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ON EXTERNAL SCOTT ALGEBRAS IN NONSTANDARD MODELS OF PEANO ARITHMETIC

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Abstract. We prove that a necessary and sufficient condition for a countable set $\mathscr{Z}$ of sets of integers to be equal to the algebra of all sets of integers definable in a nonstandard elementary extension of ω by a formula of the **PA** language which may include the standardness predicate but does not contain nonstandard parameters, is as follows: $\mathscr{Z}$ is closed under arithmetical definability and contains $0^{(\omega)}$, the set of all (Gödel numbers of) true arithmetical sentences.

Some results related to definability of sets of integers in elementary extensions of ω are included.

CONTENTS

Introduction. One of the questions which usually appear in study of definability is the question of the nature of the set of all objects definable in the sense of some fixed notion of definability. The following two results are well-known: the set of all arithmetically definable sets $X \subseteq \omega$ is Δ^1_1 but not arithmetically definable; the set of all Δ^1_1 sets $X \subseteq \omega$ is Π^1_1 but not Δ^1_1. More complicated notions of definability may lead to independence theorems. For example, the set of all *analytically* definable subsets of ω is not analytically definable in the constructible model, but is Σ^1_2 in a generic extension of L, see Harrington [1] and Kanovei [3].

Nonstandard models of **PA** lead to notions of definability other than those considered in descriptive set theory.

D. Scott gave in [5] a necessary and sufficient condition for a countable set $\mathscr{Z} \subseteq \mathscr{P}(\omega)$ to be equal to SA^M, the *Scott algebra* of a **PA** model M, that is, the family of all sets $Z \subseteq \omega$ definable in M by a parameter-free **PA** formula.

It is evident that SA^M is exactly the collection of all arithmetical $Z \subseteq \omega$ provided M is an elementary extension of ω. One may, however, extend the **PA** language $\mathscr{L}$ by an additional unary predicate, the *predicate of standardness* st

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interpreted, in all nonstandard models of **PA**, as being a member of ω. (Model theoretically, this means that we consider the structure $\langle M; \omega\rangle$ rather than M.) [1]

Let ESA^M, the *external Scott algebra* of M, denote the family of all subsets of ω definable in M by a parameter-free formula of the extended language $\mathscr{L}^{\mathrm{st}}$. We give necessary and sufficient conditions for a countable family of subsets of ω to be the external Scott algebra of an elementary extension of ω.

THEOREM 1. *Let $\mathscr{X} \subseteq \mathscr{P}(\omega)$ be countable. Conjunction of the following two conditions is necessary and sufficient for there to exist a countable $M \models$ **PA**, $M \succ \omega$, such that $\mathrm{ESA}^M = \mathscr{X}$:*

1. *$\mathscr{X}$ is arithmetically closed.*
2. *$\mathscr{X}$ contains $0^{(\omega)}$, the set of all (Gödel numbers of) true arithmetical sentences.*

Necessity. The necessity part in this theorem is quite easy. Suppose that $\mathscr{X} = \mathrm{ESA}^M$ for a countable $M \succ \omega$. Condition 1 is entirely obvious: indeed, one can distinguish ω in M by the predicate st. Condition 2 needs some care.

Let us fix a recursive coding of finite sets of natural numbers by natural numbers, so that S_j denotes the set coded by j. Let T_n be the set of all Gödel numbers of true (in ω) Σ_n sentences of $\mathscr{L}$. Let finally $\tau(J, n)$ be the $\mathscr{L}^{\mathrm{st}}$ formula which says that n is a standard number, J is a nonstandard number, and the ("hyper")finite set $S = S_J$ coded by J satisfies $S \cap \omega = T_n$. (To be more precise, τ is the conjunction of Tarski rules restricted to level Σ_n and below and relativized to ω by the predicate st.) Since $0^{(\omega)} = \bigcup_n T_n$, it remains to check that for any $n \in \omega$ there exists $J \in M$ such that $S_J \cap \omega = T_n$.

Notice that T_n is an arithmetical set. Therefore (we use the assumption $M \succ \omega$) $T_n = X \cap \omega$, where $X \subseteq M$ is definable in M by a formula of $\mathscr{L}$. We take an arbitrary nonstandard $H \in M$, put $S = \{J \in X : J < H\}$—then $S = S_J$ for some $J \in M$, and note finally that $T_n = S \cap \omega$, as required. ⊣

Sufficiency. This is the hard part of the theorem. To make every $Z \in \mathscr{X}$ definable in M by a formula of the extended language, we use a coding system, which also gives an instrument to prove a definability theorem similar to some definability theorems of Harrington [1] and Kanovei [3] in the domain of **ZFC** models.

THEOREM 2. *Let $Z \subseteq \omega$, $n \in \omega$. There exists a countable model $M \models$ **PA**, $M \succ \omega$, such that Z is $\Sigma^{\mathrm{st}}_{n+2}$ in M but every $Y \subseteq \omega$, $Y \in \Sigma^{\mathrm{st}}_{n+1}$ in M, is $\Sigma_{n+1}[0^{(\omega)}]$.* [2]

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[1] The author was acquainted with the theorem of Scott by Ali Enayat in September, 1993. Since the author had some experience in nonstandard set theories, where the standardness predicate is one of the principal notions, the idea to study the definability in the sense of the extended language was found very natural. It was soon discovered in discussions between A. Enayat and the author that conditions 1 and 2 are necessary in this case. The proof of their sufficiency takes much more effort.

[2] By Σ^{st}_n we denote the class of all $\mathscr{L}^{\mathrm{st}}$ formulas having Σ_n prefix, followed by a formula composed from recursive formulas by &, $\bigvee$, and quantifiers $\exists^{\mathrm{st}}$ and $\forall^{\mathrm{st}}$. See Section 1 below. $0^{(\omega)}$ in square brackets means that $0^{(\omega)}$ can participate as an extra parameter.

Take notice that $0^{(\omega)}$ itself is $\Delta_1^{\rm st}$ in M; moreover, any $Y \subseteq \omega$, $Y \in \Sigma_{n+1}[0^{(\omega)}]$ is $\Sigma_{n+1}^{\rm st}$ in any **PA** model $M \succ \omega$. Thus Theorem 2 tells that a set Z of natural numbers can be $\Sigma_{n+2}^{\rm st}$ but not $\Sigma_{n+1}^{\rm st}$ in an appropriate **PA** model $M \succ \omega$ unless it is essentially simple—belongs to $\Sigma_{n+1}[0^{(\omega)}]$.

Open problem. We would be interested to prove Theorem 2 also in the case $n = -1$. A more easy (?) question is as follows: Find a model M, $\omega \prec M$, such that there exists a $\Sigma_1^{\rm st}$ in M set $Z \subseteq \omega$ which is not arithmetical. (Notice that $Z \in \Sigma_0^{\rm st}$ in a model $M \succ \omega$ iff Z is arithmetical, and this implies that Z is recursive in $0^{(\omega)}$, i.e., $\Delta_0[0^{(\omega)}]$.)

The proof of both Theorem 1 and Theorem 2 is based on several forcing ideas. The first principal idea is to reduce the definability questions from nonstandard models to second order structures $\langle \omega; \mathscr{F} \rangle$, where $\mathscr{F} \subset \omega^\omega$ is a countable arithmetically closed set. It is proved in Section 1 that for any such set $\mathscr{F}$ there exists a Peano model $M \succ \omega$ such that, for all $n \geq 1$ and $X \subseteq \omega$, X is $\Sigma_n^{\rm st}$ in M iff X is Σ_n^1 in $\mathscr{F}$ (in the sense of relativization of all quantifiers of type ω^ω to $\mathscr{F}$).

This model M is constructed as an ultrapower of the form $M = \mathrm{Ult}_{\mathscr{U}}\mathscr{F}$, where $\mathscr{U}$ is a *generic* in some sense ultrafilter in the algebra of all infinite sets $X \subseteq \omega$ arithmetical in some $\gamma \in \mathscr{F}$. By a kind of Łoś Theorem, a consequence of the genericity of $\mathscr{U}$, definability in M can be expressed in terms of the corresponding forcing relation, that is, as $\exists X \in \mathscr{U}$ (X forces ...).

Furthermore it occurs that all forcing conditions force the same parameter-free formulas (to prove this we use a system of permutations of the forcing). Thus, as long as only parameter-free formulas are considered, expressions "$\exists X \in \mathscr{U}$ (X forces ...)" can be replaced by "ODD forces ... ," where ODD is the set of all odd numbers. This is how $\mathscr{U}$ is finally eliminated and the reduction to definability in $\mathscr{F}$ is made.

Thus Theorem 2 converts to the following form: find a countable arithmetically closed set $\mathscr{F} \subseteq \omega^\omega$ such that Z is Σ_{n+2}^1 in $\mathscr{F}$, but every set $Y \subseteq \omega$, $Y \in \Sigma_{n+1}^1$ in $\mathscr{F}$, is $\Sigma_{n+1}[0^{(\omega)}]$. To define such a set $\mathscr{F}$, we use in Section 2 a $\Sigma_n[0^{(\omega)}]$ generic and $\Delta_{n+1}[0^{(\omega)}]$ definable function $\alpha \in 2^\omega$ which splits in a natural way in a sequence of functions $(\alpha)_z \in 2^\omega$, $z \in \omega$. In particular α is arithmetically (Σ_m for all m) generic, therefore the collection $\mathscr{F} = \mathscr{F}(\alpha, Z)$ of all functions $\gamma \in \omega^\omega$ arithmetical in a finite number of functions $(\alpha)_z$, $z \in Z$, does not contain any of $(\alpha)_z$, $z \notin Z$. This allows to obtain a Σ_{n+2}^1 definition of Z in $\mathscr{F}$.

Second important corollary of the genericity is as follows: $\mathscr{F}$ is an elementary submodel of $\mathscr{F}(\alpha, \omega)$—the set of all functions $\gamma \in \omega^\omega$ arithmetical in a finite number of functions $(\alpha)_z$, $z \in \omega$—with respect to Σ_{n+1}^1 formulas. This implies, in particular, that every $Y \subseteq \omega$ of class Σ_{n+1}^1 in $\mathscr{F}$ is of class Σ_{n+1}^1 in $\mathscr{F}(\alpha, \omega)$ as well, therefore is $\Sigma_{n+1}[0^{(\omega)}]$ since the set $\mathscr{F}(\alpha, \omega)$, unlike $\mathscr{F}$, admits an enumeration recursive in $0^{(\omega)}$. (An enumeration of $\mathscr{F}$ would involve Z, that we want to avoid.)

This is how Theorem 2 is proved. To prove the sufficiency part of Theorem 1 in Section 3 for a given countable $\mathscr{Z} = \{Z_n : n \in \omega\}$, we make every Z_n definable by the method used for Theorem 2 at the corresponding level n.

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§1. Reduction to standard second order definability. We prove in this section that for any countable arithmetically closed set $\mathscr{F} \subseteq \omega^\omega$ there exists a nonstandard **PA** model M which satisfies the property that $\mathscr{L}^{\mathrm{st}}$ definability of subsets of ω in M is quite the same as definability in $\mathscr{F}$. To formulate this statement correctly, some definitions are necessary. The first subsection presents both standard notations, included to make the exposition more or less self-contained, and some more special definitions related to formulas containing the standardness predicate, taken partially from papers on nonstandard analysis.

Formulas and definability.

Peano arithmetic. $\mathscr{L}$ is the first order language of Peano Arithmetic **PA**. In particular, $\mathscr{L}$ is assumed to contain all (symbols for) recursive relations. We shall denote natural numbers and the corresponding variables by small Italic letters, sets of natural numbers by capital Italic letters, finite sets by u, v, w, with indices and primes, of course. Formulas of $\mathscr{L}$ are called *arithmetical* formulas.

Δ_0 *formula* is a recursive formula.

Σ_n *prefix* is a quantifier prefix of the form $\exists x_1 \, \forall x_2 \, \exists x_3 \, \ldots \, \forall (\exists) \, x_n$. The notion of Π_n *prefix* has similar meaning. Σ_n *formula* is a formula of $\mathscr{L}$ which has a Σ_n prefix followed by a Δ_0 formula. The notion of Π_n *formula* has similar meaning.

The standardness predicate. $\mathscr{L}^{\mathrm{st}}$ is the extension of $\mathscr{L}$ by the unary predicate of standardness st interpreted in nonstandard models of **PA** as being an element of ω, the set of all standard natural numbers.

We introduce a hierarchy of $\mathscr{L}^{\mathrm{st}}$ formulas. Δ_0^{st} is the class of formulas obtained from Δ_0 formulas (where st does not occur) by & and $\bigvee$ and quantifiers $\exists^{\mathrm{st}}$, $\forall^{\mathrm{st}}$ which are shortcuts for $\exists$ *standard* and $\forall$ *standard*.[3] For $n \geq 1$, Σ_n^{st} *formula* is a formula of $\mathscr{L}^{\mathrm{st}}$ which has a Σ_n prefix followed by a Δ_0^{st} formula. The notion of Π_n^{st} *formula* has similar meaning.

[3] The absence of negation is implied by some technical reasons related to the definition of forcing below. But obviously $\neg$, would it occur together with the mentioned connectives, could be eliminated since the negation of a Δ_0 formula is Δ_0, too.

Thus st may occur in these formulas only through the quantifiers $\exists^{\mathrm{st}}$ and $\forall^{\mathrm{st}}$. Take notice that any $\mathscr{L}^{\mathrm{st}}$ formula can be transformed to Σ_n^{st} for a suitable n; indeed, one can easily convert every quantifier-free formula to Δ_0^{st} by changing every occurrence of $\mathrm{st}\, x$ to $\exists^{\mathrm{st}} y\,(y = x)$.

Definability in nonstandard models. We say that a $\mathscr{L}^{\mathrm{st}}$ formula $\Phi(z)$, having z as the unique free variable, *defines* a set $Z \subseteq \omega$ in a nonstandard **PA** model M iff it is true for all $z \in \omega$ that $z \in Z \longleftrightarrow M \models \Phi(z)$.

SA^M, the *Scott algebra* of M, is the collection of all $Z \subseteq \omega$ definable in M by an $\mathscr{L}$ formula. ESA^M, the *external Scott algebra* of M, is the collection of all $Z \subseteq \omega$ definable in M by an $\mathscr{L}^{\mathrm{st}}$ formula. *Parameters,* that is, elements of M, are not allowed to enter formulas in the definition of SA^M and ESA^M. In the case when M is an elementary extension of ω, SA^M is equal to the collection of all arithmetical $Z \subseteq \omega$, but may be not equal when $\omega \not\prec M$, as it follows from the theorem of Scott [5]. On the other hand, Theorem 1 tells that ESA^M cannot be equal to the collection of all arithmetical sets provided $\omega \prec M$; indeed, $0^{(\omega)}$ is not arithmetical.

We say that a set $Z \subseteq \omega$ *is* Σ_n^{st} *in* M iff Z is definable in M by a Σ_n^{st} parameter-free formula. Then $\mathrm{ESA}^M = \bigcup_n \{Z \subseteq \omega : Z \text{ is } \Sigma_n^{\mathrm{st}} \text{ in } M\}$.

Second order formulas and definability. Sometimes we shall admit *second order* variables and parameters; this means that expressions $\alpha(t)$, where α a type ω^ω object or variable, may enter formulas as terms. Formulas obtained this way are called *analytical.* Greek letters are used to denote elements of ω^ω and variables over ω^ω.

Σ_n^1 *prefix* is a quantifier prefix of the form $\exists \alpha_1 \forall \alpha_2 \exists \alpha_3 \ldots \forall(\exists)\, \alpha_n \exists(\forall)\, m$. The notion of Π_n^1 *prefix* has similar meaning. Σ_n^1 *formula* is a formula which has a Σ_n^1 prefix followed by a Δ_0 formula (where free variables of type ω^ω may occur). The notion of Π_n^1 *formula* has similar meaning.

Assume that $\mathscr{F} \subseteq \omega^\omega$. We say that Φ, an analytical sentence having, perhaps, natural numbers and elements of $\mathscr{F}$ as parameters, *is true in* $\mathscr{F}$, $\mathscr{F} \models \Phi$ in brief, if and only if Φ is true provided all quantifiers of the type ω^ω are relativized to $\mathscr{F}$. In other words, this reflects the definability in the 2nd order structure $\langle \omega; \mathscr{F} \rangle$.

A set $X \subseteq \omega$ *is* Σ_n^1 *in* $\mathscr{F}$ iff there exists a Σ_n^1 parameter-free formula which defines X in $\mathscr{F}$.

Reduction to standard systems. Let $\omega^{<\omega} = \{s_k : k \in \omega\}$ be a fixed once and for all recursive enumeration of the set $\omega^{<\omega}$ of all finite sequences of natural numbers. Let $\mathrm{ss}\, k$ be the $\mathscr{L}^{\mathrm{st}}$ formula which tells that s_k has infinitely large length and $s_k(l)$ is standard for all standard l.

Let $M \succ \omega$ be a fixed nonstandard **PA** model. We recall that

$$\mathrm{SS}[M] = \{s_K \restriction \omega : K \in M \;\&\; M \models \mathrm{ss}\, K\}$$

is the *standard system* of M. Thus $\mathrm{SS}[M]$ is a subset of ω^ω, countable whenever M is countable. The following lemma shows that analytical definability in $\mathrm{SS}[M]$ can be reduced in a level-to-level way to $\mathscr{L}^{\mathrm{st}}$ definability in M.

LEMMA 3. *Let* $n \geq 1$, *and* $X \subseteq \omega$ *be* Σ_n^1 *in* $\mathrm{SS}[M]$. *Then* $X \in \Sigma_n^{\mathrm{st}}$ *in* M.

PROOF. Let us say $n = 3$. There exists a (parameter-free) Σ^1_3 formula $\Phi(x)$ of the form $\exists\alpha\,\forall\beta\,\exists\gamma\,\forall m\;\varphi(x,m,\alpha,\beta,\gamma)$, where φ is Δ_0, which defines X in $\mathrm{SS}[M]$, that is, $X = \{x \in \omega : \mathrm{SS}[M] \models \Phi(x)\}$. Let $\Psi(x)$ denote the $\mathscr{L}^{\mathrm{st}}$ formula

$$\exists a\,\forall b\,\exists c\,[\,\mathrm{ss}\,a\;\&\;[\,\mathrm{ss}\,b \longrightarrow \mathrm{ss}\,c\;\&\;\forall^{\mathrm{st}} m\;\varphi(x,m,s_a,s_b,s_c)\,]\,],$$

so that $X = \{x \in \omega : M \models \Psi(x)\}$ by definition. The formula in the outer square brackets is (easily transformable to) Δ^{st}_0, as required. ⊣

It is an essentially more difficult problem to obtain a reduction in the opposite direction. Perhaps this even cannot be done in the most general case because there is no idea why the construction of an arbitrary nonstandard **PA** model M can be completely traced in $\mathrm{SS}[M]$.

THEOREM 4. *Let $\mathscr{F}$ be a countable arithmetically closed subset of ω^ω. There exists a countable* **PA** *model $M \succ \omega$ such that, first, $\mathrm{SS}[M] = \mathscr{F}$, and second, for any $n \geq 1$ and $X \subseteq \omega$, X is Σ^{st}_n in M if and only if X is Σ^1_n in $\mathscr{F}$.*

PROOF. The required model M is constructed as an ultrapower of the form $\mathrm{Ult}_{\mathscr{U}}\mathscr{F}$ where $\mathscr{U}$ is a nonprincipal ultrafilter in the algebra

$$\mathscr{S} = \{U \subseteq \omega : \text{the characteristic function of } U \text{ belongs to } \mathscr{F}\} \tag{1}$$

of all subsets of ω arithmetical in some $\gamma \in \mathscr{F}$.

Generally speaking, the truth in $M = \mathrm{Ult}_{\mathscr{U}}\mathscr{F}$ depends on the ultrafilter $\mathscr{U}$. Thus $\mathscr{U}$ has to be eliminated from the truth definition in M. To get rid of $\mathscr{U}$, we define this ultrafilter as a *generic ultrafilter* via a sufficiently symmetric notion of forcing.

The ultrafilter and the model.

- $\mathscr{S}^{\mathrm{inf}} = \{U \in \mathscr{S} : U \text{ is infinite and coinfinite}\}$.
- A set $\mathscr{D} \subseteq \mathscr{S}^{\mathrm{inf}}$ is *dense* iff $\forall U \in \mathscr{S}^{\mathrm{inf}}\,\exists V \in \mathscr{D}\;(V \subseteq U)$.
- A set $\mathscr{D} \subseteq \omega^l \times \mathscr{F}^k$ is *$\mathscr{F}$-definable* iff it is definable in the structure $\langle\omega;\mathscr{F}\rangle$ by an analytical formula in which elements of $\mathscr{F}$ *may occur as parameters.* This notion extends on sets $\mathscr{D} \subseteq \omega^l \times \mathscr{F}^k \times \mathscr{S}^m$ via characteristic functions.
- An ultrafilter $\mathscr{U} \subseteq \mathscr{S}^{\mathrm{inf}}$ is *$\mathscr{F}$-generic* iff it nonempty intersects every dense $\mathscr{F}$-definable set $\mathscr{D} \subseteq \mathscr{S}^{\mathrm{inf}}$.

The collection of all $\mathscr{F}$-definable sets $\mathscr{D}$ is countable; therefore $\mathscr{F}$-generic filters exist. *Let $\mathscr{U}$ be a fixed $\mathscr{F}$-generic ultrafilter* henceforth.

We let M denote the ultrapower $\mathrm{Ult}_{\mathscr{U}}(\mathscr{F})$.

It is asserted that M satisfies the requirements of Theorem 4.

First of all the ordinary Łoś theorem holds because both $\mathscr{F}$ and $\mathscr{S}$ are arithmetically closed. Hence M is an elementary extension of ω. However to verify the other properties of M we need to exploit the genericity of $\mathscr{U}$. This investigation is based on the corresponding notion of forcing.

The forcing. We are going to force sentences related to the truth in ultrapowers of ω obtained as factors of the set $\mathscr{F}$ via an ultrafilter (say, $\mathscr{U}$) in the algebra $\mathcal{S}^{\mathrm{inf}}$. Elements of the algebra are considered as *forcing conditions*.

Let, for $a \in \omega$, $\underline{a} \in \mathscr{F}$ be defined by $\underline{a}(i) = a$ for all i .

Let $\mathscr{L}^{\mathrm{st}}[\mathscr{F}]$ denote the extension of the language $\mathscr{L}^{\mathrm{st}}$ by elements of $\mathscr{F}$ as parameters of the natural numbers' type. (Standard natural numbers a are represented by the constant functions $\underline{a}$.) $\mathscr{L}[\mathscr{F}]$ has the same meaning.

The notation $\Sigma^{\mathrm{st}}_m[\mathscr{F}]$, $\Pi^{\mathrm{st}}_m[\mathscr{F}]$, $\Delta^{\mathrm{st}}_0[\mathscr{F}]$, $\Delta_0[\mathscr{F}]$ (classes of $\mathscr{L}^{\mathrm{st}}[\mathscr{F}]$ formulas) has the same meaning. Formulas in these classes are called *prenex*. Obviously, for a prenex formula Φ , there is a certain uniquely defined way to convert the negation $\neg\,\Phi$ (which is not a prenex formula unless Φ is Δ_0) to prenex form. Let Φ^- denote the result of such a transformation. Thus if, say, Φ is a Σ^{st}_n formula then Φ^- is a Π^{st}_n formula.

Let finally Φ be an $\mathscr{L}[\mathscr{F}]$ formula, $i \in \omega$. By $\Phi[i]$ we denote the $\mathscr{L}$ formula obtained by changing every $\gamma \in \mathscr{F}$ which occurs in Φ to $\gamma(i)$.

DEFINITION 5.. The forcing relation U **forc** Φ , where $U \in \mathcal{S}^{\mathrm{inf}}$ and Φ is a prenex $\mathscr{L}^{\mathrm{st}}[\mathscr{F}]$ sentence, is introduced as follows.

1. Let Φ be a $\Delta_0[\mathscr{F}]$ sentence. We define U **forc** Φ iff $\Phi[i]$ is true for all but finite number elements $i \in U$.
2. U **forc** Φ & Ψ iff U **forc** Φ and U **forc** Ψ . Similarly for $\bigvee$.
3. U **forc** $\exists^{\mathrm{st}} a\ \Phi(a)$ iff $\exists\, a \in \omega\ U$ **forc** $\Phi(\underline{a})$.
4. U **forc** $\forall^{\mathrm{st}} a\ \Phi(a)$ iff $\forall\, a \in \omega\ U$ **forc** $\Phi(\underline{a})$.
5. U **forc** $\exists\, a\ \Phi(a)$ iff there exists $\gamma \in \mathscr{F}$ such that U **forc** $\Phi(\gamma)$.
6. Let Φ be a $\Pi^{\mathrm{st}}_n[\mathscr{F}]$ formula, $n \geq 1$. Then U **forc** Φ iff none among sets $V \subseteq U$, $V \in \mathcal{S}^{\mathrm{inf}}$ forces Φ^- . ⊣

Thus, in particular, items 2, 3, 4 cover the Δ^{st}_0 case while the last two items extend the definition to Σ^{st}_n and Π^{st}_n , $n \geq 1$.

To formulate the assertion which presents the connection between truth and forcing in this setting, one more definition is necessary. Let Φ be an $\mathscr{L}^{\mathrm{st}}[\mathscr{F}]$ formula. By $[\Phi]$ we denote the result of changing every $\gamma \in \mathscr{F}$ occurring in Φ by $[\gamma]$, the class of $\mathscr{U}$ -equivalence of γ in M . Thus $[\Phi]$ is an $\mathscr{L}^{\mathrm{st}}$ formula having, perhaps, elements of M as parameters.

THEOREM 6 (Forcing Łoś Theorem). *Let Φ be a prenex $\mathscr{L}^{\mathrm{st}}[\mathscr{F}]$ sentence. Then $[\Phi]$ is true in M if and only if there exists $U \in \mathscr{U}$ such that U* **forc** *Φ .*

This is an ordinary application of forcing technique. However the manner how we introduced **forc** (especially items 2, 3, 4) needs to verify several general facts associated with forcing in this setting.

LEMMA 7. *The forcing of $\Delta^{\mathrm{st}}_0[\mathscr{F}]$ formulas is Δ^1_1 in $\mathscr{F}$. For $n \geq 1$, the forcing of $\Sigma^{\mathrm{st}}_n[\mathscr{F}]$ or $\Pi^{\mathrm{st}}_n[\mathscr{F}]$ formulas is respectively Σ^1_n or Π^1_n in $\mathscr{F}$.*

PROOF. Thus the following is asserted. Given, say, a Σ^{st}_n formula ($n \geq 1$) $\Phi(a_1, \dots, a_k)$, there exists an analytical Σ^1_n formula $\Phi^*(\xi, \gamma_1, \dots, \gamma_k)$, such that, for all $\gamma_1, \dots, \gamma_k \in \mathscr{F}$ and $U \in \mathcal{S}^{\text{inf}}$,

$$U \textbf{ forc } \Phi(\gamma_1, \dots, \gamma_k) \quad \text{iff} \quad \mathscr{F} \models \Phi^*(h_U, \gamma_1, \dots, \gamma_k),$$

where h_U denotes the characteristic function of U.

This can be easily proved by induction on n. ⊣

We say that $U \in \mathcal{S}^{\text{inf}}$ *decides* Φ iff either U **forc** Φ or U **forc** Φ^-.

LEMMA 8. *Let Ψ be a prenex $\mathscr{L}^{\text{st}}[\mathscr{F}]$ sentence, and $U \in \mathscr{U}$. There exists $V \in \mathscr{U}$, $V \subseteq U$, which decides Ψ.*

PROOF. The set of all $V \in \mathcal{S}^{\text{inf}}$ which decide Ψ is $\mathscr{F}$-definable by Lemma 7. We shall verify that it is dense. The density is quite obvious in the most elementary case of $\Delta_0[\mathscr{F}]$ formulas Ψ, and easily goes on by induction on n. It remains to carry out the case of $\Delta_0^{\text{st}}[\mathscr{F}]$ sentences.

Let Ψ be such a sentence. Let $\mathrm{Q}_1^{\text{st}}\, a_1, \dots, \mathrm{Q}_m^{\text{st}}\, a_m$ be an enumeration of all quantifiers $\exists^{\text{st}}$, $\forall^{\text{st}}$, occurred in Ψ. It is assumed that the variables a_i are pairwise different and none of them coincides with some other free or bounded variable occurring in Ψ. Let $\Phi(a_1, \dots, a_m)$ denote the $\Delta_0[\mathscr{F}]$ formula obtained from Ψ by dropping all the quantifiers Q_i^{st}; thus if, say, Ψ is $\exists^{\text{st}} a\, [\varphi(a)\ \&\ \forall^{\text{st}} b\, \psi(a,b)]$, where φ and ψ are $\Delta_0[\mathscr{F}]$, then Φ is $\varphi(a)\ \&\ \psi(a,b)$.

It is easy to see that any $V \in \mathcal{S}^{\text{inf}}$ which decides every formula $\Phi(\underline{a_1}, \dots, \underline{a_m})$, $a_1, \dots, a_m \in \omega$, will also decide Ψ. Therefore, we have to prove the following: if $U \in \mathcal{S}^{\text{inf}}$ and $\Phi(a_1, \dots, a_m)$ is a $\Delta_0[\mathscr{F}]$ formula then there exists $V \in \mathcal{S}^{\text{inf}}$, $V \subseteq U$, such that, for all $a_1, \dots, a_m \in \omega$, V decides $\Phi(\underline{a_1}, \dots, \underline{a_m})$.

This is being demonstrated in the case $m = 1$; the general case is quite similar. Let $U_a = \{i \in U : \Phi(\underline{a})[i]\}$ for all $a \in \omega$; every U_a belongs to $\mathcal{S}$. A decreasing sequence V_a, $a \in \omega$, of infinite subsets of U is defined the following way. First $V_{-1} = U$. Let V_{a-1} be already defined. If the set $V' = V_{a-1} \bigcap U_a$ is infinite then we put $V_a = V'$; otherwise we set $V_a = V_{a-1} \setminus U_a$. Then V_a decides $\Phi(\underline{a})$.

We finally define $V = \{i_a : a \in \omega\}$ where i_a is the least element of V_a greater than i_{a-1}. It can be easily seen that $V \in \mathcal{S}^{\text{inf}}$ since $\mathcal{S}$ is arithmetically closed; on the other hand, V decides every $\Phi(\underline{a})$ because $V \setminus V_a$ is finite. ⊣

PROOF OF THEOREM 6. The proof goes on by induction on the construction of the formula Φ. Assume first that Φ is a $\Delta_0[\mathscr{F}]$ sentence. Since the standardness predicate does not occur, we apply the ordinary Łoś theorem and obtain:

$$[\Phi] \text{ is true in } M \quad \text{if and only if} \quad \{i : \Phi[i]\} \in \mathscr{U}.$$

It remains to refer to item 1 of Definition 5; indeed, if $U \in \mathscr{U}$ and U' differs from U in a finite number of elements then $U' \in \mathscr{U}$ as well.

Let Φ be a $\Delta_0^{\text{st}}[\mathscr{F}]$ sentence, say, $\exists^{\text{st}} a\, \forall^{\text{st}} b\, \varphi(a,b)$, φ being $\Delta_0[\mathscr{F}]$. Assume that some $U \in \mathscr{U}$ forces Φ. By definition and by what has been proved in the case $\Delta_0[\mathscr{F}]$, this implies $\exists a \in \omega\, \forall b \in \omega\; M \models [\varphi](a,b)$, because evidently $[\underline{a}]$, the class of $\mathscr{U}$-equivalence of $\underline{a}$ in M, is identified with a. Thus $M \models [\Phi]$.

Assume that none of $U \in \mathscr{U}$ forces Φ. By Lemma 8, there exists $U \in \mathscr{U}$ which forces Φ^-. We conclude that $M \models [\Phi^-]$, as above. This completes the $\Delta_0^{\mathrm{st}}[\mathscr{F}]$ case.

Let Φ be a $\Sigma_n^{\mathrm{st}}[\mathscr{F}]$ sentence $\exists a\ \Psi(a)$. If some $U \in \mathscr{U}$ forces Φ then by definition and the induction hypothesis we obtain $[\Psi(\gamma)]$ for some $\gamma \in \mathscr{F}$, therefore $[\Psi]([\gamma])$, and then $[\Phi]$ itself, in M. If $[\Phi]$ is true in M then $[\Psi(\gamma)]$ is true in M for some $\gamma \in \mathscr{F}$, and the induction hypothesis can be applied again.

Let finally Φ be a $\Pi_n^{\mathrm{st}}[\mathscr{F}]$ sentence. If $[\Phi]$ is true in M then $[\Phi^-]$ is false, therefore by the hypothesis, none of $U \in \mathscr{U}$ forces the $\Sigma_n^{\mathrm{st}}[\mathscr{F}]$ sentence Φ^-. Then some $U \in \mathscr{U}$ forces Φ by Lemma 8. Conversely, let $U \in \mathscr{U}$ force Φ. Assume that, on the contrary, $[\Phi]$ is false in M. Then $[\Phi^-]$ is true, therefore there exists $V \in \mathscr{U}$ which forces Φ^-. Then $V' = V \cap U \in \mathscr{U}$ and $V' \subseteq V$, therefore V' forces Φ^-. On the other hand, $V' \subseteq U$, a contradiction with the definition of U **forc** Φ. $\dashv$

PROOF OF THEOREM 4—THE FINAL VERIFICATION.

Part 1. We prove that $\mathrm{SS}[M]$, the standard system of M, is equal to $\mathscr{F}$, as required by Theorem 4.

First let $\alpha \in \mathscr{F}$. We recall that $\{s_k : k \in \omega\}$ is a recursive enumeration of all finite sequences of natural numbers. The function γ defined so that $s_{\gamma(i)} = \alpha \restriction i$ for all i, belongs to $\mathscr{F}$ since this set is arithmetically closed. On the other hand, $s_{[\gamma]} \restriction \omega = \alpha$.

Let, conversely, $\gamma \in \mathscr{F}$ be such that $\alpha = s_{[\gamma]} \restriction \omega \in \omega^\omega$; we have to show that $\alpha \in \mathscr{F}$. Let us consider $s_\gamma(a) = b$ as the $\Delta_0[\mathscr{F}]$ formula Φ in the Δ_0^{st} case in the proof of Lemma 8. Then, as it was demonstrated there, the set of all $V \in \mathscr{S}^{\mathrm{inf}}$ which decide every formula $s_\gamma(\underline{a}) = \underline{b}$, $a, b \in \omega$, is dense. Therefore, by the genericity of $\mathscr{U}$, there exists $U \in \mathscr{U}$ which decides every formula $s_\gamma(\underline{a}) = \underline{b}$. Then, by Theorem 6,

$$\alpha(a) = b \longleftrightarrow U \ \mathbf{forc}\ s_\gamma(\underline{a}) = \underline{b}$$

for all a, b. We conclude that α is arithmetical relatively to U and γ by Lemma 7, so that $\alpha \in \mathscr{F}$, as required.

Part 2. To complete the proof of Theorem 4, we consider an arbitrary set $Y \subseteq \omega$. If Y is Σ_n^1 in $\mathscr{F}$ for some $n \geq 1$ then Y is Σ_n^{st} in M by Lemma 3. Let, conversely, Y be Σ_n^{st} in M; therefore, $Y = \{y \in \omega : M \models \Phi(y)\}$ for a parameter-free Σ_n^{st} formula Φ. We shall prove that Y is Σ_n^1 in $\mathscr{F}$. The first step is quite evident: by Theorem 6,

$$Y = \{y \in \omega : \exists\, U \in \mathscr{U}\ (U \ \mathbf{forc}\ \Phi(\underline{y}))\}.$$

To get rid of $\mathscr{U}$ in the right-hand side, it suffices to demonstrate that all $U \in \mathscr{S}^{\mathrm{inf}}$ force, generally speaking, the same parameter-free formulas. This is based on a system of automorphisms of the forcing.

Let $\pi \in \mathscr{F}$ be a bijection ω onto itself. We set

$\pi U = \{\pi(i) : i \in U\}$ for all $U \subseteq \omega$;

$(\pi\gamma)(\pi(i)) = \gamma(i)$ for all $i \in \omega$ and $\gamma \in \omega^\omega$, so that $\pi\gamma \in \omega^\omega$.
For any $\mathscr{L}^{\mathrm{st}}[\mathscr{F}]$ formula Φ, let $\pi\Phi$ denote the result of changing every γ which occurs in Φ to $\pi\gamma$.

ASSERTION. *For $\pi \in \mathscr{F}$ and $U \in \mathscr{S}^{\mathrm{inf}}$, U* **forc** *Φ if and only if πU* **forc** *$\pi\Phi$.*

PROOF. Both $\mathscr{S}^{\mathrm{inf}}$ and $\mathscr{F}$ are closed under the action of π. ⊣

Let ODD denote the set of all odd numbers; evidently ODD $\in \mathscr{S}^{\mathrm{inf}}$.

COROLLARY. *Let $U \in \mathscr{S}^{\mathrm{inf}}$ and Φ be a $\mathscr{L}^{\mathrm{st}}[\mathscr{F}]$ formula which contains only functions $\underline{a}$, $a \in \omega$, as parameters. Then U* **forc** *Φ if and only if* ODD **forc** *Φ.*

PROOF. Let $\pi \in \mathscr{F}$ be a one-to-one map ω onto itself such that $\pi U = \mathrm{ODD}$. We apply the assertion, having in mind that, by the restriction related to parameters, $\pi\Phi$ coincides with Φ. ⊣

Therefore, $Y = \{y \in \omega : \mathrm{ODD}$ **forc** $\Phi(\underline{y})\}$. By Lemma 7, Y is a Σ^1_n set in $\mathscr{F}$, as required. This ends the proof of Theorem 4. ⊣

§2. Making a set of integers definable. This section ends the proof of Theorem 2. Theorem 4 reduces the question to the following form: given a natural number n and a set $Z \subseteq \omega$, find a countable arithmetically closed set $\mathscr{F} \subseteq \omega^\omega$ such that

(a) Z is Σ^1_{n+2} in $\mathscr{F}$; and
(b) any set $Y \subseteq \omega$, $Y \in \Sigma^1_{n+1}$ in $\mathscr{F}$, belongs to $\Sigma_{n+1}[0^{(\omega)}]$.

Generic functions. The construction of $\mathscr{F}$ is based on a version of *arithmetical forcing* of Feferman, see Hinman [2] for details. Here follow several relevant definitions.

- $\mathrm{Seq} = 2^{<\omega}$ is the set of all finite sequences of zeros and ones.
- A function $\alpha \in 2^\omega$ is Γ *generic,* where Γ is a definability class, if and only if, given a set $D \subseteq \mathrm{Seq}$, $D \in \Gamma$, there exists $s \in \mathrm{Seq}$, $s \subset \alpha$, such that either $s \in D$ or there are no $s' \in D$ satisfying $s \subseteq s'$.
- α is *arithmetically generic* iff it is Σ_m generic for all m.

Some additional notation is necessary.

- $\ulcorner i, j \urcorner = 2^i(2j+1) - 1$, the "arithmetical pair".
- $(\zeta)_n(i) = \zeta(\ulcorner n, i \urcorner)$ for all $\zeta \in \omega^\omega$ and $n, i \in \omega$; so that $(\zeta)_n \in \omega^\omega$.
- Let $\alpha \in 2^\omega$. We define, for any $W \subseteq \omega$,

$\mathscr{F}(\alpha, W) = \{\gamma \in \omega^\omega : \gamma$ is arithmetical in a finite number of functions $(\alpha)_z$, $z \in W\}$.

The required set $\mathscr{F}$ will have the form $\mathscr{F} = \mathscr{F}(\alpha, Z)$ for a $\Sigma_n[0^{(\omega)}]$ generic function α of class $\Delta_{n+1}[0^{(\omega)}]$. The next lemma explains how Z will be defined in $\mathscr{F}(\alpha, Z)$.

LEMMA 9. *Assume that $\alpha \in 2^\omega$ is arithmetically generic. Let $z \in \omega$ and $Z \subseteq \omega$. Then $(\alpha)_z \in \mathscr{F}(\alpha, Z)$ if and only if $z \in Z$.*

PROOF. The "if" part is obvious. To prove the "only if" part, assume on the contrary that $z \notin Z$ but $(\alpha)_z \in \mathscr{F}$, so that $(\alpha)_z = F((\alpha)_{z_1}, \ldots, (\alpha)_{z_m})$ where $z_1, \ldots, z_m \in Z$ and F is an arithmetical function. Then α belongs to the meager arithmetical set $\{\alpha' \in 2^\omega : (\alpha')_z = F((\alpha')_{z_1}, \ldots, (\alpha')_{z_m})\}$, which contradicts the genericity. ⊣

This lemma is in connection with the easy part of the proof of Theorem 2; it will only remain to compute accurately the level of definability of Z implied by the lemma. The other part, that is, property (b), needs much more effort. There are two principal ideas involved in the proof. First, it occurs that $\mathscr{F}(\alpha, Z)$ is an elementary submodel of $\mathscr{F}(\alpha, \omega)$ with respect to all Σ^1_{n+1} formulas, so that we can prove (b) in $\mathscr{F}(\alpha, \omega)$, getting rid of Z, which is a great relief. Second, we exploit the fact that $\mathscr{F}(\alpha, \omega)$ admits an enumeration simply expressible in terms of $0^{(\omega)}$ and α, and thus reduce the analytical definability in $\mathscr{F}(\alpha, \omega)$ to arithmetical definability with $0^{(\omega)}$ as an extra parameter. This is approximately how the proof will go on.

Now we come to details. It will take some space to present the arithmetical forcing, the key technical tool in this section.

The forcing. We shall force formulas of $\mathscr{L}^\bullet$, the extension of $\mathscr{L}$, the language of **PA**, by the constant $\dot{\alpha}$ for a generic function and free variables (but *not* quantifiers) of type ω^ω. These variables can be replaced by elements of ω^ω as parameters. $\Delta^\bullet_0$, $\Sigma^\bullet_m$, and $\Pi^\bullet_m$ will denote the naturally introduced classes of $\mathscr{L}^\bullet$ formulas. Formulas which belong to these classes are called *prenex* in this section.

DEFINITION 10. The forcing relation s forc φ is introduced; here $s \in \mathrm{Seq}$ while φ is a prenex $\mathscr{L}^\bullet$ sentence which may contain elements of ω^ω as parameters.

1. Let $\varphi(\dot{\alpha})$ be a $\Delta^\bullet_0$ sentence. We set s forc $\varphi(\dot{\alpha})$ iff $\varphi(\alpha)$ is true for all $\alpha \in 2^\omega$ such that $s \subset \alpha$.
2. s forc $\exists i\, \varphi(i)$ iff there exists $i \in \omega$ such that s forc $\varphi(i)$.
3. Let Φ be a $\Pi^\bullet_m$ formula, $m \geq 1$. Then s forc Φ iff none among $s' \in \mathrm{Seq}$, $s' \supseteq s$, forces Φ^-. ⊣

As above, for a prenex formula Φ, Φ^- denotes the result of straightforward transformation of $\neg\, \Phi$ to the prenex form.

The following well-known properties of the forcing are included with proofs to make the exposition self-contained at this point.

PROPOSITION 11. *Let $\Phi(\dot{\alpha}, \gamma_1, \ldots, \gamma_k)$ be a parameter-free $\Sigma^\bullet_n$ formula, $n \geq 1$. Then the set $S_\Phi = \{\langle s, \gamma_1, \ldots, \gamma_k\rangle \in \mathrm{Seq} \times (\omega^\omega)^k : s \text{ forc } \Phi(\dot{\alpha}, \gamma_1, \ldots, \gamma_k)\}$ is Σ_n.*

PROOF. It suffices to prove that S_Φ is Δ_1 (that is, Δ_0) in the case when Φ is a $\Delta^\bullet_0$ formula; the general result expands then automatically by induction on Definition 10. Thus let $\Phi(\dot{\alpha}, \gamma)$ be a $\Delta^\bullet_0$ formula. Let, for $s \in \mathrm{Seq}$, $\tilde{s} \in 2^\omega$ be the extension of s by infinitely many zeros. Let $\Phi^+(s, \gamma)$ be the formula which tells that $\Phi(\tilde{s}, \gamma)$ is "true" and only the values $\tilde{s}(k)$ with $k < \operatorname{dom} s$ (that is, values of s itself) participate in the computation of the truth value of

$\Phi(\tilde{s},\gamma)$. Let $\Phi^-(s,\gamma)$ tell the same but with "false" instead of "true". Then first $s \in S_\Phi \longleftrightarrow \forall s' \supseteq s \neg \Phi^-(s',\gamma)$, which gives a Π_1 definition of S_Φ (both Φ^+ and Φ^- are Δ_0 formulas). On the other hand, a Σ_1 definition can be given by

$$s \in S_\Phi \longleftrightarrow \exists k > \operatorname{dom} s \, \forall s' \in 2^k \, [s \subset s' \longrightarrow \Phi^+(s',\gamma)].$$

The direction $\longleftarrow$ is obvious. The opposite direction can be easily proved using the König lemma. ⊣

COROLLARY 12. *Let $\alpha \in 2^\omega$ be a $\Sigma_n[\zeta]$ generic function (or arbitrary, if $n = 0$), and $\zeta \in \omega^\omega$. Then for any $m \leq n$ and a closed $\Sigma^\bullet_m[\zeta]$ formula Φ, there exists $s \subset \alpha$ such that either s* forc *Φ or s* forc *Φ^-.*

PROOF. Only the case $m = n = 0$, that is, when no genericity assumption is made, does not follow immediately from the lemma. However if $\Phi(\dot{\alpha})$ is a $\Sigma^\bullet_0$ (that is, $\Delta^\bullet_0$) formula (where some $\zeta \in \omega^\omega$ may occur as a parameter) then the set $\{\alpha : \Phi(\alpha)\}$ is open and closed in 2^ω, which easily implies the required fact. ⊣

The following principal lemma connects the truth of $\mathscr{L}$ formulas having a generic $\alpha \in 2^\omega$ as a parameter with the forcing of $\mathscr{L}^\bullet$ formulas by initial segments of α.

LEMMA 13. *Let α be a $\Sigma_n[\zeta]$ generic function (or arbitrary, if $n = 0$), where $\zeta \in \omega^\omega$. Let $\varphi(\dot{\alpha},\zeta)$ be a $\Sigma^\bullet_{n'}[\zeta]$ formula, $n' \leq n+1$. Then $\varphi(\alpha,\zeta)$ is true iff some $s \in$* Seq, *$s \subset \alpha$, forces $\varphi(\dot{\alpha},\zeta)$.*

PROOF. The proof goes on by induction on n'. The case $n' = 0$ is evident: the set of all α satisfying $\varphi(\alpha,\zeta)$ is open and closed. To carry out the step, let $n' = m+1$, $m \leq n$. Let $\varphi(\dot{\alpha},\zeta)$ be the $\Sigma^\bullet_{m+1}[\zeta]$ formula $\exists k \, \psi(k,\dot{\alpha},\zeta)$, where ψ is a $\Pi^\bullet_m[\zeta]$ formula.

Assume that $\varphi(\alpha,\zeta)$ is true. Then $\psi(k,\alpha,\zeta)$ holds for some k, so that the $\Sigma^\bullet_m$ formula $\psi^-(k,\alpha,\zeta)$ is false and, by the induction hypothesis, none among $s \subset \alpha$ forces $\psi^-(k,\dot{\alpha},\zeta)$. By Corollary 12, there exists $s \subset \alpha$ which forces $\psi(k,\dot{\alpha},\zeta)$. Therefore s forc $\varphi(\dot{\alpha},\zeta)$ by definition.

Conversely assume that some $s \subset \alpha$ forces $\varphi(\dot{\alpha},\zeta)$, that is, forces $\psi(k,\dot{\alpha},\zeta)$ for some k. We prove that $\psi(k,\alpha,\zeta)$ is true. Assume on the contrary that $\neg \psi(k,\alpha,\zeta)$, that is, $\psi^-(k,\alpha,\zeta)$. Applying the induction hypothesis, we obtain some $s' \subset \alpha$ which forces $\psi^-(k,\dot{\alpha},\zeta)$. One may assume that $s \subseteq s'$ since s also is expanded by α. Thus we have a contradiction because s forc $\psi(k,\dot{\alpha},\zeta)$. ⊣

Reduction to arithmetical truth. We first prove a technical assertion which reduces the truth in $\mathscr{F}(\alpha,Z)$ to the truth in $\mathscr{F}(\alpha,Z'')$, where $Z'' \subseteq Z$, at the cost of introduction of the $Z \setminus Z''$-part of α as an extra parameter.

Let $\alpha \in 2^\omega$. If $Z = \{z_1,\dots,z_m\} \subseteq \omega$ is a finite set, $z_1 < \cdots < z_m$, then we define $\alpha/Z = \beta \in 2^\omega$ by $(\beta)_{i-1} = (\alpha)_{z_i}$ for $i = 1,\dots,m$, and $(\beta)_i(k) = 0$ for $i \geq m$ and all k. If $Z = \{z_i : i \in \omega\} \subseteq \omega$ is infinite, $z_i < z_{i+1}$ for all i, then we define $\beta = \alpha/Z \in 2^\omega$ by $(\beta)_i = (\alpha)_{z_i}$ for all i. Thus in both cases $\mathscr{F}(\alpha/Z,\omega) = \mathscr{F}(\alpha,Z)$.

The lemma deals with analytical definability in sets like $\mathscr{F}(\alpha, Z)$, including the case when functions which do not belong to $\mathscr{F}(\alpha, Z)$ are admitted as parameters. This is well defined, but we must be careful in this point to avoid nonlegitimate use of a rule of quantifier transformation from descriptive set theory which may be incorrect with respect to definability in sets like $\mathscr{F}(\alpha, Z)$.

In particular, the idea that a formula of the form: Σ^1_n *prefix* + *arithmetical formula* is itself Σ^1_n becomes, in general, wrong. This is the reason for the following definition.

- Σ^{+1}_n *formula* is an analytical formula of the form

$$\exists \gamma_1 \, \forall \gamma_2 \, \exists \gamma_3 \, \forall \gamma_4 \, \dots \, \exists(\forall) \, \gamma_n \, \forall(\exists) \, k \, \exists(\forall) \, m \; \varphi,$$

where φ is a Δ_0 formula.

In descriptive set theory this would be Σ^1_n, of course.

LEMMA 14. *Let $\Phi(\zeta, y)$ be a Σ^1_n formula. There exists a Σ^{+1}_n formula $\Phi'(\alpha', \zeta, y)$ such that, if $\alpha \in 2^\omega$ is arithmetically generic, $Z' \subseteq Z \subseteq \omega$, and $Z'' = Z \setminus Z'$, then for all $y \in \omega$ and $\zeta \in \omega^\omega$: $\mathscr{F}(\alpha, Z) \models \Phi(\zeta, y)$ iff $\mathscr{F}(\alpha, Z'') \models \Phi'(\alpha/Z', \zeta, y)$.* [4]

PROOF. The case $n = 0$ is elementary, so assume that $n \geq 1$. We exploit the fact that $\mathscr{F} = \mathscr{F}(\alpha, Z)$ can be "modeled" inside $\mathscr{F}'' = \mathscr{F}(\alpha, Z'')$. Let $\omega \times \omega \times \mathrm{Seq} = \{\langle k_p, l_p, t_p \rangle : p \in \omega\}$ be a recursive enumeration. For $\tau \in \omega^\omega$, we put $R_\tau = \{\langle k_p, l_p, t_p \rangle : \tau(p) = 1\}$. Let, for all $\beta \in 2^\omega$,

$$\gamma_{\tau,\beta}(k) = \begin{cases} l, & \text{if } \; l \text{ is the unique } l \in \omega \text{ such that } \exists t \subset \beta \; (\langle k, l, t \rangle \in R_\tau) \\ 0, & \text{if } \; \neg \exists! l \, [\exists t \subset \beta \; (\langle k, l, t \rangle \in R_\tau)]. \end{cases}$$

ASSERTION 15. $\mathscr{F}(\alpha, Z) = \{\gamma_{\tau, \alpha/w'} : \tau \in \mathscr{F}(\alpha, Z'') \;\&\; w' \subseteq Z' \text{ is finite}\}$.

PROOF. Since $\mathscr{F} = \mathscr{F}(\alpha, Z)$ is arithmetically closed, we have $\gamma_{\tau,\beta} \in \mathscr{F}$ for all $\tau, \beta \in \mathscr{F}$. Let, conversely, $\gamma \in \mathscr{F}$, that is, γ is arithmetical in some α/w, $w \subseteq Z$ being finite. Let $w' = w \cap Z'$ and $w'' = w \cap Z''$. We observe, applying Proposition 11 and Lemma 13, that there exists an arithmetical set $R \subseteq \omega \times \omega \times \mathrm{Seq} \times \mathrm{Seq}$ such that

$$\gamma(k) = l \;\longleftrightarrow\; \exists t' \subset \alpha/w' \; \exists t'' \subset \alpha/w'' \; (\langle k, l, t', t'' \rangle \in R)$$

for all k, l. [5] The set $R'' = \{\langle k, l, t' \rangle : \exists t'' \subset \alpha/w'' \; (\langle k, l, t', t'' \rangle \in R)\}$ is equal to some R_τ, $\tau \in \mathscr{F}''$, because $\mathscr{F}''$ is arithmetically closed. Then $\gamma = \gamma_{\tau, \alpha/w'}$, as required. ⊣

Coming back to the lemma, we let $\Phi'(\alpha', \zeta, y)$ denote the formula obtained by changing every quantifier $\mathrm{Q}\gamma \dots \gamma \dots$ of type ω^ω in Φ to $\mathrm{Q}^{\text{finite}} u \subseteq$

[4] Generally speaking, it is not assumed that $\zeta \in \mathscr{F}(\alpha, Z)$ or $\alpha/Z' \in \mathscr{F}(\alpha, Z'')$.

[5] Basically R expresses forcing for an arithmetical definition of γ from α/w. We first observe that α/w is arithmetically generic, and then divide every $t \subset \alpha/w$ which may force something about α/w onto two parts, the one related to α/w', and the one related to α/w'', thus getting R.

$\omega\ \mathrm{Q}\tau\ \ldots \gamma_{\tau,\alpha'/u} \ldots$. Then, by the assertion, $\mathscr{F} \models \Phi(\zeta, y)$ if and only if $\mathscr{F}'' \models \Phi'(\alpha/Z', \zeta, y)$, for all ζ, y .

Finally, it is asserted that Φ' is a Σ_n^{+1} formula; more exactly, Φ' can be transformed to a Σ_n^{+1} form equivalent in $\mathscr{F}''$. To avoid very long formulas, let $n = 2$ (the general case does not differ much), so that Φ is $\exists \gamma_1 \forall \gamma_2\, \varphi(\gamma_1, \gamma_2, \zeta, y)$, where φ is Σ_1 . There exists a recursive set $C \subseteq \mathrm{Seq}^3 \times \omega$ such that

$$\varphi(\gamma_1, \gamma_2, \zeta, y) \quad\longleftrightarrow\quad \exists \langle s_1, s_2, t\rangle\ [\,C(s_1, s_2, t, y)\ \&\ s_1 \subset \gamma_1\ \&\ s_2 \subset \gamma_2\ \&\ t \subset \zeta\,].$$

Let us consider what happens when we replace the variables γ_i , $i = 1, 2$, by terms $\gamma_{\tau_i,\alpha'/u_i}$. By definition the relation $\gamma_{\tau,\alpha''}(k) = l$ is a propositional combination of Σ_1 and Π_1 formulas having τ, α'', k, l as variables. Hence the relation $s \subset \gamma_{\tau,\alpha'/u}$ is a Σ_2 formula with variables s, τ, α', u . Thus one may treat $\varphi(\gamma_{\tau_1,\alpha'/u_1}, \gamma_{\tau_2,\alpha'/u_2}, \zeta, y)$ as a Σ_2 formula with variables $\alpha', \tau_1, \tau_2, u_1, u_2, \zeta, y$. However, $\Phi'(\alpha', \zeta, y)$ is

$$\exists \tau_1\ \exists^{\mathrm{finite}} u_1 \subseteq \omega\ \forall \tau_2\ \forall^{\mathrm{finite}} u_2 \subseteq \omega\ \varphi(\gamma_{\tau_1,\alpha'/u_1}, \gamma_{\tau_2,\alpha'/u_2}, \zeta, y),$$

so that, after the obvious inclusion of u_i in τ_i , we obtain a Σ_2^{+1} form for Φ' . ⊣

Mainly Lemma 14 will be used in Section 3. However a special case, $Z'' = \emptyset$, enters the reasoning right now. We assume that

- $\Delta_\infty = \{\tau_d : d \in \omega\}$ is a recursive in $0^{(\omega)}$, the set of (Gödel numbers of) true arithmetical sentences, enumeration of the set $\Delta_\infty = \{$ all arithmetical $\tau \in \omega^\omega\}$.

LEMMA 16. *Let $\Phi(\zeta, y)$ be a Σ_n^1 formula. There exists a Σ_{n+2} formula $\varphi(0^{(\omega)}$, α , ζ , $y)$ such that for any arithmetically generic function $\alpha \in 2^\omega$ and all $y \in \omega$ and $\zeta \in \omega^\omega$, we have $\mathscr{F}(\alpha, \omega) \models \Phi(\zeta, y)$ if and only if $\varphi(0^{(\omega)}, \alpha, \zeta, y)$.*

PROOF. Let $\Phi'(\alpha, \zeta, y)$ be the Σ_n^{+1} formula guaranteed by Lemma 14; in particular, in the case when $Z' = Z = \omega$ and $Z'' = \emptyset$ —then $\mathscr{F}(\alpha/Z'', \omega) = \Delta_\infty$, we obtain: $\mathscr{F}(\alpha, \omega) \models \Phi(\zeta, y)$ iff $\Delta_\infty \models \Phi'(\alpha, \zeta, y)$ for all y, ζ , and arithmetically generic α .

It remains to replace every quantifier $\mathrm{Q}\tau \ldots \tau \ldots$ of type ω^ω in Φ' by $\mathrm{Q}d \ldots \tau_d \ldots$. The obtained formula $\varphi(0^{(\omega)}, \alpha, \zeta, y)$ is Σ_{n+2} since the Σ_n^{+1} formula Φ' contains actually $(n + 2)$ quantifiers, in particular, n of type ω^ω and two of type ω . $0^{(\omega)}$ appears via the recursive in $0^{(\omega)}$ enumeration $\Delta_\infty = \{\tau_d : d \in \omega\}$. ⊣

It turns out that in the case when it is assumed that $\zeta \in \mathscr{F}(\alpha, \omega)$, one can save two levels of definability lost in the last lemma by a more serious use of $0^{(\omega)}$ as an extra parameter. In addition to the notation introduced above we put

- $\delta_{d,\beta} = \gamma_{\tau_d,\beta}$ for all $d \in \omega$ and $\beta \in \omega^\omega$.

Then $\mathscr{F}(\alpha, Z) = \{\delta_{d,\alpha/w} : d \in \omega\ \&\ w \subseteq Z\}$ for all $Z \subseteq \omega$ and arithmetically generic α , by Assertion 15 in the case $Z'' = \emptyset$.

LEMMA 17. *Let* $\Phi(\delta, y)$ *be a* Σ^1_n *formula. There exists a* Σ_n *formula* $\varphi(0^{(\omega)}, \alpha, d, w, y)$ *such that for any arithmetically generic function* $\alpha \in 2^\omega$ *, a finite* $w \subseteq \omega$ *, and* $y, d \in \omega$ *, we have* $\mathscr{F}(\alpha, \omega) \models \Phi(\delta_{d,\alpha/w}, y)$ *if and only if* $\varphi(0^{(\omega)}, \alpha, d, w, y)$ *.*

PROOF. Let as above $\Phi'(\alpha, \delta, y)$ be the Σ^{+1}_n formula given by Lemma 14, so that $\mathscr{F}(\alpha, \omega) \models \Phi(\delta, y)$ iff $\Delta_\infty \models \Phi'(\alpha, \delta, y)$ for all $\delta \in \omega^\omega$, $y \in \omega$, and arithmetically generic α . Let, e.g., $n = 2$; then Φ' is $\exists \gamma_1 \forall \gamma_2 \Psi'(\gamma_1, \gamma_2, \alpha, \delta, y)$, where Ψ' is arithmetical. Therefore, if $\delta = \delta_{d,\alpha/w}$, then we have $\mathscr{F} \models \Phi(\delta, y)$ iff

$$\exists d_1 \forall d_2 \Psi'(\tau_{d_1}, \tau_{d_2}, \alpha, \delta_{d,\alpha/w}, y) .$$

Let $\psi_{d_1 d_2 d w y}(\alpha)$ be the formula $\Psi'(\tau_{d_1}, \tau_{d_2}, \alpha, \delta_{d,\alpha/w}, y)$ where τ_{d_1} and τ_{d_2} are replaced by their arithmetical definitions and $\delta_{d,\alpha/w} = \gamma_{\tau_d,\alpha/w}$ is also replaced by its arithmetical definition; thus $\psi_{d_1 d_2 d w y}(\alpha)$ is an arithmetical formula with α as the unique variable.

Let $\rho_{d_1 d_2 d w y}(s)$ denote the formula s forc $\neg \psi_{d_1 d_2 d w y}(\dot{\alpha})$, arithmetical by Proposition 11, and $\#(s, d_1, d_2, d, w, y)$ its Gödel number. We observe, using Lemma 13 and Corollary 12, that provided α is arithmetically generic, $y \in \omega$, and $\delta = \delta_{d,\alpha/w} \in \omega^\omega$, we have $\mathscr{F}(\alpha, \omega) \models \Phi(\delta, y)$ iff

$$\exists d_1 \forall d_2 \forall m \, [\#(\alpha \restriction m, d_1, d_2, d, w, y) \notin 0^{(\omega)}] .$$

Notice that $\#$ is a recursive function. Therefore the displayed Σ_2 formula can be taken as $\varphi(0^{(\omega)}, \alpha, d, w, y)$. ⊣

This reasoning would not go on in the proof of Lemma 16 unless we suppose that α is arithmetically in ζ generic and replace $0^{(\omega)}$ by $\zeta^{(\omega)}$. This is, however, more than we can afford; actually this would mean that the set $\mathscr{Z}$ in Theorem 1 is closed under the operation $\zeta \longmapsto \zeta^{(\omega)}$, which is, generally speaking, not assumed.

Absoluteness. The already obtained results allow us to apply the forcing technique and prove the principal absoluteness lemma.

LEMMA 18. *Let* $W \subseteq \omega$ *be an infinite recursive set,* $n \in \omega$ *. Then*

1. *If* α *is a* $\Sigma_n[0^{(\omega)}]$ *generic (arithmetically generic in the case* $n = 0$ *) function then* $\mathscr{F}(\alpha, W)$ *is an elementary submodel of* $\mathscr{F}(\alpha, \omega)$ *with respect to all* Σ^1_{n+1} *formulas with parameters in* $\mathscr{F}(\alpha, W)$ *.*

2. *If* $\zeta \in \omega^\omega$ *,* α *is a* $\Sigma_{n+2}[0^{(\omega)}, \zeta]$ *generic function, then* $\mathscr{F}(\alpha, W)$ *is an elementary submodel of* $\mathscr{F}(\alpha, \omega)$ *with respect to all* Σ^1_n *formulas with parameters in* $\mathscr{F}(\alpha, W)$ *and* ζ *as an extra parameter.*

PROOF. *Part 1.* Let $\Phi(\delta)$ be a Σ^1_{n+1} formula having some $\delta = \delta_{d,\alpha/w} \in \mathscr{F}(\alpha, W)$ as parameter; $w \subseteq W$ being a finite set. We assume that $\Phi(\delta)$ is true in $\mathscr{F}(\alpha, \omega)$ and prove that it is true in $\mathscr{F}(\alpha, W)$, too.

Let $\varphi(0^{(\omega)}, \alpha, d, w)$ be the Σ_{n+1} formula which Lemma 17 in the y -free case gives for Φ , so that $\mathscr{F}(\alpha', \omega) \models \Phi(\delta_{d,\alpha'/w})$ iff $\varphi(0^{(\omega)}, \alpha', d, w)$, for all arithmetically generic α' , in particular, $\varphi(0^{(\omega)}, \alpha, d, w)$ is true. By Lemma 13 some $s \in \mathrm{Seq}$, $s \subset \alpha$, forces $\varphi(0^{(\omega)}, \dot{\alpha}, d, w)$. Let $l = \mathrm{dom}\, s$. One may without any loss of generality assume that $w \subseteq l = \{0, 1, \ldots, l-1\}$ (otherwise take a bigger part of α as s).

The next step is to define a generic function α' which also expands s , does not change α/w , and sees W the same way as α sees ω .

CLAIM. *There exists a* $\Sigma_n[0^{(\omega)}]$ *generic (arithmetically generic in the case* $n = 0$ *) function* $\alpha' \in 2^\omega$ *such that* $s \subset \alpha'$ *,* $\alpha'/w = \alpha/w$ *, and* $\mathscr{F}(\alpha', \omega) = \mathscr{F}(\alpha, W)$.

PROOF. *Step 1.* Let $u = l \setminus W = \{z_1, \ldots, z_m\}$. By the genericity of α and recursivity of W , there exists an m -element set $v = \{x_1, \ldots, x_m\} \subseteq W$, $\min v > l$, such that $(\alpha)_{x_i} \upharpoonright l = (\alpha)_{z_i} \upharpoonright l$ for all $i = 1, \ldots, m$. The function $\alpha'' \in 2^\omega$ defined by

- $(\alpha'')_{x_i} = \alpha_{z_i}$ and $(\alpha'')_{z_i} = \alpha_{x_i}$ for all i , and
- $(\alpha'')_y = \alpha_y$ for all $y \notin u \cup v$,

satisfies the same genericity condition as α does, $(\alpha'')_k = (\alpha)_k$ for all $k \in w$ — therefore $\alpha/w = \alpha''/w$, $s \subseteq \alpha''$, and $\mathscr{F}(\alpha, W) = \mathscr{F}(\alpha'', W'')$, where $W'' = W \cup u \setminus v$. Finally, $l = \{0, \ldots, l-1\} \subseteq W''$ —the real aim of this step.

Step 2. Let π be the order preserving one-to-one map W'' onto ω; take notice that π is equal to the identity on l because $l \subseteq W''$, and π is recursive since W'' is recursive. Again the function $\alpha' \in 2^\omega$ defined by $(\alpha')_n = (\alpha'')_{\pi(n)}$ (that is, $\alpha' = \alpha''/W''$) satisfies the same genericity condition, $s \subset \alpha'$, $\alpha'/w = \alpha/w$, and $\mathscr{F}(\alpha', \omega) = \mathscr{F}(\alpha'', W'')$. ⊣

We come back to the proof of the lemma (Part 1). Let α' be given by the claim. Then $\varphi(0^{(\omega)}, \alpha', d, w)$ is true by Lemma 13, therefore $\mathscr{F}(\alpha', \omega) \models \Phi(\delta_{d,\alpha'/w})$ by the choice of φ . However $\delta_{d,\alpha'/w} = \delta_{d,\alpha/w} = \delta$ and $\mathscr{F}(\alpha', \omega) = \mathscr{F}(\alpha, W)$ by the choice of α' , and we are done.

Part 2. The reasoning differs a bit from the proof of the first part. Assume that $\Phi(\delta, \zeta)$ is a Σ^1_n formula, $\delta = \delta_{d,\alpha/w} \in \mathscr{F}(\alpha, W)$, $w \subseteq W$ is finite, $\zeta \in \omega^\omega$ arbitrary. We assume that $\mathscr{F}(\alpha, \omega) \models \Phi(\delta, \zeta)$ and prove that $\Phi(\delta, \zeta)$ is also true in $\mathscr{F}(\alpha, W)$.

The new point in the proof is getting rid of δ since otherwise we cannot imply the genericity. Let $\Psi(0^{(\omega)}, \alpha, \zeta)$ be the formula $\Phi(\delta_{d,\alpha/w}, \zeta)$ ($0^{(\omega)}$ enters via the definition of $\delta_{d,\beta}$ which depends on τ_d , a recursive in $0^{(\omega)}$ element of 2^ω , d and w are considered as fixed parameters). Therefore $\Phi(\delta, \zeta) \longleftrightarrow \Psi(0^{(\omega)}, \alpha, \zeta)$ in both $\mathscr{F}(\alpha, W)$ and $\mathscr{F}(\alpha, \omega)$. The method of elimination of terms used in the end of the proof of Lemma 14 allows to treat $\Psi(0^{(\omega)}, \alpha, \zeta)$ as a Σ^{+1}_n formula, therefore as Σ^1_{n+1} formula.

Let $\varphi(0^{(\omega)}, \alpha, \zeta)$ be the Σ_{n+3} formula guaranteed by Lemma 16 in the y -free case. (The triple of variables $0^{(\omega)}$, α , ζ is treated as a single variable in the application of Lemma 16.) Therefore, $\varphi(0^{(\omega)}, \alpha, \zeta)$ is true.

The remainder of the proof is quite the same as in the proof of Part 1. ⊣

It turns out that the hypothesis that W is recursive is not essential.

COROLLARY 19. *Let $Z \subseteq \omega$ be any set containing all odd numbers, $n \in \omega$. Then*

1. *If α is a $\Sigma_n[0^{(\omega)}]$ generic (arithmetically generic in the case $n = 0$) function then $\mathscr{F}(\alpha, Z)$ is an elementary submodel of $\mathscr{F}(\alpha, \omega)$ with respect to all Σ^1_{n+1} formulas with parameters in $\mathscr{F}(\alpha, Z)$.*

2. *If $\zeta \in \omega^\omega$, α is a $\Sigma_{n+2}[0^{(\omega)}, \zeta]$ generic function, then $\mathscr{F}(\alpha, Z)$ is an elementary submodel of $\mathscr{F}(\alpha, \omega)$ with respect to all Σ^1_n formulas with parameters in $\mathscr{F}(\alpha, Z)$ and ζ as an extra parameter.*

PROOF. We set $Z(l) = \{k \in Z : k \text{ is odd or } k \le l\}$; so that every $Z(l)$ is recursive and infinite [6] subset of Z and $Z = \bigcup_{l \in \omega} Z(l)$, that implies $\mathscr{F}(\alpha, Z) = \bigcup_{l \in \omega} \mathscr{F}(\alpha, Z(l))$. To complete the proof apply Lemma 18 for $W = Z(l)$ for all l and use the model theoretic elementary chain lemma. ⊣

Part 1 of the corollary will be used right now to prove Theorem 2, Part 2 will be applied in the next section, for the proof of Theorem 1.

PROOF OF THEOREM 2. Let $n \in \omega$ and a set $Z \subseteq \omega$ be fixed; one may assume without any loss of generality that Z contains all odd numbers. As mentioned in the beginning of this section, it suffices to prove the existence of a countable arithmetically closed set $\mathscr{F} \subseteq \omega^\omega$ satisfying conditions (a) and (b).

Using a universal $\Sigma_n[0^{(\omega)}]$ set in the case $n \ge 1$, and a $\Delta_1[0^{(\omega)}]$ set universal for all arithmetical sets in the case $n = 0$, we obtain a function $\alpha \in 2^\omega$ which is :

(A) $\Delta_{n+1}[0^{(\omega)}]$, and
(B) $\Sigma_n[0^{(\omega)}]$ generic (in particular arithmetically generic)—in the case when $n \ge 1$, and arithmetically generic—in the case $n = 0$.

It is asserted that $\mathscr{F} = \mathscr{F}(\alpha, Z)$ is the required set.

FACT 1. *$\mathscr{F}$ satisfies* (a), *that is, Z is Σ^1_{n+2} in $\mathscr{F}$.*

PROOF. First of all we prove that $0^{(\omega)} \in \Delta^1_1$ in $\mathscr{F}$. Indeed, let $\natural(\varphi)$ denote the Gödel number of an arithmetical formula φ. Let

$$T_n = \{\natural(\varphi) : \varphi \text{ is a true } \Sigma_k \text{ or } \Pi_k \text{ sentence}, \ k \le n\},$$

so that $0^{(\omega)} = \bigcup_{n \in \omega} T_n$, and $\tau_n \in 2^\omega$ be the characteristic function of T_n. There exists a single arithmetical formula, say $\mathbf{t}(n, \tau)$ such that, for any n, $\mathbf{t}(n, \tau)$ is true iff $\tau = \tau_n$. (Roughly, $\mathbf{t}$ says that the set of sentences $\{\varphi : \tau(\natural(\varphi)) = 1\}$ satisfies the Tarski conditions at the level n and below.) On the other hand, every τ_n is arithmetical, so $\tau_n \in \mathscr{F}$ since $\mathscr{F}$ is arithmetically closed. Therefore

$$\natural(\varphi) \in 0^{(\omega)} \longleftrightarrow \mathscr{F} \models \exists \tau \, [\mathbf{t}(n, \tau) \ \& \ \tau(\natural(\varphi)) = 1],$$

and $0^{(\omega)} \in \Sigma^1_1$. A Π^1_1 definition can be obtained the same way. [7]

[6] This is precisely the point where we need Z to contain all odd numbers. Of course ODD could be replaced by any infinite and coinfinite recursive set.

[7] We conclude, using also Lemma 3, that, for any **PA** model $M \succ \omega$, $0^{(\omega)}$ is Δ^{st}_1 in M. By the way this implies the necessity part of Theorem 1.

We now prove that α is Δ^1_{n+1} in $\mathscr{F}$. Notice that $\alpha \in \Delta_{n+1}[0^{(\omega)}]$ by the choice of α. To shorten notation, let $n = 2$. Then α is $\Delta_3[\tau]$ where τ is the characteristic function of $0^{(\omega)}$. Therefore there exists a recursive set $C \subseteq \omega^4 \times \mathrm{Seq}$ such that

$$\alpha(k) = l \longleftrightarrow \exists a\, \forall b\, \exists s\, [\, C(k,l,a,b,s)\ \&\ s \subset \tau\,].$$

The occurrence of $s \subset \tau$ can be replaced by a Σ^1_1 formula. This proves that α is Σ^1_3 in $\mathscr{F}$. Therefore α is Δ^1_3 in $\mathscr{F}$, as required. [8]

Finally, $Z = \{z \in \omega : \exists \gamma \in 2^\omega\ \forall l\ [\gamma(l) = (\alpha)_z(l)]\}$ by Lemma 9, which can be easily converted to a Σ^1_{n+2} definition of Z in $\mathscr{F}$. ⊣

FACT 2. *$\mathscr{F}$ satisfies* (b), *that is, every $Y \subseteq \omega$, $Y \in \Sigma^1_{n+1}$ in $\mathscr{F}$, belongs to* $\Sigma_{n+1}[0^{(\omega)}]$.

PROOF. Corollary 19.1 implies $Y \in \Sigma^1_{n+1}$ in $\mathscr{F}(\alpha, \omega)$. Therefore Y is $\Sigma_{n+1}[0^{(\omega)}, \alpha]$ by Lemma 17 in the δ-free case, so that $Y = \{y : \varphi(0^{(\omega)}, \alpha, y)\}$ for a Σ_{n+1} formula φ. To eliminate α, we use Lemma 13 and obtain:

$$y \in Y \longleftrightarrow \exists s \subset \alpha\, [\, s \text{ forc } \varphi(0^{(\omega)}, \dot{\alpha}, y)\,]$$

for all y. Therefore Y is $\Sigma_{n+1}[0^{(\omega)}]$ since $\alpha \in \Delta_{n+1}[0^{(\omega)}]$ and the forcing for a given $\Sigma^\bullet_{n+1}$ formula is Σ_{n+1}, see Proposition 11. ⊣

This ends the proof of Theorem 2. ⊣

§3. External Scott algebras. The goal of this section is to prove the sufficiency part of Theorem 1. Thus, we fix a countable set $\mathscr{Z} = \{Z_n : n \in \omega\} \subseteq \mathscr{P}(\omega)$ satisfying conditions 1 and 2 of Theorem 1. By Theorem 4, it suffices to find a countable arithmetically closed set $\mathscr{F} \subseteq \omega^\omega$ such that

(i) Every Z_n is analytically definable (that is, Σ^1_m for some m) in $\mathscr{F}$, and
(ii) Every set $Y \subseteq \omega$ analytically definable in $\mathscr{F}$ is equal to some Z_n.

The principal idea is as follows. We make every Z_n coded, by the method similar to the one used to prove Theorem 2, precisely at the level $n+1$ of the construction, thus avoiding a diagonal argument which might cause trouble if all Z_n had been defined at a certain fixed level.

The set $\mathscr{F}$. This idea needs a special sequence of functions $\alpha^n \in 2^\omega$ containing functions of different levels of genericity and definability, which is to be introduced first.

- Let $\zeta_n \in 2^\omega$ be the characteristic function of Z_n, and $\zeta_{<n} = \langle \zeta_0, \dots, \zeta_{n-1} \rangle$.
- Assume that $\alpha^n \in 2^\omega$ is defined for all n. We put $\alpha^{<n} = \langle \alpha^0, \dots, \alpha^{n-1} \rangle$. We define then $\alpha^{\geq n} \in 2^\omega$ by the equalities: $(\alpha^{\geq n})_{\ulcorner m,z \urcorner} = (\alpha^{n+m})_z$ for all m, z. (We recall that $\ulcorner m, z \urcorner = 2^m(2z+1) - 1$.) Then

$$\{(\alpha^{\geq n})_k\} = \{(\alpha^p)_z : p \geq n\ \&\ z \in \omega\}.$$

[8] Here type ω quantifiers cannot be compressed to *one* type ω^ω quantifier (which would imply that α is even Δ^1_1 in $\mathscr{F}$) by the usual laws of quantifier transformation in descriptive set theory, since the law which would be involved here depends on the countable choice, perhaps, false in $\mathscr{F}$. Instead of this we replace every type ω quantifier, say, $\exists x \dots x \dots$ by $\exists \gamma \dots \gamma(0) \dots$.

LEMMA 20. *There exists a sequence* $\langle \alpha^n : n \in \omega \rangle$ *of functions* $\alpha^n \in 2^\omega$ *satisfying the following conditions:*

1. $\alpha^n \in \Delta_{n+3}[0^{(\omega)}, \zeta_{<n}, \alpha^{<n}]$ *for all* n.
2. $\alpha^{\geq n}$ *is* $\Sigma_{n+2}[0^{(\omega)}, \zeta_{<n}, \alpha^{<n}]$ *generic for all* n.

PROOF. We start with several technical definitions. We recall that $\mathrm{Seq} = 2^{<\omega}$.

- For $U \subseteq \omega$, $\mathbf{Seq}[U]$ is the set of all functions $\mathbf{s} : u \longrightarrow \mathrm{Seq}$, having a finite $u \subseteq U$ as the domain, and ordered the following way: $\mathbf{s} \leq \mathbf{s}'$ iff $\mathrm{dom}\,\mathbf{s} \subseteq \mathrm{dom}\,\mathbf{s}'$ and $\mathbf{s}(n) \subseteq \mathbf{s}'(n)$ for all $n \in \mathrm{dom}\,\mathbf{s}$.
- $\mathbf{Seq}$ is $\mathbf{Seq}[\omega]$. $\mathbf{Seq}[> n]$ is $\mathbf{Seq}[\{m : m > n\}]$.
- For $\mathbf{s} \in \mathbf{Seq}$, $\mathbf{s}[> n]$ is the restriction of $\mathbf{s}$ to $\{m \in \mathrm{dom}\,\mathbf{s} : m > n\}$.
  $\mathbf{s}[\geq n]$, $\mathbf{Seq}[\geq n]$ etc. have similar meaning.
- $\mathbf{s}[k] = \mathbf{s}(k)$ for $k \in \mathrm{dom}\,\mathbf{s}$; $\mathbf{s}[k] = \Lambda$ (the empty sequence) for $k \notin \mathrm{dom}\,\mathbf{s}$.
- We say that $s \in \mathrm{Seq}$ *decides* a set $S \subseteq \mathrm{Seq}$ iff either $s \in S$ or none among $s' \in S$ satisfies $s \subseteq s'$. We say that $\mathbf{s} \in \mathbf{Seq}$ *decides* a set $\mathbf{S} \subseteq \mathbf{Seq}$ iff either $\mathbf{s} \in \mathbf{S}$ or none among $\mathbf{s}' \in \mathbf{S}$ satisfies $\mathbf{s} \leq \mathbf{s}'$.

The construction of functions α^n goes on by steps. Each step $\nu \in \omega$ defines a number $r_\nu > r_{\nu-1}$ and adds a finite group of functions α^m, $r_{\nu-1} < m \leq r_\nu$.

At the beginning, $r_{-1} = -1$.

Let $\nu \in \omega$. We assume that all functions α^m, $m \leq n = r_{\nu-1}$, have been defined so that condition 1 of the lemma holds, that is, $\alpha^m \in \Delta_{m+3}[0^{(\omega)}, \zeta_{<m}, \alpha^{<m}]$ for all $m \leq n$, and condition 2 holds in the local form: every α^m, $m \leq n$, is $\Sigma_{m+2}[0^{(\omega)}, \zeta_{<m}, \alpha^{<m}]$ generic. We show how a number $r_\nu > n = r_{\nu-1}$ and a finite sequence of functions α^j, $n < j \leq r_\nu$, can be added to those already defined to decide a certain set $\mathbf{S} \subseteq \mathbf{Seq}$.

Let $\nu = \ulcorner m, k \urcorner = 2^m(2k+1) - 1$, so that $m \leq \nu$. Let $\{\mathbf{S}_m(k') : k' \in \omega\}$ be an enumeration of all $\Sigma_{m+2}[0^{(\omega)}, \zeta_{<m}, \alpha^{<m}]$ sets $\mathbf{S} \subseteq \mathbf{Seq}$, fixed at the first step ν' such that $\nu' = \ulcorner m, k' \urcorner$ for some k'. The set $\mathbf{S} = \mathbf{S}_m(k)$ is to be decided at the step ν. The reasoning is based on the following fact:

CLAIM. *Assume that* $m \leq n$ *and* $\mathbf{S} \subseteq \mathbf{Seq}[\geq m]$ *is a* $\Sigma_{m+2}[0^{(\omega)}, \zeta_{<m}, \alpha^{<m}]$ *set. Then there exists* $\mathbf{s} \in \mathbf{Seq}[\geq m]$, *deciding* $\mathbf{S}$ *and such that* $\mathbf{s}[j] \subset \alpha^j$ *for all* j, $m \leq j \leq n$.

PROOF. The proof goes on by induction on $n - m$. That is, we prove the claim in the case $m = n$ and then demonstrate how the case of m, n follows from $m+1, n$.

Thus let first $m = n$. The set $S = \{s \in \mathrm{Seq} : \exists\, \mathbf{s} \in \mathbf{S}\ (\mathbf{s}[n] = s)\}$ is obviously $\Sigma_{n+2}[0^{(\omega)}, \zeta_{<n}, \alpha^{<n}]$, therefore by the genericity of α^n some $s \in \mathrm{Seq}$, $s \subset \alpha^n$, decides S.

Case 1 : $s \in S$. Let this be sertified by $\mathbf{s} \in \mathbf{S}$. Then $\mathbf{s}$ decides $\mathbf{S}$ and satisfies $\mathbf{s}[n] = s \subset \alpha^n$, as required.

Case 2 : none among $s' \in S$ expands s. We define $\mathbf{s} \in \mathbf{S}$ by $\mathrm{dom}\,\mathbf{s} = \{n\}$ and $\mathbf{s}(n) = s$. It is asserted that none among $\mathbf{s}' \in \mathbf{S}$ satisfies $\mathbf{s} \leq \mathbf{s}'$, so that $\mathbf{s}$

decides $\mathbf{S}$. Suppose on the contrary that, $\mathbf{s}' \in \mathbf{S}$ and $\mathbf{s} \leq \mathbf{s}'$. Then $s' = \mathbf{s}[n] \in S$ and $s \subset s'$, a contradiction.

We now carry out the induction step. Thus it is assumed that $m < n$. The set

$$\mathbf{T} = \{\mathbf{t} \in \mathbf{Seq}[> m] : \exists \mathbf{s} \in \mathbf{S}\ (\mathbf{s}[m] \subset \alpha^m\ \&\ \mathbf{t} = \mathbf{s}[> m])\}$$

is obviously $\Sigma_{m+3}[0^{(\omega)}, \zeta_{<m+1}, \alpha^{<m+1}]$, even $\Sigma_{m+2}[0^{(\omega)}, \zeta_{<m}, \alpha^{<m+1}]$, therefore, by the induction hypothesis, there exists $\mathbf{t} \in \mathbf{Seq}[> m]$ which decides $\mathbf{T}$ and satisfies $\mathbf{t}[j] \subset \alpha^j$ for all j, $m < j \leq n$.

Case 1: $\mathbf{t} \in \mathbf{T}$. Let this be demonstrated by some $\mathbf{s} \in \mathbf{S}$. Then $\mathbf{s}$ decides $\mathbf{S}$ and satisfies $\mathbf{s}[j] \subset \alpha^j$, all $j = m, \dots, n$, as required.

Case 2: none among $\mathbf{t}' \in \mathbf{T}$ satisfies $\mathbf{t} \leq \mathbf{t}'$. The set

$$S = \{s \in \mathrm{Seq} : \exists \mathbf{s} \in \mathbf{S}\ (\mathbf{s}[m] = s\ \&\ \mathbf{t} \leq \mathbf{s}[> m])\}$$

is $\Sigma_{m+2}[0^{(\omega)}, \zeta_{<m}, \alpha^{<m}]$. Therefore there exists $s \in \mathrm{Seq}$, $s \subset \alpha^m$, deciding S.

Case 2.1: $s \in S$. Let this be demonstrated by $\mathbf{s} \in \mathbf{S}$; then $\mathbf{t}' = \mathbf{s}[> m] \in \mathbf{T}$ and $\mathbf{t} \leq \mathbf{t}'$, a contradiction with the Case 2 assumption.

Case 2.2: none among $s' \in S$ expands s. We define $\mathbf{s} \in \mathbf{Seq}[\leq m]$ by $\mathbf{s}(m) = s$ and $\mathbf{s}[> m] = t$. It is asserted that none among $\mathbf{s}' \in \mathbf{S}$ satisfies $\mathbf{s} \leq \mathbf{s}'$—so that $\mathbf{s}$ decides $\mathbf{S}$ and is as required. Assume, on the contrary, $\mathbf{s}' \in \mathbf{S}$ and $\mathbf{s} \leq \mathbf{s}'$. Then $s \subset s' = \mathbf{s}'[m]$ and $s' \in S$, a contradiction with the Case 2.2 assumption. ⊣

We return to the proof of the lemma. Let $\mathbf{s} \in \mathbf{Seq}[\geq m]$ be given by the claim, that is, $\mathbf{s}$ decides $\mathbf{S} = \mathbf{S}_m(k)$ and satisfies $\mathbf{s}[j] \subset \alpha^j$ for all $j = m, \dots, n$. We put $r_\nu = 1 + \max\{n, \max \mathrm{dom}\, \mathbf{s}\}$. For every j, $n < j \leq r_\nu$, let $\alpha^j \in 2^\omega$ be an arbitrary $\Sigma_{j+2}[0^{(\omega)}, \zeta_{<j}, \alpha^{<j}]$ generic function of the class $\Delta_{j+3}[0^{(\omega)}, \zeta_{<j}, \alpha^{<j}]$ which expands $\mathbf{s}[j]$. This ends the step ν.

It is asserted that $\langle \alpha^n : n \in \omega \rangle$ is the required sequence. Condition 2 of the lemma is guaranteed by the construction. Thus we have to check that every $\alpha^{\geq m}$ is $\Sigma_{m+2}[0^{(\omega)}, \zeta_{<m}, \alpha^{<m}]$ generic.

Evidently it suffices to prove that, for any $\Sigma_{m+2}[0^{(\omega)}, \zeta_{<m}, \alpha^{<m}]$ set $\mathbf{S} \subseteq \mathbf{Seq}[\geq m]$, there exists $\mathbf{s} \in \mathbf{Seq}[\geq n]$ which decides $\mathbf{s}$ and satisfies $\mathbf{s}(j) \subset \alpha^j$ for all $j \in \mathrm{dom}\, \mathbf{s}$. To get such $\mathbf{s}$, let $\mathbf{S} = \mathbf{S}_m(k)$, $\nu = \ulcorner m, k \urcorner$, and $n = r_{\nu-1}$. By the construction at the step ν, a certain $\mathbf{s} \in \mathbf{Seq}[\geq m]$ decides $\mathbf{S}$ and satisfies, first $\mathbf{s}[j] \subset \alpha^j$ for all $j = m, \dots, n$, and second $\mathbf{s}[j] \subset \alpha^j$ for all $j = n+1, \dots, r_\nu$. It follows that $\mathbf{s}[j] \subset \alpha^j$ for all $j \in \mathrm{dom}\, \mathbf{s}$, as required. ⊣

Proof of Theorem 1. Thus let $\langle \alpha^n : n \in \omega \rangle$ be a sequence of elements of 2^ω satisfying conditions 1 and 2 of Lemma 20. In particular, every α^n is the characteristic function of some $Z \in \mathscr{Z}$ by condition 1 and the closure properties of $\mathscr{Z}$. We put, for all n and z,

$$\alpha^n_z = (\alpha^n)_z, \text{so that} \{\alpha^n_z : n, z \in \omega\} = \{(\alpha^{\geq 0})_k : k \in \omega\}.$$

Then we put $Z'_n = \{2z : z \in Z_n\} \bigcup \mathrm{ODD}$, where $\mathrm{ODD} = \{\text{all odd numbers}\}$. To make every Z'_n (therefore every Z_n) definable, we put

$$\mathscr{F} = \{\gamma \in \omega^\omega : \gamma \text{ is arithmetical in a finite number of functions } \alpha^n_z,\ n \in \omega \text{ and } z \in Z'_n\}.$$

FACT 1. *For all n, both Z_n and α^n are analytically definable in $\mathscr{F}$.*

PROOF. First, by the definability properties of functions α^n, α^n is analytically definable in $\mathscr{F}$ as soon as $Z_0, \dots, Z_{n-1}$ and $\alpha^0, \dots, \alpha^{n-1}$ are analytically definable in $\mathscr{F}$. Furthermore, $Z'_n = \{z \in \omega : \exists \gamma\ (\gamma = \alpha^n_z)\}$ in $\mathscr{F}$ by Lemma 9 applied to $\alpha^{\geq n}$. Therefore, Z'_n is analytically definable in $\mathscr{F}$ as soon as α^n is analytically definable in $\mathscr{F}$. After this remark, the proof goes on automatically by induction. ⊣

FACT 2. *Let $Y \subseteq \omega$ be analytically definable in $\mathscr{F}$. Then $Y \in \mathscr{Z}$.*

PROOF. Let Y be, say, Σ^1_{n-1} in $\mathscr{F}$. We are going to prove that Y is arithmetical in $0^{(\omega)}, \zeta_{<n}, \alpha^{<n}, \alpha^n$. This would imply $Y \in \mathscr{Z}$ by the closure properties of the set $\mathscr{Z}$ and the definability properties of the functions α^m.

To reduce definability in $\mathscr{F}$ to definability in a more convenient set, we define

$$\mathscr{F}_1 = \{\gamma \in \omega^\omega : \gamma \text{ is arithmetical in a finite number of functions } \alpha^m_z,\ m \geq n \text{ and } z \in Z'_m\}.$$

Lemma 14 implies that Y is $\Sigma^1_n[\zeta_{<n}, \alpha^{<n}]$ in $\mathscr{F}_1$. Then we set

$$\mathscr{F}_2 = \{\gamma \in \omega^\omega : \gamma \text{ is arithmetical in a finite number of functions } \alpha^m_z,\ m \geq n \text{ and } z \text{ arbitrary}\}.$$

Since $\alpha^{\geq n}$ is $\Sigma_{n+2}[0^{(\omega)}, \zeta_{<n}, \alpha^{<n}]$ generic, Corollary 19 (Part 2) implies that Y is $\Sigma^1_n[\zeta_{<n}, \alpha^{<n}]$ in $\mathscr{F}_2$, too. We finally put

$$\mathscr{F}_3 = \{\gamma \in \omega^\omega : \gamma \text{ is arithmetical in a finite number of functions } \alpha^n_z,\ z \text{ arbitrary}\}.$$

Corollary 19.2 again implies that Y is $\Sigma^1_n[\zeta_{<n}, \alpha^{<n}]$ in $\mathscr{F}_3$. We observe that $\mathscr{F}_3 = \mathscr{F}(\alpha^n, \omega)$ in the sense of Section 2. Thus Y is $\Sigma_{n+2}[0^{(\omega)}, \zeta_{<n}, \alpha^{<n}, \alpha^n]$ by Lemma 16, as required. ⊣

This ends the proof of Theorem 1. ⊣

REFERENCES

[1] L. HARRINGTON, *The constructible reals can be anything*, a preprint dated May 1974 with several addendums dated up to October 1975.

[2] P. HINMAN, ***Recursion theoretic hierarchies***, Springer-Verlag, Berlin, 1978.

[3] V. KANOVEI, *The set of all analytically definable sets of natural numbers can be defined analytically*, ***Mathematics of the USSR-Izvestiya***, vol. 15 (1980), pp. 469–500.

[4] R. W. KAYE, *Models of Peano arithmetic*, ***Oxford logic guides***, vol. 15, Oxford Science Publications, 1991.

[5] D. SCOTT, *Algebras of sets binumerable in complete extensions of arithmetic*, ***Recursive function theory***, Proceedings of the Fifth Symposium in Pure Mathematics of the American Mathematical Society, 1962, pp. 117–121.

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