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AN ULM-TYPE CLASSIFICATION THEOREM FOR EQUIVALENCE RELATIONS IN SOLOVAY MODEL

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Abstract. We prove that in the Solovay model, every OD equivalence relation, E, over the reals, either admits an OD reduction to the equality relation on the set of all countable (of length $< \omega_1$) binary sequences, or continuously embeds E_0, the Vitali equivalence.

If E is a Σ^1_1 (resp. Σ^1_2) relation then the reduction above can be chosen in the class of all Δ_1 (resp. Δ_2) functions.

The proofs are based on a topology generated by OD sets.

Introduction. The solution of the continuum problem leaves open a variety of related questions. For instance, if one works in descriptive set theory then one may be interested to know how different uncountable cardinals can be presented in the real line. This research direction can be traced as far in the past as the beginning of the century; indeed Lebesgue [7] found such a presentation for $\aleph_1$, the least uncountable cardinal.

The construction given in [7] merits a brief review. One can associate, in an effective way, a set of rationals Q_x, to each real x, so that every set Q of rationals has the form Q_x for some (perhaps, not unique) x. Let, for a countable ordinal α,

$$X_\alpha = \{x : Q_x \text{ is wellordered as a set of rationals and has the order type } \alpha\}.$$

Then the sets X_α, $\alpha < \omega_1$, are nonempty and pairwise disjoint; therefore we represent $\aleph_1$ in the reals, as the sequence of the sets X_α.

This example is a particular case of a much more general construction.

Let E be an equivalence relation on the reals. Let κ be the cardinal of the set of all E-equivalence classes; then $\kappa \leq 2^{\aleph_0}$. One may view the partition of the real line on E-equivalence classes as a presentation of κ in the reals.

For instance, in Lebesgue's example, the equivalence relation can be defined as follows: $x \mathrel{\mathscr{L}} y$ iff either (1) both Q_x and Q_y are wellordered and have the same order type, or (2) both Q_x and Q_y are *not* wellordered. The $\mathscr{L}$-equivalence classes are the sets X_α, $\alpha < \omega_1$, plus one more "default" class of all reals x such that Q_x is not wellordered.

Of course, one can present *every* cardinal $\kappa \leq 2^{\aleph_0}$ in this way by a suitable equivalence relation on reals. But the problem becomes much more difficult when

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one intends to involve only those equivalence relations which belong to a certain type of pointsets, for instance are Borel, or analytic etc. relations. (The Lebesgue equivalence $\mathscr{L}$ is analytic.)

This leads us to the following question: let $\mathbf{\Gamma}$ be a class of pointsets; how many equivalence classes relations in $\mathbf{\Gamma}$ may have ?

An associated question is how to "count" the classes. Generally speaking, counting is a numbering of a given set of mathematical objects by mathematical objects of another type, usually more primitive in some sense. In particular, one could try to use ordinals (e.g., natural numbers) to count the classes. This works well as long as we are not interested in the "effectiveness" of the counting. Otherwise we face problems even with very simple relations. (Consider the equality as an equivalence relation. Then one cannot define in **ZFC** an "effective" in any reasonable sense counting of the equivalence classes, *alias* reals, by ordinals.)

The other natural possibility is to use *sets of* ordinals (e.g., reals) to count the equivalence classes.[1] Note that the next step, that is, counting by *sets of sets* of ordinals, would be silly because the classes themselves are of this type.

DEFINITION (Informal). An equivalence relation is *discrete* iff it admits an "effective" enumeration of the equivalence classes by ordinals. An equivalence relation is *smooth* iff it admits an "effective" enumeration of the equivalence classes by *sets of* ordinals. $\dashv$

Of course the definition has a precise meaning only provided one makes clear the meaning of "effective". However in any reasonable case we have the following two counterexamples:

EXAMPLE 1. The equality relation on a perfect set of reals is *not* discrete.

EXAMPLE 2. The Vitali equivalence relation is *not* smooth.

(*Not* here means that one cannot prove in **ZFC** the existence of the required enumerations among real-ordinal definable functions. However different additional axioms, for instance the axiom of constructibility, make all equivalence relations discrete in a certain sense.)

At the first look, there should be plenty of other counterexamples. However, in certain particular but quite representative cases one can prove a *dichotomy theorem* which says that an equivalence relation is not discrete (resp. not smooth) iff it somehow includes Example 1 (resp. Example 2).

To be more exact, let us review some basic notation. We refer to [2, 4, 6] for a more substantial review with details and explanations.

Let E and E' be equivalence relations on resp. sets X, X'.

A function $U : X' \longrightarrow X$ is a *reduction* of E' to E iff the equivalence $x \mathrel{\mathsf{E}'} y \longleftrightarrow U(x) \mathrel{\mathsf{E}} U(y)$ holds for all $x, y \in X'$.

For any set X, $\mathsf{D}(X)$ (the *diagonal*) will denote the equality relation on X. An *enumeration* of the E'-equivalence classes (by elements of X) is a reduction of E' to $\mathsf{D}(X)$. In other words $U : X' \longrightarrow X$ enumerates E'-classes iff we have $x \mathrel{\mathsf{E}'} y \longleftrightarrow U(x) = U(y)$ for all $x, y \in X'$.

[1]There exist mathematical examples, in probability and the measure theory, based on this type of enumeration of the equivalence classes, see Harrington, Kechris, and Louveau [2].

A 1-1 reduction is called an *embedding*. E *continuously embeds* E' iff there exists a continuous embedding of E' to E. In the case when X' is the *Cantor set* $\mathscr{D} = 2^\omega$ (with the usual topology), E continuously embeds E' if and only if there exists a perfect set $P \subseteq X$ such that $\langle P; \mathsf{E} \restriction P\rangle$ is homeomorphic to $\langle X'; \mathsf{E}'\rangle$. In other words, embedding E continuously means in this case that E contains a homeomorphic copy of E'.

In particular E continuously embeds the equality relation on $\mathscr{D}$ iff there exists a perfect set of E-inequivalent points.

Finally, E_0 is the *Vitali equivalence relation* on $\mathscr{D} = 2^\omega$, defined as follows: $x \mathrel{\mathsf{E}_0} y$ iff $x(n) = y(n)$ for almost all (i.e., all but finitely many) $n \in \omega$.

The main theorem. This paper intends to complete the pattern suggested by the following three classical theorems on equivalence relations.

"Borel-1". *Each Borel equivalence relation on the reals, either has countably many equivalence classes or admits a perfect set of pairwise inequivalent reals.* (Silver [9], in fact for $\mathbf{\Pi}^1_1$-relations.)

"Borel-2". *Each Borel equivalence relation on the reals, either admits a Borel enumeration of the equivalence classes by reals*[2], *or continuously embeds the Vitali equivalence relation* E_0. (Harrington, Kechris, and Louveau [2].)

"Solovay-1". *In the Solovay model*[3], *each* R-OD (real-ordinal definable) *equivalence relation on reals, either has* $\le \aleph_1$ *equivalence classes and admits a* R-OD *enumeration of them, or admits a perfect set of pairwise inequivalent reals.* (Stern [11].) Thus, the results "Borel-1" and "Solovay-1" say (informally) that an equivalence relation either is discrete or contains a continuous copy of Example 1 above. Similarly "Borel-2" says that an equivalence relation either is smooth or contains a continuous copy of Example 2 above.

THEOREM 1 ("Solovay-2"). *The following is true in the Solovay model. Let* E *be a* R-OD *equivalence relation on reals. Then one and only one of the following two statements holds:*

(I) E *admits a* R-OD *enumeration of the equivalence classes by elements of* $2^{<\omega_1}$.[4] *If moreover* E *is a* $\mathbf{\Sigma}^1_1$ *(resp.* $\mathbf{\Sigma}^1_2$*) equivalence relation then the enumeration exists in the class* $\mathbf{\Delta}^{\mathrm{HC}}_1$ *(resp.* $\mathbf{\Delta}^{\mathrm{HC}}_2$*)*[5]*;*
(II) E *continuously embeds* E_0.

This is the main result of this paper.

REMARK 1. Hjorth [3] obtained a similar theorem in a strong determinacy hypothesis (**AD** holds 11 in L[`reals`]), yet with a weaker part (I): an OD reduction to the equality relation on a set 2^κ, $\kappa \in \mathtt{Ord}$.

[2] That is, admits a Borel reduction to the equality relation on the reals. Relations of this kind are called *smooth.*

[3] By the *Solovay model* we mean a generic extension $\mathrm{L}[G]$ of L, the class of all constructible sets, by a generic over L subset of a certain notion of forcing $\mathscr{P}^\Omega \in \mathrm{L}$ which provides the collapse of all cardinals in L, smaller than a fixed inaccessible cardinal Ω, to ω, see Solovay [10] or Section 1 below. In this model, all projective sets are Lebesgue measurable.

[4] $2^{<\omega_1} = \bigcup_{\alpha<\omega_1} 2^\alpha$ is the set of all countable (of any length $< \omega_1$) binary sequences.

[5] $\mathbf{\Delta}^{\mathrm{HC}}_n$ denotes the class of all subsets of HC (the family of all hereditarily countable sets) which are Δ_n in HC by formulas which may contain arbitrary *reals* as parameters.

REMARK 2. The statements (I) and (II) are incompatible. Indeed otherwise there would exist a R-OD enumeration $U : \mathscr{D} \longrightarrow 2^{<\omega_1}$ of E_0-equivalence classes by elements of $2^{<\omega_1}$. Let U be OD$[z]$, $z \in \mathscr{D}$. Then for each $p \in P = \operatorname{ran} U$ (note that $P \subseteq 2^{<\omega_1}$), $U^{-1}(p)$ is an E_0-equivalence class, a countable OD$[p, z]$ subset of $\mathscr{D}$. In the Solovay model, this implies $U^{-1}(p) \subseteq \mathrm{L}[z, p]$ for all p. We obtain an OD$[z]$ choice function $g : P \longrightarrow \mathscr{D}$ such that $g(p) \in U^{-1}(p)$ for all p. Then $\operatorname{ran} g$ is an R-OD selector for E_0, hence a nonmeasurable R-OD set, which is a contradiction with known properties of the model.

REMARK 3. $2^{<\omega_1}$ cannot be replaced in Theorem 1 by an essentially smaller set. To see this let us consider the OD equivalence relation C on pairs of reals, defined as follows: $\langle z, x\rangle \mathrel{\mathsf{C}} \langle z', x'\rangle$ iff

— *either* z and z' code the same countable ordinal and x and x' code, in the sense of z and z' respectively, the same subset of the ordinal,

— *or* both z and z' do not code an ordinal while x and x' are arbitrary.

Clearly C-equivalence classes can be put in a 1-1 OD correspondence with all elements of $2^{<\omega_1}$. Therefore C does not embed E_0 continuously in the Solovay model (see Remark 2). Moreover any set W such that C admits a R-OD enumeration of the classes by elements of W has a subset $W' \subseteq W$ which is in 1-1 R-OD correspondence with $2^{<\omega_1}$. In particular, the reals do not satisfy this condition in the Solovay model. (Indeed $2^{<\omega_1}$ has R-OD subsets of cardinality exactly $\aleph_1$ while the reals do not have those in the Solovay model.)

REMARK 4. Even in the case of $\mathbf{\Sigma}^1_1$ equivalence relations, $2^{<\omega_1}$ cannot be replaced by the reals in (I). Indeed the Σ^1_1 equivalence relation $x \mathrel{\mathsf{E}} y$ iff either the reals x, y code the same (countable) ordinal or both x and y do not code an ordinal (an example in [4]) neither admits a $\mathbf{\Delta}^1_2$ enumeration of the classes by reals nor embeds E_0 via a $\mathbf{\Delta}^1_2$ function, in $\mathbf{ZFC} + \forall$ real x $(\omega_1^{\mathrm{L}[x]} < \omega_1)$. (In the Solovay model, $\mathbf{\Delta}^1_2$ can be strengthened to R-OD.) This shows that the "Glimm-Effros" dichotomy theorem of [2] (theorem "Borel-2" above) cannot be generalized from Borel to $\mathbf{\Sigma}^1_1$ equivalence relations in **ZFC**.

REMARK 5. On the other hand, $\mathbf{\Sigma}^1_1$ equivalence relations tend to satisfy a looser "*Ulm*" dichotomy.[6] In particular, Hjorth and Kechris [4] proved that every $\mathbf{\Sigma}^1_1$ equivalence relation *with Borel classes* either admits a $\mathbf{\Delta}_1$ enumeration of the classes by elements of $2^{<\omega_1}$, or embeds E_0 continuously; furthermore the requirement that the E-classes are Borel can be dropped in the assumption $\forall$ real x ($x^{\#}$ exists).[7] Thus, Theorem 1 proves that the "Ulm" dichotomy is available in the Solovay model. This yields a partial answer to a question of Hjorth and Kechris [4].[8] It

[6]The notion introduced in [4]. Hjorth and Kechris refer to the Ulm classification of countable abelian p-groups.

[7]The author proved in [5] the "furthermore" result assuming that each real belongs to a generic extension of L (which is incompatible with the "sharps" hypothesis). S. D. Friedman and B. Velickovic proved the result from the hypothesis of existence of a weakly compact cardinal in every class of the form $\mathrm{L}[x]$, x being a real, see [1] for an exposition of his idea with respect to another problem.

[8]"We do not know how to prove that at least one of (I) (with a Δ_1^{HC} enumeration) or (II) must hold (for $\mathbf{\Sigma}^1_1$ relations) *without making use of the assumption of sharps.*" (*The end of Section 5 in* [4].) *Since* the sharps hypothesis fails in the Solovay model, we observe that the hypothesis is not necessary for the dichotomy.

would be very interesting to prove a dichotomy theorem of this type for Σ^1_1 relations in **ZFC** without any additional hypothesis.

The remaining structure of this article is organized as follows.

Section 1 outlines the proof of Theorem 1. A topology $\mathscr{T}$ generated by OD sets in the Solovay model (a counterpart of the Gandy-Harrington topology) is introduced. Similarly to Harrington, Kechris, and Louveau [2], we have two cases: either the equivalence relation E of consideration is closed in the product topology $\mathscr{T}^2$ or it is not closed. The plan of the proof of Theorem 1 is to demonstrate that the first case provides (I) while the second leads to (II).

Section 2 reviews some important properties of the Solovay model.

Section 3 proves that $\mathscr{T}^2$-closed equivalence relations E satisfy the requirements of Item (I) of Theorem 1. The argument for the "moreover" part of Item (I) includes the idea of forcing the equivalence of mutually generic reals over countable models, due to Stern [11] and Hjorth and Kechris [4].

Section 4 begins consideration of the case when the given equivalence relation is not $\mathscr{T}^2$-closed. We define forcing notions $\mathbb{X}$ and $\mathbb{P}$ associated with $\mathscr{T}$ and $\mathscr{T}^2$ respectively. In particular it is demonstrated that the intersection of an $\mathbb{X}$-generic set is nonempty. The set $H = \{x : [x]_{\mathsf{E}} \subsetneqq [x]_{\overline{\mathsf{E}}}\}$, nonempty as soon as we assume $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$, is considered. ($\overline{\mathsf{E}}$ is the $\mathscr{T}^2$-closure of E.)

We accomplish the case when the given relation E is not $\mathscr{T}^2$-closed in Section 5. It is demonstrated that in this case, E continuously embeds E_0. The construction of the embedding is based on a technical idea of Harrington, Kechris, and Louveau [2], but we shall proceed differently, making use of straightforward forcing arguments rather than Choquet games, which yields a little bit more elementary construction.

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Notation. We shall use the *Cantor set* $\mathscr{D} = 2^\omega$ rather than the Baire space $\mathscr{N} = \omega^\omega$ as the principal space in this paper. Elements of $\mathscr{D}$ will be called *reals* below. In the rest we shall follow the ordinary notation. Sometimes the F-image $\{F(x) : x \in X\}$ of a set X will be denoted by $F"X$.

V will denote the universe of all sets, L the constructible universe.

We shall use *sans serif* characters like E to denote equivalence relations and other binary relations.

By *dense* we shall always mean: *open dense* (for subsets of p.o. sets).

§1. Approach to the main theorem. The proof of Theorem 1 resembles the proof of the "Borel" Glimm-Effros theorem in Harrington, Kechris, and Louveau [2]; in particular the dichotomy will be determined by an answer to the question whether the given relation E is closed in a certain topology on $\mathscr{D}^2$.

First of all, we review the definition and some properties of the Solovay model.

For an ordinal α, $\mathscr{P}_\alpha = \alpha^{<\omega} = \bigcup_{n\in\omega} \alpha^n$ denotes the forcing to collapse α down to ω. It is ordered as follows: $u \le v$ (u is *stronger* than v) iff $v \subseteq u$.

For $\lambda \in \mathtt{Ord}$, we let[9] $\mathscr{P}_{<\lambda}$ be the "finite support" product of all sets $\mathscr{P}_\alpha$, $\alpha < \lambda$. In other words, $\mathscr{P}_{<\lambda}$ is the set of all functions p such that $\mathtt{dom}\, p$ is a finite subset of λ and $p(\alpha) \in \mathscr{P}_\alpha$ for each $\alpha < \lambda$, $\alpha \in \mathtt{dom}\, p$, with the order: $p \le q$ iff $\mathtt{dom}\, q \subseteq \mathtt{dom}\, p$ and $p(\alpha) \le q(\alpha)$ for all $\alpha \in \mathtt{dom}\, q$.

DEFINITION. Let M (a set or a proper class) be a transitive model of **ZFC**, containing Ω, an inaccessible cardinal in M. By *Ω-Solovay extension* of M we shall understand a generic extension of the form $M[G]$, where $G \subseteq \mathscr{P}_{<\Omega}$ is $\mathscr{P}_{<\Omega}$-generic over M. $\dashv$

DEFINITION. Ω-SM is the following hypothesis:

Ω-SM : *Ω is inaccessible in* L, *the class of all constructible sets, and the universe* V *of all sets is a $\mathscr{P}_{<\Omega}$-generic extension of* L. $\dashv$

The following definition introduces the OD topology.

DEFINITION. $\mathscr{T}$ will denote the topology generated on a given set X (typically $X = \mathscr{D} = 2^\omega$, the Cantor set) by all OD subsets of X. By $\mathscr{T}^2$ we shall denote the product of two copies of $\langle \mathscr{D}\,;\, \mathscr{T} \rangle$, a topology on $\mathscr{D}^2$. $\dashv$

Let us consider an OD equivalence relation E on $\mathscr{D}$.

For any set $X \subseteq \mathscr{D}$, we put $[X]_{\mathsf{E}} = \{y : \exists x \in X\, (x \mathrel{\mathsf{E}} y)\}$, the E-*saturation* of X. A set X is E-*saturated* iff $[X]_{\mathsf{E}} = X$.

We define $\overline{\mathsf{E}}$ to be the $\mathscr{T}^2$-closure of E in $\mathscr{D}^2$. Thus, $x \mathrel{\overline{\not\mathsf{E}}} y$ iff there exist OD sets X and Y containing resp. x and y and such that $x' \mathrel{\not\mathsf{E}} y'$ for all $x' \in X$, $y' \in Y$. Clearly X can be chosen as an E-invariant set (otherwise change X to $[X]_{\mathsf{E}}$), and then Y can be replaced by the complement of X, so that

$$x \mathrel{\overline{\mathsf{E}}} y \;\longleftrightarrow\; \forall X\, [\, X \text{ is OD and } \mathsf{E}\text{-invariant} \;\longrightarrow\; (x \in X \;\longleftrightarrow\; y \in X)\,].$$

Therefore $\overline{\mathsf{E}}$ is an equivalence relation, too.

We now come to the splitting point of the dichotomy: either $\mathsf{E} = \overline{\mathsf{E}}$ or $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$.

THEOREM 2. *Assume* Ω-SM. *Suppose that* E *is an* OD *equivalence relation on* $\mathscr{D}$. *Then*

- $\langle \mathrm{I} \rangle$ *If* $\mathsf{E} = \overline{\mathsf{E}}$ *then* E *admits an* OD *enumeration of the equivalence classes by elements of* $2^{<\omega_1}$. *If moreover* E *is a* Σ^1_1 *(resp.* Σ^1_2*) equivalence relation then the enumeration exists in the class* Δ^{HC}_1 *(resp.* Δ^{HC}_2*)*;
- $\langle \mathrm{II} \rangle$ *If* $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$ *then* E *continuously embeds* E_0.

PROOF OF THEOREM 1 FROM THEOREM 2. Theorem 2 is a re-formulation of the "lightface" case in Theorem 1. In the case when the relation E is OD[z] (resp. $\Sigma^1_1[z]$ or $\Sigma^1_2[z]$ in the "moreover" part of $\langle \mathrm{I} \rangle$) for a real z, one simply introduces z as a parameter through the reasoning. In particular one considers $\mathscr{T}[z]$, the topology generated by OD[z] sets, rather than $\mathscr{T}$, etc. $\dashv$ (of Theorem 1)

We prove part $\langle \mathrm{I} \rangle$ of Theorem 2 in Section 3. Part $\langle \mathrm{II} \rangle$ will be considered in the two following sections.

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[9]The forcing notion $\mathscr{P}_{<\lambda}$ is equivalent to $\mathscr{P}^\lambda$ in Solovay [10].

§2. **Reals and sets of reals in Solovay model.** This section presents some properties of the Solovay model important for the proof of Theorem 2. They are mostly related to reals and sets of reals.

DEFINITION. Let Ω be an ordinal. A set x will be called *Ω-weak over M* (M is a model of **ZFC**, possibly a proper class) iff x belongs to a $\mathscr{P}_\alpha$-generic extension of M for some $\alpha < \Omega$. (Recall that $\mathscr{P}_\alpha = \alpha^{<\omega}$.) ⊣

PROPOSITION 3. *Assume* Ω-SM. *Then* $\Omega = \omega_1$. *Furthermore, suppose that* $S \subseteq$ Ord *is Ω-weak over* L. *Then*

1. *Ω is inaccessible in* L$[S]$ *and* V *is an Ω-Solovay extension of* L$[S]$.
2. *If Φ is a sentence containing only sets in* L$[S]$ *as parameters then Λ decides Φ in the sense of $\mathscr{P}_{<\Omega}$ as a forcing notion over* L$[S]$.
3. *If a set $X \subseteq$* L$[S]$ *is* OD$[S]$ *then $X \in$* L$[S]$.

(Λ is the empty function. OD$[S]$ means: *S-ordinal-definable*, that is, definable by an $\in$-formula containing S and ordinals as parameters.)

Thus, the Solovay model is seen from each subclass L$[S]$ generated by an Ω-weak set in one and the same regular way. The proof (essentially a copy of the proof of Theorem 4.1 in Solovay [10]) is based on several lemmas, including the following crucial lemma:

LEMMA 4 (Lemma 4.4 in [10]). *Let M be a transitive model of* **ZFC**, $\lambda \in \mathtt{Ord} \cap M$. *Suppose that M' is a $\mathscr{P}_\lambda$-generic extension of M and M'' is a $\mathscr{P}_\lambda$-generic extension of M'. Let $S \in M'$, $S \subseteq$ Ord. Then M'' is a $\mathscr{P}_\lambda$-generic extension of $M[S]$.* ⊣

PROOF OF THE PROPOSITION. In accordance with the assumption Ω-SM, we have V $=$ L$[G]$ where $G \subseteq \mathscr{P}_{<\Omega}$ is generic over L.

Item 1. By definition, S belongs to a $\mathscr{P}_\alpha$-generic extension of L where $\alpha < \Omega$. Then $S \in$ L$[x]$ for a real x. It follows (Corollary 3.4.1 in [10]) that there exists an ordinal $\lambda < \Omega$ such that S belongs to the model $M' =$ L$[G_{\le\lambda}]$, for some $\lambda < \Omega$, where $G_{\le\lambda} = G \cap \mathscr{P}_{\le\lambda}$ ($\le\lambda$ means $<\lambda+1$).

Note that $G_{\le\lambda}$ is $\mathscr{P}_{\le\lambda}$-generic over L. Therefore by Lemma 4.3 in Solovay [10], M' is a $\mathscr{P}_\lambda$-generic extension of L.

Let us consider the next step $\lambda + 1$. Obviously the model $M_{\lambda+1} =$ L$[G_{\le\lambda+1}]$ is a $\mathscr{P}_{\lambda+1}$-generic extension of M'. Since $\mathscr{P}_{\lambda+1}$ is order isomorphic to the product[10] $\mathscr{P}_\lambda \times \mathscr{P}_{\lambda+1}$, we conclude that $M_{\lambda+1}$ is a $\mathscr{P}_{\lambda+1}$-generic extension of a certain $\mathscr{P}_\lambda$-generic extension M'' of M'.

Now M'' is a $\mathscr{P}_\lambda$-generic extension of L$[S]$ by Lemma 4, therefore a $\mathscr{P}_{\le\lambda}$-generic extension of L$[S]$ as well by Lemma 4.3 in [10].

It follows that $M_{\lambda+1}$ is a $\mathscr{P}_{\le\lambda+1}$-generic extension of L$[S]$.

Finally $M =$ L$[G]$ is a $\mathscr{P}_{\ge\lambda+2}$-generic extension of $M_{\lambda+1}$. This ends the proof of Item 1 of the proposition.

Items 2 *and* 3. It suffices to refer to Item 1 and apply resp. Lemma 3.5 and Corollary 3.5 in [10] for L$[S]$ as the initial model. ⊣

Coding of reals and sets of reals. If $G \subseteq \mathscr{P}_\alpha = \alpha^{<\omega}$ is $\mathscr{P}_\alpha$-generic over a transitive model M (M is a set or a class) then $f = \bigcup G$ maps ω onto α, so that α is countable

[10] $\mathscr{P}_\lambda \times \mathscr{P}_{\lambda+1}$ is understood here as the set of all pairs $\langle p, q\rangle$ such that $p \in \mathscr{P}_\lambda$, $q \in \mathscr{P}_{\lambda+1}$, and $\operatorname{dom} p = \operatorname{dom} q$.

in $M[G] = M[f]$. Functions $f : \omega \longrightarrow \alpha$ obtained this way will be called $\mathscr{P}_\alpha$-*generic over* M.

We let $\mathbb{F}_\alpha(M)$ be the set of all $\mathscr{P}_\alpha$-generic over M functions $f \in \alpha^\omega$. We put $\mathbb{F}_\alpha[S] = \mathbb{F}_\alpha(\mathrm{L}[S])$ and $\mathbb{F}_\alpha = \mathbb{F}_\alpha(\mathrm{L}) = \mathbb{F}_\alpha[\emptyset]$.

The following definitions introduce a useful coding system for reals (i.e., points of the Cantor space $\mathscr{D} = 2^\omega$ in this paper) and sets of reals.

Let $\alpha \in \mathtt{Ord}$. By $\mathbb{T}_\alpha$ we denote the set of all "terms"—indexed sets $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle$ such that $t_n \subseteq \mathscr{P}_\alpha$ for each n.

We put $\mathbb{T} = \bigcup_{\alpha<\omega_1} \mathbb{T}_\alpha$. (Recall that $\omega_1 = \Omega$ assuming Ω-SM.)

"Terms" $t \in \mathbb{T}_\alpha$ are used to code functions $C : \alpha^\omega \longrightarrow \mathscr{D} = 2^\omega$; namely, for every $f \in \alpha^\omega$ we define $x = \mathsf{C}_t(f) \in \mathscr{D}$ by the condition: $x(n) = 1$ iff $f \restriction m \in t_n$ for some m.

Suppose that $\alpha < \omega_1$, $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle \in \mathbb{T}_\alpha$, $u \in \mathscr{P}_\alpha$, M an arbitrary model. We introduce the sets $\mathscr{X}_{tu}(M) = \{\mathsf{C}_t(f) : u \subset f \in \mathbb{F}_\alpha(M)\}$ and $\mathscr{X}_t(M) = \mathscr{X}_{t\Lambda}(M) = \mathsf{C}_t``\mathbb{F}_\alpha(M)$. As above, we define $\mathscr{X}_t[S] = \mathscr{X}_t(\mathrm{L}[S])$ and $\mathscr{X}_t = \mathscr{X}_t[\emptyset] = \mathscr{X}_t(\mathrm{L})$; the same for $\mathscr{X}_{tu}$.

PROPOSITION 5. *Assume* Ω-SM. *Let* $S \subseteq \mathtt{Ord}$ *be* Ω-*weak over* L. *Then*

1. *If* $\alpha < \omega_1 = \Omega$, $F \subseteq \mathbb{F}_\alpha[S]$ *is* OD$[S]$, *and* $f \in F$, *then there exists* $m \in \omega$ *such that each* $f' \in \mathbb{F}_\alpha[S]$ *satisfying* $f' \restriction m = f \restriction m$ *belongs to* F.
2. *For each* $x \in \mathscr{D}$, *there exist* $\alpha < \omega_1$, $f \in \mathbb{F}_\alpha[S]$, *and* $t \in \mathbb{T}_\alpha \cap \mathrm{L}[S]$ *such that* $x = \mathsf{C}_t(f)$.
3. *Every* OD$[S]$ *set* $X \subseteq \mathscr{D}$ *is a union of sets of the form* $\mathscr{X}_t[S]$, *where* $t \in \mathbb{T} \cap \mathrm{L}[S]$.
4. *Suppose that* $t \in \mathbb{T}_\alpha \cap \mathrm{L}[S]$, $\alpha < \omega_1$, *and* $u \in \mathscr{P}_\alpha$. *Then every* OD$[S]$ *set* $X \subseteq \mathscr{X}_{tu}[S]$ *is a union of sets* $\mathscr{X}_{tv}[S]$, *where* $u \subseteq v \in \mathscr{P}_\alpha$.

PROOF OF ITEM 1. We observe that $F = \{f' \in \alpha^\omega : \Phi(S, f')\}$ for an $\in$-formula Φ. Let $\Psi(S, f')$ denote the formula: "Λ $\mathscr{P}_{<\Omega}$-forces $\Phi(S, f')$ over the universe", so that $F = \{f' \in \alpha^\omega : \Psi(S, f')$ is true in $\mathrm{L}[S, f']\}$ by Proposition 3. Since $f \in F \subseteq \mathbb{F}_\alpha[S]$, there exists $m \in \omega$ such that the restriction $u = f \restriction m \in \mathscr{P}_\alpha$ $\mathscr{P}_\alpha$-forces $\Psi(S, \mathbf{f})$ over $\mathrm{L}[S]$, where $\mathbf{f}$ is the name of the α-collapsing function. The m is as required.

Item 2. Since the universe is a Solovay extension of $\mathrm{L}[S]$ (Proposition 3), x belongs to a $\mathscr{P}_\alpha$-generic extension of $\mathrm{L}[S]$, for some $\alpha < \omega_1$. Thus, $x \in \mathrm{L}[S, f]$ where $f \in \mathbb{F}_\alpha[S]$. We put $t_n = \{u \in \mathscr{P}_\alpha : u$ $\mathscr{P}_\alpha$-forces $\mathbf{x}(n) = 1$ over $\mathrm{L}[S]\}$, where $\mathbf{x}$ is a name for x.

Item 3. Consider $x \in X$. We use Item 2 to obtain $\alpha < \omega_1$, $f \in \mathbb{F}_\alpha[S]$, and $t \in \mathbb{T}_\alpha \cap \mathrm{L}[S]$ such that $x = \mathsf{C}_t(f)$. Then we apply Item 1 to the OD$[S]$ set $F = \{f' \in \mathbb{F}_\alpha[S] : \mathsf{C}_t(f') \in X\}$ and the given function f. This results in a condition $u = f \restriction m \in \mathscr{P}_\alpha$ such that $x \in \mathscr{X}_{tu}[S] \subseteq X$. Finally, the set $\mathscr{X}_{tu}[S]$ is equal to $\mathscr{X}_{t'}[S]$ for some other $t' \in \mathbb{T}_\alpha \cap \mathrm{L}[S]$.

Item 4. Similar to the previous item. ⊣

§3. The case of a closed relation. In this section, we prove Item ⟨I⟩ of Theorem 2. Thus, let us suppose Ω-SM and consider an OD equivalence relation E on $\mathscr{D}$ satisfying $\mathsf{E} = \overline{\mathsf{E}}$.

First of all we obtain a useful characterization lemma for E.

We recall that $\Omega = \omega_1$ in the assumption Ω-SM, and $\mathbb{T} = \bigcup_{\alpha<\omega_1} \mathbb{T}_\alpha$.

Let us fix an Δ_1^{HC} enumeration $\mathbb{T} \cap \mathrm{L} = \{t(\xi) : \xi < \omega_1\}$ such that each "term" $t \in \mathbb{T} \cap \mathrm{L}$ has uncountably many numbers ξ, and $t(\xi) \in \mathbb{T}_\alpha$ for some $\alpha \leq \xi$ whenever $\xi < \omega_1$.

LEMMA 6. *Assume* Ω-SM *and* $\mathsf{E} = \overline{\mathsf{E}}$. *Let* $x, y \in \mathscr{D}$. *Then* $x \mathsf{E} y$ *is equivalent to the following condition*:

$$(*) \qquad x \in [\mathscr{X}_{t(\xi)}(\mathrm{L}_\xi)]_{\mathsf{E}} \longleftrightarrow y \in [\mathscr{X}_{t(\xi)}(\mathrm{L}_\xi)]_{\mathsf{E}} \quad \text{for all} \quad \xi < \omega_1 .$$

PROOF. It is clear that $x \mathsf{E} y$ implies $(*)$.

To prove the opposite direction, assume that $x \not\mathsf{E} y$. Then $x \not\overline{\mathsf{E}} y$ as well, hence there exists an OD set X such that $x \in [X]_{\mathsf{E}}$ but $y \notin [X]_{\mathsf{E}}$. By Proposition 5, $x \in \mathscr{X}_t(\mathrm{L}) \subseteq [X]_{\mathsf{E}}$ for a "term" $t \in \mathbb{T}_\alpha \cap \mathrm{L}$, $\alpha < \omega_1$. Then $y \notin [\mathscr{X}_t(\mathrm{L})]_{\mathsf{E}}$. It remains to check that $\mathscr{X}_t(\mathrm{L}) = \mathscr{X}_{t(\xi)}(\mathrm{L}_\xi)$ for some $\xi < \omega_1$.

Let $\gamma = \alpha^{++}$ in L, so that $\gamma < \omega_1 = \Omega$ and $\mathbb{F}_\alpha(\mathrm{L}) = \mathbb{F}_\alpha(\mathrm{L}_\gamma)$. Then $\mathscr{X}_t(\mathrm{L}) = \mathscr{X}_t(\mathrm{L}_\xi)$ whenever $\gamma \leq \xi < \omega_1$. Finally, $t = t(\xi)$ for an ordinal ξ, $\gamma \leq \xi < \omega_1$, and then $\mathscr{X}_t(\mathrm{L}) = \mathscr{X}_{t(\xi)}(\mathrm{L}_\xi)$. ⊣

3.1. The OD subcase. We have to prove that the $\mathscr{T}^2$-closed relation $\mathsf{E} = \overline{\mathsf{E}}$ admits an OD enumeration of the equivalence classes by elements of $2^{<\omega_1}$, assuming Ω-SM.

For every $x \in \mathscr{D}$, we define the set $\Xi(x) = \{\xi < \omega_1 : x \in [\mathscr{X}_{t(\xi)}(\mathrm{L}_\xi)]_{\mathsf{E}}\}$ and let $\varphi_x \in 2^{\omega_1}$ be the characteristic function of $\Xi(x)$. Then the OD map $x \longmapsto \varphi_x$ enumerates the E-equivalence classes by sequences in 2^{ω_1} by Lemma 6. To get an enumeration by shorter sequences, we prove

LEMMA 7. *Assume* Ω-SM. *If* $h \in 2^{\omega_1}$ *is* R-OD *then there exists* $\lambda < \omega_1$ *such that* $h \in \mathrm{L}[h \restriction \lambda]$.

PROOF. By Ω-SM, there exists $\alpha < \omega_1$ such that $h \in \mathrm{L}[f]$ for a $\mathscr{P}_\alpha$-generic over L function $f \in \alpha^\omega$. Let $\mathbf{h}$ be a $\mathscr{P}_\alpha$-name for h in $\mathrm{L}[f]$.

We argue in L. Let $H_\xi = \{s \in \mathscr{P}_\alpha : s \ \mathscr{P}_\alpha\text{-forces } \mathbf{h}(\xi) = 1\}$ for all $\xi < \Omega$. Since $\alpha < \Omega$, there exist $< \Omega$ different sets H_ξ. We have (in L) an ordinal $\lambda < \Omega$ and a function $\rho : \Omega \longrightarrow \lambda$ such that $H_\xi = H_{\rho(\xi)}$ for all $\xi < \Omega$.

In the universe, this implies $h \in \mathrm{L}[h \restriction \lambda]$, as required. ⊣

To continue the proof of Theorem 2 (Item ⟨I⟩), we let λ_x denote the least ordinal $\lambda < \Omega = \omega_1$ such that $\varphi_x \in \mathrm{L}[\varphi_x \restriction \lambda_x]$, and put $\psi_x = \varphi_x \restriction \lambda_x$—for each real $x \in \mathscr{D}$. Obviously, $x \mathsf{E} y$ implies $\psi_x = \psi_y$, but we do not know whether conversely $\psi_x = \psi_y$ implies $x \mathsf{E} y$. We utilize a more sophisticated idea.

Let $x \in \mathscr{D}$. Then $\psi_x \in 2^{\lambda_x}$. The set $[x]_{\mathsf{E}} = \{x' : \varphi_x = \varphi_{x'}\}$ is $\mathrm{OD}[\varphi_x]$, therefore $\mathrm{OD}[\psi_x]$ because $\varphi_x \in \mathrm{L}[\psi_x]$. It follows from Proposition 5 that $[x]_{\mathsf{E}}$ includes a nonempty subset of the form $\mathscr{X}_t(\mathrm{L}[\psi_x])$, where $t \in \mathbb{T} \cap \mathrm{L}[\psi_x]$.

Let t_x be the least, in the Gödel $\mathrm{OD}[\psi_x]$ wellordering of $\mathrm{L}[\psi_x]$, among the "terms" $t \in \mathbb{T} \cap \mathrm{L}[\psi_x]$ such that $\emptyset \neq \mathscr{X}_t(\mathrm{L}[\psi_x]) \subseteq [x]_{\mathsf{E}}$.

The map $x \longmapsto \langle \psi_x, t_x \rangle$ is OD, of course. Furthermore $x \mathsf{E} y$ implies $\psi_x = \psi_y$ and $t_x = t_y$ since the definition is E-invariant. To prove the converse assume that $\psi_x = \psi_y$ and $t_x = t_y$. Then one and the same nonempty set $\mathscr{X}_{t_x}(\mathrm{L}[\psi_x]) = \mathscr{X}_{t_y}(\mathrm{L}([\psi_y])$ is a subset of both $[x]_{\mathsf{E}}$ and $[y]_{\mathsf{E}}$, so $x \mathsf{E} y$. It follows that the map

$x \longmapsto \langle \psi_x, t_x \rangle$ enumerates E-classes by elements of the set

$$\{\langle \psi, t \rangle : \psi \in 2^{<\omega_1} \text{ and } t \in \mathbb{T} \cap \mathrm{L}[\psi]\}.$$

This set admits an OD injection in $2^{<\omega_1}$. Therefore we can obtain an OD enumeration of the E-equivalence classes by elements of $2^{<\omega_1}$. This ends the proof of the principal assertion in Item $\langle$I$\rangle$ of Theorem 2.

3.2. The Σ^1_2 and Σ^1_1 subcases. Let us consider the case when E is a Σ^1_2 (resp. Σ^1_1) equivalence relation in Item $\langle$I$\rangle$ of Theorem 2. We have to engineer a Δ^{HC}_2 (resp. Δ^{HC}_1) enumeration of the E-equivalence classes by elements of $2^{<\omega_1}$.

The most natural plan would be to prove that the OD enumeration $x \longmapsto \langle \psi_x, t_x \rangle$ defined above is e.g., Δ^{HC}_2 provided E is Σ^1_2. However, there is no obvious method to convert the definition of ψ_x to Δ^{HC}_2, or even to formalize it in HC. Fortunately we do not need in fact the minimality of λ_x; all that we exploited is the existence of a term $t \in \mathbb{T} \cap \mathrm{L}[\psi_x]$ such that $\emptyset \neq \mathscr{X}_t(\mathrm{L}[\psi_x]) \subseteq [x]_{\mathsf{E}}$.

We could now define $\psi_x = \varphi_x \restriction \gamma$, where $\gamma = \gamma_x$ is the least ordinal $\gamma < \omega_1$ such that $\mathbb{T} \cap \mathrm{L}[\varphi_x \restriction \gamma]$ contains the required "term". This can be formalized in HC, but hardly as a Δ^{HC}_2 definition: indeed, in particular the requirement $\mathscr{X}_t(\mathrm{L}[\psi_x]) \subseteq [x]_{\mathsf{E}}$ does not look better than Π^{HC}_2 because E is Σ^1_2.

The actual plan includes one more idea, forcing of the equivalence over submodels, used earlier by Silver [9], Stern [11], and Hjorth and Kechris [4].

Let us consider the details. We recall that Ω-SM is assumed.

Definition. We let T_{E} be the set of all triples $\langle x, \psi, t \rangle$ such that $x \in \mathscr{D}$, $\psi \in 2^{<\omega_1}$, $t \in \mathbb{T}_\alpha \cap \mathrm{L}_\gamma[\psi]$,[11] where $\alpha < \gamma = \mathtt{dom}\,\psi < \omega_1$, and the following conditions (a) through (d) are satisfied.

(a) $\mathrm{L}_\gamma[\psi]$ models **ZFC**$^-$ (minus power set) so that ψ can occur as an extra class parameter in the **ZFC** schemata.

(b) The pair $\langle \Lambda, \Lambda \rangle$ $(\mathscr{P}_\alpha \times \mathscr{P}_\alpha)$-forces $\mathsf{C}_t(\mathbf{f})\ \mathsf{E}\ \mathsf{C}_t(\mathbf{g})$ in $\mathrm{L}_\gamma[\psi]$, where **f** and **g** are the names for the generic functions in α^ω. (Recall that $\mathscr{P}_\alpha = \alpha^{<\omega}$.)

(c) $\psi = \varphi_x \restriction \gamma$.

(d) x belongs to $[\mathscr{X}_t(\mathrm{L}_\gamma[\psi])]_{\mathsf{E}}$.

A real $x \in \mathscr{D}$ is E-*classifiable* iff there exist ψ and t such that $\langle x, \psi, t \rangle \in T_{\mathsf{E}}$.

Lemma 8. *Assume* Ω-SM. *If* E *is a* Σ^1_2 *equivalence relation and* $\mathsf{E} = \overline{\mathsf{E}}$ *then all reals* $x \in \mathscr{D}$ *are* E-*classifiable.*

Proof. Let $x \in \mathscr{D}$. Then φ_x is OD$[x]$, so $\varphi_x \in \mathrm{L}[x]$ by Proposition 3. Lemma 6 implies that the set $[x]_{\mathsf{E}}$ is OD$[\varphi_x]$. It follows from Proposition 5 that $x \in \mathscr{X}_t(\mathrm{L}[\varphi_x]) \subseteq [x]_{\mathsf{E}}$ for some $t \in \mathbb{T}_\alpha \cap \mathrm{L}[\varphi_x]$, $\alpha < \omega_1$.

The model $\mathrm{L}_{\omega_1}[\varphi_x]$ has an elementary submodel $\mathrm{L}_\gamma[\psi]$, where $\gamma < \omega_1$ and $\psi = \varphi_x \restriction \gamma$, containing t and α. We prove that $\langle x, \psi, t \rangle \in T_{\mathsf{E}}$. Since $\mathrm{L}_\gamma[\psi]$ obviously satisfies (a) and (c), let us focus on requirements (b), (d).

We check (b). Indeed otherwise there exist conditions $u, v \in \mathscr{P}_\alpha = \alpha^{<\omega}$ such that $\langle u, v \rangle$ forces $\mathsf{C}_t(\mathbf{f})\ \not\mathsf{E}\ \mathsf{C}_t(\mathbf{g})$ in $\mathrm{L}_\gamma[\psi]$ in the sense of $\mathscr{P}_\alpha \times \mathscr{P}_\alpha$ as the notion of forcing. Then $\langle u, v \rangle$ also forces $\mathsf{C}_t(\mathbf{f})\ \not\mathsf{E}\ \mathsf{C}_t(\mathbf{g})$ in $\mathrm{L}_{\omega_1}[\varphi_x]$. Let us consider a $(\mathscr{P}_\alpha \times \mathscr{P}_\alpha)$-generic over $\mathrm{L}[\varphi_x]$ pair $\langle f, g \rangle \in \alpha^\omega \times \alpha^\omega$ such that $u \subset f$ and $v \subset g$.

[11] By $\mathrm{L}_\gamma[\psi]$ we understand the result of the Gödel construction of length γ arranged so that only $\psi \restriction \delta$ is available at each step $\delta < \gamma$. Note that $\psi \notin \mathrm{L}_\gamma[\psi]$.

Then both $y = \mathsf{C}_t(f)$ and $z = \mathsf{C}_t(g)$ belong to $\mathscr{X}_t(\mathrm{L}[\varphi_x])$, so $y \mathsf{E} z$ holds in the universe because $\mathscr{X}_t(\mathrm{L}[\varphi_x]) \subseteq [x]_{\mathsf{E}}$.

Note that $\langle f, g\rangle$ also is generic over $\mathrm{L}_{\omega_1}[\varphi_x]$. We observe that $y \mathsf{E} z$ is *false* in $\mathrm{L}_{\omega_1}[\varphi_x, f, g]$, then in $\mathrm{L}[\varphi_x, f, g]$, by the choice of u and v. But $y \mathsf{E} z$ is a Σ^1_2 formula, therefore absolute for transitive models containing all ordinals by Shoenfield, which is a contradiction.

We check (d). Take any $\mathscr{P}_\alpha$-generic over $\mathrm{L}[\varphi_x]$ function $f \in \alpha^\omega$. Then $y = \mathsf{C}_t(f)$ belongs to $\mathscr{X}_t(\mathrm{L}[\varphi_x])$, hence $y \mathsf{E} x$. On the other hand, f is generic over $\mathrm{L}_\gamma[\psi]$ as well, so we have $y \in \mathscr{X}_t(\mathrm{L}_\gamma[\psi])$ and $x \in [\mathscr{X}_t(\mathrm{L}_\gamma[\psi])]_{\mathsf{E}}$.

Thus $\langle x, \psi, t\rangle \in T_{\mathsf{E}}$. This means that x is E-classifiable, as required. ⊣

Let $x \in \mathscr{D}$. It follows from Lemma 8 that there exists the least ordinal $\gamma = \gamma_x < \Omega = \omega_1$ such that $T_{\mathsf{E}}(x, \varphi_x \restriction \gamma, t)$ for some t.

We put $\psi_x = \varphi_x \restriction \gamma_x$ and let t_x denote the least, in the sense of the Gödel $\mathrm{OD}[\psi_x]$ wellordering of $\mathrm{L}_\gamma[\psi_x]$, "term" $t \in \mathbb{T} \cap \mathrm{L}_\gamma[\psi_x]$ which satisfies $T_{\mathsf{E}}(x, \psi_x, t)$. We finally set $U(x) = \langle \psi_x, t_x\rangle$.

Lemma 9. *Assume* Ω-SM. *If* E *is a* Σ^1_2 *equivalence relation, and* $\mathsf{E} = \overline{\mathsf{E}}$ *then the map* U *enumerates the* E*-classes.*

Proof. If $x \mathsf{E} y$ then $U(x) = U(y)$ because the definitions are E-invariant.

Let us prove the converse. Assume that $U(x) = U(y)$, in particular, $\psi_x = \psi_y = \psi \in 2^{<\omega_1}$ and $t_x = t_y = t \in \mathbb{T}_\alpha \cap \mathrm{L}_\gamma[\psi]$, where $\alpha < \gamma = \mathrm{dom}\,\psi < \omega_1$.

By (d) we have $\mathsf{C}_t(f) \mathsf{E} x$ and $\mathsf{C}_t(g) \mathsf{E} y$ for some $\mathscr{P}_\alpha$-generic over $\mathrm{L}_\gamma[\psi]$ functions $f, g \in \alpha^\omega$. Let us consider a $\mathscr{P}_\alpha$-generic over both $\mathrm{L}_\gamma[\psi, f]$ and $\mathrm{L}_\gamma[\psi, g]$ function $h \in \alpha^\omega$. Then, by (b), $\mathsf{C}_t(h) \mathsf{E} \mathsf{C}_t(f)$ holds in $\mathrm{L}_\gamma[\psi, f, h]$, therefore in the universe because E is Σ^1_2. Similarly, we have $\mathsf{C}_t(h) \mathsf{E} \mathsf{C}_t(g)$. It follows that $\mathsf{C}_t(f) \mathsf{E} \mathsf{C}_t(g)$, hence $x \mathsf{E} y$, as required. ⊣

Lemma 10. *Suppose that* E *is* Σ^1_2 *(resp.* Σ^1_1*) and* $\mathsf{E} = \overline{\mathsf{E}}$. *Then* U *is a function of class* Δ^{HC}_2 *(resp.* Δ^{HC}_1*).*

Proof. It suffices to check that the set T_{E} is Δ^{HC}_2 (resp. Δ^{HC}_1).

Requirements (a) and (b) are Δ^{HC}_1 because they reflect truth within $\mathrm{L}_\gamma[\psi]$.

Suppose that E is Σ^1_2, that is, Σ^{HC}_1. Then requirement (d) is obviously Σ^{HC}_1. Requirement (c) can be converted to Δ^{HC}_2 : indeed (c) is equivalent to

$$(\dagger) \qquad \forall \xi < \gamma\ (\psi(\xi) = 1 \longleftrightarrow x \in [\mathscr{X}_{t(\xi)}(\mathrm{L}_\xi)]_{\mathsf{E}}),$$

and, we recall, the enumeration $t(\xi)$ was chosen in Δ^{HC}_1.

The case when E belongs to Σ^1_1 is more difficult.

Let us first consider condition (d). Immediately, it is Σ^1_2, therefore Σ^{HC}_1, so it remains to convert it also to a Π^{HC}_1 form. Notice that the set $X = \mathscr{X}_t(\mathrm{L}_\gamma[\psi])$ consists of pairwise E-equivalent points in the assumption of (a) and (b): this was actually shown in the proof of Lemma 9. Therefore (d) is equivalent to the formula $\forall y \in \mathscr{X}_t(\mathrm{L}_\gamma[\psi])\ (x \mathsf{E} y)$ because obviously $\mathscr{X}_t(\mathrm{L}_\gamma[\psi]) \neq \emptyset$. This is clearly Π^{HC}_1 provided E is at least Π^1_2.

Let us consider (c). The right-hand side of the equivalence (†) is Σ^1_1 (recall that now E is Σ^1_1) with inserted Δ^{HC}_1 functions, therefore Δ^{HC}_1. It follows that (†) itself is Δ^{HC}_1, as required. ⊣

This completes the proof of the additional part (Σ^1_1 and Σ^1_2 relations) in Item ⟨I⟩ of Theorem 2.

§4. OD topology and the forcing. This section starts the proof of Item ⟨II⟩ of Theorem 2 for a given OD equivalence relation E in the assumption Ω-SM.

We have to embed E_0 in E continuously, assuming that $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$. The embedding will be defined in the next section; here we obtain some preliminary results related to the topology $\mathscr{T}$, an associated forcing, and the relevant product forcing. At the end of the section, we introduce the set H of all points $x \in \mathscr{D}$ whose $\overline{\mathsf{E}}$-classes are strictly bigger than E-classes.

The reasoning is based on special properties of the topology $\mathscr{T}$, having a semblance of the Gandy-Harrington topology (in a simplified form as some specific Σ^1_1 details vanish). In particular, the topology is strongly Choquet (see [2] or [6]). However we shall not utilize this property (and shall not prove it). The reasoning will be organized as a sequence of forcing arguments. This manner of treating of equivalence relations is taken from Miller [8].

4.1. 1st countable sets and the forcing. The topology $\mathscr{T}$ (see Section 1) obviously does not have a countable base; but it has one in a local sense, in the assumption Ω-SM.

DEFINITION. A set X is OD-1*st-countable* if the OD power set $\mathscr{P}^{\mathrm{OD}}(X) = \mathscr{P}(X) \cap \mathrm{OD}$ is at most countable. (In this case, $\mathscr{P}^{\mathrm{OD}}(X)$ has only countably many different OD subsets because it is a general property of the Solovay model that $\mathscr{P}^{\mathrm{OD}}(\mathscr{X})$ is countable for any countable OD set $\mathscr{X} \subseteq \mathrm{OD}$.) ⊣

LEMMA 11. *Assume* Ω-SM. *Let* $t \in \mathbb{T} \cap \mathrm{L}$. *Then the set* $X = \mathscr{X}_t(\mathrm{L})$ *is* OD-1*st-countable.*

PROOF. Let $t \in \mathbb{T}_\alpha$, $\alpha < \omega_1 = \Omega$. By Proposition 5 every OD subset of X is determined by an OD subset of $\mathscr{P}_\alpha = \alpha^{<\omega}$. Let α^+ be the next cardinal in L. Since all OD sets $S \subseteq \mathscr{P}_\alpha$ are constructible (Proposition 3), X has $\leq \alpha^+$-many OD subsets. However $\alpha^+ < \omega_1 = \Omega$ because Ω is inaccessible in L. ⊣

Let $\mathbb{X} = \{X \subseteq \mathscr{D} : X \text{ is OD and nonempty}\}$.

Let us consider $\mathbb{X}$ as a forcing notion (smaller sets are stronger conditions). We say that a set $G \subseteq \mathbb{X}$ is OD-*generic* iff it is pairwise compatible in $\mathbb{X}$ (that is for any pair of $X, Y \in G$ there exists $Z \in G$, $Z \subseteq X \cap Y$) and nonempty intersects every dense[12] OD subset of $\mathbb{X}$.

COROLLARY 12. *Assume* Ω-SM. *If* $X \in \mathbb{X}$ *then there exists an* OD-*generic set* $G \subseteq \mathbb{X}$ *containing* X.

PROOF. We can suppose, by Proposition 5, that $X = \mathscr{X}_t(\mathrm{L})$ where $t \in \mathbb{T} \cap \mathrm{L}$. Now apply Lemma 11. ⊣

LEMMA 13. *Assume* Ω-SM. *Let* $\Phi(\cdot)$ *be an* $\in$-*formula containing only ordinals as parameters. Suppose that* $G \subseteq \mathbb{X}$ *is* OD-*generic and* $\Phi(G)$ *is true* (*in the universe*). *Then there exists a condition* $X \in G$ *such that* $\Phi(G')$ *is true for* <u>*every*</u> OD-*generic set* $G' \subseteq \mathbb{X}$ *containing* X.

[12] By *dense* we shall always understand *open dense*.

PROOF. Let us show that he $\mathbb{X}$-genericity can be transformed to an ordinary forcing over L. Of course formally $\mathbb{X} \notin \mathrm{L}$, but $\mathbb{X}$ is OD order isomorphic to a p.o. set $\mathbb{X}' \in \mathrm{L}$ by Proposition 3 because $\mathbb{X}$ itself, the order on $\mathbb{X}$, and all elements of $\mathbb{X}$ are OD. Let $\pi : \mathbb{X}$ onto $\mathbb{X}'$ be the isomorphism.

Then π sends each dense OD set $D \subseteq \mathbb{X}$ to a dense OD, therefore constructible by Proposition 3, set $D' = \pi\text{"}D \subseteq \mathbb{X}'$. It follows that $G \subseteq \mathbb{X}$ is OD-generic iff $G' = \pi\text{"}G$ is $\mathbb{X}'$-generic over L in the ordinary sense.

We assert that G' is Ω-weak over L. Indeed by the genericity, Lemma 11, and Proposition 5 G contains an OD-1st-countable condition $X \in G$. Then

$$\mathscr{P} = \{X' \in \mathbb{X}' : X' \text{ is stronger than } \pi(X) \text{ in } \mathbb{X}'\}$$

is a countable OD, therefore constructible, subset of $\mathbb{X}'$, $\mathscr{G} = G' \cap \mathscr{P}$ is $\mathscr{P}$-generic over L, and $G' \in \mathrm{L}[\mathscr{G}]$. Finally $\mathscr{P}$ has a cardinality $\alpha < \Omega$ in L, so that $\mathscr{G}$ is Ω-weak over L because Ω is inaccessible in L.

Furthermore any OD property of G in the main Ω-SM universe is an OD property of G' as well simply because π is OD. Therefore such a property admits an appropriate relativization to $\mathrm{L}[G']$ by Proposition 3. We conclude that OD properties of G in the universe are $\mathbb{X}$-forced, as required. ⊣

LEMMA 14. *Assume* Ω-SM. *If $G \subseteq \mathbb{X}$ is an* OD-*generic set then the intersection $\bigcap G$ is a singleton $\{x\} = \{x_G\}$.*

PROOF. Otherwise by Lemma 13 there exists a condition $X \in \mathbb{X}$ such that $\bigcap G$ is *not* a singleton for *every* OD-generic set $G \subseteq \mathbb{X}$ containing X. We can assume that $X = \mathscr{X}_t(\mathrm{L})$, where $t \in \mathbb{T}_\alpha \cap \mathrm{L}$, $\alpha < \omega_1$. Then X is OD-1st-countable; let $\{\mathscr{X}_n : n \in \omega\}$ be an enumeration of all OD dense subsets of $\mathscr{P}^{\mathrm{OD}}(X)$. Using Proposition 5 (Item 1), we obtain an increasing $\mathscr{P}_\alpha$-generic over L sequence $u_0 \subset u_1 \subset u_2 \subset \dots$ of $u_n \in \mathscr{P}_\alpha = \alpha^{<\omega}$ such that $X_n = \mathscr{X}_{tu_n}(\mathrm{L}) \in \mathscr{X}_n$. Obviously this yields an OD-generic set $G \subseteq \mathbb{X}$ containing X and all X_n.

Now let $f = \bigcup_{n\in\omega} u_n$; $f \in \alpha^\omega$ and f is $\mathscr{P}_\alpha$-generic over L. Then $x = \mathsf{C}_t(f) \in X_n$ for all n, so $x \in \bigcap G$. Since $\bigcap G$ cannot contain more than one element, it is a singleton, which is a contradiction with the choice of X. ⊣

Reals of the form x_G will be called OD-*generic*.

4.2. The product forcing. The classical proof of the "Glimm-Effros" dichotomy for Borel sets in Harrington et al. [2] is based on interactions between E and its $\mathscr{T}^2$-closure $\overline{\mathsf{E}}$. In the forcing setting, we have to fix a restriction by $\overline{\mathsf{E}}$ directly in the definition of the product forcing. Thus, we consider

$$\mathbb{P} = \mathbb{P}(\overline{\mathsf{E}}) = \{P \subseteq \overline{\mathsf{E}} : P \text{ is OD and nonempty and } P = (\mathrm{pr}_1 P \times \mathrm{pr}_2 P) \cap \overline{\mathsf{E}}\}$$

as a forcing notion (smaller sets are stronger conditions), where the projections are defined by $\mathrm{pr}_1 P = \{x : \exists y\ P(x,y)\}$ and $\mathrm{pr}_2 P = \{y : \exists x\ P(x,y)\}$ for every $P \subseteq \mathscr{D}^2$. (Note that if P is OD then so are $\mathrm{pr}_1 P$ and $\mathrm{pr}_2 P$.)

We say that a set $\mathrm{G} \subseteq \mathbb{P}$ is $\mathbb{P}$-*generic* iff it is pairwise compatible in $\mathbb{P}$ and has nonempty intersection with every dense OD subset of $\mathbb{P}$.

We recall that a set P is OD-1st-countable if the OD power set $\mathscr{P}^{\mathrm{OD}}(P)$ has only countably many different OD subsets. Now we introduce a similar notion which reflects the product character of $\mathbb{P}$.

DEFINITION. $P \in \mathbb{P}$ is $\mathbb{P}$-1*st-countable* iff the set $\mathbb{P}_{\subseteq P} = \{Q \in \mathbb{P} : Q \subseteq P\}$ of all stronger conditions is at most countable. (Then $\mathbb{P}_{\subseteq P}$ contains at most countably many OD subsets, assuming Ω-SM.) ⊣

ASSERTION 15. *Assume* Ω-SM. *Then*

1. *If* $P \in \mathbb{P}$ *then* $\mathrm{pr}_1 P$ *and* $\mathrm{pr}_2 P$ *belong to* $\mathbb{X}$.
2. *If* $X, Y \in \mathbb{X}$ *and* $P = (X \times Y) \cap \overline{\mathsf{E}} \neq \emptyset$ *then* $P \in \mathbb{P}$.
3. *If* X, Y *are* OD-1*st-countable in Item 2 then* P *is* $\mathbb{P}$-1*st-countable.*
4. *If* $P \in \mathbb{P}$, $X \in \mathbb{X}$, *and* $X \subseteq \mathrm{pr}_1 P$, *then there exists* $Q \in \mathbb{P}$, $Q \subseteq P$, *such that* $X = \mathrm{pr}_1 Q$. *Similarly for* pr_2.

PROOF. Set $Q = \{\langle x, y\rangle \in P : x \in X\}$ in Item 4. ⊣

LEMMA 16. *Assume* Ω-SM. *Let* $\mathsf{G} \subseteq \mathbb{P}$ *be* $\mathbb{P}$-*generic. Then the intersection* $\bigcap \mathsf{G}$ *contains a single point* $\langle a, b\rangle$. *In this pair, a and b are* OD-*generic reals and* $a \,\overline{\mathsf{E}}\, b$.

PROOF. Both $G_1 = \{\mathrm{pr}_1 P : P \in \mathsf{G}\}$ and $G_2 = \{\mathrm{pr}_2 P : P \in \mathsf{G}\}$ are OD-generic sets by Assertion 15, so by Lemma 14 there exist unique OD-generic points $a = x_{G_1}$ and $b = x_{G_2}$. It remains to show that $a \,\overline{\mathsf{E}}\, b$.

Indeed, otherwise there exists an E-invariant OD set A such that $x \in A$ and $y \in B = \mathscr{D} \setminus A$. Then $A \in G_1$ and $B \in G_2$ by the genericity. There exists a condition $P \in \mathsf{G}$ such that $\mathrm{pr}_1 P \subseteq A$ and $\mathrm{pr}_2 P \subseteq B$, therefore $P \subseteq (A \times B) \cap \overline{\mathsf{E}} = \emptyset$, which is impossible. ⊣

Pairs $\langle a, b\rangle$ as in Lemma 16 will be called $\mathbb{P}$-*generic*.

As further notation, we write $X \mathrel{\mathsf{C}} Y$, for sets X and Y and a binary relation C, to mean $\forall x \in X\, \exists y \in Y\, (x \mathrel{\mathsf{C}} y)$ and $\forall y \in Y\, \exists x \in X\, (x \mathrel{\mathsf{C}} y)$. This is the same as $[X]_{\mathsf{C}} = [Y]_{\mathsf{C}}$ in the case when C is an equivalence relation.

LEMMA 17. *Assume* Ω-SM. *Suppose that* $P_0 \in \mathbb{P}$, *reals* $a, a' \in X_0 = pr_1 P_0$ *are* OD-*generic, and* $a \,\overline{\mathsf{E}}\, a'$. *Then there exists a real b such that both* $\langle a, b\rangle$ *and* $\langle a', b\rangle$ *belong to* P_0 *and are* $\mathbb{P}$-*generic pairs.*

PROOF. It follows from Proposition 5, Lemma 11, and Assertion 15 that there exists a $\mathbb{P}$-1st-countable set $P_1 \in \mathbb{P}$, $P_1 \subseteq P_0$ such that $a \in X_1 = \mathrm{pr}_1 P_1$. We define $Y_1 = \mathrm{pr}_2 P_1$; then $X_1 \,\overline{\mathsf{E}}\, Y_1$ and $P_1 = (X_1 \times Y_1) \cap \overline{\mathsf{E}}$.

We let $P' = \{\langle x, y\rangle \in P_0 : y \in Y_1\}$. Then $P' \in \mathbb{P}$ and $P_1 \subseteq P' \subseteq P_0$. Furthermore $a' \in X' = \mathrm{pr}_1 P'$. (Indeed, since $a \in X_1$ and $X_1 \,\overline{\mathsf{E}}\, Y_1$, there exists $y \in Y_1$ such that $a \,\overline{\mathsf{E}}\, y$; then $a' \,\overline{\mathsf{E}}\, y$ as well because $a \,\overline{\mathsf{E}}\, a'$, hence $\langle a', y\rangle \in P'$.) As above, there exists a $\mathbb{P}$-1st-countable set $P'_1 \in \mathbb{P}$, $P'_1 \subseteq P'$ such that a' contains in $X'_1 = \mathrm{pr}_1 P'_1$. Then $Y'_1 = \mathrm{pr}_2 P'_1 \subseteq Y_1$.

By definition, $\mathbb{P}$ admits only countably many different dense OD sets below P_1 and below P'_1. Let $\{\mathscr{P}_n : n \in \omega\}$ and $\{\mathscr{P}'_n : n \in \omega\}$ be enumerations of both families of dense sets. We define sets $P_n, P'_n \in \mathbb{P}$ $(n \in \omega)$ satisfying:

(i) $a \in X_n = \mathrm{pr}_1 P_n$ and $a' \in X'_n = \mathrm{pr}_1 P'_n$;
(ii) $Y'_n = \mathrm{pr}_2 P'_n \subseteq Y_n = \mathrm{pr}_2 P_n$ and $Y_{n+1} \subseteq Y'_n$;
(iii) $P_{n+1} \subseteq P_n$, $P'_{n+1} \subseteq P'_n$, $P_n \in \mathscr{P}_n$, and $P'_n \in \mathscr{P}'_n$.

By (iii) both $\{P_n : n \in \omega\}$ and $\{P'_n : n \in \omega\}$ are generic sequences in $\mathbb{P}$, so by Lemma 16 they result in two $\mathbb{P}$-generic pairs, $\langle a, b\rangle \in P_0$ and $\langle a', b\rangle \in P_0$, having the first terms equal to a and a' by (i) and second terms equal to each other by (ii). Thus it suffices to carry out the construction of P_n and P'_n.

The construction goes on by induction on n.

Assume that P_n and P'_n have been defined. We define P_{n+1}. By (ii) and Assertion 15, the set $P = (X_n \times Y'_n) \cap \overline{\mathsf{E}} \subseteq P_n$ belongs to $\mathbb{P}$ and $a \in X = \mathrm{pr}_1 P$. (Indeed, $\langle a, y \rangle \in P$, where y satisfies $\langle a', y \rangle \in P'_n$, because $a \mathbin{\overline{\mathsf{E}}} a'$.) However $\mathscr{P}_{n+1}$ is dense in $\mathbb{P}$ below $P \subseteq P_0$; therefore $\mathrm{pr}_1 \mathscr{P}_{n+1} = \{\mathrm{pr}_1 P' : P' \in \mathscr{P}_{n+1}\}$ is dense in $\mathbb{X}$ below $X = \mathrm{pr}_1 P$ by Assertion 15. Since a is OD-generic, we have $a \in \mathrm{pr}_1 P'$ for some $P' \in \mathscr{P}_{n+1}$, $P' \subseteq P$. It remains to put $P_{n+1} = P'$, and then $X_{n+1} = \mathrm{pr}_1 P_{n+1}$ and $Y_{n+1} = \mathrm{pr}_2 P_{n+1}$.

After this, to define P'_{n+1} we let $P = (X'_n \times Y_{n+1}) \cap \overline{\mathsf{E}}$, etc. ⊣

4.3. The key set. We recall that Ω-SM is assumed, E is an OD equivalence relation on $\mathscr{D}$, and $\overline{\mathsf{E}}$ is the $\mathscr{T}^2$-closure of E in $\mathscr{D}^2$. We also suppose that $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$, as in Item ⟨II⟩ of Theorem 2. Then there exist $\overline{\mathsf{E}}$-classes which include more than one E-class. We define the union of all those $\overline{\mathsf{E}}$-classes,

$$H = \{x \in \mathscr{D} : \exists\, y \in \mathscr{D}\ (x \mathbin{\overline{\mathsf{E}}} y \ \&\ x \mathbin{\not\mathsf{E}} y)\},$$

the "key set" from the title. The role of this set in the reasoning below is entirely similar to the role of the corresponding set V in Harrington et al. [2].

LEMMA 18. *Assume* Ω-SM. *If* $a, b \in H$ *and* $\langle a, b \rangle$ *is* $\mathbb{P}$*-generic then* $a \mathbin{\not\mathsf{E}} b$.

PROOF. Otherwise there exists a set $P \in \mathbb{P}$, $P \subseteq H \times H$, such that $a \mathsf{E} b$ holds for *all* $\mathbb{P}$-generic pairs $\langle a, b \rangle \in P$. (Lemma 13 is true for $\mathbb{P}$ as well as for $\mathbb{X}$.) We conclude that then $a \overline{\mathsf{E}} a' \longrightarrow a \mathsf{E} a'$ for all OD-generic points a, $a' \in X = \mathrm{pr}_1 P$; indeed, take b such that both $\langle a, b \rangle \in P$ and $\langle a', b \rangle \in P$ are $\mathbb{P}$-generic, by Lemma 17. In other words the relations E and $\overline{\mathsf{E}}$ coincide on the set $Y = \{x \in X : x \text{ is OD-generic}\} \in \mathbb{X}$.

Note that $Y \neq \emptyset$ by Corollary 12 and Lemma 14. Let $y \in Y$. Then $y \in H$ because $Y \subseteq X \subseteq H$. By definition, there exists a real x such that $x \mathbin{\overline{\mathsf{E}}} y$ but $x \mathbin{\not\mathsf{E}} y$. Then $x \in [Y]_{\mathsf{E}}$ because otherwise y and x would belong to the OD E-invariant disjoint sets $[Y]_{\mathsf{E}}$ and $D \setminus [Y]_{\mathsf{E}}$, a contradiction with $x \mathbin{\overline{\mathsf{E}}} y$. We have $x \mathsf{E} y'$ for some $y' \in Y$. Then $y \mathbin{\overline{\mathsf{E}}} y'$, hence $y \mathsf{E} y'$ because E and $\overline{\mathsf{E}}$ coincide on Y, and finally $x \mathsf{E} y$, which is a contradiction. ⊣

Lemma 18 is a counterpart of a proposition in Harrington et al. [2] which says that $\mathsf{E} \restriction H$ is meager in $\overline{\mathsf{E}} \restriction H$. But in fact the main content of this argument in [2] was implicitly included in Lemma 17.

LEMMA 19. *Assume* Ω-SM. *Let* $X, Y \subseteq H$ *be nonempty* OD *sets satisfying* $X \mathbin{\overline{\mathsf{E}}} Y$. *Then there exist nonempty* OD *sets* $X' \subseteq X$ *and* $Y' \subseteq Y$ *such that still* $X' \mathbin{\overline{\mathsf{E}}} Y'$ *but* $X' \cap Y' = \emptyset$.

PROOF. There exist points $x_0 \in X$ and $y_0 \in Y$ such that $x_0 \neq y_0$ but $x_0 \mathbin{\overline{\mathsf{E}}} y_0$. (Otherwise $X = Y$, and $\overline{\mathsf{E}}$ is the equality relation on X, which is impossible, see the previous proof.) Let U and V be disjoint Baire intervals in $\mathscr{D}$ containing resp. x_0 and y_0. We put $X' = X \cap U \cap [Y \cap V]_{\overline{\mathsf{E}}}$ and $Y' = Y \cap V \cap [X \cap U]_{\overline{\mathsf{E}}}$. ⊣

§5. The embedding. In this section we accomplish the proof of Item ⟨II⟩ of Theorem 2, therefore Theorem 1 as well (see Section 1). Thus, we prove, assuming

Ω-SM and $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$, that E, the given OD equivalence relation on $\mathscr{D}$, continuously embeds E_0.

5.1. The embedding. During the construction of the embedding, 2^m will denote the set of all binary m-sequences, and $2^{<\omega} = \bigcup_{m\in\omega} 2^m$. 0^k will be the sequence of k terms each equal to 0. By $^\wedge$ we denote the concatenation of sequences and numbers 0, 1.

By the assumption of $\mathsf{E} \subsetneqq \overline{\mathsf{E}}$, the set H of Subsection 4.3 is nonempty; obviously H is OD and E-invariant. It follows from Proposition 5 and Lemma 11 that there exists a nonempty OD-1st-countable OD set $X_0 \subseteq H$. Then the set $P_0 = (X_0 \times X_0) \cap \overline{\mathsf{E}}$ belongs to $\mathbb{P}$ and is $\mathbb{P}$-1st-countable by Assertion 15.

We shall define a family of sets X_u $(u \in 2^{<\omega})$ satisfying

(a) $X_u \subseteq X_0$, X_u is nonempty and OD, and $X_{u^\wedge i} \subseteq X_u$, for all u and i.

In addition to the sets X_u, we shall define binary relations Q_{uv} for *some* pairs $\langle u, v\rangle$, to provide important interconnections between different sets X_u.

Let $u, v \in 2^n$. We say that $\langle u, v\rangle$ is a *crucial pair* in 2^n iff $u = 0^{k\wedge}0^\wedge w$ and $v = 0^{k\wedge}1^\wedge w$ where $k < n$ and $w \in 2^{n-k-1}$ (possibly $k = n-1$, that is, $w = \Lambda$). Note that if $\langle u, v\rangle$ is crucial and $i = 0, 1$ then $\langle u^\wedge i, v^\wedge i\rangle$ is crucial, but $\langle u^\wedge i, v^\wedge j\rangle$ is not crucial for $i \neq j$ unless $u = v = 0^k$ for some k.

Thus, we define sets $\mathsf{Q}_{uv} \subseteq X_u \times X_v$ for all crucial pairs $\langle u, v\rangle$ so that the following requirements (b) and (c) are satisfied.

(b) Q_{uv} is OD, $\mathrm{pr}_1\mathsf{Q}_{uv} = X_u$, $\mathrm{pr}_2\mathsf{Q}_{uv} = X_v$, and $\mathsf{Q}_{u^\wedge i,\, v^\wedge i} \subseteq \mathsf{Q}_{uv}$ for every crucial pair $\langle u, v\rangle$ and each $i \in \{0, 1\}$.

(c) For any k, the set $\mathsf{Q}_k = \mathsf{Q}_{0^{k\wedge}0,\, 0^{k\wedge}1}$ is OD-1st-countable, and $\mathsf{Q}_k \subseteq \mathsf{E}$.

This implies $X_u \; \mathsf{Q}_{uv} \; X_v$, therefore $X_u \; \mathsf{E} \; X_v$, for all crucial pairs $\langle u, v\rangle$.[13]

REMARK 20. Every pair of u, $v \in 2^n$ can be tied in 2^n by a finite chain of crucial pairs. It follows that (b) + (c) implies $X_u \; \mathsf{E} \; X_v$ and $X_u \; \overline{\mathsf{E}} \; X_v$ for *all* pairs $\langle u, v\rangle$ in 2^n. ⊣

Three more requirements, (g1), (g2), and (g3), will concern genericity.

In accordance with the 1st-countability of X_0 and P_0, $\{\mathscr{X}_n : n \in \omega\}$ will be a fixed (not necessarily OD) enumeration of all dense in $\mathbb{X}$ below X_0 OD subsets of $\mathbb{X}$ while $\{\mathscr{P}_n : n \in \omega\}$ will be a fixed enumeration of all dense in $\mathbb{P}$ below P_0 OD subsets of $\mathbb{P}$. It is assumed that $\mathscr{X}_{n+1} \subseteq \mathscr{X}_n$ and $\mathscr{P}_{n+1} \subseteq \mathscr{P}_n$. Note that $\mathscr{P}' = \{P \in \mathbb{P} : P \subseteq P_0 \;\&\; \mathrm{pr}_1 P \cap \mathrm{pr}_2 P = \emptyset\}$ is dense in $\mathbb{P}$ below P_0 by Lemma 19, so we can suppose in addition that $\mathscr{P}_0 = \mathscr{P}'$.

In general, for any OD-1st-countable OD set Q let $\{\mathscr{X}_n(\mathsf{Q}) : n \in \omega\}$ be an enumeration of all dense OD subsets in the algebra $\mathscr{P}^{\mathrm{OD}}(\mathsf{Q}) \setminus \{\emptyset\}$. It is assumed that $\mathscr{X}_{n+1}(\mathsf{Q}) \subseteq \mathscr{X}_n(\mathsf{Q})$. We now formulate:

(g1) $X_u \in \mathscr{X}_n$ whenever $u \in 2^n$.

(g2) If u, $v \in 2^n$ and $u(n-1) = 0$, $v(n-1) = 1$, then $P_{uv} = (X_u \times X_v) \cap \overline{\mathsf{E}}$ belongs to $\mathscr{P}_n$.

(g3) If $u = 0^{k\wedge}0^\wedge w$, $v = 0^{k\wedge}1^\wedge w$ is a crucial pair in 2^n and $k < n-1$ (then w is not equal to Λ), then $\mathsf{Q}_{uv} \in \mathscr{X}_n(\mathsf{Q}_k)$. (Recall that $\mathsf{Q}_k = \mathsf{Q}_{0^{k\wedge}0,\, 0^{k\wedge}1}$.)

[13] We recall that $X \; \mathsf{Q} \; Y$ means that $\forall x \in X \; \exists y \in Y \; (x \; \mathsf{Q} \; y)$ and $\forall y \in Y \; \exists x \in X \; (x \; \mathsf{Q} \; y)$.

In particular (g1) implies by Lemma 14 that for any $a \in 2^\omega$ the intersection $\bigcap_{n\in\omega} X_{a\restriction n}$ contains a single point, denoted by $\phi(a)$, which is OD-generic, and the map ϕ is continuous in the sense of the usual (Polish) topology.

ASSERTION 21. *If* (a), (b), (c), *and* (g1), (g2), (g3) *are satisfied then* ϕ *is a continuous* 1-1 *embedding of* E_0 *to* E.

PROOF. Let us prove that ϕ is 1-1. Suppose that $a \neq b \in 2^\omega$. Then, for instance $a(n-1) = 0$ and $b(n-1) = 1$ for some n. Let $u = a\restriction n$, $v = b\restriction n$, so that we have $x = \phi(a) \in X_u$ and $y = \phi(b) \in X_v$. The set $P = (X_u \times X_v) \cap \bar{\mathsf{E}}$ belongs to $\mathscr{P}_n$ by (g2), therefore to $\mathscr{P}_0$. This implies $X_u \cap X_v = \emptyset$ by the assumption that $\mathscr{P}_0 = \mathscr{P}'$, hence $\phi(a) \neq \phi(b)$, as required.

Furthermore if $a \not\mathsf{E}_0\, b$ (this means that $a(k) \neq b(k)$ for infinitely many numbers k) then $\langle \phi(a), \phi(b)\rangle$ is $\mathbb{P}$-generic by (g2), so $\phi(a) \not\mathsf{E}\, \phi(b)$ by Lemma 18.

Let us finally verify that $a\; \mathsf{E}_0\; b$ implies $\phi(a)\; \mathsf{E}\; \phi(b)$. It is sufficient to prove that $\phi(0^k{}^\wedge 0^\wedge c)\; \mathsf{E}\; \phi(0^k{}^\wedge 1^\wedge c)$ holds for all $k \in \omega$ and $c \in 2^\omega$, simply because every pair of u, $v \in 2^n$ is tied in 2^n by a chain of crucial pairs, for any n.

The sequence of sets $W_m = \mathsf{Q}_{0^k{}^\wedge 0^\wedge(c\restriction m),\, 0^k{}^\wedge 1^\wedge(c\restriction m)}$ $(m \in \omega)$ is OD-generic by (g3) in the sense of the forcing $\mathscr{P}^{\text{OD}}(\mathsf{Q}_k) \setminus \{\emptyset\}$ (we recall that $\mathsf{Q}_k = \mathsf{Q}_{0^k{}^\wedge 0,\, 0^k{}^\wedge 1} \subseteq \mathsf{E}$), which is simply a copy of $\mathbb{X}$, so that by Corollary 14 the intersection of all sets W_m is a singleton, which obviously can be equal only to $\langle \phi(0^k{}^\wedge 0^\wedge c)\,,\, \phi(0^k{}^\wedge 1^\wedge c)\rangle$. This yields $\phi(0^k{}^\wedge 0^\wedge c)\; \mathsf{E}\; \phi(0^k{}^\wedge 1^\wedge c)$, as required. ⊣

5.2. Restriction lemma. Thus, part $\langle$II$\rangle$ of Theorem 2 is reduced to the construction of sets X_u and relations Q_{uv} satisfying (a), (b), (c), and (g1), (g2), (g3) (in the assumption Ω-SM). The following combinatorial lemma will be used in the construction.

LEMMA 22. *Let* $n \in \omega$ *and* X_u *be a nonempty* OD *set for each* $u \in 2^n$. *Assume that an* OD *binary relation* $\mathsf{S}_{uv} \subseteq \mathscr{D}^2$ *is given for every crucial pair* $\langle u, v\rangle$ *in* 2^n *so that* $X_u\; \mathsf{S}_{uv}\; X_v$.

1. *If* $u_0 \in 2^n$ *and* $X' \subseteq X_{u_0}$ *is an* OD *and nonempty set then there exists a system of* OD *nonempty sets* $Y_u \subseteq X_u$ $(u \in 2^n)$ *such that still* $Y_u\; \mathsf{S}_{uv}\; Y_v$ *holds for all crucial pairs* $\langle u, v\rangle$, *and in addition* $Y_{u_0} = X'$.

2. *Suppose that* $\langle u_0, v_0\rangle$ *is a crucial pair in* 2^n *and nonempty* OD *sets* $X' \subseteq X_{u_0}$ *and* $X'' \subseteq X_{v_0}$ *satisfy* $X'\; \mathsf{S}_{u_0v_0}\; X''$. *Then there exists a system of* OD *nonempty sets* $Y_u \subseteq X_u$ $(u \in 2^n)$ *such that still* $Y_u\; \mathsf{S}_{uv}\; Y_v$ *holds for all crucial pairs* $\langle u, v\rangle$, *and in addition* $Y_{u_0} = X'$, $Y_{v_0} = X''$.

PROOF. Note that 1 follows from 2. Indeed take an arbitrary v_0 such that either $\langle u_0, v_0\rangle$ or $\langle v_0, u_0\rangle$ is crucial, and put $X'' = \{y \in X_{v_0} : \exists x \in X'\; (x\; \mathsf{S}_{u_0v_0}\; y)\}$, or resp. $X'' = \{y \in X_{v_0} : \exists x \in X'\; (y\; \mathsf{S}_{v_0u_0}\; x)\}$.

To prove Item 2, we use induction on n.

If $n = 1$ then simply take $Y_{u_0} = Y'$ and $Y_{v_0} = Y''$.

The step. We prove the lemma for $n+1$ provided it has been proved for n ; $n \geq 1$. The principal idea is to split 2^{n+1} in two copies of 2^n, namely $U_0 = \{s^\wedge 0 : s \in 2^n\}$ and $U_1 = \{s^\wedge 1 : s \in 2^n\}$, and handle them more or less separately, using the induction hypothesis and the fact that the only crucial pair that connects U_0 and U_1 is the pair of $\hat{u} = 0^n{}^\wedge 0$ and $\hat{v} = 0^n{}^\wedge 1$.

If now $u_0 = \hat{u}$ and $v_0 = \hat{v}$ then we apply the induction hypothesis (Item 1) independently for the families of sets $\{X_u : u \in U_0\}$ and $\{X_u : u \in U_1\}$ and the given sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$. Assembling the results, we get nonempty OD sets $Y_u \subseteq X_u$ $(u \in 2^{n+1})$ such that $Y_u \mathsf{S}_{uv} Y_v$ for all crucial pairs $\langle u, v\rangle$.

Suppose that u_0 and v_0 belong to one and the same domain, say to U_0. Then we first apply the induction hypothesis (Item 2) to the family $\{X_u : u \in U_0\}$ and the sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$. This results in a system of nonempty OD sets $Y_u \subseteq X_u$ $(u \in U_0)$, in particular an OD nonempty set $Y_{\hat{u}} \subseteq X_{\hat{u}}$. We put $Y_{\hat{v}} = \{y \in X_{\hat{v}} : \exists x \in Y_{\hat{u}}\,(x \mathsf{S}_{\hat{u}\hat{v}}\, y)\}$, so that $Y_{\hat{u}} \mathsf{S}_{\hat{u}\hat{v}} Y_{\hat{v}}$, and apply the induction hypothesis (Item 1) to the family $\{X_u : u \in U_1\}$ and the set $Y_{\hat{v}} \subseteq X_{\hat{v}}$. ⊣

5.3. The construction. We put $X_\Lambda = X_0$.

Now assume that the sets X_s $(s \in 2^{n-1})$ and relations Q_{st} for all crucial pairs $\langle s, t\rangle$ in $2^{<n}$ have been defined, and expand the construction at level n.

We first put $A_{s^\wedge i} = X_s$ for all $s \in 2^{n-1}$ and $i \in \{0, 1\}$. We also define $\mathsf{S}_{uv} = \mathsf{Q}_{st}$ for any crucial pair of $u = s^\wedge i$, $v = t^\wedge i$ in 2^n other than the pair of $\hat{u} = 0^{n-1\wedge}0$ and $\hat{v} = 0^{n-1\wedge}1$. For the latter one (note that $A_{\hat{u}} = A_{\hat{v}} = X_{0^{n-1}}$) we put $\mathsf{S}_{\hat{u}\hat{v}} = \overline{\mathsf{E}}$, so that $A_u \mathsf{S}_{uv} A_v$ holds for all crucial pairs $\langle u, v\rangle$ in 2^n including the pair $\langle \hat{u}, \hat{v}\rangle$.

The sets A_u and relations S_{uv} will be reduced in several steps to satisfy requirements (a), (b), (c) and (g1), (g2), (g3) of Subsection 5.1.

Part 1. After 2^n steps of the procedure of Lemma 22 (Item 1), we obtain a system of nonempty OD sets $B_u \subseteq A_u$ $(u \in 2^n)$ such that still $B_u \mathsf{S}_{uv} B_v$ for all crucial pairs $\langle u, v\rangle$ in 2^n, and $B_u \in \mathscr{X}_n$ for all u. Thus, (g1) is guaranteed.

Part 2. To fix (g2), consider an arbitrary pair of $u_0 = s_0{}^\wedge 0$, $v_0 = t_0{}^\wedge 1$, where $s_0, t_0 \in 2^{n-1}$. By Remark 20 and the density of the set $\mathscr{P}_n$ there exist nonempty OD sets $B' \subseteq B_{u_0}$ and $B'' \subseteq B_{v_0}$ s. t. $P = (B' \times B'') \cap \overline{\mathsf{E}} \in \mathscr{P}_n$ and $\mathrm{pr}_1 P = B'$, $\mathrm{pr}_2 P = B''$, so in particular $B' \,\overline{\mathsf{E}}\, B''$. Now we apply Lemma 22 (Item 1) for the two systems of sets, $\{B_{s^\wedge 0} : s \in 2^{n-1}\}$ and $\{B_{t^\wedge 1} : t \in 2^{n-1}\}$, separately (compare with the proof of Lemma 22 !), and the sets $B' \subseteq B_{s_0{}^\wedge 0}$, $B'' \subseteq B_{t_0{}^\wedge 1}$ respectively. This results in a system of nonempty OD sets $B'_u \subseteq B_u$ (where $u \in 2^n$) satisfying $B'_{u_0} = B'$ and $B'_{v_0} = B''$, so that we have $(B'_{u_0} \times B'_{v_0}) \cap \overline{\mathsf{E}} = P \in \mathscr{P}_n$, and still $B'_u \mathsf{S}_{uv} B'_v$ for all crucial pairs $\langle u, v\rangle$ in 2^n, perhaps with the exception of the pair of $\hat{u} = 0^{n-1\wedge}0$, $\hat{v} = 0^{n-1\wedge}1$, which is the only one that connects the two domains. To handle this pair, note that $B'_{\hat{u}} \,\overline{\mathsf{E}}\, B'_{u_0}$ and $B'_{\hat{v}} \,\overline{\mathsf{E}}\, B'_{v_0}$ (Remark 20 is applied to each of the two domains), so that $B'_{\hat{u}} \,\overline{\mathsf{E}}\, B'_{\hat{v}}$ since $B' \,\overline{\mathsf{E}}\, B''$. However $\mathsf{S}_{\hat{u}\hat{v}}$ is so far equal to $\overline{\mathsf{E}}$.

After 4^{n-1} steps (the number of pairs $\langle u_0, v_0\rangle$ to be considered here) we get a system of nonempty OD sets $C_u \subseteq B_u$ $(u \in 2^n)$ such that $(C_u \times C_v) \cap \overline{\mathsf{E}}$ belongs to $\mathscr{P}_n$ whenever $u(n) \neq v(n)$, and still $C_u \mathsf{S}_{uv} C_v$ for all crucial pairs $\langle u, v\rangle$ in 2^n. Thus, (g2) is fixed.

Part 3. We fix (c) for the special crucial pair of $\hat{u} = 0^{n-1\wedge}0$, $\hat{v} = 0^{n-1\wedge}1$. As E is $\mathscr{T}^2$-dense in $\overline{\mathsf{E}}$ and $C_{\hat{u}} \,\overline{\mathsf{E}}\, C_{\hat{v}}$, the set $\mathsf{R} = (C_{\hat{u}} \times C_{\hat{v}}) \cap \mathsf{E}$ is nonempty. Then some nonempty OD set $\mathsf{S} \subseteq \mathsf{R}$ is OD-1st-countable by Lemma 11. Consider the OD sets $C' = \mathrm{pr}_1 \mathsf{S}$ $(\subseteq C_{\hat{u}})$ and $C'' = \mathrm{pr}_2 \mathsf{S}$ $(\subseteq C_{\hat{v}})$; obviously $C' \mathsf{S} C''$, so that $C' \mathsf{S}_{\hat{u}\hat{v}} C''$. (We recall that at the moment $\mathsf{S}_{\hat{u}\hat{v}} = \overline{\mathsf{E}}$.) Using Lemma 22 (Item 2) again, we obtain a system of nonempty OD sets $Y_u \subseteq C_u$ $(u \in 2^n)$ such that still $Y_u \mathsf{S}_{uv} Y_v$ for all crucial pairs $\langle u, v\rangle$ in 2^n, and $Y_{\hat{u}} = C'$, $Y_{\hat{v}} = C''$. We redefine $\mathsf{S}_{\hat{u}\hat{v}}$ by $\mathsf{S}_{\hat{u}\hat{v}} = \mathsf{S}$, but this keeps $Y_{\hat{u}} \mathsf{S}_{\hat{u}\hat{v}} Y_{\hat{v}}$.

Part 4. We fix (g3). Consider a crucial pair of $u_0 = 0^{k\wedge}0^\wedge w$ and $v_0 = 0^{k\wedge}1^\wedge w$ in 2^n such that $k < n - 1$. The relation $\mathsf{R} = \mathsf{S}_{u_0v_0} \cap (Y_{u_0} \times Y_{v_0})$ is a nonempty (since $Y_{u_0}\ \mathsf{S}_{u_0v_0}\ Y_{v_0}$) OD subset of $\mathsf{Q}_k = \mathsf{Q}_{0^{k\wedge}0,\,0^{k\wedge}1}$ by the construction. Let $\mathsf{S} \subseteq \mathsf{R}$ be a nonempty OD set in $\mathscr{X}_n(\mathsf{Q}_k)$. Now put $Y' = \mathrm{pr}_1\mathsf{S}$ and $Y'' = \mathrm{pr}_2\mathsf{S}$ (then $Y'\ \mathsf{S}\ Y''$ and $Y'\ \mathsf{S}_{u_0v_0}\ Y''$) and apply Lemma 22 to the system of sets Y_u (where $u \in 2^n$) and the sets $Y' \subseteq Y_{u_0}$, $Y'' \subseteq Y_{v_0}$. After this define the "new" relation $\mathsf{S}_{u_0v_0}$ by $\mathsf{S}_{u_0v_0} = \mathsf{S}$.

Do this consecutively for all crucial pairs; the finally obtained sets—let us denote them by X_u $(u \in 2^n)$—are as required. The final relations Q_{uv} ($\langle u, v\rangle$ being crucial pairs in 2^n) can be obtained as the restrictions of the relations S_{uv} to $X_u \times X_v$.

This ends the construction.

$\dashv$ (Theorem 2 and Theorem 1, see Section 1.)

REFERENCES

[1] S. D. Friedman and B. Velickovic, *Nonstandard models and analytic equivalence relations*, ***Bulletin of the American Mathematical Society***, to appear.

[2] L. A. Harrington, A. S. Kechris, and A. Louveau, *A Glimm-Effros dichotomy for Borel equivalence relations*, ***Journal of the American Mathematical Society***, vol. 3 (1990), pp. 903–928.

[3] G. Hjorth, *A dichotomy for the definable universe*, this Journal, vol. 60 (1995), pp. 1199–1207.

[4] G. Hjorth and A. S. Kechris, *Analytic equivalence relations and Ulm-type classification*, this Journal, vol. 60 (1995), pp. 1273–1300.

[5] V. Kanovei, *Ulm classification of analytic equivalence relations in generic universes*, ***Mathematical Logic Quarterly***, to appear.

[6] A. S. Kechris, ***Classical descriptive set theory***, Springer-Verlag, 1995.

[7] H. Lebesgue, *Sur les fonctions représentables analytiquement*, ***Journal de Mathématiques Pures et Appliqués, Neuviéme Série*** (1905), pp. 139–216, t. 1, fasc. 2.

[8] A. W. Miller, ***Descriptive set theory and forcing: how to prove theorems about Borel sets the hard way***, University of Wisconsin, Madison, January 1994.

[9] J. Silver, *Counting the number of equivalence classes of Borel and coanalytic equivalence relations*, ***Annals of Mathematical Logic***, vol. 18 (1980), pp. 1–28.

[10] R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, ***Annals of Mathematics***, vol. 92 (1970), pp. 1–56.

[11] J. Stern, *On Lusin's restricted continuum problem*, ***Annals of Mathematics***, vol. 120 (1984), pp. 7–37.

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