view of the determinacy one of the players has a WS in the game $G$.

CASE 1: Player II has a WS $\tau$ in the game $G$. (We prefer to examine this case in more detail, since it is the more revealing case). We shall show that $X$ is of first category.

For what follows we shall agree that the letters $u$ and $v$ (with indices) denote only finite sequences from $FC$, not equal to $\Lambda$. The strategy $\tau$ is defined on finite sequences of the form

$$\langle\langle a_0, u_0\rangle, \langle a_1, u_1\rangle, \ldots, \langle a_{k-1}, u_{k-1}\rangle\rangle \tag{1}$$

and assumes values among the pairs $\langle b, v\rangle \quad (b \in \omega)$. We shall call a sequence of the form

$$t = \langle a_0, u_0, b_0, v_0, \ldots, a_{k-1}, u_{k-1}, b_{k-1}, v_{k-1}\rangle \tag{2}$$

a *$\tau$-compatible sequence* (a $\tau$-CS for short), if we have $\langle b_i, v_i\rangle = \tau(\langle a_0, u_0,\rangle, \ldots, \langle a_i, u_i\rangle)$ for any $i \leqslant k$. If in this case $a \in \omega$, $\gamma \in \mathcal{N}$, the finite sequence

$$w(t) = u_0\hat{\ }v_0\hat{\ }u_1\hat{\ }v_1\hat{\ }\cdots\hat{\ }u_{k-1}\hat{\ }v_{k-1}$$

satisfies the relation $w(t) \subset \gamma$, and there is no $\tau$-CS $t'$ of the form $t\hat{\ }\langle a_k, u_k, b_k, v_k\rangle$ such that $w(t') \subset \gamma$ and $a_k = a$, then we shall call the pair $\langle a, t\rangle$ *$\gamma$-maximal.*

We claim that for every point $\gamma \in X$ there exists a $\gamma$-maximal pair $\langle a, t\rangle$. The idea is the same as in the proof of theorem 1, only each number $a_k$ (the analog of $l_k$) is computed from the formula $a_k = \sigma(b_0, \ldots, b_{k-1})$, where $\sigma$ is a WS for player I (fixed in advance) in the game $G(B/\gamma)$, which exists in view of the fact that $\gamma \in \mathfrak{D}B$. We omit the details.

Having accepted the assertion about the existence of maximal pairs, let us deduce that $X$ is of first category. Let $\gamma \in X$, and let the pair $\langle a, t\rangle$ be $\gamma$-maximal; let $t$ be a $\tau$-CS of the form (2). Let $u \in FC$, $u \neq \Lambda$ be arbitrary, and

$$\langle b, v\rangle = \tau(\langle a_0, u_0\rangle, \ldots, \langle a_{k-1}, u_{k-1}\rangle, \langle a, u\rangle).$$

The finite sequence $v \in FC - \{\Lambda\}$ defined by this equation will be denoted $v(u)$.

The sequence $t' = t\hat{\ }\langle a, u, b, v(u)\rangle$ will be a $\tau$-CS, and thus, in view of the $\gamma$-maximality of the pair $\langle a, t\rangle$ we obtain: $w(t') = w(t)\hat{\ }u\hat{\ }v(u) \not\subset \gamma$. Thus the point $\gamma$ belongs to the set

$$W_t = \mathcal{N}_{w(t)} - \bigcup_{u \in FC, u \neq \Lambda} \mathcal{N}_{w(t)\hat{\ }u\hat{\ }v(u)}\,.$$

In view of the arbitrariness of the point $\gamma \in X$ in this reasoning we can conclude that $X \subset \bigcup_t W_t$. But each of the sets $W_t$ is nowhere dense.

CASE 2: Player I has a WS $\sigma$ in the game $G$. Let $\langle a, u\rangle = \sigma(\Lambda)$ be the opening move according to the strategy $\sigma$. Calculations similar to those carried out in case 1 make it possible to deduce that $\mathcal{N}_u - X$ is a set of first category. An additional point here is that each section $B/\gamma$, $\gamma \in \mathcal{N}$ of the set $B$ belongs to the class $\Gamma$, as does $B$ itself (rule 3 of §1). Consequently, by the assumption $\Gamma$-Det, if $\gamma \in \mathcal{N} - X$, then player II has a WS in the game $G(B/\gamma)$. □

In concluding this section, we note one major consequence of theorem 1. Novikov has shown [11] that, using the axioms of ZFC, it is impossible to prove that even one of the three regularity properties we are considering holds for all sets of the class $\Sigma^1_2$. Consequently the hypothesis $\Sigma^1_1$-Det is also not deducible in the system ZFC, for by theorem 1 it implies all three properties (actually it suffices to consider any one of them, say the perfect kernel property) for all sets of the class $\Sigma^1_2$.

## §4. Separation and reduction theorems in determinate universes. Normed classes

The concept of separability was introduced into descriptive set theory by Luzin. Consider a pair of disjoint sets $X, Y$ of some point space. How simple can a set $Z$ separating $X$ from $Y$ be? (This means that

$X \subseteq Z$ and $Y \cap Z = \emptyset$.) Luzin [48] suggested taking as the index of simplicity of existing separating sets (among which is the set $X$) a special descriptive measure of the "distance" between the sets $X$ and $Y$.

The question that aroused the greatest interest in studies of separability in the projective hierarchy was the following. For which projective classes $\Gamma$ does the following assertion—the separation principle[5]—hold?

$\Gamma$-Sep: For each pair of disjoint $\Gamma$-sets $X$ and $Y$ there exists a set $Z$ belonging to the class $\Gamma$ along with its complement and separating $X$ from $Y$.

Separability played a very important role in the development of the descriptive theory in the 20's and 30's. In current work the following more convenient reduction principle, which first appeared in an article [40] of Kuratowski, is more frequently considered:

$\Gamma$-Red: For each pair of $\Gamma$-sets $X$ and $Y$ there exists a pair of disjoint $\Gamma$-sets $X' \subseteq X$ and $Y' \subseteq Y$ whose union $X' \cup Y'$ coincides with the union $X \cup Y$.

(Such a pair $X'$, $Y'$ is said to *reduce* the given pair $X$, $Y$.)

The classes $\Delta^1_n$, being closed under the operation of set difference, obviously satisfy both Sep and Red. The situation is significantly more complicated with the classes $\Sigma^1_n$ and $\Pi^1_n$. The research of Luzin [48], Novikov [63], and Kuratowski [40] showed that for the classes $\Sigma^1_0$, $\Pi^1_1$, and $\Sigma^1_2$ the reduction principle holds, but not the separation principle, while the classes $\Pi^1_0$, $\Sigma^1_1$, and $\Pi^1_2$, in contrast, satisfy Sep but not Red. And the higher projective levels absolutely defeated all attempts to ascertain their separation and reduction laws. In addition the separation and reduction problem—together with the problem of regularity properties—has traditionally been given the foremost place among the classical problems of the descriptive theory. It is precisely for this reason that the following theorem, proved independently by Martin [51] and Moschovakis [15], made a very strong impression on specialists in this area:

SEPARATION AND REDUCTION THEOREM ($\Sigma^1_{2n}$-DET). *The classes*

$$\Sigma^1_0, \Pi^1_1, \Sigma^1_2, \ldots, \Sigma^1_{2n}, \Pi^1_{2n+1}, \Sigma^1_{2n+2}$$

*satisfy the reduction principle, but not the separation principle. In contrast the classes*

$$\Pi^1_0, \Sigma^1_1, \Pi^1_2, \ldots, \Pi^1_{2n}, \Sigma^1_{2n+1}, \Pi^1_{2n+2}$$

*satisfy the separation principle, but not the reduction principle.*

Thus in a projectively determinate universe (i.e., one with the axiom PD) there is a series of "reducible" but not "separable" classes

$$\Sigma^1_0, \Pi^1_1, \Sigma^1_2, \ldots, \Sigma^1_{2n}, \Pi^1_{2n+1}, \Sigma^1_{2n+2}, \ldots$$

and a series of "separable" but not "reducible" classes

$$\Pi^1_0, \Sigma^1_1, \Pi^1_2, \ldots, \Pi^1_{2n}, \Sigma^1_{2n+1}, \Pi^1_{2n+2}, \ldots$$

The hypothesis $\Sigma^1_{2n}$-Det turns out to be strong enough to extend both series to level $2n+2$. And for $n = 0$ we arrive by means of the Gale–Stewart theorem of §2 at a new proof of the classical results noted above on separation and reduction, a proof based on games. (As a matter of fact, such a possibility of using games was demonstrated by Blackwell [16] even before [15] and [51] appeared.)

The reason for the "alternation" of reduction and separation between $\Sigma$-classes and $\Pi$-classes remained unexplained for some time. This question became answerable by use of the "game operator" of which we spoke in §3. The point is that the hypothesis $\Sigma^1_m$-Det implies that the equations

$$\mathfrak{D}\,\Pi^m_1 = \Sigma^1_{m+1}, \qquad \mathfrak{D}\,\Sigma^1_m = \Pi^1_{m+1},$$

hold, so that in a projectively determinate universe the "reduced" classes are obtained from the initial class $\Sigma^1_0$ and the "separated" classes from $\Pi^1_0$ by successive applications of the operator $\mathfrak{D}$.

[5] More precisely, the first principle. A second principle of separability has also been studied, for which see [2, 10, 14, 6].

We now give the proof of these equalities. It suffices to do the first one, since the second is easily obtained from the first under the assumption $\Sigma^1_m$-Det by the same method that was used to deduce the corollary from the Gale–Stewart theorem in §2.

RIGHT-TO-LEFT INCLUSION. Suppose $X \subseteq \mathcal{X}$ and $X \in \Sigma^1_{m+1}$. Then $X = \{x : \exists\alpha Q(x,\alpha)\}$ for a suitable $\Pi^1_m$-set $Q \subseteq \mathcal{X} \times \mathcal{N}$. then $X = \mathfrak{D}B$, where

$$B = \{\langle x, \alpha, \beta\rangle : Q(x,\alpha) \wedge \beta \in \mathcal{N}\} \in \Pi^1_m.$$

LEFT-TO-RIGHT INCLUSION. Let $X = \mathfrak{D}B$, where $B \subseteq \mathcal{X} \times \mathcal{N}^2$ is a set of class $\Pi^1_m$. Thus

$$X = \{x : \exists\sigma : FC \to \omega\forall\beta B(\sigma * \beta, \beta)\}.$$

To represent the quantifier $\exists\sigma$ in terms of a quantifier on $\mathcal{N}$, fix once and for all an enumeration by the natural numbers of the finite sequences in $FC$. Let $[k]$ be the finite sequence corresponding to the number $k \in \omega$; thus, $FC = \{[k] : k \in \omega\}$. Now each point $\varepsilon \in \mathcal{N}$ gives a strategy $[\varepsilon]$: to be specific, $[\varepsilon]([k]) = \varepsilon(k)$ for all $k$.

The mapping $\langle\varepsilon, \beta\rangle \mapsto [\varepsilon] * \beta$ is of course continuous, and therefore our set

$$X = \{\alpha : \exists\varepsilon \in \mathcal{N}\forall\beta B([\varepsilon] * \beta, \beta)\}$$

has class $\Sigma^1_{m+1}$ (here we are using rules 2 and 6 of § 1).

To begin our exposition of the proof of the separation and reduction theorem we note the following: if the reduction principle holds for a class $\Gamma$, then the separation principle holds for the class $\neg\Gamma$ of complementary sets (simply pass to the complements). The elegant reasoning discovered by Novikov [62] involving "doubly universal pairs" shows that Sep and Red cannot both hold for the same class $\Gamma = \Sigma^1_m$ or $\Pi^1_m$ (see the proof of theorem 3.2 in the book [3, p. 249]). Thus it suffices to verify only the reduction principle for the classes $\Sigma^1_0, \Pi^1_1, \ldots, \Pi^1_{2n+1}, \Sigma^1_{2n+2}$ in a $\Sigma^1_{2n}$-determinate universe.

The utilization (in one form or another) of the set of all at-most-countable ordinals was the crucial step in the classical proofs of separation and reduction theorems. The connection of the first and second levels of the projective hierarchy with the ordinals found its most natural expression in the principles of index comparison discovered by Novikov (see [10], [63], [6, §3]). In contemporary research two related concepts derived from these principles are used—the norm and the prewellordering.

A *norm* on a set $X$ is any function $\varphi$ mapping $X$ into the ordinals. To each norm $\varphi : X \to \mathrm{Ord}$ defined on a point set $X \subseteq \mathcal{X}$ there correspond binary relations $\leqslant_\varphi$ and $<_\varphi$ (on $X$) and $\leqslant^*_\varphi$ and $<^*_\varphi$ (on $\mathcal{X}$):

$$x \leqslant_\varphi y \leftrightarrow \varphi(x) \leqslant \varphi(y), \quad x <_\varphi y \leftrightarrow \varphi(x) < \varphi(y);$$

$$x \leqslant^*_\varphi y \leftrightarrow x \in X \wedge (y \in X \to \varphi(x) \leqslant \varphi(y));$$
$$x <^*_\varphi y \leftrightarrow x \in X \wedge (y \in X \to \varphi(x) < \varphi(y)).$$

If the relations $\leqslant^*_\varphi$ and $<^*_\varphi$ (which are more important than $\leqslant_\varphi$ and $<_\varphi$) belong (as sets of ordered pairs) to a given class $\Gamma$, then $\varphi$ is called a $\Gamma$-norm.

A non-strict partial ordering $\leqslant$ on a set $X$ is called a prewellordering of $X$ if it satisfies the following two conditions:

1) if $x, y \in X$ then $x \leqslant y$ or $y \leqslant x$ (but it is not assumed that $x \leqslant y \wedge y \leqslant x$ implies $x = y$); and
2) there are no infinite chains $x_0 > x_1 > x_2 > \cdots$ of elements $x_i$ of the set $X$ ($x < y$ means that $x \leqslant y$ but it is not the case that $y \leqslant x$).

For example, if $\varphi$ is a norm on $X$, then the relation $\leqslant_\varphi$ is a prewellordering of $X$. Conversely it is easy to see that for any prewellordering $\leqslant$ of the set $X$ there is a norm $\varphi$ on $X$ such that $\leqslant$ coincides with $\leqslant_\varphi$.

Finally the key definition. The class $\Gamma$ is called *normed* if every set of $\Gamma$ supports a $\Gamma$-norm. The application of these concepts to the proof of the separation and reduction theorem is obtained using the following proposition:

*If a projective class $\Gamma$ is normed, then it satisfies the reduction principle.*

The proof is very simple. Consider a pair of $\Gamma$-sets $X, Y \subseteq \mathcal{X}$. The set

$$P = \{\langle x, i\rangle : (x \in X \wedge i = 0) \vee (x \in Y \wedge i = 1)\}$$

also belongs to the class $\Gamma$, being obtained from the $\Gamma$-sets $X, Y, \{0\}, \{1\}$ using $\wedge$ and $\vee$ (rule 5 of §1). Therefore there exists a $\Gamma$-norm $\varphi : P \to \mathrm{Ord}$. Now the pair of $\Gamma$-sets

$$X' = \{x : \langle x, 0\rangle <^*_\varphi \langle x, 1\rangle\}, \quad Y' = \{x : \langle x, 1\rangle \leqslant^*_\varphi \langle x, 0\rangle\}$$

guarantees the reduction of the initial pair (the sets $X'$ and $Y'$ belong to $\Gamma$, being obtained from the $\Gamma$-relations $<^*_\varphi$ and $\leqslant^*_\varphi$ by a continuous substitution—rule 2 of §1).

From all that has been said, it now suffices for the proof of the separation and reduction theorem to prove the following theorem:

First Periodicity Theorem. *($\Sigma^1_{2n}$-Det). The classes $\Sigma^1_0$, $\Pi^1_1$, $\Sigma^1_2, \dots, \Sigma^1_{2n}$, $\Pi^1_{2n+1}$, $\Sigma^1_{2n+2}$ are normed.*

The known proofs of this first fundamental proposition of the descriptive theory of determinate universes run according to the following scheme: First it is verified that the class $\Sigma^1_0$ is normed, and then an induction $\Sigma^1_m \to \Pi^1_{m+1} \to \Sigma^1_{m+2}$ holds. Having in mind the action of the operator $\mathfrak{D}$ described above, we can express the essence of this scheme by saying that the property of being normed passes from the class $\Gamma$ to the class $\mathfrak{D}\Gamma$. A proof of such a general proposition has been obtained by Moschovakis [58, Ch. 6]. However, it is too complicated to be included in the present article. We shall present instead the original proof of the first periodicity theorem due to Martin [51] and Moschovakis [15]. It consists of three lemmas:

Lemma 4.1. *The class $\Sigma^1_0$ of open sets is normed.*

Lemma 4.2. *If the class $\Pi^1_m$ is normed, so is the class $\Sigma^1_{m+1}$.*

Lemma 4.3. *($\Sigma^1_m$-Det). If the class $\Sigma^1_m$ is normed, then the class $\Pi^1_{m+1}$ is also normed.*

We emphasize that lemma 4.2 assumes no hypotheses of determinacy.

Proof of lemma 4.1: Every open point set $X$ is the union $X = \bigcup_{k\in\omega} X_k$ of open-closed sets $X_k$ (recall that we are considering only sets of spaces of the form $\omega^l \times \mathcal{N}^m$; for Euclidean spaces this assertion, of course, is false). If $x \in X$, then we define the value of the norm $\varphi(x)$ to be equal to the smallest number $k$ for which $x \in X_k$. It can be shown without difficulty that $\varphi$ is an "open" norm. □

Proof of lemma 4.2: The proof of lemma 4.2 is based on Novikov's idea of minimal index [63]. We construct a $\Sigma^1_{m+1}$-norm on a $\Sigma^1_{m+1}$-set $X = \{x \in \mathcal{X} : \exists\alpha P(x,\alpha)\}$, where $P \subseteq \mathcal{X} \times \mathcal{N}$ is a set of class $\Pi^1_m$, supporting a $\Pi^1_m$-norm $\varphi$. We set

$$\psi(x) = \min_{\langle x,\alpha\rangle\in P} \varphi(x,\alpha)$$

for each point $x \in X$. The norm $\psi : X \to \mathrm{Ord}$ so constructed is a $\Sigma^1_{m+1}$-norm: for example

$$x \leqslant^*_\varphi y \leftrightarrow x \in X \wedge (y \in X \to \psi(x) \leqslant \psi(y)) \leftrightarrow \exists\alpha\forall\beta(P(x,\alpha) \wedge (P(y,\beta) \to$$

$$\to \varphi(x,\alpha) \leqslant \varphi(y,\beta)) \leftrightarrow \exists\alpha\forall\beta(\langle x,\alpha\rangle \leqslant^*_\varphi \langle y,\beta\rangle),$$

i.e., the relation $\leqslant^*_\psi$ has class $\Sigma^1_{m+1}$ (rule 6 of §1). □

Proof of lemma 4.3: We must construct a $\Pi^1_{m+1}$-norm on the $\Pi^1_{m+1}$-set $X = \{x \in \mathcal{X} : \forall\alpha P(x,\alpha)\}$, where $P \subseteq \mathcal{X} \times \mathcal{N}$ is a set of class $\Sigma^1_m$ on which is defined a $\Sigma^1_m$-norm $\varphi$. To each pair of points $x, y \in \mathcal{X}$ there correspond games $G_{xy}$ and $G'_{xy}$ with game sets

$$A_{xy} = \{\langle\alpha,\beta\rangle \in \mathcal{N}^2 : \neg(\langle x,\alpha\rangle \leqslant^*_\varphi \langle y,\beta\rangle)\},$$

and

$$A'_{xy} = \{\langle\alpha,\beta\rangle \in \mathcal{N}^2 : \langle x,\beta\rangle <^*_\varphi \langle y,\alpha\rangle\}$$

of classes $\Pi^1_m$ and $\Sigma^1_m$ (according to the choice of $\varphi$). All these games are determinate, since the hypothesis $\Sigma^1_m$-Det implies $\Pi^1_m$-Det (see the deduction of the corollary of the Gale–Stewart theorem in §2).

We consider the following binary relations on $\mathcal{X}$:

$$x \leqslant^* y \leftrightarrow \text{player II has a WS in the game } G_{xy};$$

$$x <^* y \leftrightarrow \text{player I has a WS in the game } G'_{xy}.$$

We claim that $\leqslant^*$ and $<^*$ are relations of class $\Pi^1_{m+1}$. Indeed, $\leqslant^*$, say, is identical to the complement of the set $\mathfrak{D}B$, where

$$B = \{\langle x, y, \alpha, \beta\rangle : \neg(\langle x, \alpha\rangle \leqslant^*_\phi \langle y, \beta\rangle\}$$

is a set of class $\Pi^1_m$. But $\mathfrak{D}B \in \Sigma^1_{m+1}$ according to what was said above about the action of the operator $\mathfrak{D}$ on sets of the class $\Pi^1_m$. The relation $<^*$ is handled in exactly the same way.

It now remains to verify that the relations $\leqslant^*$ and $<^*$ coincide with $\leqslant^*_\varphi$ and $<^*_\varphi$ for a suitable norm $\varphi$ on $X$. To prove the existence of such a norm, it is quite sufficient to prove the following seven assertions:

(1) If $x \in X$ while $y \in \mathcal{X} - X$, then $x <^* y$.
(2) If $x \leqslant^* y$, then $x \in X$.
(3) If $x \in X$, then the relation $x \leqslant^* x$ holds.
(4) If $x \leqslant^* y$ and $y \leqslant^* z$, then $x \leqslant^* z$.
(5) If $x <^* y$, then $x \leqslant^* y$.
(6) If $x, y \in X$, then $x \leqslant^* y \leftrightarrow \neg(y <^* x)$
(7) There are no infinite chains of elements of the set $X$ which are decreasing in the sense of $<^*$.

The winning strategy $\sigma$ for player I, which provides a proof of assertion (1), consists of the following: Player I, independently of the moves of player II, makes his moves $a_i$ according to the equation $a_i = \alpha(i)$, where $\alpha$ is a point of $\mathcal{N}$ fixed in advance and such that $\langle y, a\rangle \notin P$ ($\alpha$ exists, since $y \notin X$). At the same time, no matter how player II makes his sequence of moves $\beta$, player I obtains $\langle x, \beta\rangle \in P$, since $x \in X$. Hence, $\langle x, \beta\rangle <^*_\varphi \langle y, \alpha\rangle$.

Assertion (2) is proved in exactly the same way. If $x \notin X$, then player I guarantees himself a win in the game $G_{xy}$ by making his sequence of moves $\alpha$ so that $\langle x, \alpha\rangle \notin P$.

Assertion (3) is completely trivial: it suffices for player II to repeat the moves of player I, thus guaranteeing $\beta = \alpha$ and $\langle x, \alpha\rangle \leqslant^*_\varphi \langle x, \beta\rangle$ (it is important that $x \in X$—this gives $\langle x, \alpha\rangle \in P$ for all $\alpha$.)

To prove (4) suppose player II has a WS $\tau_1$ in the game $G_{xy}$ and a WS $\tau_2$ in the game $G_{yz}$. A WS $\tau$ in the game $G_{xz}$ for player II is obtained by a special composition of the strategies $\tau_1$ and $\tau_2$:

$$\tau(a_0, \ldots, a_k) = \tau_2(b_0, \ldots, b_k),$$

where $b_i = \tau_1(a_0, \ldots, a_i)$ for all $i$. This strategy satisfies the equation $\alpha * \tau = (\alpha * \tau_1) * \tau_2$ for all $\alpha \in \mathcal{N}$.

Let us now prove (5). Suppose player I has a WS $\sigma$ in the game $G'_{xy}$. Then a WS $\tau$ for player II in the game $G_{xy}$ can be defined by the equation

$$\tau(a_0, \ldots, a_k) = \sigma(a_0, \ldots, a_{k-1}).$$

It is easy to see that for any point $\alpha \in \mathcal{N}$ if $\beta = \alpha * \tau$, then $\sigma * \alpha = \beta$ and $\langle x, \alpha\rangle <^*_\varphi \langle y, \beta\rangle$ by the choice of $\sigma$.

PROOF OF (6). LEFT-TO-RIGHT: Suppose to the contrary that players I and II have winning strategies $\sigma$ and $\tau$ in the games $G'_{yx}$ and $G_{xy}$ respectively. Consider the match $\langle \alpha, \beta\rangle$ in which the players adhere to the indicated strategies: i.e., $\alpha = \sigma * \beta$ and $\beta = \alpha * \tau$. We immediately arrive at a contradiction:

$$\langle y, \beta\rangle <^*_\varphi \langle x, \alpha\rangle \quad \text{and} \quad \langle x, \alpha\rangle \leqslant^*_\varphi \langle y, \beta\rangle.$$

RIGHT-TO-LEFT: Since $y <^* x$ does not hold, it follows in view of the determinacy of the games $G'_{xy}$ that player II has a WS $\tau$ in the game $G'_{xy}$. We shall show that this strategy is also a WS in the game $G_{xy}$ for player II. Let $\alpha \in \mathcal{N}$ be arbitrary, and let $\beta = \alpha * \tau$. Then the relation $\langle y, \beta \rangle <^*_\varphi \langle x, \alpha \rangle$ does not hold. But $\langle x, \alpha \rangle$ and $\langle y, \beta \rangle$ belong to $P$, since $x, y \in X$. Consequently $\langle x, \alpha \rangle \leqslant^*_\varphi \langle y, \beta \rangle$.

Finally, let us prove assertion (7). Suppose the contrary: there exists an infinite $<^*$-decreasing sequence of points $x_i \in X$ (where $i \in \omega$). For any $i$ player I has a WS $\sigma_i$ in the game $G'_{x_{i+1}\, x_i}$. Let us prescribe a sequence of points $\alpha_i \in \mathcal{N}$ by the equations $\alpha_i(l) = \sigma_i(\alpha_{i+1} \restriction l)$ by induction on $l$ simultaneously for all $i$. Thus $\alpha_i(0) = \sigma_i(\Lambda)$, $\alpha_i(1) = \sigma_i(\alpha_{i+1}(0))$, $\alpha_i(2) = \sigma_i(\alpha_{i+1}(0), \alpha_{i+1}(1))$, and so forth. We thereby obtain $\alpha_i = \sigma_i * \alpha_{i+1}$, i.e. $\langle x_{i+1}, \alpha_{i+1} \rangle <^*_\varphi \langle x_i, \alpha_i \rangle$ for all $i$. In other words, $\varphi(x_0, \alpha_0) > \varphi(x_1, \alpha_1) > \varphi(x_2, \alpha_2) \cdots$, which is impossible, since the values of $\varphi$ are ordinals. □

A recent study of Steel [73] yielded the following result: if $\Gamma$ is a class of pointsets satisfying certain rather elementary conditions and not coinciding with the class $\neg\Gamma$ of complementary sets, then under the assumption of the full axiom of determinacy AD precisely one of the classes $\Gamma$ and $\neg\Gamma$ satisfies the separation principle.

Norms and prewellorderings are now by no means used only to prove separation theorems. For various applications of these concepts in the theory of determinate universes, see the collection [35].

## §5. Uniformization and scales in determinate universes

Studies connected with the uniformization problem constitute one of the most important areas of both classical and modern descriptive set theory. We shall now present some of the necessary definitions. Let $\mathcal{X}$ and $\mathcal{Y}$ be a pair of point spaces.

A set $P \subseteq \mathcal{X} \times \mathcal{Y}$ is called *single-valued* or *uniform* (in the sense of $\mathcal{X} \times \mathcal{Y}$) when each of its sections

$$P/x = \{y \in \mathcal{Y} : P(x, y)\},$$

where $x \in \mathcal{X}$ contains at most one point. The projection $\pi P$ of a set $P \subseteq \mathcal{X} \times \mathcal{Y}$ (on the space $\mathcal{X}$) is defined by the equation

$$\pi P = \{x \in \mathcal{X} : P/x \neq \varnothing\} = \{x \in \mathcal{X} : \exists y P(x, y)\},$$

If $P \subseteq Q \subseteq \mathcal{X} \times \mathcal{Y}$, the set $P$ is single-valued, and $\pi P = \pi Q$, then the set $P$ is said to *uniformize* $Q$.

The uniformization problem in descriptive set theory consists of choosing for a given projective class $\Gamma$ the smallest class $\Gamma'$ such that every $\Gamma$-set admits uniformization by a set of class $\Gamma'$. (In such a case $\Gamma'$ is said to be a uniformization basis of the class $\Gamma$.) Usually the problem is to decide whether the following uniformization principle holds for some projective class $\Gamma$ or other:

$\Gamma$-Unif: Every $\Gamma$-set can be uniformized by a set of the same class $\Gamma$.

The uniformization problem, like the name itself, was introduced into the descriptive theory by Luzin [47]. However, even before [47] appeared several interesting results on uniformization of Borel- and $\Sigma^1_1$-sets had been obtained by Luzin and Novikov (see the next section). But the most important classical uniformization theorem, proved by Kondô [39] using a uniformization method introduced by Novikov [49], asserts that the uniformization principle holds for the class $\Pi^1_1$. This result is known as the Novikov–Kondô Theorem.

Uniformization in a class implies reduction for the same class. Indeed, to reduce a pair of $\Gamma$-sets $X, Y \subseteq \mathcal{X}$ we carry out a uniformization of the $\Gamma$-set

$$Q = \{\langle x, i \rangle : (x \in X \wedge i = 0) \vee (x \in Y \wedge i = 1)\}$$

by a set $P \subseteq Q$ of the same class $\Gamma$ and take the sets

$$X' = \{x : P(x, 0)\}, \quad Y' = \{x : P(x, 1)\}.$$

They belong to $\Gamma$ and reduce the pair $X, Y$. In this argument it is assumed that $\Gamma$ is a projective class.

Thus by what was said in the last section, the uniformization principle does not hold for the class $\Sigma^1_1$. In fact Novikov [69] constructed a $\Pi^1_0$-set in $\mathcal{N} \times \mathcal{N}$ which does not admit uniformization by any set of the class $\Sigma^1_1$, so that even the class $\Pi^1_0$ of closed sets does not satisfy Unif. From obvious considerations this holds also for the class of open sets $\Sigma^1_0$.

Uniformization passes from the class $\Pi^1_1$ to $\Sigma^1_2$ and in general from any $\Pi^1_m$ to $\Sigma^1_{m+1}$. Indeed, consider an arbitrary $\Sigma^1_{m+1}$-set $Q \subseteq \mathcal{X} \times \mathcal{Y}$. Then $Q = \{\langle x, y\rangle : \exists \alpha Q'(x, y, \alpha)\}$ for a suitable $\Pi^1_m$-set $Q' \subseteq \mathcal{X} \times \mathcal{Y} \times \mathcal{N}$. Using $\Pi^1_1$-Unif, we uniformize the set $Q'$ in the sense of $\mathcal{X} \times (\mathcal{Y} \times \mathcal{N})$ by means of a $\Pi^1_m$-set $P'$ and choose as $P$ the set $\{\langle x, y\rangle : \exists \alpha P'(x, y, \alpha)\}$.

Thus, $\Sigma^1_2$-Unif and the negation of $\Pi^1_2$-Unif are corollaries of the Novikov–Kondô theorem.

In the light of the results of the preceding section and what has been said here, the conjecture naturally arises that in a $\Sigma^1_{2n}$-determinate universe all classes of the first series (see the separation and reduction theorem in §4) satisfy the uniformization principle, except, of course, the class $\Sigma^1_0$. Such is indeed the case; however, the concept of norm and the first periodicity theorem turn out to be too weak to work with uniformization. It is necessary to use the more complicated concept of a scale, derived by Moschovakis [55] from the classical works [39,49] on uniformization.

A *scale* on a point set $X$ is a collection $\varphi = \langle \varphi_k : k \in \omega\rangle$ of norms $\varphi_k : X \to \mathrm{Ord}$, satisfying the following condition:

(A) If $x_0, x_1, x_2, \cdots \in X$, and $\lim x_i = x$, while for each $k$ there is an ordinal $\lambda_k$ such that for almost all $i$ (i.e., except for a finite number of indices $i$) the ordinals $\varphi_k(x_i)$ coincide with $\lambda_k$, then $x \in X$ and $\varphi_k(x) \leqslant \lambda_k$ for each $k$.

Like norms, scales are classified from the point of view of determinacy. A scale $\varphi = \langle \varphi_k : k \in \omega\rangle$ on the set $X \subseteq \mathcal{X}$ is called a $\Gamma$-scale when both of the sets

$$\{\langle k, x, y\rangle : x \leqslant^*_{\varphi_k} y\}, \quad \{\langle k, x, y\rangle : x <^*_{\varphi_k} y\}$$

in the space $\omega \times \mathcal{X}^2$ have class $\Gamma$. In the case when $\Gamma$ is a projective class a necessary and sufficient condition for this to hold is that each of the norms $\varphi_k$ be a $\Gamma$-norm.

If there exists a $\Gamma$-scale on each set of the given class $\Gamma$, the class $\Gamma$ is said to have the *scale property*. In this case at least for projective classes $\Gamma$, it is possible to define a $\Gamma$-scale $\varphi = \langle \varphi_k : k \in \omega\rangle$ on each $\Gamma$-set $X$ satisfying two additional conditions:

(B) If $x_0, x_1, x_2, \cdots \in X$ and for any $k$ the ordinals $\varphi_k(x_i)$ coincide with some $\lambda_k \in \mathrm{Ord}$ for almost all $i$, then there exists a point $x \in X$ such that $\lim x_i = x$;

(C) If $j < k$ and $\varphi_k(x) \leqslant \varphi_k(y)$, then $\varphi_j(x) \leqslant \varphi_j(y)$.

Scales satisfying (B) and (C) are called *good*.

Let us prove this assertion. On the $\Gamma$-set $X \subseteq \mathcal{N}$ (it suffices to consider only subsets of this space) let there be defined a $\Gamma$-scale $\psi = \langle \psi_k : k \in \omega\rangle$. We must construct a good $\Gamma$-scale on $X$. Choose an ordinal $\lambda$ such that $\psi_k(\alpha) < \lambda$ for all $k \in \omega$ and $\alpha \in \mathcal{N}$. To each $k$ we assign an order isomorphism $f_k$ of the set

$$C_k = \{\langle \xi_0, l_0, \xi_1, l_1, \ldots, \xi_k, l_k\rangle : \xi_i \leqslant \lambda \wedge l_i \in \omega\},$$

lexicographically ordered, to a corresponding (unique) ordinal $\varkappa_k$. We set

$$\varphi_k(\alpha) = f_k(\psi_0(\alpha), \alpha(0), \ldots, \psi_k(\alpha), \alpha(k))$$

for $\alpha \in X$ and $k \in \omega$. It is easily verified that the norms $\varphi_k$ form the desired good scale $\varphi$ on $X$.

The role of good scales is revealed by the following

PROPOSITION [55]. (Based on [39, 49].) *Suppose $m \geqslant 1$. If the class $\Pi^1_m$ has the scale property, then it satisfies the uniformization principle.*

PROOF: We shall uniformize a $\Pi^1_m$-set $Q \subseteq \mathcal{X} \times \mathcal{Y}$. According to the preceding there is a good $\Pi^1_m$-scale $\varphi = \langle \varphi_k : k \in \omega\rangle$ on the set $Q$. We shall set

$$S = \{\langle k, x, y\rangle \in \omega \times \mathcal{X} \times \mathcal{Y} : \forall y'(\langle x, y\rangle \leqslant^*_{\varphi_k} \langle x, y'\rangle)\},$$

$$P = \{\langle x, y\rangle : \forall k S(k, x, y)\},$$

and show that $P$ uniformizes our set $Q$ (that the set $P$ belongs to the class $\Pi^1_m$ is guaranteed by rules 6 and 7 of §1 and by the fact that $\varphi$ is a $\Pi^1_m$-scale.) First of all

$$\langle x, y\rangle \in P \to \langle x, y\rangle \leqslant^*_{\varphi_0} \langle x, y\rangle \to \langle x, y\rangle \in Q,$$

i.e., $P \subseteq Q$. It remains to verify that for each point $x \in \mathcal{X}$ if the section $Q/x$ is nonempty, then $P/x$ contains exactly one point. To this end we set $B_0 = Q/x$ and $\lambda_k = \inf_{y' \in B_0} \varphi_k(x, y')$, and

$$B_{k+1} = \{y : S(k, x, y)\} = \{y : \varphi_k(x, y) = \lambda_k\}$$

for all $k$. Each of the sets $B_k$ of course, is nonempty, and $P/x$ coincides with the intersection of all $B_k$. It remains to verify that this intersection contains exactly one point.

We remark that $B_1 \subseteq B_0$ by definition. For $k \geqslant 1$ the inclusion $B_{k+1} \subseteq B_k$ is given by the "goodness" of the scale $\varphi$. Consequently for $i \geqslant k$ we have $B_i \subseteq B_k$ and, choosing an arbitrary point $y_i$ in each $B_i$, we obtain $\varphi_k(x, y_i) = \lambda_k$ whenever $i > k$. Again, in view of the "goodness" there exists a point $y = \lim y_i$ such that $\langle x, y\rangle \in Q$, i.e., $y \in Q/x$ and $\varphi_k(x, y) \leqslant \lambda_k$—in fact $= \lambda_k$—for all $k$. But this means that $y \in \bigcap_{k \in \omega} B_k$.

If $y'$ is another point of the intersection of all $B_k$, we repeat the reasoning just carried out for the sequence $y, y', y, y', \ldots$, and obtain the convergence of the latter, whence $y' = y$. □

Thus the scale property for the class $\Pi^1_m$ leads to a proof of the uniformization principle for the classes $\Pi^1_m$ and $\Sigma^1_{m+1}$, i.e., scales play approximately the role with respect to uniformization that norms play with respect to reduction. This analogy is extended by the two following theorems of Moschovakis [55]:

UNIFORMIZATION THEOREM ($\Sigma^1_{2n}$-Det). *The classes*

$$\Pi^1_1, \Sigma^1_2, \ldots, \Sigma^1_{2n}, \Pi^1_{2n+1}, \Sigma^1_{2n+2}$$

*satisfy the uniformization principle, while the classes*

$$\Sigma^1_1, \Pi^1_2, \ldots, \Pi^1_{2n}, \Sigma^1_{2n+1}, \Pi^1_{2n+2},$$

*as well as* $\Sigma^1_0$ *and* $\Pi^1_0$ *do not.*

SECOND PERIODICITY THEOREM ($\Sigma^1_{2n}$-Det). *The classes*

$$\Sigma^1_0, \Pi^1_1, \Sigma^1_2, \ldots, \Sigma^1_{2n}, \Pi^1_{2n+1}, \Sigma^1_{2n+2}$$

*have the scale property.*

As has been shown, the second theorem implies the first. The second periodicity theorem, in turn, is proved by the same method as the first periodicity theorem in §4:

LEMMA 5.1. *The class* $\Sigma^1_0$ *of open sets has the scale property.*

LEMMA 5.2. *If the class* $\Pi^1_m, m \geqslant 1$ *has the scale property, then the class* $\Sigma^1_{m+1}$ *also has this property.*

LEMMA 5.3. ($\Sigma^1_m$-Det) *If the class* $\Sigma^1_m$ *has the scale property, then the class* $\Pi^1_{m+1}$ *also has this property.*

PROOF OF LEMMA 5.1: We shall construct a $\Sigma^1_0$-scale on the open set $X = \bigcup_{u \in c} \mathcal{N}_u$, where $c \subseteq FC$ and $\mathcal{N}_u$ (for $u \in FC$) is a Baire interval: $\mathcal{N}_u = \{\alpha \in \mathcal{N} : u \subset \alpha\}$. For each $\alpha \in X$ we denote by $m_\alpha$ the smallest number $m$ such that $\alpha \restriction m \in c$.

If $k \in \omega$, there exists a unique order isomorphism $f_k$ of the set $\omega^k$ with lexicographic ordering onto a corresponding (also unique) ordinal. Set

$$\varphi_k(\alpha) = f_{k+1}\left(\langle m_\alpha\rangle^\frown(\alpha \restriction k)\right)$$

for all $\alpha \in X$ and $k \in \omega$. The norms $\varphi_k$ form a $\Sigma^1_0$-scale on $X$. $\square$

PROOF OF LEMMA 5.2: We shall construct a $\Sigma^1_{m+1}$-scale on the $\Sigma^1_{m+1}$-set $X = \{x \in \mathcal{X} : \exists\alpha P(x,\alpha)\}$, where $P \subseteq \mathcal{X} \times \mathcal{N}$ is a set of class $\Pi^1_m$. By the proposition proved before the uniformization theorem we may assume that the set $P$ is single-valued, i.e., for every $x \in X$ there exists a unique point $\alpha_x \in \mathcal{N}$ such that $P(x,\alpha_x)$. Further there exists a good $\Pi^1_m$-scale $\varphi = \langle \varphi_k : k \in \omega \rangle$ on $X$. Set $\psi_k(x) = \varphi_k(x,\alpha_x)$ for $x \in X$ and $k \in \omega$. The norms $\psi_k$ give the desired $\Sigma^1_{m+1}$-scale on $X$. $\square$

PROOF OF LEMMA 5.3: Let $P \subseteq \mathcal{X} \times \mathcal{N}$, $P \in \Sigma^1_m$. We shall construct a $\Pi^1_{m+1}$-scale on the set $X = \{x : \forall\alpha P(x,\alpha)\}$. The scale property for the class $\Sigma^1_m$ provides a good $\Sigma^1_m$-scale $\varphi = \langle \varphi_k : k \in \omega \rangle$ on the set $P$.

Now we shall again need the enumeration of finite sequences of natural numbers introduced in §4, in which to each $k \in \omega$ there corresponds in a one-to-one manner a finite sequence $[k] \in FC$. In what follows we shall assume the following holds: if $[k_1] \subset [k_2]$, then $k_1 < k_2$.

Let $x, y \in \mathcal{X}$ and $k \in \omega$. Consider the game $G_{kxy} = G([k];[k];A_{kxy})$ with game set

$$A_{kxy} = \{\langle \alpha,\beta \rangle : \neg(\langle x,\alpha \rangle \leqslant^*_{\varphi_k} \langle y,\beta \rangle)\},$$

beginning at position $[k];[k]$ (see §2). As in the proof of lemma 4.3, for each $k$ there exists a $\Pi^1_{m+1}$-norm $\psi_k$ on $X$ satisfying for any $x, y$ the equivalence

$$x \leqslant^*_{\psi_k} y \leftrightarrow \text{player II has a WS in the game } G_{kxy}\,.$$

It remains to verify that these norms form a scale on $X$.

Suppose $x_0, x_1, x_2, \dots \in X$, $x = \lim x_i$, and for any $k$ the ordinals $\psi_k(x_i)$ coincide for almost all $i$ with an ordinal $\lambda_k$ depending on $k$. We must verify that $x \in X$ and $\psi_k(x) \leqslant \lambda_k$ for all $k$. Without loss of generality we may assume that $\psi_k(x_i) = \lambda_k$ for any pair $i \geqslant k$ (otherwise, we simply pass to a suitable subsequence). Then for $k \leqslant m$ we obtain $\psi_k(x_i) = \psi_k(x_m) = \lambda_k$, i.e., player II has some WS $\tau_{km}$ in the game $G_{kx_m x_k}$.

To prove $x \in X$ we fix an arbitrary point $\alpha \in \mathcal{N}$ and verify that $\langle x,\alpha \rangle \in P$. If $i \in \omega$, there exists a unique natural number $k_i$ such that $[k_i] = \alpha \restriction i$. Here $k_i < k_{i+1}$ for all $i$ in view of the requirement imposed on the enumeration of finite sequences, so that $\tau_i = \tau_{k_i k_{i+1}}$ is a WS for II in the game $G_i = G_{k_i x_{k_{i+1}} x_{k_i}}$. There exists a sequence of points $\alpha_i \in \mathcal{N}$ such that $\alpha_i = \alpha_{i+1} * \tau_i$ for all $i$: the values $\alpha_i(l)$ are determined by the equations $\alpha_i(l) = \alpha(l)$ for $l < i$ and $\alpha_i(l) = \tau_i(\alpha_{i+1} \restriction l+1)$ for $l \geqslant i$. (The game $G_i$ begins at position $\alpha \restriction i; \alpha \restriction i$ and hence the equation $\alpha_i(l) = \tau_i(\alpha_{i+1} \restriction l+1)$ is automatically fulfilled for $l < i$ also.)

Thus $\langle x_{k_{i+1}}, \alpha_{i+1} \rangle \leqslant^*_{\varphi_{k_i}} \langle x_{k_i}, \alpha_i \rangle$ for all $i$; and since $x_k \in X, \forall k$, it follows that for any $i$ we obtain

$$\langle x_{k_i}, \alpha_i \rangle \in P \quad \text{and} \quad \varphi_{k_i}(x_{k_{i+1}}, \alpha_{i+1}) \leqslant \varphi_{k_i}(x_{k_i}, \alpha_i).$$

In view of the "goodness" of the scale $\varphi$, it follows from this that

$$\varphi_j(x_{k_{i+1}}, \alpha_{i+1}) \leqslant \varphi_j(x_{k_i}, \alpha_i) \tag{1}$$

whenever $j \leqslant k_i$. Therefore for each $j$ there is an ordinal $\mu_j$ such that $\varphi_j(x_{k_i}, \alpha_i) = \mu_j$ for almost all $i$. Again by the "goodness" we obtain $\lim\langle x_{k_i}, \alpha_i \rangle \in P$. However, $\lim x_k = x$ and $\lim \alpha_i = \alpha$. Thus $\langle x,\alpha \rangle \in P$, and since $\alpha$ is arbitrary in this reasoning, it follows that $x \in X$.

We remark that by definition of a scale in the situation under consideration we have $\varphi_j(x,\alpha) \leqslant \mu_j$ for all $j$. Combining this with inequality (1) and taking account of the choice of $\mu_j$, we obtain

$$\varphi_k(x,\alpha) \leqslant \varphi_k(x_k, \alpha_i) \quad \text{i.e.,} \quad \langle x,\alpha \rangle \leqslant^*_{\varphi_k} \langle x_k, \alpha_i \rangle \tag{2}$$

for any pair $i, k$ such that $k = k_i$ (it is necessary to take $j = k$). In addition, analyzing the construction of the sequence of points $\alpha_i$, it is not difficult to notice that each value $\alpha_i(l)$ requires for its determination only a knowledge of the numbers $\alpha(l')$, $l' \leqslant l$. Consequently we can write $\alpha_i = \alpha * \tau^i$, where $\tau^i$ is a strategy (for player II) depending only on $i$.

We can now prove the inequality $\psi_k(x) \leqslant \lambda_k$ without difficulty, where $k \in \omega$ is arbitrary. Since $\psi_k(x_k) = \lambda_k$, it suffices to verify that player II has a WS in the game $G_{kxx_k}$. Denoting the length of the finite sequence $[k]$ by $i$, we show that $\tau^i$ is the desired strategy. Let $\alpha \in \mathcal{N}$ be arbitrary. If $[k] \not\subset \alpha$, then player I loses no matter what response II gives, so that we may assume that $[k] \subset \alpha$. Then, repeating the calculations of the first part of the proof of the lemma, we have $k = k_i$, $\alpha * \tau^i = \alpha_i$, and, finally, $\langle x, \alpha\rangle \leqslant^*_{\varphi_k} \langle x_k, \alpha * \tau^i\rangle$, according to (2). But this is tantamount to a win for player II in the game $G_{kxx_k}$. $\square$

The importance of scales in the descriptive theory of determinate universes is by no means exhausted by the second periodicity theorem and applications to uniformization. Together with norms and preorderings, the concept of a scale occupies a central place in descriptive investigations. In particular, the majority of articles in the recently published collection [36] are devoted to scales in determinate universes (as pointed out in the foreword by the editors of the collection). We note the interesting article [74] in this collection, where the question of scales on sets of class $\Sigma^1_1$ is considered. Naturally the aforementioned class does not have the scale property, i.e., it cannot be said that on any $\Sigma^1_1$-set there is a $\Sigma^1_1$-scale (or even a $\Sigma^1_1$-norm). The second periodicity theorem (for $n = 0$) gives the following result: on any $\Sigma^1_1$-set there exists a $\Sigma^1_2$-scale. As shown in [74], this result is by no means optimal: in fact every $\Sigma^1_1$-set supports a scale of class $\mathbf{B}_{\omega_1}(\Sigma^1_1)$, where the symbol $\mathbf{B}_{\omega_1}$ denotes the closure of the class in parentheses with respect to the Borel operations of complementation and countable union.

## §6. Projective sets with special sections in determinate universes

Let $\mathcal{X}$, $\mathcal{Y}$ be two point spaces and $P \subseteq \mathcal{X} \times \mathcal{Y}$. We recall that each point $x \in \mathcal{X}$ defines a section $P/x = \{y : P(x,y)\}$ of the set $P$. We may distinguish the sets $P$ such that each section $P/x$ contains at most one point—the single-valued, or uniform, sets. A weaker requirement—that each $P/x$ be at most countable—distinguishes the countably-valued sets. Single- and countably-valued sets form the simplest categories of sets with special sections; besides them, sets with compact and $\sigma$-compact sections, sets with sections of positive measure, and others are also studied (see [20, 17, 43]). The classical studies of projective sets with special sections began in the second half of the 20's (a survey of the results obtained can be found in [2], [4, §2], and [6, §4]).

The limited scope of the present article permits us to examine in detail only one problem from this field. We shall study the problem of partitioning a countably-valued set of a given projective class into a countable number of single-valued sets of the same class. We shall say that a projective class $\Gamma$ has the partition property if any planar countably-valued $\Gamma$-set (in the sense of the given pair of point spaces $\mathcal{X}$, $\mathcal{Y}$) is a countable union of single-valued $\Gamma$-sets.

The investigations of Luzin [48] and Novikov [62] showed that the classes $\Delta^1_1$ (of Borel sets) and $\Sigma^1_1$ possess the partition property. This result, like the separation, reduction, and uniformization theorems, generalizes in determinate universes to higher odd levels:

PARTITION THEOREM. [77] ($\Sigma^1_{2n}$-Det) *The classes $\Delta^1_{2n+1}$ and $\Sigma^1_{2n+1}$ have the partition property.*

The known proofs of the partition theorem all include in one form or another the use of one of the variants of a theorem [57, 58] on the choice of a winning strategy. We have preferred to use a variant that avoids having recourse to "effective" classes, although it is not the most natural.

THEOREM ON THE CHOICE OF A WINNING STRATEGY. ($\Sigma^1_{2n}$-Det) *If $B \subseteq \mathcal{X} \times \mathcal{N}^2$ and $B \in \Sigma^1_{2n}$, then there exists a function $\Phi : \mathfrak{D}B \to \mathcal{N}$ of class $\Sigma^1_{2n+1}$ on $\mathfrak{D}B$ (i.e., the graph of $\Phi$ is the intersection of $\mathfrak{D}B \times \mathcal{N}$ with some subset of $\mathcal{X} \times \mathcal{N}$ of class $\Sigma^1_{2n+1}$), possessing the property that for any $x \in \mathfrak{D}B$ the strategy $[\Phi(x)]$ is a WS for player I in the game $G(B/x)$. (For the definition of the strategy $[\varepsilon]$ for $\varepsilon \in \mathcal{N}$ see §4.)*

The proof of the theorem on the choice of a winning strategy begins by defining two families of games. Let $x \in \mathcal{X}$, and $k \in \omega$, and let $u$ and $v$ belong to $\omega^k$ (i.e., they are finite sequences of length $k$ consisting of natural numbers.) and finally let $a \in \omega$. The symbol $G_{xkuv}(a)$ denotes the game with game set $B/x$ starting at position $u^\frown\langle a\rangle; v$. All these games are determinate by the hypothesis $\Sigma^1_{2n}$-Det.

Before defining the second family of games we first associate a point $\prec \alpha\beta \succ \in \mathcal{N}$ with each pair of points $\alpha$, $\beta \in \mathcal{N}$ by the definitions

$$\prec \alpha\beta \succ (2k) = \alpha(k), \quad \prec \alpha\beta \succ (2k+1) = \beta(k)$$

for all $k$. According to the second periodicity theorem there is a $\Sigma^1_{2n}$-scale on our $\Sigma^1_{2n}$-set $B$; and then, as was shown in §5, there is also a good $\Sigma^1_{2n}$-scale $\varphi = \langle \varphi_k : k \in \omega\rangle$. Using this good scale we assign to each collection $k \in \omega$, $x \in \mathcal{X}$; $u, v \in \omega^k$, and $a, a' \in \omega$, the game $G_{xkuv}(a', a)$ with game set

$$\{\langle \prec \alpha\beta' \succ, \prec \alpha'\beta \succ\rangle : \neg(\langle x, \alpha', \beta'\rangle \leqslant^*_{\varphi_k} \langle x, \alpha, \beta\rangle)\},$$

starting at the position $\prec uv \succ {}^\frown\langle a\rangle; \prec uv \succ {}^\frown\langle a'\rangle$, where the concatenation $\prec uv \succ$ of two finite sequences $u$, $v$ of length $k$ is defined in a way similar to the concatenation $\prec \alpha\beta \succ$ of the points $\alpha$, $\beta \in \mathcal{N}$, i.e., $\prec uv \succ$ is a finite sequence of length $2k$ and

$$\prec uv \succ (2i) = u(i) \quad \prec uv \succ (2i+1) = v(i)$$

for all $i < k$. The game sets of these games have class $\Pi^1_{2n}$ by the choice of the scale $\varphi$ and are therefore determinate, since $\Pi^1_{2n}$-Det follows from the hypothesis $\Sigma^1_{2n}$-Det (see §2).

We give three more definitions:

$$W_{xk} = \{\langle u, a, v\rangle : u, v \in \omega^k \wedge a \in \omega \wedge \text{player I has a WS in the game } G_{xkuv}(a)\};$$

$$a' \leqslant_{xkuv} a \leftrightarrow \text{II has a WS in the game } G_{xkuv}(a', a);$$

$$M_{xk} = \{\langle u, a', v\rangle \in W_{xk} : \forall a \in \omega (a' \leqslant_{xkuv} a)\}.$$

LEMMA 6.1. *Let $x \in \mathcal{X}$, $k \in \omega$, and let the points $\alpha$, $\beta \in \mathcal{N}$ be such that $\langle \alpha \restriction k, \alpha(k), \beta \restriction k\rangle \in M_{xk}$ for all $k$. Then $\langle \alpha, \beta\rangle \in B/x$.*

PROOF: For $k = 0$ we have $\langle \Lambda, \alpha(0), \Lambda\rangle \in M_{x0}$, i.e., player I has a WS $\sigma$ in the game $G_{x0\Lambda\Lambda}(\alpha(0))$. Further, if $k \in \omega$, then $\alpha(k) \leqslant_{xk(\alpha\restriction k)(\beta\restriction k)} a$ for any $a \in \omega$ by the definition of $M_{xk}$, i.e., player II has a WS $\tau_{ka}$ in the game $G^{ka} = G_{xk(\alpha\restriction k)(\beta\restriction k)}(\alpha((k), a)$. Using this system of strategies we shall define a sequence of points $\alpha_k$, $\beta_k \in \mathcal{N}$ and strategies $\tau_k$ by means of the following system of equations:

(1) $\alpha_k(l) = \alpha(l)$ and $\beta_k(l) = \beta(l)$ for $l < k$;
(2) $\tau_k = \tau_{k\alpha_k(k)}$ for all $k$;
(3) $\alpha_0(l) = \sigma(\beta_0(0), \ldots, \beta_0(l-1))$ for all $l$;
(4) $\alpha_{k+1}(l) = \tau_k(\alpha_k(0), \beta_{k+1}(0), \ldots, \alpha_k(l-1), \beta_{k+1}(l-1), \alpha_k(l))$ and
$\beta_k(l) = \tau_k(\alpha_k(0), \beta_{k+1}(0), \ldots, \alpha_k(l), \beta_{k+1}(l))$ for $l \geqslant k$.

Relation (3) gives $\alpha_0 = \sigma * \beta_0$, i.e., $\langle \alpha_0, \beta_0\rangle \in B/x$ by the choice of $\sigma$. Further, relation (1) shows that equations (4) hold also for $l < k$; for the games $G^{ka}$ start at the positions $\prec \alpha \restriction k, \beta \restriction k \succ {}^\frown\langle a\rangle; \prec \alpha \restriction k, \beta \restriction k \succ {}^\frown\langle \alpha(k)\rangle$. Hence $\prec \alpha_{k+1}\beta_k \succ = \prec \alpha_k\beta_{k+1} \succ * \tau_k$, i.e., $\langle x, \alpha_{k+1}, \beta_{k+1}\rangle \leqslant^*_{\varphi_k} \langle x, \alpha_k, \beta_k\rangle$ for all $k$. Combining this with the proven relation $\langle \alpha_0, \beta_0\rangle \in B/x$, we obtain:

$$\langle \alpha_k, \beta_k\rangle \in B/x \quad \text{and} \quad \varphi_k(x, \alpha_{k+1}, \beta_{k+1}) \leqslant \varphi_k(x, \alpha_k, \beta_k)$$

for all $k$. The "goodness" of the scale $\varphi$ allows us to deduce from this (see the proof of lemma 5.3) that the sequence of points $\langle x, \alpha_k, \beta_k\rangle$ converges to some point of $B$. But this point can only be the point $\langle x, \alpha, \beta\rangle$. □

LEMMA 6.2. *Let $x \in \mathcal{X}$. If $x \in \mathfrak{D}B$, then there exists $a \in \omega$ such that $\langle \Lambda, a, \Lambda \rangle \in M_{x0}$. If $k \in \omega$ and $\langle u, a, v \rangle \in M_{xk}$, $b \in \omega$, then there exists $c \in \omega$ such that $\langle u^\frown\langle a \rangle, c, v^\frown\langle b \rangle \rangle \in M_{x,k+1}$.*

PROOF: For sets $W_{xk}$ in place of $M_{xk}$ the lemma is obvious. Therefore the contrary assumption provides us with a point $x \in \mathcal{X}$, a natural number $k$, a pair of finite sequences $u$, $v \in \omega^k$, and a sequence of natural numbers $a_i$, $i \in \omega$, such that $\langle u, a_0, v \rangle \in W_{xk}$ and $\neg(a_i \leqslant_{xkuv} a_{i+1})$ for all $i \in \omega$. Thus, player I obtains a WS $\sigma$ in the game $G_{xkuv}(a_0)$ and a WS $\sigma_i$ in each game $G_{xkuv}(a_i, a_{i+1})$, $i \in \omega$. Using this system of strategies we can construct a sequence of points $\alpha_i$, $\beta_i \in \mathcal{N}$ satisfying the relations: $\alpha_i \restriction k = u$, $\beta_i \restriction k = v$ for all $i$,

$$\alpha_0 = \sigma * \beta_0, \quad \text{and} \quad \prec \alpha_{i+1}, \beta_i \succ = \sigma_i * \prec \alpha_i \beta_{i+1} \succ \quad \text{for } i \in \omega.$$

By the choice of the strategies $\sigma$ and $\sigma_i$ we obtain $\langle \alpha_0, \beta_0 \rangle \in B/x$ and

$$\neg(\langle x, \alpha_i, \beta_i \rangle \leqslant^*_{\varphi_k} \langle x, \alpha_{i+1}, \beta_{i+1} \rangle)$$

for all $i \in \omega$, whence by induction on $i$ it is not difficult to deduce that

$$\langle \alpha_i, \beta_i \rangle \in B/x \quad \text{and} \quad \varphi_k(x, \alpha_{i+1}, \beta_{i+1}) < \varphi_k(x, \alpha_i, \beta_i)$$

for all $i$, which is impossible, since $\varphi_k$ is a norm. □

We now continue the proof of the theorem on the choice of a winning strategy. If $x \in \mathfrak{D}B$, then the strategy $\sigma_x = [\Phi(x)]$ must operate on an arbitrary finite sequence $v = \langle b_0, \ldots, b_{k-1} \rangle \in FC$ in such a way that $\langle u, \sigma_x(v), v \rangle \in M_{xk}$, where $u = \langle a_0, \ldots, a_{k-1} \rangle$ and $a_i = \sigma_x(b_0, \ldots, b_{i-1})$ for $i < k$. Lemma 6.2 guarantees that such a strategy is possible, and according to lemma 6.1 this strategy (or, more precisely, any of the strategies satisfying the indicated condition) is winning for player I in the game $G(B/x)$. It now remains to guarantee the construction of $\Phi$ as a $\Sigma^1_{2n+1}$-function on the set $\mathfrak{D}B$.

To this end consider the set

$$M = \{\langle x, k, l, j, a \rangle : x \in \mathcal{X} \wedge \langle [l], a, [j] \rangle \in M_{xk}\}$$

(where, we recall, [i] is the finite sequence with index $i$). It belongs to the class $\Pi^1_{2n+1}$: we use what was said in §4 about the action of the operator $\mathfrak{D}$ (the existence of a WS for player II in the definition of $\leqslant_{xkuv}$ is expressed through the absence of a WS for player I). Thus according to the uniformization theorem of §5 the set $M$ can be uniformized by a $\Pi^1_{2n+1}$-set $C \subseteq M$. In essence $C$ is a function defined on some subset of $\mathcal{X} \times \omega^3$ (containing all of $\mathfrak{D}B \times \omega^3$) with values in $\omega$, and $\langle x, k, l, j, C(x, k, l, j) \rangle \in M$ for any quadruple $\langle x, k, l, j \rangle$ in the domain of definition of $C$, and the graph of $C$ is a set of class $\Pi^1_{2n+1}$.

We can now give the construction of $\Phi$. Let $x \in \mathfrak{D}B$, and $j \in \omega$ be arbitrary. The value $a = \Phi(x)(j)$ is defined as follows. Let $v = [j] = \langle b_0, \ldots, b_{k-1} \rangle$. By induction on $i \leqslant k$ we define a collection of numbers $a_j$ using the equations

$$a_i = C(x, i, \operatorname{num}\langle a_0, \ldots, a_{i-1} \rangle, \operatorname{num}\langle b_0, \ldots, b_{i-1} \rangle),$$

where, for each finite sequence $w$, $\operatorname{num} w$ denotes the index, i.e., $w = [\operatorname{num} w]$. Finally set $\Phi(x)(j) = a_k$.

To verify that the function $\Phi$ completes the proof of the theorem on the choice of a winning strategy, we need only verify that it is a $\Sigma^1_{2n+1}$-function on $\mathfrak{D}B$; the fact that $[\Phi(x)]$ is a WS for I in the game $G(B/x)$ for any $x \in \mathfrak{D}B$ is guaranteed, as we have seen above, by lemma 6.1. Using the choice of the set $C$ (in the class $\Pi^1_{2n+1}$) and various rules from §1, it is easy to show that the set

$$U = \{\langle x, j, a \rangle : x \in \mathfrak{D}B \wedge \Phi(x)(j) = a\}$$

also belongs to $\Pi^1_{2n+1}$. However,

$$\Phi(x) = \varepsilon \leftrightarrow x \in \mathfrak{D}B \wedge \forall j \forall a (U(x, j, a) \rightarrow \varepsilon(j) = a),$$

whence the required fact about the function $\Phi$ follows immediately. □

Let us now return to the proof of the partition theorem. Consider the countably-valued $\Sigma^1_{2n+1}$-set $P \subseteq \mathcal{X} \times \mathcal{Y}$. We may assume that the second axis is $\mathcal{N}$ and in addition (see the beginning of the proof of theorem 1 in §3) that $P \subseteq \mathcal{X} \times \mathcal{D}$ (where $\mathcal{D} = 2^\omega$ is the Cantor discontinuum). In this situation we shall show that $P$ is covered by the union of a countable number of single-valued $\Delta^1_{2n+1}$-sets.

Let $P = \{\langle x, \delta\rangle : \exists \gamma Q(x, \delta, \gamma)\}$, where $Q \subseteq \mathcal{X} \times \mathcal{D} \times \mathcal{N}$ is a $\Pi^1_{2n}$-set. Using the functions $D$ and $H$ from the proof of theorem 1 of §3, we define a set

$$B = \{\langle x, \alpha, \beta\rangle \in \mathcal{X} \times \mathcal{N}^2 : \langle x, D(\alpha, \beta), H(\beta)\rangle \notin Q\}.$$

It belongs to $\Sigma^1_{2n}$ (rule 2 of §1), and also, in accordance with the propositions of the abovementioned proof, we have:

$$\mathfrak{D} B = \{x \in \mathcal{X} : \text{the section} \quad P/x \quad \text{is at most countable}\},$$

so that $\mathfrak{D} B = \mathcal{X}$ in view of the countablevaluedness of $P$. The theorem on the choice of a winning strategy provides a $\Sigma^1_{2n+1}$-function $\Phi : \mathcal{X} \to \mathcal{N}$ such that the strategy $[\Phi(x)]$ is winning for I in the game $G(P/x)$ for any $x \in \mathcal{X}$. In addition we may assume that $\Phi(x) \in \mathcal{D}$ for any $x \in \mathcal{X}$—see the remark at the beginning of the analysis of case 1 in the proof of theorem 1 in §3. Thus $\Phi$ is a function from $\mathcal{X}$ to $\mathcal{D}$.

Returning to the analysis of the current case, let us denote by $T$ the set of all sequences of the form $t = \langle a_0, b_0, \ldots, a_{k-1}, b_{k-1}, a_k\rangle$ of numbers $a_i = 0$ or 1 and $b_i \in \omega$, of arbitrary odd length $2k+1$. To each sequence $t \in T$ we assign a function $F_t : \mathcal{D} \to \mathcal{D}$ operating as follows. Let $\varepsilon \in \mathcal{D}$. The strategy $\sigma = [\varepsilon]$, together with $t$, determines via the relations $w(t) \subset \delta$ and (*) from the analysis of the abovementioned case a unique point $\delta \in \mathcal{D}$ which we denote by $F_t(\varepsilon)$. All the functions $F_t : \mathcal{D} \to \mathcal{D}$ are continuous.

The key property of the family of functions $F_t$ so defined resides in the fact that for any point $x \in \mathcal{X}$, if $\varepsilon \in \mathcal{D}$ and the strategy $[\varepsilon]$ is winning for player I in the game $G(B/x)$, then $P/x \subseteq \{F_t(\varepsilon) : t \in T\}$. Consequently, defining

$$P_t = \{\langle x, F_t(\Phi(x))\rangle : x \in \mathcal{X}\},$$

we obtain a family of single-valued sets $P_t \subseteq \mathcal{X} \times \mathcal{D}$, whose union covers $P$. In addition each $P_t$ has class $\Sigma^1_{2n+1}$, according to the choice of $\Phi$ and the continuity of $F_t$, and hence also class $\Delta^1_{2n+1}$, since

$$\langle x, \delta\rangle \in P_t \leftrightarrow \forall \delta'(\delta' \neq \delta \to \langle x, \delta'\rangle \notin P_t). \square$$

Having finished the proof of the partition theorem, we now give several other consequences of the construction exhibited here which relate to single- and countably-valued sets in a $\Sigma^1_{2n}$-determinate universe (for a fixed natural number $n$).

1. A countably-valued $\Sigma^1_{2n+1}$-set $P$ is covered by a countably-valued $\Delta^1_{2n+1}$-set, namely the union of all the sets $P_t$. Thus in a $\Sigma^1_{2n}$-determinate universe every countably-valued set of class $\Sigma^1_{2n+1}$ can be covered by a countably-valued $\Delta^1_{2n+1}$-set. When $n = 0$, as in similar cases above, we obtain a classical result—a theorem of Luzin in [48].

It is interesting that single-valued $\Sigma^1_{2n+1}$-sets are covered by $\Delta^1_{2n+1}$-sets that are also single-valued. This fact can also be proved using the theorem on the choice of a winning strategy.

2. Suppose that our countably-valued set $P$ belongs to the class $\Delta^1_{2n+1}$. Then its projection $\pi P = \{x : \exists \delta P(x, \delta)\}$ belongs to the same class $\Delta^1_{2n+1}$. In fact it is trivial that $\pi P \in \Sigma^1_{2n+1}$ (apply rule 6 of §1). But the class $\Pi^1_{2n+1}$ is given by the equation

$$\pi P = \bigcup_{t \in T} \{x \in \mathcal{X} : \forall \delta(\langle x, \delta\rangle \in P_t \to \langle x, \delta\rangle \in P)\}.$$

Thus the projections of countably-valued $\Delta^1_{2n+1}$-sets are (assuming $\Sigma^1_{2n}$-Det) sets of class $\Delta^1_{2n+1}$. With $n = 0$ this result gives a classical theorem of Novikov [62]. It is interesting that every $\Delta^1_{2n+1}$-set $X \subseteq \mathcal{X}$ is the projection of a suitable single-valued set $P \subseteq \mathcal{X} \times \mathcal{N}$ of class $\Pi^1_{2n}$ (for $n = 0$ this is a theorem of Luzin [45, 48]). This converse result is also proved using the theorem on the choice of a winning strategy.

3. Uniformization. We again assume that our countably-valued $P$ has class $\Delta^1_{2n+1}$. Then $P$ uniformizes by a $\Delta^1_{2n+1}$-set $P' \subseteq P$. Indeed, by the partition theorem $P$ is the union of single-valued $\Delta^1_{2n+1}$-sets

$P_k$, $k \in \omega$. and the projection $\pi P_k$ of any of them also belongs to $\Delta^1_{2n+1}$ (see above). Therefore each of the sets

$$P'_k = \{\langle x, \alpha\rangle \in P_k : x \notin \bigcup_{i<k} \pi P_i\}$$

has once again the class $\Delta^1_{2n+1}$. It remains only to take as $P'$ the union of all the $P_k$.

For $n = 0$ the theorem that each countably-valued set of class $\Delta^1_1$ (i.e., each Borel set, for the class $\Delta^1_1$ is identical to the class of Borel sets by a theorem of Suslin [71]) uniformizes by a $\Delta^1_1$-set was proved by Novikov in [62].

We note the essential meaning of countable-valuedness of a uniformizable set: Under the assumption $\Sigma^1_{2n}$-Det there exists a $\Pi^1_{2n}$-set that does not admit uniformization by any set of class $\Sigma^1_{2n+1}$ (and of course is not countably-valued). Such a set is not difficult to obtain starting from a pair of $\Pi^1_{2n+1}$-sets of the space $\mathcal{N}$ for which the reduction principle is violated (which exist according to the separation and reduction theorem of §4). With $n = 0$ the construction was carried out in the paper [62] of Novikov.

The construction giving a uniformization of countably-valued $\Delta^1_{2n+1}$-sets can be used to uniformize countably-valued sets of class $\Sigma^1_{2n+1}$. The following result is obtained: In a $\Sigma^1_{2n}$-determinate universe every countably-valued $\Sigma^1_{2n+1}$-set can be uniformized by a set that is a countable union of differences of $\Sigma^1_{2n+1}$-sets.

As of the present it remains an open problem to uniformize $\Sigma^1_{2n+1}$-sets of general form by sets that are essentially simpler than sets of class $\Sigma^1_{2n+2}$ (which guarantees the uniformization even of $\Sigma^1_{2n+2}$-sets by the uniformization theorem of §5). For $n = 0$ there is an important result of Luzin–Yankov (see [48] or [2, §11]): Countable intersections of countable unions of differences of $\Sigma^1_1$-sets constitute a uniformization basis for the class $\Sigma^1_1$.

For other interesting applications of the theorem on the choice of a winning strategy in the descriptive theory see [57] or [58, Ch. 6]. These works also contain more detailed information on the results given by us in sections 1, 2, and 3. We note that the theorem on the choice of a winning strategy itself is proved in [58] for "effective" projective classes, where it can be formulated more simply and naturally: Under the assumption $\Sigma^1_{2n}$-Det, if $\varepsilon \in \mathcal{N}$, $A$ is a set of class $\Sigma^{1,\varepsilon}_{2n}$, and player I has a WS in the game $G(A)$, then player I has a WS in the indicated game determined by a point of the class $\Delta^{1,\varepsilon}_{2n+1}$.

## §7. Generalized Borel and Suslin representations of projective sets in determinate universes. Projective ordinals

The theorems of Suslin [71] on the coincidence of the projective classes $\Sigma^1_1$ and $\Delta^1_1$ respectively with the classes of $A$-sets and Borel sets, mentioned in §1 of this survey, have also been generalized in determinate universes to higher projective levels. But these generalizations, unlike those considered in the preceding sections, presume a definite generalization of the concepts considered. Let $\varkappa$ be an infinite ordinal, and supposed fixed some topological space (for example, one of the point sets in the sense of §1).

We denote by $\mathbf{B}_\varkappa$ the smallest family of sets of the given space containing all open sets and closed under the operations of complementation and taking a union of fewer than $\varkappa$ sets. It is customary to call sets of $\mathbf{B}_\varkappa$ $\varkappa$-Borel sets. The usual Borel sets are precisely the sets of $\mathbf{B}_{\omega_1}$.

The concept of a Suslin set generalizes as follows. Suppose that to each finite sequence $u$ composed of ordinals less than $\varkappa$, there is associated a closed set $X_u$ of the space under consideration (i.e., there is defined a $\varkappa$-branching system of sets). Form the set $X = \bigcup_f \bigcap_{m\geqslant 1} X_{f\restriction m}$, where the union is taken over all $\omega$-sequences $f$, composed of ordinals less than $\varkappa$, and let $f \restriction m$ denote the finite sequence formed by the first $m$ terms of the infinite sequence $f$. All sets $X$ so constructed are called $\varkappa$-Suslin sets; the family of all $\varkappa$-Suslin sets is denoted $\mathbf{S}_\varkappa$[6]. It is obvious that for $\varkappa = \omega$ we obtain the definition of the usual Suslin sets (these are the $A$-sets), see [1].

[6]In the book [3, Ch. 8, §6] a different, but equivalent, definition of $\varkappa$-Suslin sets is given. The equivalence is proved, for example, in [58, Ch. 2]

The results of Suslin can now be formulated in the form of equalities: $\Delta^1_1 = \mathbf{B}_{\omega_1}$, and $\Sigma^1_1 = \mathbf{S}_\omega$.

Interesting results on the Borel and Suslin representations have been obtained also for the second projective level: $\Sigma^1_2 \subseteq \bigcup_{\omega_1} \Delta^1_1$—i.e., every $\Sigma^1_2$-set is the union of $\aleph_1$ sets of class $\Delta^1_1$—Sierpinski [66], and $\Sigma^1_2 \subseteq \mathbf{S}_{\omega_1}$—Schoenfield [65].

And now one more definition. To each natural number $m$ we assign an ordinal $\delta^1_m$—the upper bound of the lengths of the $\Delta^1_m$-norms (the length of a norm $\varphi$ is taken as the order type of the set of all values assumed by $\varphi$). It is precisely these "projective ordinals" introduced by Moschovakis which constitute the collection of ordinals that make possible a generalization of the theorems of Suslin, Sierpinski, and Schoenfield to higher projective levels.

THEOREM (AD). *Let $n \in \omega$. Then*

*(a) $\delta^1_{2n+1}$ and $\delta^1_{2n+2}$ are cardinals, and $\delta^1_{2n+2} = (\delta^1_{2n+1})^+$ (the symbol $\lambda^+$ denotes the cardinal following the cardinal $\lambda$). Moreover there exists a (unique, of course) cardinal $\varkappa_{2n+1}$ such that $\delta^1_{2n+1} = \varkappa^+_{2n+1}$.*

*(b) $\Delta^1_{2n+1} = \mathbf{B}_{\delta^1_{2n+1}}$ and $\Sigma^1_{2n+1} = \mathbf{S}_{\varkappa_{2n+1}}$.*

*(c) $\Sigma^1_{2n+2} = \bigcup_{\delta^1_{2n+1}} \Delta^1_{2n+1} = \mathbf{S}_{\delta^1_{2n+1}}$.*

The research that resulted in this theorem was begun in the work of Moschovakis [54] and then continued by Kunen, Martin, and Kechris; the final result appeared in the article [29].

Some quite simple computations (see, for example, [29]) show that $\delta^1_1 = \omega_1$ and $\varkappa_1 = \omega$. Hence it is clear that for $n = 0$ assertion (b) of the theorem just stated reduces to the results of Suslin (and here the hypothesis AD is no longer needed), while assertion (c) even strengthens the results of Sierpinski and Schoenfield, turning the inclusions contained in them into exact equalities.

In a projectively determinate universe (more precisely, using the hypothesis $\Sigma^1_{2n}$-Det) it is possible to prove only the left-to-right inclusion in (b) and (c). It is as yet unknown whether all the ordinals $\delta^1_m$ are necessarily cardinals under the assumption PD, but it follows from $\Sigma^1_{2n}$-Det that there exists a cardinal $\varkappa_{2n+1}$ such that $\varkappa_{2n+1} < \delta^1_{2n+1} \leqslant \varkappa^+_{2n+1}$.

The problem of the location of the cardinals $\delta^1_m$ in the series of cardinals in a completely determinate universe presents considerable interest. We have already pointed out that $\delta^1_1 = \omega_1$ and $\varkappa_1 = \omega$. Further it has been established that $\delta^1_2 = \omega_2$, $\varkappa_3 = \omega_\omega$, $\delta^1_3 = \omega_{\omega+1}$, and $\delta^1_4 = \omega_{\omega+2}$ under the hypothesis AD [29, 58]. Quite recently the computation of the following "triad" was carried out: $\varkappa_5 = \omega_\lambda$, where $\lambda = \omega^{(\omega^\omega)}$, and consequently $\delta^1_5 = \omega_{\lambda+1}$ and $\delta^1_6 = \omega_{\lambda+2}$. This is all reported in [36, conclusion].

It is interesting that if we take the axiom of projective determinacy PD and the full axiom of choice AC instead of AD, different relations are obtained for the initial projective ordinals: $\delta^1_2 \leqslant \omega_2$, $\varkappa_3 \leqslant \omega_2$, $\delta^1_3 \leqslant \omega_3$, $\delta^1_4 \leqslant \omega_4$. Moreover it follows from PD + AC that $\Sigma^1_3 \subseteq \bigcup_{\omega_2} \Delta^1_1$ and $\Sigma^1_4 \subseteq \bigcup_{\omega_3} \Delta^1_1$. We note that the first of these inclusions also follows from the hypothesis $\Sigma^1_1$-Det, and then (see §2) also from the axiom of the existence of a measurable cardinal.

All these results (with proofs and references to the original works) are expounded in the survey [29] and the book [58].

Of the later works we note the investigations [32] and [27]. In the article [32] Suslin cardinals are studied—any infinite cardinal $\varkappa$ such that $\mathbf{S}_\varkappa \not\subseteq \bigcup_{\lambda<\varkappa} \mathbf{S}_\lambda$ is called a Suslin cardinal. It is proved that under the hypothesis AD the first $\omega$ Suslin cardinals form the series $\varkappa_1, \delta^1_1, \varkappa_3, \delta^1_3, \varkappa_5, \delta^1_5, \ldots$, and the Suslin classes $\mathbf{S}_\varkappa$ corresponding to these cardinals form respectively the series of projective classes $\Sigma^1_m$, $m \geqslant 1$. The subsequent Suslin cardinals turn out to be connected with the classes of the so-called hyperprojective hierarchy.

The article [27] contains some applications of games on ordinals, i.e., games in which the moves can be ordinals less than some infinite ordinal $\lambda$ fixed in advance (the games in §2 correspond to the case $\lambda = \omega$). Even for $\lambda = \omega_1$ it can be shown without using the axiom of choice that there exist indeterminate games, but some important types of games on the projective ordinals $\lambda = \delta^1_m$ turn out to be determinate. Hence arises a variety of interesting applications to "projective" subsets of projective ordinals. Projectivization is achieved here as follows. Let $m \geqslant 1$. Assuming PD, we can construct a $\Pi^1_m$-norm $\phi : Z$ onto $\delta^1_m$ defined on a suitable $\Pi^1_m$-set $Z \subseteq \mathcal{N}$. Now to each set $X \subseteq \delta^1_m$ we assign its "code" $\{\alpha \in Z : \varphi(\alpha) \in X\}$, by whose membership in some projective class we define the membership of the set $X$ in the same class. We can study the closure of the classes so obtained, consisting of sets of ordinals, with respect to various operations, and other questions. For details see [27].

## §8. Some applications of Borel games

It is natural that the theorem on the determinacy of Borel games studied in §2 above should be applied primarily to study Borel sets themselves. By its use it became possible to clarify some important questions on the properties of these sets.

We recall the construction of the Borel hierarchy. It is formed by the Borel classes $\Sigma^0_\xi$, $\Pi^0_\xi$, and $\Delta^0_\xi$, where $1 \leqslant \xi < \omega_1$. The initial class $\Sigma^0_1$ is formed by all open sets of the space under consideration. For any $\xi$ the class $\Pi^0_\xi$ consists of the complements of $\Sigma^0_\xi$-sets and $\Delta^0_\xi$ is the intersection of the classes $\Sigma^0_\xi$ and $\Pi^0_\xi$. Finally if $\xi \geqslant 2$, the class $\Sigma^0_\xi$ contains all countable sums of sets belonging to the classes $\Pi^0_\eta$, where $1 \leqslant \eta < \xi_1$. Every Borel set belongs to one of the Borel classes, and hence to all classes with large indices, since a growth condition holds: $\Sigma^0_\xi \cup \Pi^0_\xi \subset \Delta^0_\varsigma$ for $\xi < \varsigma$; as it happens the inclusion here is strict.

The Borel hierarchy allows us to classify the Borel sets according to their complexity: The simpler sets are those occurring earlier. It is appropriate to make use of the following definitions. A set belonging to the class $\Sigma^0_\xi$ but not to $\Pi^0_\xi$ is called a *strictly* $\Sigma^0_\xi$*-set*. In exactly the same way we can introduce the concept of a strictly $\Pi^0_\xi$-set (it belongs to $\Pi^0_\xi$ but not to $\Sigma^0_\xi$) and a strictly $\Delta^0_\xi$-set (it belongs to $\Delta^0_\xi$ but not to any class $\Sigma^0_\eta$ or $\Pi^0_\eta$ with $\eta < \xi$). Thus every Borel set is either a strictly $\Sigma^0_\xi$-set or a strictly $\Pi^0_\xi$-set or a strictly $\Delta^0_\xi$-set for some (unique) ordinal $\xi < \omega_1$. Now, for example, a strictly $\Sigma^0_5$-set can be regarded as more complicated than a strictly $\Pi^0_4$-set, and we shall say that two strictly $\Sigma^0_\xi$-sets (with the same $\xi$) are to be regarded as equally complicated.

But here a natural question arises: Is there any internal relation between point sets corresponding to the "hierarchical" complexity just described? The most natural idea appears to be to study homeomorphy. From results of Lavrent'ev [41] it follows that for $\xi \geqslant 3$ (and for the class $\Pi^0_\xi$ also for $\xi = 2$) every point set homeomorphic to a strictly $\Sigma^0_\xi$-set [resp. $\Pi^0_\xi$-set, $\Delta^0_\xi$-set] will itself be a set of the same type. Conversely, will any two, say, strictly $\Sigma^0_\xi$-sets be homeomorphic to each other? For the classes $\Sigma^0_\xi$ and $\Pi^0_\xi$ no definitive answer has yet been obtained; however, there are some very important partial results. Steel [72] has established that for $\xi \geqslant 3$ any two strictly $\Sigma^0_\xi$-sets (and also any two strictly $\Pi^0_\xi$-sets) of first category are homeomorphic if they remain strictly $\Sigma^0_\xi$-sets [resp. $\Pi^0_\xi$-sets] under intersection with every Baire interval. At the same time, as noted in [34, §5], for $\xi \geqslant 3$ there is an isomorphism of class 1 between any two strictly $\Sigma^0_\xi$-sets (or strictly $\Pi^0_\xi$-sets) $X$ and $Y$, i.e., a one-to-one mapping preserving the class $F_\sigma$ (a weaker requirement than preserving openness, which occurs in the definition of a homeomorphism). Both these results use the theorem of Borel determinacy.

As we can see, the "strict" classes $\Sigma^0_\xi$ and $\Pi^0_\xi$ are at least close to being topologically homogeneous. On the other hand the "strict" $\Delta$-classes are essentially nonhomogeneous, at least for nonlimit indices. Lavrent'ev pointed out [42] that for $\xi \geqslant 1$ the strictly $\Delta^0_{\xi+1}$-sets decompose into $\aleph_1$ nonempty subclasses in such a way that sets of different subclasses are not mutually homeomorphic (and not isomorphic in the sense of an isomorphism of class 1).

Altogether the topological classification of the Borel sets remains an open problem. The situation is significantly better with another classification, introduced by Wadge. Let $X$ and $Y$ be sets located in the point spaces $\mathcal{X}$ and $\mathcal{Y}$. We write $X \leqslant_W Y$ (the Wadge ordering) when there exists a continuous function $F : \mathcal{X} \to \mathcal{Y}$ such that $X = F^{-1}(Y)$. We also introduce the corresponding equivalence relation: $X \sim_W Y$ when $X \leqslant_W Y$ and $Y \leqslant_W X$. With each point set $X$ we associate its Wadge degree $[X] = \{Y : Y \sim_W X\}$ and the modified degree $[X]^* = [X] \cup [CX]$, where $CX$ is the complement of the set $X$. The degrees and modified degrees are ordered in the following natural manner:

$$[X] < [Y], \quad \text{when} \quad [X] \neq [Y] \quad \text{and} \quad X \leqslant_W Y;$$

$$[X]^* < [Y]^*, \quad \text{when} \quad [X] < [Y] \quad \text{or} \quad [X] < [CY].$$

Using a very simple game (see [3, Ch. 8, §6]) Wadge proved that in a completely determinate universe either $X \leqslant_W Y$ or $Y \leqslant_W CX$ holds for any pair of point sets $X$, $Y$. In other words from AD there follows a linear ordering, and—as was established by Martin—also a wellordering of the modified Wadge degrees of point sets (see [78]). The axiom PD is sufficient for a wellordering of the modified Wadge degrees of projective sets, and the theorem of Borel determinacy guarantees a wellordering of the modified degrees of the Borel sets.

In this ordering all strictly $\Sigma^0_\xi$-sets and all strictly $\Pi^0_\xi$-sets fall into a single modified degree (depending on $\xi$), consisting of two different Wadge degrees—the degrees of the strictly $\Sigma^0_\xi$-sets and the degrees of the strictly $\Pi^0_\xi$-sets. In contrast the strictly $\Delta^0_\xi$-sets (for any fixed $\xi$) form uncountably many Wadge degrees (for nonlimit $\xi$ this is revealed already by the Lavrent'ev subclasses) whose structure and methods of construction were studied by van Wesep [78].

Every projective class (rule 2 of §1) and every Borel class $\Gamma$ possesses the following Wadge closure property: If $X \in \Gamma$ and $Y \leqslant_W X$, then $Y \in \Gamma$. The structure of Wadge closed classes consisting of Borel sets only was studied in [44].

Borel games have found very interesting applications in the theory of $C$-sets and $R$-sets. These types of point sets have been intensively studied by Selivanovskii, Kantorovich, and Livenson [28], by Novikov and Lyapunov [9], and by others[7] using the methods of the theory of operations on sets developed by Kolmogorov [7]. It has been discovered [28, 9] that $C$-sets form a proper subset of the $R$-sets, and that the latter all belong to the projective class $\Delta^1_2$. Moreover Kolmogorov proved a theorem (see [9, Introduction]) to the effect that all $R$-sets are absolutely measurable and have the Baire property—the best classical result for these two regularity properties.

Recently Burgess [19] has shown that $C$-sets and $R$-sets can be obtained by the action of a game-operator:

$$\{C\text{-sets}\} = \mathfrak{D}\Delta^0_2, \qquad \{R\text{-sets}\} = \mathfrak{D}\Delta^0_3$$

and $\mathfrak{D}\Delta^0_1 = \Delta^1_1$ is the class of Borel sets. Thus the Borel sets, the $C$-sets, and the $R$-sets form the first three steps in the hierarchy of classes $\mathfrak{D}\Delta^0_\xi$, $1 \leqslant \xi < \omega_1$. The union of all these classes gives the class $\mathfrak{D}\Delta^1_1 (\subset \Delta^1_2)$; and the classes expand with increasing $\xi$: $\mathfrak{D}\Delta^0_\xi \subset \mathfrak{D}\Delta^0_\varsigma$ for $\xi < \varsigma$. We note that all sets belonging to $\mathfrak{D}\Delta^1_1$ have the Baire property (theorem 2 of §3 plus the theorem of Borel determinacy) and are absolutely measurable (an assertion analogous to theorem 2 of §3 holds also for measurability).

For applications of Borel games to the theory of operations on sets itself see the article [64].

The determinacy of Borel sets has found applications also in the study of the properties of equivalence relations. Silver showed in [68] that every $\Pi^1_1$-equivalence relation (i.e., a relation whose graph $\{\langle x, y\rangle :$ $x$ equivalent to $y\}$ belongs to $\Pi^1_1$) on any point space either has an at most countable number of equivalence classes or admits a perfect set (which therefore has cardinality of the continuum) consisting of of pairwise inequivalent elements. Relations of the first form are appropriately called countable, those of the second form, continuous. Burgess [17] has established that for $\Sigma^1_1$-equivalence relations another possibility arises: a noncontinuous relation having exactly $\aleph_1$ equivalence classes. The rather complicated calculations of Silver and Burgess include the use of the theorem of Borel determinacy.

Nonborel games also find interesting applications in the study of equivalence relations. For example, Stern has discovered (see [75]) that in a projectively determinate universe every projective equivalence relation all of whose equivalence classes are Borel sets of bounded rank (i.e., all belong to some one Borel class $\Delta^0_\xi$, $\xi < \omega_1$) is either countable or continuous. In [76] games with game sets of class $\Pi^1_1$ are used to study $\Sigma^1_1$-equivalence relations (in particular, possessing the indicated property of boundedness in rank).

## LITERATURE CITED


1. P. S. Aleksandrov, *Introduction to Set Theory and General Topology* [in Russian], Nauka, Moscow (1977).


[7]For a survey of the results on $C$-sets, $R$-sets, and the theory of operations, see [9, 10], [6, §5].

2. V. Ya. Arsenin and A. A. Lyapunov, "The theory of $A$-sets," Usp. Mat. Nauk, **5**, No. 5(39), 45–108 (1950).
3. J. Barwise, ed., *Handbook of Mathematical Logic. Part II: Set Theory* North-Holland Pub. Co., Amsterdam–New York (1977).
4. V. G. Kanovei, "N. N. Luzin's projective hierarchy: the current status of the theory," in: *Handbook of Mathematical Logic. Part II, Set Theory* [Russian translation], Nauka, Moscow, (1982), pp. 273–364.
5. V. G. Kanovei, *The Axiom of Choice and the Axiom of Determinacy* [in Russian], Nauka, Moscow (1984).
6. V. G. Kanovei, "The development and current status of descriptive set theory in the light of the works of N. N. Luzin," Usp. Mat. Nauk, **40**, No. 3(243), 117–155 (1985).
7. A. N. Kolmogorov, "On operations on sets," Mat. Sb. **35**, No. 3–4, 414–422 (1928).
8. N. N. Luzin, *On Some New Results in Descriptive Function Theory* [in Russian], Akad. Nauk SSSR, Moscow (1935).
9. A. A. Lyapunov, "$R$-sets," Tr. Mat. Instituta im. V. A. Steklova Akad. Nauk SSSR, **40**, 1–68 (1953).
10. A. A. Lyapunov and P. S. Novikov, "Descriptive set theory," in: *Mathematics in the USSR after 30 Years* [in Russian], Gostekhizdat, Moscow, (1948), pp. 243–255.
11. P. S. Novikov, "On the consistency of some propositions of descriptive set theory," Tr. Mat. Instituta im. V. A. Steklova Acad. Nauk SSSR, **38**, 279–316 (1951).
12. H. Rogers, *Theory of Recursive Functions and Effective Computability*, McGraw-Hill, New York (1967).
13. V. A. Uspenskii "N. N. Luzin's contribution to the descriptive theory of sets and functions: concepts, problems, predictions," Usp. Mat. Nauk, **40**, No. 3(243), 85–116 (1985).
14. J. W. Addison, "Separation principles in the hierarchies of classical and effective descriptive set theory," Fund. Math., **46**, No. 2, 123–135 (1958).
15. J. W. Addison and Y. M. Moschovakis, "Some consequences of the axiom of definable determinateness," Proc. Nat. Acad. Sci. USA, **59**, No. 3, 708–712 (1968).
16. D. Blackwell, "Infinite games and analytic sets," Proc. Nat. Acad. Sci. USA, **58**, No. 6, 1836–1837 (1967).
17. J. Burgess, "Effective enumeration of classes of $\Sigma^1_1$ equivalence relations," Indiana Univ. Math. J., **28**, No. 3, 353-364 (1979).
18. J. Burgess, "Careful choices: a last word on Borel selectors," Notre Dame J. Form. Log., **22**, No. 3, 219–226 (1981).
19. J. Burgess, "What are $R$-sets?" Patras Logic Symposium. Proc. Log. Symp., Patras, 18–22 Aug. 1980, 307–324, Amsterdam (1982).
20. D. Cenzer and R. Mauldin, "Inductive definability: measure and category," Adv. Math., **38**,No. 1, 55–90 (1980).
21. D. Cenzer and R. Mauldin, "Borel equivalence and isomorphism of coanalytic sets," Rozpr. Math., No. 228 (1984).
22. M. Davis, "Infinite games of perfect information," Ann. Math. Stud., **52**, 85–101 (1964).
23. H. Friedman, "Higher set theory and mathematical practice," Ann. Math. Log., **2**, No. 3, 326–357 (1971).
24. D. Gale and F. Stewart, "Infinite games with perfect information," Ann. Math. Stud., **2**, No. 3, 245–256 (1953).
25. D. Guaspari, "Trees, norms, and scales," London Math. Soc. Lecture Notes Series, No. 87, 135–161 (1983).
26. L. A. Harrington, "Analytic determinacy and $0^{\#}$," J. Symbolic Log., **43**, No. 4, 685–693 (1978).
27. L. A. Harrington and A. S. Kechris, "On the determinacy of games on ordinals," Ann. Math. Log., **20**, No. 2, 109–154 (1981).
28. L. Kantorovitch and E. Livenson, "Memoir on the analytical operations and projective sets," Fund. Math., **18**, 214–279 (1932); **20**, 54–97, (1933).
29. A. S. Kechris, "AD and projective ordinals," Lecture Notes Math., **689**, 91–132 (1978).

30. A. S. Kechris, "On transfinite sequences of projective sets with an application to $\Sigma^1_2$ equivalence relations," Logic Colloquium **77**, 155–160, Amsterdam (1978).
31. A. S. Kechris, "Forcing in analysis," Lect. Notes Math., **669**, 277–302 (1978).
32. A. S. Kechris, "Souslin cardinals, $\varkappa$-souslin sets and the scale property in the hyperprojective hierarchy," Lect. Notes Math., **839**, 127–146 (1981).
33. A. S. Kechris, "The axiom of determinacy implies dependent choices in $L(R)$," J. Symbol. Log., **49**, No. 1, 161–175 (1984).
34. A. S. Kechris and D. A. Martin, "Infinite games and effective descriptive set theory," in: *Analytic Sets*, London, (1980), 403–470.
35. A. S. Kechris, D. A. Martin, and Y. M. Moschovakis, eds., Cabal Seminar 77–79, Lect. Notes Math., **839**, 1–274 (1981).
36. A. S. Kechris, D. A. Martin, and Y. M. Moschovakis, eds., Cabal Seminar 79–81, Lect. Notes Math., 1–288 (1983).
37. A. S. Kechris and Y. M. Moschovakis, eds., Cabal Seminar 76–77, Lect. Notes Math., **689**, 1–322 (1978).
38. E. M. Kleinberg, "Infinitary combinatorics and the axiom of determinateness," Lect. Notes Math., **612**, 1–150 (1977).
39. M. Kondô, "Sur l'uniformisation des complémentaires analytiques et les ensembles projectifs de seconde classe," Jap. J. Math., **15**, 197–230 (1938).
40. K. Kuratowski, "Sur les théorèmes de séparation dans la théorie des ensembles," Fund. Math., **26**, 183–191 (1936).
41. M. Lavrent'ev (M. Lavrentieff), "Contribution à la théorie des ensembles homéomorphes," Fund. Math., **6**, 149–160 (1924).
42. M. Lavrent'ev (M. Lavrentieff), "Sur les sous-classes et la classification de M. Baire," C. R. Acad. Sci. Paris, **180**, 111–114 (1925).
43. A. Louveau, "A separation theorem for $\Sigma^1_1$ sets," Trans. Amer. Math. Soc., **260**, No. 2, 363–378 (1980).
44. A. Louveau, "Some results in the Wadge hierarchy of Borel sets," Lect. Notes Math., **1019**, 28–55 (1983).
45. N. Luzin (N. Lusin), "Sur la classification de M. Baire," C. R. Acad. Sci. Paris, **164**, 91–94 (1917).
46. N. Luzin (N. Lusin), "Sur les ensembles projectifs de M. Henri Lebesgue," C. R. Acad. Sci. Paris, **180**, 1572–1574 (1925).
47. N. Luzin (N. Lusin), "Sur le problème de M. Hadamard d'uniformisation des ensembles," Mathematica, Cluj, 54–66 (1930).
48. N. Luzin (N. Lusin), *Leçons sur les ensembles analytiques et leurs applications*, Paris (1930).
49. N. Luzin and P. Novikov (N. Lusin and P. Novikoff), "Choix effectif d'un point dans un complémentaire analytique donné par un crible," Fund. Math., **25**, 559–560 (1935).
50. R. Mansfield, "A Souslin operation for $\Pi^1_2$," Isr. J. Math., **9**, No. 3, 367–369 (1971).
51. D. A. Martin, "The axiom of determinateness and reduction principles in the analytical hierarchy," Bull. Amer. Math. Soc., **74**, No. 4, 687–689 (1968).
52. D. A. Martin, "Measurable cardinals and analytic games," Fund. Math., **66**, No. 3, 287–291 (1970).
53. D. A. Martin, "Borel determinacy," Ann. Math., **102**, No. 2, 363–371 (1975).
54. Y. M. Moschovakis, "Determinacy and prewell orderings of the continuum," Math. Logic and Foundations of Set Theory, Amsterdam–London, 24–62 (1970).
55. Y. M. Moschovakis, "Uniformization in a playful universe," Bull. Amer. Math. Soc., **77**, No. 5, 731–736 (1971).
56. Y. M. Moschovakis, "The game quantifier," Proc. Amer. Math. Soc., **31**, No. 1, 245–250 (1972).
57. Y. M. Moschovakis, "Analytical definability in a playful universe," in: *Logic, Methodology and Philosophy of Science*, vol. IV, Amsterdam, (1973), pp. 77–85.
58. Y. M. Moschovakis, *Descriptive Set Theory*, Amsterdam (1980).
59. J. Mycielski, "On the axiom of determinateness," Fund. Math., **53**, No. 2, 205–224 (1964); **59**, No. 2, 203–212 (1966).

60. J. Mycielski and H. Steinhaus, "A mathematical axiom contradicting the axiom of choice," Bull. Acad. pol. sci. Sér. sci. math., astron. et phys., **10**, No. 1, 1–3 (1962).
61. J. Mycielski and S. Swierczkowski, "On the Lebesgue measurability and the axiom of determinateness," Fund. Math., **54**, No. 1, 67–71 (1964).
62. P. Novikov (P. Novikoff), "Sur les fonctions implicites mesurables $B$," Fund. Math., **17**, 8–25 (1931).
63. P. Novikov (P. Novikoff), "Sur la séparabilité des ensembles projectifs de seconde classe," Fund. Math., **25**, 459–466 (1935).
64. K. Shilling and R. L. Vaught, "Borel games and the Baire property," Trans. Amer. Math. Soc., **279**, No. 1, 411–428 (1983).
65. J. Shoenfield, "The problem of predicativity," in: *Essays on the Foundations of Mathematics*, Jerusalem, (1961), pp. 132–139.
66. W. Sierpinski, "Sur une classe d'ensembles," Fund. Math., **7**, 237–243 (1925).
67. J. Silver, "Some applications of model theory in set theory," Ann. Math. Log., **3**, No. 1, 45–110 (1970).
68. J. Silver, "Counting the number of equivalence classes of Borel and coanalytic equivalence relations," Ann. Math. Log., **18**, No. 1, 1–28 (1980).
69. R. M. Solovay, "A model of set theory in which every set of reals is Lebesgue measurable," Ann. Math. **92**, No. 1, 1–56 (1970).
70. R. M. Solovay, "The independence of DC from AD," Lect. Notes Math., **689**, 171–184 (1978).
71. M. Suslin (M. Souslin), "Sur une définition des ensembles mesurables $B$ sans nombres transfinis," C. R. Acad. Sci. Paris, **164**, 89–91 (1917).
72. J. R. Steel, "Analytic sets and Borel isomorphisms," Fund. Math., **108**, No. 2, 83–88 (1980).
73. J. R. Steel, "Determinateness and the separation property," J. Symbol. Log., **46**, No. 1, 41–44 (1981).
74. J. R. Steel, "Scales on $\Sigma_1^1$ sets," Lect. Notes Math., **1019**, 72–76 (1983).
75. J. Stern, "Effective partitions of the real line into Borel sets of bounded rank," Ann. Math. Log., **18**, No. 1, 29–60 (1980).
76. J. Stern, "Analytic equivalence relations and coanalytic games," Patras Logic Symp., Proc. Log. Symp., Patras, 18–22 Aug., 1980, 239–260, Amsterdam (1982).
77. H. Tanaka, "Recursion theory in analytical hierarchy," Comment. Math. Univ. St. Pauli, **27**, No. 2, 113–132 (1978).
78. R. van Wesep, "Wadge degrees and descriptive set theory," Lect. Notes Math., **689**, 151–170 (1978).