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ON NON-WELLFOUNDED ITERATIONS OF THE PERFECT SET FORCING

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Abstract. We prove that if I is a partially ordered set in a countable transitive model $\mathfrak{M}$ of **ZFC** then $\mathfrak{M}$ can be extended by a generic sequence of reals $\mathbf{a}_i$, $i \in I$, such that $\aleph_1^{\mathfrak{M}}$ is preserved and every $\mathbf{a}_i$ is Sacks generic over $\mathfrak{M}[\langle \mathbf{a}_j : j < i \rangle]$. The structure of the degrees of $\mathfrak{M}$-constructibility of reals in the extension is investigated.

As applications of the methods involved, we define a cardinal invariant to distinguish product and iterated Sacks extensions, and give a short proof of a theorem (by Budinas) that in ω_2-iterated Sacks extension of L the Burgess selection principle for analytic equivalence relations holds.

Introduction. It is the usual practice in set theory that one is interested to consider a generic extension M_1 of a given model M, then a generic extension M_2 of M_1, and so on, including the case of infinite or transfinite number of steps. *Iterated forcing* of Solovay and Tennenbaum [8] converts this iterated construction in an ordinary one-step generic extension.

In many cases, iterated forcing is used to define transfinite sequences of models such that every model is a generic extension of the preceding model. (We do not consider here sophisticated details at limit steps.) Identifying the steps of this construction with ordinals, and interpreting the set of the ordinals involved as the *length* of the iteration, we may say that the classical iterated forcing has *wellordered* length of iteration.

In principle it does not require a principal improvement of the basic iterated forcing method to define iterations with *wellfounded*, but not wellordered, "length" of iteration. This version is much rarely used than the ordinary iterated forcing. (However see Groszek and Jech [6].)

It is a much more challenging question (we refer to Groszek and Jech [6], p. 6) to carry out "ill"founded iterations. No general method is known, at least.

For a few number of rather simple forcing notions, "ill"founded iterations can be obtained without any use of the idea of iteration at all. For example if $a \in 2^\omega$ is a Cohen generic real over a model $\mathfrak{M}$, and $a_m \in 2^\omega$ is defined for any m by $a_m(k) = a(2^m 3^k)$, $\forall k$, then the sequence of reals a_n realizes iterated Cohen forcing with ω^* (the order of negative integers) as the length of iteration: every a_n is Cohen

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generic over the model $\mathfrak{M}[\langle a_m : m > n \rangle]$. This construction also applies to Solovay random reals.

An idea how to define iterated forcing with a linear but not wellordered length of iteration I can be as follows. Consider first a usual iteration of a length $\lambda \in \mathrm{Ord}$ as a pattern. The forcing conditions in this case are functions p defined on $\lambda = \{ \alpha : \alpha < \lambda \}$ and satisfying certain property $P(p, \alpha)$ for every $\alpha < \lambda$. Now, to proceed with the I-case, one may want to use functions p defined on I and satisfying $P(p, i)$ for all $i \in I$.

The principal problem in this argument is that the property $P(p, \alpha)$ is itself defined, in the wellordered setting, by induction on α in quite a sophisticated way. So we first have to eliminate the induction and extend the property P to "ill"ordered sets I taken as the length of iteration.

We do not know how this can be realized at least for a more or less representative category of forcing notions. There is, however, a forcing which allows us to express the property P in simple "geometric" terms, so that "ill"founded iterations become available. This is the perfect set forcing of Sacks [7]. (See Baumgartner and Laver [1] as the basic reference on the iterated Sacks forcing, and Groszek [4] on some further applications.)

THEOREM 1. *Let $\mathfrak{M}$ be a countable transitive model of* **ZFC**, *I a partially ordered set in $\mathfrak{M}$. Then there exists a generic $\aleph_1$-preserving extension $\mathfrak{N} = \mathfrak{M}[\langle \boldsymbol{a}_i : i \in I \rangle]$ of $\mathfrak{M}$ such that for every $i \in I$, $\boldsymbol{a}_i$ is a Sacks-generic real over the model $\mathfrak{M}[\langle \boldsymbol{a}_j : j < i \rangle]$, and in addition*

1. *If $i, j \in I$ and $i < j$ then $\boldsymbol{a}_i \in \mathfrak{M}[\boldsymbol{a}_j]$. But if $J \in \mathfrak{M}$ is an initial segment in I and $i \in I \smallsetminus J$ then $\boldsymbol{a}_i$ does not belong to $\mathfrak{M}[\langle \boldsymbol{a}_j : j \in J \rangle]$.*
2. *If r, r' are reals in $\mathfrak{N}$ then either $r \in \mathfrak{M}[r']$ or there exists $i \in I$ such that $\boldsymbol{a}_i \in \mathfrak{M}[r] \smallsetminus \mathfrak{M}[r']$. In the "either" case there exists a continuous function $F : \mathtt{reals} \to \mathtt{reals}$, coded in $\mathfrak{M}$, such that $r = F(r')$.*
3. *Suppose that all initial segments $J \subseteq I$ belong to $\mathfrak{M}$. Then for any real $r \in \mathfrak{N}$ there is a countable in $\mathfrak{M}$ set $\xi \in \mathfrak{M}$, $\xi \subseteq I$ such that*

$$\mathfrak{M}[r] = \mathfrak{M}[\langle \boldsymbol{a}_i : i \in \xi \rangle].$$

The set I is not necessarily wellfounded or linearly ordered in $\mathfrak{M}$. Items 1, 2, and 3[1] say that the degrees of $\mathfrak{M}$-constructibility of reals in the extension

$$\mathfrak{N} = \mathfrak{M}[\langle \mathbf{a}_i : i \in I \rangle]$$

are essentially determined by the order structure of I. For instance if all initial segments of I belong to $\mathfrak{M}$ (this includes, in particular, the cases when I is an ordinal or an inverse ordinal), the structure of $\mathfrak{M}$-degrees of reals in $\mathfrak{N}$ is isomorphic, by the theorem, to the structure of all *countably generated* (that is of the form

$$\bigcup_{k \in \omega} \{ i : i \leq i_k \}$$

in $\mathfrak{M}$) initial segments of I.

[1] All of them are known for Sacks iterations of wellordered or even wellfounded length, although it is not easy to give a comprehensive reference.

A construction of iterated Sacks generic extensions, having inverse ordinals as the "length" of iteration, was introduced by Groszek [5]. We make different technical arrangements to obtain "ill"founded Sacks iterations.

Let I be a partially ordered set in $\mathfrak{M}$. Let $\mathscr{D} = 2^\omega$, the Cantor space. A typical forcing condition is, in $\mathfrak{M}$, a set $X \subseteq \mathscr{D}^\zeta$, where $\zeta \subseteq I$ is countable, of the form $X = \{ H(x) : x \in \mathscr{D}^\zeta \}$, where $H : \mathscr{D}^\zeta \to \mathscr{D}^\zeta$ is a one-to-one continuous function such that

$$x \upharpoonright \xi = y \upharpoonright \xi \iff H(x) \upharpoonright \xi = H(y) \upharpoonright \xi$$

for all $x, y \in \mathscr{D}^\zeta$ and any initial segment ξ of ζ. Section 1 contains the definition and several basic lemmas on the forcing conditions.

Sections 2 and 3 show how the forcing conditions split and gather via a kind of fusion technique common for the Sacks forcing. Section 4 considers the behaviour of continuous real functions on forcing conditions. The results of this study are involved in the proof of additional items of Theorem 1.

Section 5 formally defines the generic extension and proves the preservation of $\aleph_1$ and an important theorem saying that the reals in the extension can be presented by continuous functions coded in the ground model and applied to generic objects. This leads to the proof of Theorem 1 in Section 6.

Two applications of the technique of Sacks iterations are presented in the final part of the paper.

Iterated vs. product Sacks models. Section 7 is devoted to a cardinal invariant which distinguish "long" product and iterated Sacks extensions. J. Steprāns gave some invariants in a talk on this matter at LC '95 (Haifa, August 1995). We present a simpler invariant.

Every collection $\mathscr{F}$ of continuous functions $f : \mathscr{N} \to \mathscr{N}$ determines a partial order $\leq_{\mathscr{F}}$ on the reals as follows: $x \leq_{\mathscr{F}} y$ if and only if

$$x = f_1(f_2(\dots f_n(y)\dots))$$

for some functions $f_1, f_2, \dots, f_n \in \mathscr{F}$. Let $\mathfrak{l}$ (the *linear order cardinal*) denote the least cardinality of a family $\mathscr{F}$ such that $\leq_{\mathscr{F}}$ linearly orders the reals.

THEOREM 2. *Let $\mathfrak{M}$ be a countable transitive model of* **ZFC**. *Then we have $\mathfrak{l} >$ $\operatorname{card}(\mathfrak{c}^{\mathfrak{M}})$ in each countable support* ***product*** *Sacks extension of $\mathfrak{M}$ with strictly more than $\mathfrak{c}^{\mathfrak{M}}$-many factors but we have $\mathfrak{l} \leq \operatorname{card}(\mathfrak{c}^{\mathfrak{M}})$ in each countable support* ***iterated*** *Sacks extension of $\mathfrak{M}$.*

In particular, if $\mathfrak{c} = \aleph_1$ in $\mathfrak{M}$ then $\mathfrak{l} = \mathfrak{c} > \aleph_1^{\mathfrak{M}}$ in countable support product Sacks extensions of $\mathfrak{M}$, the ground model, provided we have at least $\aleph_2^{\mathfrak{M}}$-many factors, but $\mathfrak{l} = \aleph_1^{\mathfrak{M}} < \mathfrak{c}$ in countable support iterated Sacks extensions of $\mathfrak{M}$, provided the length of the extension is an ordinal of cofinality $\geq \aleph_2^{\mathfrak{M}}$ in $\mathfrak{M}$. In the second case, the collection $\mathscr{F}$ of all continuous real functions coded in $\mathfrak{M}$ witnesses that $\mathfrak{l} \leq \operatorname{card}(\mathfrak{c}^{\mathfrak{M}})$ in the extension.

The selection principle is consistent with the negation of CH. Burgess [3] introduced the following *selection principle*:

SP: every $\mathbf{\Sigma}^1_1$ equivalence relation on the reals has a $\mathbf{\Sigma}^1_2$ selector.

(A *selector* for an equivalence relation E is a subset of the domain of E which has exactly one element in common with each E-class.)

Clearly **SP** follows from the axiom of constructibility $\mathrm{V} = \mathrm{L}$, and, more generally, from $\mathrm{V} = \mathrm{L}[a]$ for a real a. But actually a "good" $\mathbf{\Sigma}^1_2$ wellordering of the reals is applied. Burgess asked whether **SP** implies the existence of a $\mathbf{\Sigma}^1_2$ wellordering of the reals. Budinas [2] answered the question negatively:

THEOREM 3 (Budinas [2]). **SP**+ *"there does not exist a real-ordinal definable well-ordering of the reals"* $+2^{\aleph_0} = \aleph_2$ *is consistent with* **ZFC**.

Sacks iterations of the constructible universe, of length ω_2, were applied in [2] to prove the theorem. It is demonstrated in Section 8 how our general technique of exploration of iterated Sacks models produces another proof of Theorem 3, considerably shorter than the proof given by Budinas.

§1. **The forcing.** Let **CPO** be the class of all countable (including finite) partially ordered sets $\zeta = \langle \zeta; < \rangle$. Greek letters $\xi, \eta, \zeta, \vartheta$ will denote sets in **CPO**. Characters i, j are used for *elements* of sets in **CPO**. For any $\zeta \in \mathbf{CPO}$, $\mathbf{IS}_\zeta$ is the collection of all initial segments of ζ. For instance $\emptyset$ and ζ itself belong to $\mathbf{IS}_\zeta$.

Usually a "basic" p.o. set $\zeta \in \mathbf{CPO}$ will be fixed, so that the other p.o. sets involved in the reasoning are subsets of ζ and even members of $\mathbf{IS}_\zeta$. In this case, for any $i \in \zeta$ we shall consider special initial segments

$$[{<}i] = \{ j \in \zeta : j < i \} \qquad \text{and} \qquad [{\not\geq} i] = \{ j \in \zeta : j \not\geq i \},$$

and $[{\leq} i]$, $[{\not>} i]$ defined similarly.

As usual, $\mathscr{N} = \omega^\omega$ is the *Baire space*; points of $\mathscr{N}$ will be called *reals*.

$\mathscr{D} = 2^\omega$ is the *Cantor space*. For any countable set ξ, $\mathscr{D}^\xi$ is the product of ξ-many copies of $\mathscr{D}$ with the product topology. Then every $\mathscr{D}^\xi$ is a compact space, homeomorphic to $\mathscr{D}$ itself unless $\xi = \emptyset$.

Assume that $\eta \subseteq \xi$. If $x \in \mathscr{D}^\xi$ then let $x \restriction \eta \in \mathscr{D}^\eta$ denote the usual restriction. If $X \subseteq \mathscr{D}^\xi$ then let

$$X \restriction \eta = \{ x \restriction \eta : x \in X \}.$$

But if $Y \subseteq \mathscr{D}^\eta$ then we set

$$Y \restriction^{-1} \xi = \{ x \in \mathscr{D}^\xi : x \restriction \eta \in Y \}.$$

To save space, let $X \restriction_{<i}$ mean $X \restriction [{<}i]$, $X \restriction_{\not\geq i}$ mean $X \restriction [{\not\geq} i]$, etc.

DEFINITION (The forcing). For any set $\zeta \in \mathbf{CPO}$, $\mathtt{Perf}_\zeta$ is the collection of all sets $X \subseteq \mathscr{D}^\zeta$ such that there exists a homeomorphism $H : \mathscr{D}^\zeta$ onto X satisfying

$$x_0 \restriction \xi = x_1 \restriction \xi \iff H(x_0) \restriction \xi = H(x_1) \restriction \xi$$

for all $x_0, x_1 \in \operatorname{dom} H$ and $\xi \in \mathbf{IS}_\zeta$. Homeomorphisms H satisfying this requirement will be called *projection-keeping*. To conclude, sets in $\mathtt{Perf}_\zeta$ are images of $\mathscr{D}^\zeta$ via projection-keeping homeomorphisms.

PROPOSITION 4. *Every set* $X \in \mathtt{Perf}_\zeta$ *is closed and satisfies the following*:[2]

P-1. *If* $i \in \zeta$ *and* $z \in X \restriction_{<i}$ *then the set*

$$D_{Xz}(i) = \{ x(i) : x \in X \;\&\; x \restriction_{<i} = z \}$$

is a perfect subset of $\mathscr{D}$.

[2]This could be taken as the base for an independent definition of the forcing; however in fact the properties P-1, P-2, P-3 do not fully characterize $\mathtt{Perf}_\zeta$.

P-2. *If* $\xi \in \mathbf{IS}_\zeta$ *and a set* $X' \subseteq X$ *is open in* X *(in the relative topology) then the projection* $X' \restriction \xi$ *is open in* $X \restriction \xi$.[3]

P-3. *If* $\xi,\ \eta \in \mathbf{IS}_\zeta,\ \ x \in X \restriction \xi,\ \ y \in X \restriction \eta$, *and* $x \restriction (\xi \cap \eta) = y \restriction (\xi \cap \eta)$, *then* $x \cup y \in X \restriction (\xi \cup \eta)$.

PROOF. Obviously $\mathscr{D}^\zeta$ satisfies P-1, P-2, and P-3. On the other hand, one easily sees that projection-keeping homeomorphisms preserve the properties. ⊣

The following lemma shows how P-3 works.

LEMMA 5. *Suppose that* $X \in \mathtt{Perf}_\zeta,\ \ \xi, \eta \in \mathbf{IS}_\zeta$, *and* $Y \subseteq X \restriction \eta$ *is any set. Let* $Z = X \cap (Y \restriction^{-1} \zeta)$. *Then*

$$Z \restriction \xi = (X \restriction \xi) \cap (Y \restriction (\xi \cap \eta) \restriction^{-1} \xi).$$

PROOF. To prove the nontrivial direction $\supseteq$ let x belong to the right-hand side. Then in particular $x \restriction (\xi \cap \eta) = y \restriction (\xi \cap \eta)$ for some $y \in Y$. On the other hand, $x \in X \restriction \xi$ and $y \in X \restriction \eta$. Property P-3 of X implies $x \cup y \in X \restriction (\xi \cup \eta)$. Thus $x \cup y \in Z \restriction (\xi \cup \eta)$ since $y \in Y \subseteq X \restriction \eta$, so $x \in Z \restriction \xi$. ⊣

DEFINITION. If $H : \mathscr{D}^\zeta$ onto X is a projection-keeping homeomorphism then for any $\xi \in \mathbf{IS}_\zeta$ we define an associated projection-keeping homeomorphism $H_\xi : \mathscr{D}^\xi$ onto $X \restriction \xi$ by $H_\xi(x \restriction \xi) = H(x) \restriction \xi$ for all $x \in \mathscr{D}^\zeta$.

LEMMA 6. *If* $X \in \mathtt{Perf}_\zeta$ *and* $\xi \in \mathbf{IS}_\zeta$ *then* $X \restriction \xi \in \mathtt{Perf}_\xi$.

PROOF. If H witnesses that $X \in \mathtt{Perf}_\zeta$ then $X \restriction \xi \in \mathtt{Perf}_\xi$ via H_ξ. ⊣

LEMMA 7. *Suppose that* H *is a projection-keeping homeomorphism, defined on* $X \in \mathtt{Perf}_\zeta$. *Then the image* $H"X = \{ H(x) : x \in X \}$ *belongs to* $\mathtt{Perf}_\zeta$.

PROOF. A superposition of projection-keeping homeomorphisms is a projection-keeping homeomorphism. ⊣

LEMMA 8. *Assume that* $X \in \mathtt{Perf}_\zeta$, *a set* $X' \subseteq X$ *is open in* X, *and* $x_0 \in X'$. *There exists a clopen in* X *set* $X'' \in \mathtt{Perf}_\zeta,\ X'' \subseteq X'$, *containing* x_0.

PROOF. By the previous lemma, it suffices to prove the result for $X = \mathscr{D}^\zeta$. Note that if $x_0 \in X' \subseteq \mathscr{D}^\zeta$ and X' is open in $\mathscr{D}^\zeta$ then there exists a basic clopen set $C \subseteq X'$ containing x_0. (*Basic clopen sets* are sets of the form

$$C = \{ x \in \mathscr{D}^\zeta : u_1 \subset x(i_1)\ \&\ \cdots\ \&\ u_m \subset x(i_m) \},$$

where $m \in \omega,\ i_1, \dots, i_m \in \zeta$ are pairwise different, and $u_1, \dots, u_m \in 2^{<\omega}$.) One easily proves that every set C of this type actually belongs to $\mathtt{Perf}_\zeta$. ⊣

LEMMA 9. *Suppose that* $X \in \mathtt{Perf}_\zeta,\ \eta \in \mathbf{IS}_\zeta,\ \ Y \in \mathtt{Perf}_\eta$, *and* $Y \subseteq X \restriction \eta$. *Then the set* $Z = X \cap (Y \restriction^{-1} \zeta)$ *belongs to* $\mathtt{Perf}_\zeta$.

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[3]In other words, it is required that the projection from X to $X \restriction \xi$ is an *open* map.

PROOF. Let $F : \mathscr{D}^\zeta$ onto X and $G : \mathscr{D}^\eta$ onto Y witness that, respectively, $X \in \mathtt{Perf}_\zeta$ and $Y \in \mathtt{Perf}_\eta$. Define a map $H : \mathscr{D}^\zeta \to Z$ by

$$H(z) = F\big(F_\eta^{-1}(G(z\restriction\eta)) \cup (z\restriction(\zeta \smallsetminus \eta))\big) \qquad \text{for all } z \in \mathscr{D}^\zeta.$$

Prove that H maps $\mathscr{D}^\zeta$ onto Z. Let $z \in \mathscr{D}^\zeta$. Then $H(z) \in X$ by the choice of F. Furthermore

$$H(z)\restriction\eta = F_\eta\big(F_\eta^{-1}(G(z\restriction\eta))\big) = G(z\restriction\eta) \in Y,$$

so $H(z) \in Z$. Let conversely $z' \in Z$, so that $z' = F(x)$ for some $x \in \mathscr{D}^\zeta$. We define $z \in \mathscr{D}^\zeta$ by:

$$z = G^{-1}(F_\eta(x\restriction\eta)) \cup (x\restriction(\zeta \smallsetminus \eta)).$$

(To be sure that G^{-1} is applicable note that $F_\eta(x\restriction\eta) = F(x)\restriction\eta = z'\restriction\eta \in Z\restriction\eta = Y$.) Then $H(z) = F(x) = z'$.

Prove that H is projection-keeping. Let $z_0, z_1 \in \mathscr{D}^\zeta$ and $\xi \in \mathbf{IS}_\zeta$. Suppose that $z_0\restriction\xi = z_1\restriction\xi$, and prove $H(z_0)\restriction\xi = H(z_1)\restriction\xi$. Let us define $x_e \in \mathscr{D}^\zeta$ $(e = 0, 1)$ by

$$x_e = F_\eta^{-1}(G(z_e\restriction\eta)) \cup (z_e\restriction(\zeta \smallsetminus \eta)).$$

Then, first, $H(z_e) = F(x_e)$ and, second, since both F and G are projection-keeping, we have $x_0\restriction\xi = x_1\restriction\xi$ and then $F(x_0)\restriction\xi = F(x_1)\restriction\xi$, as required. The converse is proved similarly. $\dashv$

LEMMA 10. *Assume that $\zeta \subseteq \vartheta \in$ **CPO** and $X \in \mathtt{Perf}_\zeta$. Then the set $X' = X\restriction^{-1}\vartheta$ belongs to $\mathtt{Perf}_\vartheta$.*

PROOF. If $X \in \mathtt{Perf}_\zeta$ is witnessed by some $H : \mathscr{D}^\zeta$ onto X then the homeomorphism H', defined on $\mathscr{D}^\vartheta$ by the equalities

$$H'(x')\restriction(\vartheta \smallsetminus \zeta) = x'\restriction(\vartheta \smallsetminus \zeta) \qquad \text{and} \qquad H'(x')\restriction\zeta = H(x'\restriction\zeta)$$

for all $x' \in \mathscr{D}^\vartheta$, witnesses that $X' \in \mathtt{Perf}_\vartheta$. $\dashv$

Let a *perfect tree* be any (nonempty) tree $T \subseteq 2^{<\omega}$ such that the set of all splitting points of T,

$$B(T) = \{\, t \in T : t^\frown 0 \in T \,\&\, t^\frown 1 \in T \,\},$$

is cofinal in T. Suppose T is such a tree. Define the following:

- $[T] = \{\, a \in 2^\omega : \forall m\ (a\restriction m \in T) \,\}$, a perfect set in $\mathscr{D} = 2^\omega$.
- An order isomorphism $\beta_T : 2^{<\omega}$ onto $B(T)$. We define $\beta_T(u) \in B(T)$ for every $u \in 2^{<\omega}$ by induction on $\operatorname{dom} u$, putting $\beta_T(u^\frown e)$ to be the least $s \in B(T)$ such that $\beta_T(u)^\frown e \subseteq s$, for $e = 0, 1$.
- A homeomorphism $H_T : \mathscr{D}$ onto $[T]$ by

$$H_T(a) = \bigcup_{m\in\omega} \beta_T(a\restriction m)$$

for all $a \in \mathscr{D}$.

LEMMA 11. *Assume that i is the largest element in $\zeta \in \mathbf{CPO}$, $\eta = \zeta \setminus \{i\}$, $Y \in \mathtt{Perf}_\eta$, a function $\mathscr{T}$ continuously maps Y into $\mathscr{P}(2^{<\omega})$ so that $\mathscr{T}(y)$ is a perfect tree for all $y \in Y$. Then*

$$X = \{ x \in \mathscr{D}^\zeta : x \restriction \eta \in Y \,\&\, x(i) \in [\mathscr{T}(x\restriction\eta)] \}$$

belongs to $\mathtt{Perf}_\zeta$.

PROOF. The set $Z = Y \restriction^{-1} \zeta$ belongs to $\mathtt{Perf}_\zeta$ by Lemma 10, so it suffices by Lemma 7 to define a projection-keeping homeomorphism $H: Z$ onto X. Let $z \in Z$. Then $y = z\restriction\eta \in Y$ while $a = z(i) \in \mathscr{D}$ is arbitrary. We define $x = H(z) \in \mathscr{D}^\zeta$ so that $x\restriction\eta = y$ and $x(i) = H_{\mathscr{T}(y)}(a)$. Then H maps Z onto X because every $H_{\mathscr{T}(y)}$ maps $\mathscr{D}$ onto

$$[\mathscr{T}(y)] = \{ x(i) : x \in X \,\&\, x\restriction\eta = y \}.$$

H is one-to-one since each H_T is one-to-one, and H is continuous since so is the map $\mathscr{T}$. It remains to prove that H is projection-keeping, i.e., the equivalence

$$z_0\restriction\xi = z_1\restriction\xi \iff H(z_0)\restriction\xi = H(z_1)\restriction\xi$$

for all $z_0, z_1 \in Z$ and $\xi \in \mathbf{IS}_\zeta$. If $i \notin \xi$ then $\xi \subseteq \eta$ and $z\restriction\xi = H(z)\restriction\xi$ by definition. If $i \in \xi$ then $\xi = \zeta$, so the result is obvious as well. ⊣

§2. Splitting. We shall use the construction of sets in $\mathtt{Perf}_\zeta$ as

$$X = \bigcap_{m\in\omega} \bigcup_{u\in 2^m} X_u$$

where all X_u belong to $\mathtt{Perf}_\zeta$. This and the next sections introduce the technique.

First of all let us specify requirements which imply an appropriate behaviour of the sets $X_u \in \mathtt{Perf}_\zeta$ with respect to projections. We need to determine, for any pair of finite binary sequences u, $v \in 2^m$ $(m \in \omega)$, the largest initial segment $\xi = \zeta[u, v]$ of ζ such that the projections $X_u\restriction\xi$ and $X_v\restriction\xi$ have to be equal, to run the construction in proper way.

Let us fix $\zeta \in \mathbf{CPO}$ and an arbitrary function $\phi: \omega \to \zeta$.

We define, for any pair of finite sequences u, $v \in 2^m$, an initial segment

$$\left.\begin{aligned}\zeta_\phi[u, v] &= \bigcap_{l<m,\ u(l)\neq v(l)} [\not\geq \phi(l)] \\ &= \{ j \in \zeta : \neg\exists\, l < m\ (u(l) \neq v(l) \,\&\, j \geq \phi(l)) \}\end{aligned}\right| \in \mathbf{IS}_\zeta.$$

DEFINITION. A *ϕ-splitting system* (rather $(\phi\restriction m)$-splitting as the notion depends only on $\phi\restriction m$) in $\mathtt{Perf}_\zeta$ is a family $\langle X_u : u \in 2^m \rangle$ of sets $X_u \in \mathtt{Perf}_\zeta$ such that

S-1 $X_u\restriction\zeta_\phi[u, v] = X_v\restriction\zeta_\phi[u, v]$ and
S-2 if $i \in \zeta \setminus \zeta_\phi[u, v]$ then $X_u\restriction_{\leq i} \cap X_v\restriction_{\leq i} = \emptyset$

for all u, $v \in 2^m$. A splitting system $\langle X_{u'} : u' \in 2^{m+1} \rangle$ is an *expansion* of a splitting system $\langle X_u : u \in 2^m \rangle$ if and only if $X_{u^\frown e} \subseteq X_u$ for all $u \in 2^m$ and $e = 0, 1$.

We consider two ways how an existing splitting system can be transformed to another splitting system. One of them treats the case when one of the sets changes to a smaller set in $\mathtt{Perf}_\zeta$, the other expands to the next level.

LEMMA 12. *Assume that* $\langle X_u : u \in 2^m \rangle$ *is a* ϕ*-splitting system in* $\mathtt{Perf}_\zeta$, $u_0 \in 2^m$, *and* $X \in \mathtt{Perf}_\zeta$, $X \subseteq X_{u_0}$. *Let us re-define the sets* X_u *by*

$$X'_u = X_u \cap (X \restriction \zeta_\phi[u, u_0] \restriction^{-1} \zeta)$$

for all $u \in 2^m$. *Then the re-defined family is again a* ϕ*-splitting system.* (Notice that $X'_{u_0} = X$.)

PROOF. Each set X'_u belongs to $\mathtt{Perf}_\zeta$ by Lemmas 6 and 9. We have to check only requirement S-1. Thus let $u, v \in 2^m$ and $\xi = \zeta_\phi[u, v]$. We prove that $X'_u \restriction \xi = X'_v \restriction \xi$. Let in addition $\zeta_u = \zeta_\phi[u, u_0]$ and $\zeta_v = \zeta_\phi[v, u_0]$. Then

$$X'_u \restriction \xi = (X_u \restriction \xi) \cap (X_0 \restriction (\xi \cap \zeta_u) \restriction^{-1} \xi), \quad X'_v \restriction \xi = (X_v \restriction \xi) \cap (X_0 \restriction (\xi \cap \zeta_v) \restriction^{-1} \xi)$$

by Lemma 5. Thus it remains to prove that $\xi \cap \zeta_u = \xi \cap \zeta_v$ (the "triangle" equality). Assume on the contrary that, e.g., $i \in \xi \cap \zeta_u$ but $i \notin \zeta_v$. The latter means that $i \geq \phi(l)$ in ζ for some $l < m$ such that $v(l) \neq u_0(l)$. But then either $u(l) \neq u_0(l)$—so $i \notin \zeta_u$, or $u(l) \neq v(l)$—so $i \notin \xi$, contradiction. ⊣

We are going to prove that each splitting system has an expansion. This needs to define first a special splitting construction.

Let $i \in \zeta$ and $X \in \mathtt{Perf}_\zeta$. Let us say that a pair of sets $X_0, X_1 \in \mathtt{Perf}_\zeta$ is an *i-splitting* of X if and only if

$$X_0 \cup X_1 \subseteq X, \qquad X_0 \restriction_{\not\geq i} = X_1 \restriction_{\not\geq i}, \qquad \text{and} \qquad X_0 \restriction_{\leq i} \cap X_1 \restriction_{\leq i} = \emptyset.$$

The splitting will be called *complete* if $X_0 \cup X_1 = X$—in this case we have

$$X_0 \restriction_{\not\geq i} = X_1 \restriction_{\not\geq i} = X \restriction_{\not\geq i}.$$

ASSERTION. Let $i \in \zeta$. Every $X \in \mathtt{Perf}_\zeta$ admits a complete i-splitting.

PROOF. If $X = \mathscr{D}^\zeta$ then we define

$$X_e = \{ x \in X : x(i)(0) = e \}, \qquad e = 0, 1.$$

Lemma 7 extends the result on the general case. ⊣

LEMMA 13. *Every* ϕ*-splitting system* $\langle X_u : u \in 2^m \rangle$ *in* $\mathtt{Perf}_\zeta$ *can be expanded to a* ϕ*-splitting system* $\langle X_{u'} : u' \in 2^{m+1} \rangle$ *in* $\mathtt{Perf}_\zeta$.

PROOF. As ϕ is fixed, we shall write $\zeta[u, v]$ instead of $\zeta_\phi[u, v]$. Let $i = \phi(m)$.

Let us consider, one by one in an arbitrary but fixed order, all sequences $u \in 2^m$. At each step u, we shall i-split X_u in one of two different ways.

CASE A. There does *not* exist $w \in 2^m$, considered earlier than u, such that $i \in \zeta[u, w]$. Then let $X_{u^\frown 0}, X_{u^\frown 1}$ be an arbitrary complete i-splitting of X_u.

CASE B. Otherwise, let w be the one encountered first among all sequences w of the mentioned type. We put

$$X_{u^\frown e} = X_u \cap (X_{w^\frown e} \restriction_{\leq i} \restriction^{-1} \zeta)$$

for $e = 0, 1$.

Let us prove that $X_{u^\frown 0}, X_{u^\frown 1}$ is a complete i-splitting of X_u in this case. First of all, $X_u \restriction \zeta[u, w] = X_w \restriction \zeta[u, w]$ by S-1; it follows that

$$X_{w^\frown e} \restriction_{\leq i} \subseteq X_w \restriction_{\leq i} = X_u \restriction_{\leq i},$$

so the sets $X_{u^\frown e}$ belong to $\mathtt{Perf}_\zeta$ by Lemmas 6 and 9.

By the choice of w, we had Case A at step w. (Indeed, if otherwise $i \in \zeta[w, w']$ for some $w' \in 2^m$ considered even earlier, then $i \in \zeta[u, w']$ by the "triangle" equality in the proof of Lemma 12, contradiction.) Therefore for sure $X_{w^\frown 0}, X_{w^\frown 1}$ is a complete i-splitting of X_w. In particular, $X_{w^\frown e}{\restriction}_{<i} = X_w{\restriction}_{<i}$. On the other hand, Lemma 5 implies

$$X_{u^\frown e}{\restriction}_{\not\geq i} = X_u{\restriction}_{\not\geq i} \cap (X_{w^\frown e}{\restriction}_{<i}{\restriction}^{-1}[\not\geq i])$$

for $e = 0, 1$, since $[\not\geq i] \cap [\leq i] = [<i]$. This implies $X_{u^\frown 0}{\restriction}_{\not\geq i} = X_{u^\frown 1}{\restriction}_{\not\geq i}$.

By definition, $X_{u^\frown e}{\restriction}_{\leq i} = X_{w^\frown e}{\restriction}_{\leq i}$ for $e = 0, 1$, so

$$X_{u^\frown 0}{\restriction}_{\leq i} \cap X_{u^\frown 1}{\restriction}_{\leq i} = \emptyset$$

since $X_{w^\frown 0}, X_{w^\frown 1}$ is a splitting of X_w. Finally, since $X_{w^\frown 0}, X_{w^\frown 1}$ is a complete i-splitting of X_w, and $X_w{\restriction}_{\leq i} = X_u{\restriction}_{\leq i}$, we have $X_{u^\frown 0} \cup X_{u^\frown 1} = X_u$.

Thus $X_{u^\frown 0}, X_{u^\frown 1}$ is a complete i-splitting of X_u for all $u \in 2^m$. It remains to prove that $\langle X_{u'} : u' \in 2^{m+1} \rangle$ is a splitting system. To prove S-1 and S-2, let $u' = u^\frown d$ and $v' = v^\frown e$ belong to 2^{m+1}; $d, e \in \{0, 1\}$; $\xi = \zeta[u, v]$, $\xi' = \zeta[u', v']$, and $Y = X_u{\restriction}\xi = X_v{\restriction}\xi$. Consider three cases.

Case 1. $i \notin \xi$. Then by definition $\xi = \xi' \subseteq [\not\geq i]$. We have $X_{u'}{\restriction}\xi = X_u{\restriction}\xi = Y$ because $X_{u^\frown 0}, X_{u^\frown 1}$ is a complete i-splitting of X_u. Similarly $X_{v'}{\restriction}\xi = Y$. This proves S-1 for the sets $X_{u'}, X_{v'}$, while S-2 is inherited from the pair X_u, X_v because $\xi = \xi'$ and $X_{u'} \subseteq X_u$, $X_{v'} \subseteq X_v$.

Case 2. $i \in \xi$ and $d = e$, say $d = e = 0$. Then again $\xi = \xi'$ by definition, so S-2 is clear, but $i \in \xi'$. To prove S-1, let $w \in 2^m$ be the first (in the order fixed at the beginning of the proof) sequence in 2^m such that $i \in \zeta[u, w] \cup \zeta[v, w]$ (e.g., w can be one of u, v). Then, since $i \in \xi = \zeta[u, v]$, we have $i \in \zeta[u, w] \cap \zeta[v, w]$ by the "triangle" equality. Finally it follows from the construction (Case B) that

$$X_{u^\frown 0}{\restriction}\xi = (X_u{\restriction}\xi) \cap (X_{w^\frown 0}{\restriction}_{\leq i}{\restriction}^{-1}\xi), \qquad X_{v^\frown 0}{\restriction}\xi = (X_v{\restriction}\xi) \cap (X_{w^\frown 0}{\restriction}_{\leq i}{\restriction}^{-1}\xi).$$

However $X_u{\restriction}\xi = X_v{\restriction}\xi = Y$; this proves $X_{u^\frown 0}{\restriction}\xi' = X_{v^\frown 0}{\restriction}\xi'$. (Note that $\xi' = \xi$.)

Case 3. $i \in \xi$ but $d \neq e$, say $d = 0$, $e = 1$. Now $\xi' = \xi \cap [\not\geq i]$ is a proper subset of ξ. Let w be introduced as in Case 2. Note that $\xi' \cap [\leq i] = [<i]$, so

$$X_{u^\frown 0}{\restriction}\xi' = (X_u{\restriction}\xi') \cap (X_{w^\frown 0}{\restriction}_{<i}{\restriction}^{-1}\xi'), \qquad X_{v^\frown 1}{\restriction}\xi' = (X_v{\restriction}\xi') \cap (X_{w^\frown 1}{\restriction}_{<i}{\restriction}^{-1}\xi')$$

by the construction and Lemma 5. However $X_{w^\frown 0}{\restriction}_{<i} = X_{w^\frown 1}{\restriction}_{<i}$ because the pair $X_{w^\frown 0}, X_{w^\frown 1}$ is an i-splitting of X_w. Furthermore, $X_u{\restriction}\xi' = X_v{\restriction}\xi' = Y{\restriction}\xi'$ because $X_u{\restriction}\xi = X_v{\restriction}\xi = Y$. We conclude that $X_{u^\frown 0}{\restriction}\xi' = X_{v^\frown 1}{\restriction}\xi'$.

Let us prove S-2 for some $i' \in \zeta \smallsetminus \xi'$. If $i' \notin \xi$ then already $X_u{\restriction}_{\leq i'} \cap X_v{\restriction}_{\leq i'} = \emptyset$. If $i' \in \xi \smallsetminus \xi'$ then $i' \geq i$, so that it suffices to prove S-2 only for $i' = i = \phi(m)$. To prove S-2 in this case, note that $X_{u^\frown 0}{\restriction}_{\leq i} = X_{w^\frown 0}{\restriction}_{\leq i}$ and $X_{v^\frown 1}{\restriction}_{\leq i} = X_{w^\frown 1}{\restriction}_{\leq i}$ by the construction. But $X_{w^\frown 0}{\restriction}_{\leq i} \cap X_{w^\frown 1}{\restriction}_{\leq i} = \emptyset$ as the pair $X_{w^\frown 0}, X_{w^\frown 1}$ is an i-splitting, so $X_{u^\frown 0}{\restriction}_{\leq i} \cap X_{v^\frown 1}{\restriction}_{\leq i} = \emptyset$. ⊣

§3. Fusion lemma.

DEFINITION. An indexed family of sets $X_u \in \mathtt{Perf}_\zeta$, $u \in 2^{<\omega}$, is a *ϕ-fusion sequence* in $\mathtt{Perf}_\zeta$ if, for every $m \in \omega$, the subfamily $\langle X_u : u \in 2^m \rangle$ is a ϕ-splitting system, expanded by $\langle X_u : u \in 2^{m+1} \rangle$ to the next level, and

S-3 For any $\varepsilon > 0$ there exists $m \in \omega$ such that $\operatorname{diam} X_u < \varepsilon$ for all $u \in 2^m$. (A Polish metric on $\mathscr{D}^\zeta$ is assumed to be fixed.)

A function $\phi \colon \omega \to \zeta$ is called ζ-complete if and only if it takes each value $i \in \zeta$ infinitely many times.

THEOREM 14 (Fusion lemma). *Let ϕ be a ζ-complete function. Suppose that $\langle X_u : u \in 2^{<\omega} \rangle$ is a ϕ-fusion sequence in $\mathtt{Perf}_\zeta$. Then the set*

$$X = \bigcap_{m \in \omega} \bigcup_{u \in 2^m} X_u$$

belongs to $\mathtt{Perf}_\zeta$.

PROOF. The idea of the proof is to obtain a parallel presentation of the set $D = \mathscr{D}^\zeta$ as the "limit" of a ϕ-fusion sequence, and associate the points in D and X generated by one and the same branch in $2^{<\omega}$. So let us define a fusion sequence of sets $D_u \in \mathtt{Perf}_\zeta$ such that

$$\mathscr{D}^\zeta = D = \bigcap_{m \in \omega} \bigcup_{u \in 2^m} D_u.$$

Lemma 13 cannot be used: we would face problems with requirement S-3. We rather maintain a direct construction. For $m \in \omega$, we put $\zeta_m = \{ \phi(l) : l < m \}$. Let $i \in \zeta_m$, and

$$\{ l < m : \phi(l) = i \} = \{ l_0^i, \dots, l_{k(i)-1}^i \}$$

in the increasing order. If $u \in 2^m$ then we define $u_i \in 2^{k(i)}$ by $u_i(k) = u(l_k^i)$ for all $k < k(i)$, and put

$$D_u = \{ y \in D = \mathscr{D}^\zeta : \forall i \in \zeta_m \, (u_i \subset y(i)) \},$$

so that D_u is a basic clopen set in $\mathscr{D}^\zeta$. (Note that $y(i) \in \mathscr{D}$ whenever $y \in \mathscr{D}^\zeta$ and $i \in \zeta$.) One easily sees that the sets D_u form a ϕ-fusion sequence (S-3 follows from the ζ-completeness of ϕ) and

$$\bigcup_{u \in 2^m} D_u = \mathscr{D}^\zeta$$

for all m.

Now for each $a \in 2^\omega = \mathscr{D}$ the intersections

$$\bigcap_m X_{a \restriction m} \quad \text{and} \quad \bigcap_m D_{a \restriction m}$$

contain single points by S-3, say, respectively, $x_a \in X$ and $d_a \in D$, and the maps $a \mapsto x_a$, $a \mapsto d_a$ are continuous. We put

$$\zeta_\phi[a, b] = \bigcap_{m \in \omega} \zeta_\phi[a \restriction m, b \restriction m].$$

(In particular $\zeta_\phi[a,b] = \zeta$ if and only if $a = b$.) It follows from S-1 and S-2 that

$$(*)\left\{\begin{array}{ll} \left.\begin{array}{l} x_a\restriction\zeta_\phi[a,b] = x_b\restriction\zeta_\phi[a,b] \quad \text{and} \\ d_a\restriction\zeta_\phi[a,b] = d_b\restriction\zeta_\phi[a,b] \end{array}\right\} & \text{for all } a,b \in 2^\omega \\ x_a\restriction_{\le i} \ne x_b\restriction_{\le i} \text{ and } d_a\restriction_{\le i} \ne d_b\restriction_{\le i} & \text{whenever } i \notin \zeta_\phi[a,b]. \end{array}\right.$$

This allows us to define a homeomorphism $H\colon D = \mathscr{D}^\zeta$ onto X by $F(d_a) = x_a$ for all $a \in 2^\omega$. To see that H is projection-keeping let $\xi \in \mathbf{IS}_\zeta$ and, for instance, d_a, $d_b \in \mathscr{D}^\zeta$ and $d_a\restriction\xi = d_b\restriction\xi$. Then $\xi \subseteq \zeta_\phi[a,b]$ by the second part of $(*)$, so we get $x_a\restriction\xi = x_b\restriction\xi$ by the first part of $(*)$, as required. ⊣

The classical theorem, that any uncountable Borel or $\mathbf{\Sigma}^1_1$ set includes a perfect subset, does not directly generalize on sets in $\mathtt{Perf}_\zeta$: if card $\zeta \ge 2$ then one easily defines an uncountable closed set $W \subseteq \mathscr{D}^\zeta$ which does not include a subset in $\mathtt{Perf}_\zeta$. However a more weak statement survives.

Corollary 15. *Assume that $X \in \mathtt{Perf}_\zeta$, and $B \subseteq \mathscr{D}^\zeta$ is a Borel set. There exists a set $Y \in \mathtt{Perf}_\zeta$, $Y \subseteq X$ such that either $Y \subseteq B$ or $Y \cap B = \emptyset$.*

Proof. Argue by induction on α, where $B \in \mathbf{\Sigma}^0_\alpha$. If $\alpha = 1$, so that B is open, apply Lemma 8. Otherwise $B = \bigcup_m B_m$ where $B_m \in \mathbf{\Pi}^0_{\alpha_m}$ for some $\alpha_m < \alpha$. If there is a set $Y \in \mathtt{Perf}_\zeta$, $Y \subseteq X \cap B_m$ for some m, then $Y \subseteq B$. Otherwise, by the inductive hypothesis, we get, using Lemmas 12 and 13, a fusion sequence $\langle X_u : u \in 2^{<\omega}\rangle$ of sets $X_u \in \mathtt{Perf}_\zeta$ such that $X_\Lambda \subseteq X$ and $X_u \cap B_m = \emptyset$ for all $m \in \omega$ and $u \in 2^m$. The set

$$Y = \bigcap_{m\in\omega}\bigcup_{u\in 2^m} X_u$$

is as required. ⊣

The result can be strengthened!

Corollary 16. *Assume that $X \in \mathtt{Perf}_\zeta$, and $A \subseteq \mathscr{D}^\zeta$ is an analytic set. There exists a set $Y \in \mathtt{Perf}_\zeta$, $Y \subseteq X$ such that either $Y \subseteq A$ or $Y \cap A = \emptyset$.*

Proof. Consider a $\mathtt{Perf}_\zeta$-generic extension V^+ of the universe V. For a Borel or analytic set C in V, let C^+ denote the set defined in V^+ by the same construction. There is a condition $Y' \in \mathtt{Perf}_\zeta$ which decides $\mathbf{x} \in A^+$, where $\mathbf{x}$ is the name for the $\mathtt{Perf}_\zeta$-generic element of $\mathscr{D}^\zeta$. Suppose that, e.g., Y' forces $\mathbf{x} \in A^+$. As we shall see in Section 5, $\aleph_1$ remains uncountable in V^+. Therefore there is a Borel set $B \subseteq A$ (a *constituent* of A) and a condition $Y \in \mathtt{Perf}_\zeta$, $Y \subseteq Y'$, which forces $\mathbf{x} \in B^+$. Now, by Corollary 15, we can assume that either $Y \subseteq B$ or $Y \cap B = \emptyset$. The "or" case is impossible by the Shoenfield absoluteness, because Y forces $\mathbf{x} \in Y^+$. Therefore $Y \subseteq B$, as required. ⊣

§4. Continuous functions. This section studies the behaviour of continuous functions on sets in $\mathtt{Perf}_\zeta$ from the point of view of a certain reducibility.

Definition. For each set ζ, $\mathtt{Cont}_\zeta$ is the set of all continuous functions $F\colon \mathscr{D}^\zeta \to \mathscr{N} = \omega^\omega$. Assume that $F, G \in \mathtt{Cont}_\zeta$, $\xi \subseteq \zeta$, $i \in \zeta$, $X \subseteq \mathscr{D}^\zeta$.
F reduces to ξ on X if and only if

$$x\restriction\xi = y\restriction\xi \Longrightarrow F(x) = F(y)$$

for all $x, y \in X$.

F reduces to G on X if and only if

$$G(x) = G(y) \Longrightarrow F(x) = F(y)$$

for all $x, y \in X$.

F captures i on X if and only if

$$F(x) = F(y) \Longrightarrow x(i) = y(i)$$

for all $x, y \in X$.

It follows from the compactness of the spaces we consider that if F reduces to ξ on a closed set X then there is a continuous function $F' \in \mathtt{Cont}_\xi$ such that $F(x) = F'(x \restriction \xi)$ for all $x \in X$, while if F captures $i \in \zeta$ on a closed set X then there is a continuous $H : \mathscr{N} \to \mathscr{D}$ such that $x(i) = H(F(x))$ for all $x \in X$.

LEMMA 17. *Let $\xi, \eta \in \mathbf{IS}_\zeta$. If F reduces to both ξ and η on $X \in \mathtt{Perf}_\zeta$ then F reduces to $\vartheta = \xi \cap \eta$ on X.*

PROOF. Let $x, y \in X$ and $x \restriction \vartheta = y \restriction \vartheta$. By Proposition 4 (P-3) there is $z \in X$ such that $z \restriction \xi = x \restriction \xi$ and $z \restriction \eta = y \restriction \eta$. Now $F(x) = F(z) = F(y)$. ⊣

LEMMA 18. *Suppose that $\xi \in \mathbf{IS}_\zeta$, the sets X_1 and X_2 belong to $\mathtt{Perf}_\zeta$, and $X_1 \restriction \xi = X_2 \restriction \xi$. Then either F reduces to ξ on $X_1 \cup X_2$—and then obviously $F"X_1 = F"X_2$,—or there exist sets $X_1', X_2' \in \mathtt{Perf}_\zeta$, $X_1' \subseteq X_1$ and $X_2' \subseteq X_2$, such that still $X_1' \restriction \xi = X_2' \restriction \xi$, but $F"X_1' \cap F"X_2' = \emptyset$.*

(We recall that $F"X = \{ F(x) : x \in X \}$ is the image of X via F.)

PROOF. We assume that the function F does not reduce to ξ on $X_1 \cup X_2$, and prove the "or" alternative. By the assumption, there are points $x_1, x_2 \in X_1 \cup X_2$ satisfying $x_1 \restriction \xi = x_2 \restriction \xi$ and $F(x_1) \neq F(x_2)$. It may be supposed that $x_1 \in X_1$ and $x_2 \in X_2$, because $X_1 \restriction \xi = X_2 \restriction \xi$. By the continuity of F there exist clopen neighbourhoods U_1 and U_2 of, respectively, x_1 and x_2 such that $F"U_1 \cap F"U_2 = \emptyset$. By Lemma 8, there is a set $X_1'' \in \mathtt{Perf}_\zeta$, $X_1'' \subseteq X_1 \cap U_1$ containing x_1.

The set $X_2'' = X_2 \cap (X_1'' \restriction \xi \restriction^{-1} \zeta)$ belongs to $\mathtt{Perf}_\zeta$ by Lemma 9, and contains x_2 since $x_1 \restriction \xi = x_2 \restriction \xi$. By Lemma 8 again, there is a set $X_2' \in \mathtt{Perf}_\zeta$ satisfying $X_2' \subseteq X_2'' \cap U_2$. It remains to define $X_1' = X_1'' \cap (X_2' \restriction \xi \restriction^{-1} \zeta)$. ⊣

LEMMA 19. *Assume that $F \in \mathtt{Cont}_\zeta$ reduces to $\xi \in \mathbf{IS}_\zeta$ on $X \in \mathtt{Perf}_\zeta$. Let $i \in \zeta \setminus \xi$. Then F does not capture i on X.*

PROOF. Suppose on the contrary that F captures some $i \in \zeta \setminus \xi$ on X. Then the co-ordinate function $C_i(x) = x(i)$ reduces to ξ on X. Since i does not belong to ξ, and on the other hand C_i reduces to $[\le i]$, we conclude that C_i reduces to $[<i]$ on X by Lemma 17. But this clearly contradicts property P-1 of X (see Proposition 4). ⊣

THEOREM 20. *Assume that $X \in \mathtt{Perf}_\zeta$ and $F, G \in \mathtt{Cont}_\zeta$. Then there exists $Y \in \mathtt{Perf}_\zeta$, $Y \subseteq X$, such that either F reduces to G on Y or there exists $i \in \zeta$ such that G reduces to $[\not\geq i]$ but F captures i on Y.*

PROOF. Let us fix a ζ-complete function ϕ and define the initial segments $\zeta[u,v] = \zeta_\phi[u,v]$ (as in Section 2) for every pair of finite sequences $u, v \in 2^{<\omega}$ of equal length. The notions of splitting system and fusion sequence are understood in the sense of ϕ.

We define a fusion sequence $\langle X_u : u \in 2^{<\omega} \rangle$ satisfying $X_\Lambda = X$ and:

(†) If $m \in \omega$, $i = \phi(m)$, and $u \in 2^m$ then $\operatorname{diam}(F"X_u) \le m^{-1}$ and either F reduces to $[\not\geq i]$ on X_u or there does not exist $X' \in \mathtt{Perf}_\zeta$, $X' \subseteq X_u$, such that F reduces to $[\not\geq i]$ on X'. *The same (independently) for* G.

(‡) If $m \in \omega$ and $u, v \in 2^m$ then either
  (1) F reduces to $\zeta[u,v]$ on the set $X_u \cup X_v$, or
  (2) $F"X_u \cap F"X_v = \emptyset$.
  The same (independently) for G.

We first put $X_\Lambda = X$, as indicated.

Assume that sets X_u $(u \in 2^{m-1})$ are defined for some $m > 0$. We use Lemma 13 to get a splitting system $\langle Z_u : u \in 2^m \rangle$ which expands the splitting system $\langle X_u : u \in 2^{m-1} \rangle$ to the level m. We can suppose that $\operatorname{diam} Z_u \le m^{-1}$ for all $u \in 2^m$. (Otherwise apply Lemmas 8 and 12 consecutively 2^m times to shrink the sets.) We need this property to provide requirement S-3.

We now consider consecutively all pairs $u, v \in 2^m$. For every such a pair we first apply Lemma 18, getting sets $S_u, S_v \in \mathtt{Perf}_\zeta$ such that $S_u \subseteq Z_u$, $S_v \subseteq Z_v$, $S_u \restriction \zeta[u,v] = S_v \restriction \zeta[u,v]$, and either the function F reduces to $\zeta[u,v]$ on $S_u \cup S_v$ or $F"S_u \cap F"S_v = \emptyset$.

We set $S'_w = Z_w \cap (S_u \restriction \zeta[w,u] \restriction^{-1} \zeta)$ for all $w \in 2^m$; $\langle S'_w : w \in 2^m \rangle$ is a splitting system by Lemma 12. Note that $S_v \subseteq S'_v$ as $S_u \restriction \zeta[u,v] = S_v \restriction \zeta[u,v]$. This allows us to repeat the operation: putting

$$Z'_w = S'_w \cap (S_v \restriction \zeta[w,v] \restriction^{-1} \zeta)$$

for all $w \in 2^m$, we obtain a new splitting system of sets $Z'_w \subseteq S'_w$ $(w \in 2^m)$ such that $Z'_u = S_u$ and $Z'_v = S_v$. This ends the consideration of the particular pair of tuples $u, v \in 2^m$, and one comes to the next pair.

Let $X_u \subseteq Z_u$ $(u \in 2^m)$ be the sets finally obtained after 2^{m+1} steps of this construction (the number of pairs u, v to consider). One easily verifies that this is a splitting system in $\mathtt{Perf}_\zeta$ satisfying (‡) for the function F.

A simple application of Lemma 12 allows to consecutively shrink (2^m times) sets X_u so that they also satisfy (†) for F.

After this we repeat the same two-stage construction for G, the other function, getting finally the sets X_u $(u \in 2^m)$ of mth level.

Thus we obtain a fusion sequence of sets X_u $(u \in 2^{<\omega})$ satisfying (†) and (‡). The set $Y = \bigcap_m \bigcup_{u \in 2^m} X_u$ belongs to $\mathtt{Perf}_\zeta$ by Theorem 14.

CASE 1. for all m and $u, v \in 2^m$, the following holds: if $F"X_u \cap F"X_v = \emptyset$ then $G"X_u \cap G"X_v = \emptyset$. We prove that F reduces to G on Y in this case, so that Y satisfies the "either" requirement of the theorem.

Let $x, y \in Y$. Suppose that $F(x) \neq F(y)$ and prove $G(x) \neq G(y)$.

Note that $x = x_a$ and $y = x_b$ for some $a, b \in 2^\omega$, i.e., $\{x\} = \bigcap_{m\in\omega} X_{a \restriction m}$ and $\{y\} = \bigcap_{m\in\omega} X_{b \restriction m}$, see the proof of Theorem 14. Since $F(x) \neq F(y)$, it follows from

(†) and (‡) that for some m we have $F"X_u \cap F"X_v = \emptyset$ where $u = a{\restriction}m$ and $v = b{\restriction}m$. Then $G"X_u \cap G"X_v = \emptyset$ by the assumption, which implies $G(x) \neq G(y)$.

CASE 2. otherwise. There exist $m \in \omega$ and a pair of $u, v \in 2^{m+1}$ such that $F"X_u \cap F"X_v = \emptyset$, but G reduces to $\xi = \zeta[u, v]$ on $X_u \cup X_v$. It can be assumed that m is the least possible, so that by (‡) F reduces to $\eta = \zeta[s, t]$ on $X_s \cup X_t$ where $s = u{\restriction}m$ and $t = v{\restriction}m$.

Let $d = u(m)$, $e = v(m)$, so that $u = s^\frown d$, $v = t^\frown e$.

We observe that $i = \phi(m) \in \eta$ and $d \neq e$, as otherwise $\xi = \eta$ which easily leads to contradiction with the assumptions on F. Let say $d = 0$ and $e = 1$, so that $u = s^\frown 0$ and $v = t^\frown 1$. We have $\xi = \eta \cap [\not\geq i]$. Therefore *$G$ reduces to $[\not\geq i]$ on X_s* by an assumption above.

Now the main part of (†) enters the play. We assert that *there does not exist a set $X' \in \mathtt{Perf}_\zeta$, $X' \subseteq X_s$, such that F reduces to $[\not\geq i]$ on X'.*

(Indeed otherwise F would reduce to $[\not\geq i]$ already on X_s by (†). Then F reduces to $\xi = \eta \cap [\not\geq i]$ on X_s by Lemma 17. It follows that F reduces to ξ on a bigger set $X_s \cup X_t$ simply because F reduces to η on $X_s \cup X_t$ and $X_s{\restriction}\eta = X_t{\restriction}\eta$ by S-1. But this contradicts the assumption $F"X_u \cap F"X_v = \emptyset$ since $X_u \subseteq X_s$ and $X_v \subseteq X_t$ are nonempty sets satisfying $X_u{\restriction}\xi = X_v{\restriction}\xi$.)

Let us check that the set $Y' = Y \cap X_s = \bigcap_{k\in\omega} \bigcup_{w\in 2^k} X_{s^\frown w}$ and the element $i = \phi(m) \in \zeta$ chosen above satisfiy the "or" requirement of the theorem.

First of all $Y' \in \mathtt{Perf}_\zeta$ by Theorem 14 (via the corresponding shift of the function ϕ). Furthermore G reduces to $[\not\geq i]$ on Y' by the above. It remains to verify that F captures i on Y'.

Let $x, y \in Y'$. Suppose that $F(x) = F(y)$ and prove $x(i) = y(i)$.

Note that $x = x_a$ and $y = x_b$ for some $a, b \in 2^\omega$ satisfying $s \subset a$, $s \subset b$, i.e.,

$$\{x\} = \bigcap_{k\in\omega} X_{a{\restriction}k} \qquad \text{and} \qquad \{y\} = \bigcap_{k\in\omega} X_{b{\restriction}k},$$

see the proof of Theorem 14. We put

$$\zeta[a, b] = \bigcap_k \zeta[a{\restriction}k, b{\restriction}k];$$

then $x{\restriction}\zeta[a, b] = y{\restriction}\zeta[a, b]$ (see assertion $(*)$ in the proof of Theorem 14), so it suffices to check $i \in \zeta[a{\restriction}k, b{\restriction}k]$ for all k.

Suppose on the contrary that $i \notin \vartheta = \zeta[a{\restriction}k, b{\restriction}k]$ for some k; necessarily $k > m$ because $a{\restriction}m = b{\restriction}m = s$. Note that F reduces to ϑ on $X' = X_{a{\restriction}k}$ by (‡) because $F(x) = F(y)$. It follows that F also reduces to $[\not\geq i]$ on the set $X' \subseteq X_s$ as $i \notin \vartheta$, which is a contradiction with the above. ⊣

COROLLARY 21. *Assume that $X \in \mathtt{Perf}_\zeta$. If $i, j \in \zeta$ and $i < j$ then there is $Y \in \mathtt{Perf}_\zeta$, $Y \subseteq X$, such that the co-ordinate function C_j defined on $\mathscr{D}^\zeta$ by $C_j(x) = x(j)$ captures i on Y.*

PROOF. Otherwise Theorem 20, applied to the co-ordinate functions C_i and C_j leads to contradiction with Lemma 19. ⊣

COROLLARY 22. *Suppose that ζ has only countably many initial segments, $X \in \mathtt{Perf}_\zeta$, and $F \in \mathtt{Cont}_\zeta$. Then there exist $Y \in \mathtt{Perf}_\zeta$, $Y \subseteq X$, and an initial segment η of ζ such that F one-to-one reduces to η on Y.* (In the sense that

$$x \restriction \eta = y \restriction \eta \iff F(x) = F(y)$$

for all $x, y \in Y$.)

PROOF. Let $\{\eta_m : m \in \omega\}$ enumerate all initial segments of ζ so that each of them has infinitely many numbers. For each m let us fix once and for all a function $H_m \in \mathscr{D}^\zeta$ such that

$$x \restriction \eta_m = y \restriction \eta_m \iff H_m(x) = H_m(y)$$

for all $x, y \in \mathscr{D}^\zeta$. Applying Theorem 20 consecutively for F and different functions H_m we obtain a decreasing sequence

$$X = X_0 \supseteq X_1 \supseteq X_2 \supseteq \dots$$

of sets $X_m \in \mathtt{Perf}_\zeta$ satisfying, for each m, one of the following three conditions:

(A) there exists $i \notin \eta_m$ such that F captures i on X_m;
(B) F one-to-one reduces to η_m on X_m;
(C) there exists $i \in \eta_m$ such that F one-to-one reduces to $[\not\geq i] \cap \eta_m$ on X_m.

Let η be the intersection of all η_m such that (B) or (C) holds at step m. Then $\eta = \eta_{m_0}$ for some m_0. It remains to prove that we have (B) at step m_0.

Indeed if $i \notin \eta_{m_0}$ then by definition $i \notin \eta_m$ for some m such that F reduces to η_m on X_m. Now F cannot capture i on X_{m_0} by Lemma 21. Therefore (A) cannot hold at step m_0.

As for (C), suppose on the contrary that $i \in \eta_{m_0}$ and F one-to-one reduces to $\eta' = [\not\geq i] \cap \eta_{m_0}$ on X_{m_0}. Since $\eta' = \eta_m$ for some $m \geq m_0$, we have (B) or (C) at step m by Lemma 19, so that $\eta \subseteq \eta'$ by definition, which is a contradiction as $i \in \eta \setminus \eta'$. ⊣

§5. Introduction to generic models. This section introduces generic models obtained by forcing conditions in different sets $\mathtt{Perf}_\zeta$. This approach will then be detailized towards particular applications.

We fix a countable transitive model $\mathfrak{M}$ of **ZFC**, the *ground model*, and a partially ordered set $I \in \mathfrak{M}$ (generally speaking, uncountable in $\mathfrak{M}$)—the intended "length" of the planned iterated Sacks generic extension of $\mathfrak{M}$.

We let $\Xi = \mathbf{CPO}^{\mathfrak{M}}(I) \in \mathfrak{M}$ be the collection of all finite and $\mathfrak{M}$-countable sets $\xi \in \mathfrak{M}$, $\xi \subseteq I$, therefore $\Xi \subseteq \mathbf{CPO}$ in $\mathfrak{M}$.[4]

For any $\zeta \in \Xi$, let $\mathbb{P}_\zeta = (\mathtt{Perf}_\zeta)^{\mathfrak{M}}$. The set

$$\mathbb{P} = \mathbb{P}_I = \bigcup_{\zeta \in \Xi} \mathbb{P}_\zeta$$

will be the *forcing notion*. To define the order, we first put $\|X\| = \zeta$ whenever $X \in \mathbb{P}_\zeta$. Now define $X \leq Y$ (X is *stronger* than Y) if and only if $\zeta = \|Y\| \subseteq \|X\|$ and $X \restriction \zeta \subseteq Y$.

[4] If all initial segments of I except possibly for I itself are countable in $\mathfrak{M}$ it would be technically easier to define Ξ to be the set of all $\mathfrak{M}$-countable initial segments of I in $\mathfrak{M}$.

Notice that every set in $\mathbb{P}_\zeta$ is then a countable subset of $\mathscr{D}^\zeta$ in the universe. However it transforms to a perfect set in the universe by the closure operation: the topological closure $X^\#$ of a set $X \in \mathbb{P}_\zeta$ belongs to $\mathtt{Perf}_\zeta$ from the point of view of the universe.

Let $G \subseteq \mathbb{P}$ be a $\mathbb{P}$-generic ultrafilter over $\mathfrak{M}$. It easily follows from Lemma 8 that there exists unique indexed set $\mathbf{x} = \langle \mathbf{a}_i : i \in I \rangle \in \mathscr{D}^I$, all $\mathbf{a}_i = \mathbf{x}(i)$ being elements of $\mathscr{D}$, such that $\mathbf{x}\restriction\zeta \in X^\#$ whenever $X \in G$ and $\|X\| = \zeta \in \Xi$. Then

$$\mathfrak{M}[G] = \mathfrak{M}[\mathbf{x}] = \mathfrak{M}[\langle \mathbf{a}_i : i \in I \rangle]$$

is a $\mathbb{P}$-*generic extension of* $\mathfrak{M}$.

Suppose that $J \in \mathfrak{M}$ is an initial segment of I. It often happens in similar cases that sentences relativized to $\mathfrak{M}[\mathbf{x}\restriction J]$ are decided by forcing conditions X satisfying $\|X\| \subseteq J$. Let us prove this fact for the forcing notion $\mathbb{P}$.

THEOREM 23. *Suppose that $J \in \mathfrak{M}$ is an initial segment of I and Φ is a sentence relativized to $\mathfrak{M}[\mathbf{x}\restriction J]$. Assume that $\zeta \in \Xi$, $\zeta' = \zeta \cap J$, and a condition $X \in \mathbb{P}_\zeta$ forces Φ. Then $X' = X\restriction\zeta'$ forces Φ too.*

PROOF. Assume that this is not the case. Then there is a condition Y, stronger than X', which forces $\neg\Phi$. Applying Lemmas 6, 9, and 10, we get $\mathbb{P}$-generic over $\mathfrak{M}$ sets G_X and G_Y, containing, respectively, X and Y and such that

$$(\star) \qquad \{ X' \in G_X : \|X'\| \subseteq J \} = \{ Y' \in G_Y : \|Y'\| \subseteq J \}.$$

Let $\mathbf{x}, \mathbf{y} \in \mathscr{D}^I$ be obtained from G_X and G_Y as above. Then $\mathbf{x}\restriction J = \mathbf{y}\restriction J$ by $(\star)$, so that $\Phi(\mathbf{x})$ is true in $\mathfrak{M}[G_X]$ if and only if $\Phi(\mathbf{y})$ is true in $\mathfrak{M}[G_Y]$. But this contradicts the choice of X and Y. ⊣

In the remainder of this section, we prove a cardinal preservation theorem for the extension $\mathfrak{N} = \mathfrak{M}[G]$ and an important technical theorem which will allow us to study reals in $\mathfrak{N}$ using continuous functions in the ground model $\mathfrak{M}$. The results will be applied in the next section for the proof of Theorem 1.

THEOREM 24. *$\aleph_1^{\mathfrak{M}}$ remains a cardinal in $\mathfrak{N}$.*[5]

PROOF. Let $\underline{f}$ be a name of a function mapping ω to $\omega_1^{\mathfrak{M}}$. It would be enough, given $X_0 \in \mathbb{P}$, find a condition $X \in \mathbb{P}$, stronger than X_0, and a countable in $\mathfrak{M}$ set W such that X forces $\operatorname{ran} \underline{f} \subseteq W$.

We argue in $\mathfrak{M}$. Let $\xi_0 = \|X_0\|$. We define the following objects:

(1) *a sequence*

$$\zeta_0 \subseteq \zeta_1 \subseteq \zeta_2 \subseteq \cdots$$

of sets $\zeta_m \in \Xi$ such that $\xi_0 \subseteq \zeta_0$;

(2) *the set*

$$\zeta = \bigcup_{m \in \omega} \zeta_m \in \Xi,$$

and a ζ-complete function $\phi \colon \omega \to \zeta$, such that $\phi(m) \in \zeta_m$ for all m;

[5]We are not going to investigate the behaviour of other cardinals in $\mathfrak{N}$, which depends on the cardinal structure in $\mathfrak{M}$ and some cardinal characteristics of I.

(3) *for any* m, *a* ϕ*-splitting system* $\langle X_u : u \in 2^m \rangle$ *of sets* $X_u \in \mathtt{Perf}_{\zeta_m}$ *such that* $X_\Lambda \subseteq X_0 {\restriction}^{-1} \zeta_0$ *and*
  (a) $X_{u^\frown e} \subseteq X_u {\restriction}^{-1} \zeta_{m+1}$ *for all* $u \in 2^m$ *and* $e = 0, 1$;
  (b) *every set* X_u $(u \in 2^m)$ *has* $\operatorname{diam} X_u \le m^{-1}$;
  (c) *every condition* X_u $(u \in 2^m)$ *forces* $\underline{f}(m) = \gamma_u$ *for an ordinal* γ_u.

This solves the problem. Indeed, the family of sets $Y_u = X_u {\restriction}^{-1} \zeta$ is a ϕ-fusion sequence[6] in $\mathtt{Perf}_\zeta$, therefore

$$X = \bigcap_{m \in \omega} \bigcup_{u \in 2^m} Y_u \in \mathtt{Perf}_\zeta$$

by Theorem 14, and X is stronger than X_0 by the construction. Finally, X forces that the range of $\underline{f}$ is a subset of the countable in $\mathfrak{M}$ set $W = \{ \gamma_u : u \in 2^{<\omega} \}$.

To start the construction, we pick up a condition X_Λ, stronger than the given X_0, which decides the value $\underline{f}(0)$, and put $\zeta_0 = \|X_\Lambda\|$.

Suppose that $\phi{\restriction}m$, ζ_m, and the sets X_u $(u \in 2^m)$ have been defined. Let $u_0 \in 2^m$. There is a condition $Z \in \mathtt{Perf}_{\zeta'}$ for some $\zeta' \in \Xi$, $\zeta' \supseteq \zeta_m$, which is stronger than X_{u_0}, decides the value $\underline{f}(m+1)$, and has $\operatorname{diam} Z \le (m+1)^{-1}$. (We use Lemma 8 to fulfill the last inequality.) Define $Y'_u = X_u {\restriction}^{-1} \zeta'$ for all $u \in 2^m$; then $\langle Y'_u : u \in 2^m \rangle$ is a $(\phi{\restriction}m)$-splitting system in $\mathtt{Perf}_{\zeta'}$ and $Z \subseteq Y'_{u_0}$. Using Lemma 12, we obtain a $(\phi{\restriction}m)$-splitting system $\langle X'_u : u \in 2^m \rangle$ in $\mathtt{Perf}_{\zeta'}$ such that

$$X'_u \subseteq Y'_u = X_u {\restriction}^{-1} \zeta'$$

for all $u \in 2^m$ and the condition $X'_{u_0} = Z$ decides the value $\underline{f}(m+1)$.

Iterating this procedure 2^m times, we get a set $\zeta_{m+1} \in \Xi$, $\zeta_{m+1} \supseteq \zeta_m$, and a $(\phi{\restriction}m)$-splitting system $\langle X'_u : u \in 2^m \rangle$ in $\mathtt{Perf}_{\zeta_{m+1}}$ such that

$$X'_u \subseteq X_u {\restriction}^{-1} \zeta_{m+1}, \qquad \operatorname{diam} X'_u \le (m+1)^{-1},$$

and X'_u decides the value $\underline{f}(m+1)$ for all $u \in 2^m$.

At this moment, we define $\phi(m) \in \zeta_m$ appropriately, with the aim to provide the final ζ-completeness of ϕ, and use Lemma 13 to get a $(\phi{\restriction}(m+1))$-splitting system $\langle X_{u'} : u' \in 2^{m+1} \rangle$ in $\mathtt{Perf}_{\zeta_{m+1}}$ such that

$$X_{u^\frown e} \subseteq X'_u \subseteq X_u {\restriction}^{-1} \zeta_{m+1}$$

for all $u \in 2^m$ and $e = 0, 1$. This ends the recursive step. $\dashv$

Continuous functions. It is a principal property of several forcing notions (including the Sacks forcing and for instance the Solovay-random forcing) that reals in the generic extensions can be obtained by application of continuous functions (having a code) in the ground model, to generic sequences of reals. As we shall prove, this is also a property of the generic models considered here.

We put $\mathbb{F}_\zeta = (\mathtt{Cont}_\zeta)^{\mathfrak{M}}$ for $\zeta \in \Xi$. Obviously every $F \in \mathbb{F}_\zeta$ is a countable subset of $\mathscr{D}^\zeta \times \omega^\omega$ in the universe, but since the domain of F in $\mathfrak{M}$ is the compact set $\mathscr{D}^\zeta$, the topological closure $F^\#$ is a continuous function mapping $\mathscr{D}^\zeta$ into the reals (i.e., elements of the set $\mathscr{N} = \omega^\omega$, as usual) in the universe.

[6]We assume that $\operatorname{diam}(Z{\restriction}^{-1}\zeta) \le \operatorname{diam} Z$ whenever $Z \subseteq \mathscr{D}^\xi$ and $\xi \subseteq \zeta$. This suffices to prove requirement S-3 for the sets X_u by $\operatorname{diam} Y_u \le \operatorname{diam} X_u \le m^{-1}$ for $u \in 2^m$.

THEOREM 25. *Let $J \in \mathfrak{M}$ be an initial segment of I and r a real in $\mathfrak{M}[\mathbf{x}\restriction J]$. There exists $\zeta \in \Xi$, $\zeta \subseteq J$, and a function $F \in \mathbb{F}_\zeta$ such that $r = F^\#(\mathbf{x}\restriction\zeta)$.*

(Clearly the equality is absolute for any model containing r, $\mathbf{x}\restriction\zeta$, and F.)

PROOF. Let $\underline{r}$ be a name for r, containing an explicit absolute construction of r from $\mathbf{x}\restriction J$ and some parameter $p \in \mathfrak{M}$. Let $X_0 \in \mathbb{P}$, $\xi_0 = \|X_0\|$.

We argue in $\mathfrak{M}$.

By Theorem 23 the forcing of statements about $\underline{r}$ can be reduced to J: if $X \in \mathtt{Perf}_\zeta$ forces $\underline{r}(m) = k$ then $X\restriction(\zeta \cap J)$ also forces $\underline{r}(m) = k$.

Having this in mind and arguing as in the proof of Theorem 24, one gets a system of objects satisfying (1), (2), and (3), with the following corrections: in (1), additionally, $\zeta_m \subseteq J$—hence $\zeta \subseteq J$, and in (3) (c), each condition X_u, $u \in 2^m$, forces $\underline{r}(m) = k_u$ for some $k_u \in \omega$. We set $Y_u = X_u\restriction^{-1}\zeta$ for all $u \in 2^{<\omega}$.

Define a continuous function F' on the set

$$X = \bigcap_m \bigcup_{u \in 2^m} Y_u \in \mathtt{Perf}_\zeta$$

as follows. Let $x \in X$, $m \in \omega$. There exists unique $u \in 2^m$ such that $x \in Y_u$. We put $F'(x)(m) = k_u$. The function F' can be expanded to a function $F \in \mathtt{Cont}_\zeta$ (i.e., defined on $\mathscr{D}^\zeta$). Then X forces $\underline{r} = F'^\#(\mathbf{x}\restriction\zeta) = F^\#(\mathbf{x}\restriction\zeta)$. ⊣

§6. Proof of the main theorem. We prove in this section that any $\mathbb{P}_I$-generic model

$$\mathfrak{N} = \mathfrak{M}[G] = \mathfrak{M}[\mathbf{x}] = \mathfrak{M}[\langle \mathbf{a}_i : i \in I \rangle]$$

satisfies Theorem 1. This includes two parts: the "Sacksness" of the reals $\mathbf{a}_i$ and the properties of $\mathfrak{M}$-degrees of reals.

We keep the notation of the previous section.

6.1. The "Sacksness". Prove that $\mathbf{a}_i$ is Sacks generic over

$$\mathfrak{M}[\mathbf{x}\restriction_{<i}] = \mathfrak{M}[\langle \mathbf{a}_j : j < i \rangle]$$

for any $i \in I$.

Let $\tau \in \mathfrak{M}[\mathbf{x}\restriction_{<i}]$ be, in $\mathfrak{M}[\mathbf{x}\restriction_{<i}]$, a dense subset in the set of all perfect trees in $2^{<\omega}$; we have to prove that $\mathbf{a}_i \in [T]$ for some $T \in \tau$. Suppose on the contrary that a condition $X_0 \in G$ forces the opposite. As the forced statement is relativized to $\mathfrak{M}[\mathbf{x}\restriction_{\le i}]$, we may assume that $\|X_0\| = [\le i]$ by Theorem 23.

We argue in $\mathfrak{M}$. The set

$$D(y) = D_{X_0 y}(i) = \{ x(i) : x \in X_0 \text{ and } x\restriction_{<i} = y \}$$

is a perfect subset of $\mathscr{D} = 2^\omega$ for all $y \in Y_0 = X_0\restriction_{<i}$ by Proposition 4 (P-1).

We argue in $\mathfrak{M}[\mathbf{x}\restriction_{<i}]$. Note that $\mathbf{y} = \mathbf{x}\restriction_{<i}$ belongs to $Y_0^\#$. Therefore $D^\#(\mathbf{y}) = D_{X_0^\# \mathbf{y}}(i)$ is a perfect set in the universe. Thus there exists a tree $T \in \tau$ satisfying $[T] \subseteq D^\#(\mathbf{y})$. By the assumption, $\mathbf{a}_i = \mathbf{x}(i) \notin [T]$.

By Theorem 25, there is, in $\mathfrak{M}$, a continuous map $\mathscr{T} : \mathscr{D}^{[<i]} \to \mathscr{P}(2^{<\omega})$, satisfying $T = \mathscr{T}^\#(\mathbf{x}\restriction_{<i})$. Then $T = \mathscr{T}^\#(\mathbf{y})$, so $[\mathscr{T}^\#(\mathbf{y})] = [T] \subseteq D^\#(\mathbf{y})$.

Now "$\mathscr{T}^\#(\mathbf{y})$ is a perfect tree, $\mathscr{T}^\#(\mathbf{y}) \in \tau$, and $[\mathscr{T}^\#(\mathbf{y})] \subseteq D^\#(\mathbf{y})$" is a statement formally relativized to $\mathfrak{M}[\mathbf{y}] = \mathfrak{M}[\mathbf{x}\restriction_{<i}]$; therefore it is forced by a condition Y_1 stronger than Y_0 and satisfying $\|Y_1\| \subseteq [<i]$, by Theorem 23, hence $\|Y_1\| = [<i]$, so that $Y_1 \subseteq Y_0$.

We argue in $\mathfrak{M}$.

The set

$$B = \{ y \in Y_1 : \mathscr{T}(y) \text{ is a perfect tree and } [\mathscr{T}(y)] \subseteq D(y) \}$$

is a Borel subset of Y_1 because the map $\mathscr{T}$ is continuous. (The proof of this statement in fact involves Proposition 4—item P-2.) By Corollary 15, there is a condition $Y \in \mathtt{Perf}_{<i}$ such that either $Y \subseteq B$ or $Y \cap B = \emptyset$.

Suppose that $Y \cap B = \emptyset$. Then by the Shoenfield absoluteness theorem Y forces that either $\mathscr{T}^{\#}(\mathbf{y})$ is not a perfect tree or $[\mathscr{T}^{\#}(\mathbf{y})] \not\subseteq D^{\#}(\mathbf{y})$—contradiction with the choice of Y_1. We conclude that $Y \subseteq B$.

In particular $\mathscr{T}(y)$ is a perfect tree for all $y \in Y$. it follows that the set

$$X = \{x \in \mathscr{D}^{[\leq i]} : x{\restriction}_{<i} \in Y \;\&\; x(i) \in [\mathscr{T}(x{\restriction}_{<i})]\}$$

belongs to $\mathtt{Perf}_{\leq i}$ by Lemma 11. Furthermore

$$[\mathscr{T}(y)] \subseteq D(y) = D_{X_0 y}(i)$$

for all $y \in Y$, so that $X \subseteq X_0$.

Since X is also stronger than Y_1, X forces everything which is forced by X_0 and/or Y_1, and everything which logically follows from the mentioned.

In particular, as X_0 forces that $\mathbf{a}_i$ does not belong to a set of the form $[T]$ where $T \in \tau$ while Y_1 forces that $\mathscr{T}^{\#}(\mathbf{y}) \in \tau$, we observe that X forces $\mathbf{a}_i \notin [\mathscr{T}^{\#}(\mathbf{y})]$. It follows that X forces $\mathbf{a}_i \notin D_{X^{\#}\mathbf{y}}(i)$ because by definition $D_{Xy}(i) = [\mathscr{T}(y)]$. We conclude that X forces $\mathbf{x}{\restriction}_{\leq i} \notin X^{\#}$ (indeed, clearly $\mathbf{x}{\restriction}_{\leq i} = \mathbf{y} \cup \{\langle i, \mathbf{a}_i \rangle\}$), which is a contradiction. ⊣

A more careful[7] reasoning leads to the following reduction of the models we consider to ordinary product and iterated Sacks extensions in some cases.

PROPOSITION 26. *If $I = \Lambda$ is an ordinal in $\mathfrak{M}$ then $\mathbb{P}_I$-generic extensions of $\mathfrak{M}$ are equal to countable support* ***iterated*** *Sacks extensions of $\mathfrak{M}$ of length λ.*

If I is an unordered set of cardinality κ in $\mathfrak{M}$ then $\mathbb{P}_I$-generic extensions of $\mathfrak{M}$ are equal to countable support ***κ-product*** *Sacks extensions of $\mathfrak{M}$.*

6.2. Degrees of constructibility of reals in the extension. Items 1, 2, 3 of Theorem 1 follow from, respectively, Lemma 19 plus Corollary 21, Theorem 20, and Corollary 22, by essentially one and the same method based on Theorem 25. Therefore we present proof of item 2 and, partially, item 1, leaving the remaining content for the reader. (A remark on item 3. It is a standard fact that if all initial segments of a countable in $\mathfrak{M}$ p.o. set $\zeta \in \mathfrak{M}$ belong to $\mathfrak{M}$ then ζ has only countably many initial segments in $\mathfrak{M}$.)

PROOF OF A PART OF ITEM 1 OF THEOREM 1. We prove that if $J \in \mathfrak{M}$ is an initial segment in I and $i \in I \setminus J$ then $\mathbf{a}_i$ does not belong to $\mathfrak{M}[\langle \mathbf{a}_j : j \in J \rangle]$. Suppose on the contrary that $\mathbf{a}_i \in \mathfrak{M}[\mathbf{x}{\restriction}J]$. Then by Theorem 25 there exist: a set $\zeta \in \Xi$, a function $F \in \mathbb{F}_\xi$, where $\xi = \zeta \cap J$, and a condition $X \in \mathbb{P}_\zeta$ which forces $\mathbf{a}_i = F^{\#}(\mathbf{x}{\restriction}\xi)$.

We argue in $\mathfrak{M}$. We have $x(i) = F(x{\restriction}\xi)$ for all $x \in X$. (Otherwise there exist $m \in \omega$ and a condition $Y \subseteq X$, $Y \in \mathtt{Perf}_\zeta$ such that $x(i)(m) = 0$ but $F(x{\restriction}\xi)(m) = 1$, or vice versa, for all $x \in Y$, by Lemma 8. One easily gets a

[7]Unfortunately more cumbersome as well, therefore we do not include the proof.

contradiction with the choice of X.) Thus the co-ordinate function C_i reduces to ξ on X, a contradiction with Lemma 19 because $i \notin \xi$. $\dashv$

PROOF OF ITEM 2 OF THEOREM 1. Let $\mathscr{F}$ denote the set of all continuous functions $H : \mathscr{N} \to \mathscr{N}$ coded in $\mathfrak{M}$. Then, for a pair of reals r, r', the relation $r \leq_{\mathscr{F}} r'$ means that there is a function $H \in \mathscr{F}$ such that $r = H(r')$, see Introduction. This obviously implies $r \in \mathfrak{M}[r']$.

We have to prove the following: *for any two reals r, $r' \in \mathfrak{N}$, either $r \leq_{\mathscr{F}} r'$ or there exists $i \in I$ such that $\mathbf{a}_i \in \mathfrak{M}[r] \setminus \mathfrak{M}[r']$.*

Assume on the contrary that *the opposite is forced by some $X \in \mathbb{P}$.* We may suppose, by Lemma 10 and Theorem 25, that there exist functions F, $F' \in \mathbb{F}_\zeta$, where $\zeta = \|X\|$, such that $r = F^{\#}(\mathbf{x}{\restriction}\zeta)$ and $r' = F'^{\#}(\mathbf{x}{\restriction}\zeta)$.

We argue in $\mathfrak{M}$. Applying Theorem 20, we find a condition $Y \in \mathtt{Perf}_\zeta$, $Y \subseteq X$, such that either F reduces to F' on Y or there exists $i \in \zeta$ such that F' reduces to $\eta = \zeta \cap [\not\geq i]$ while F captures i on Y.

In the "either" case we have a continuous map $H : \mathscr{N} \to \mathscr{N}$ such that $F(x) = H(F'(x))$ for all $x \in Y$. Then Y forces

$$F^{\#}(\mathbf{x}{\restriction}\zeta) = H^{\#}(F'^{\#}(\mathbf{x}{\restriction}\zeta)),$$

which is a contradiction with the choice of X.

To get a contradiction in the "or" case, it suffices to prove $\mathbf{a}_i \notin \mathfrak{M}[\mathbf{x}{\restriction}\eta]$. But this follows from the already proved part of item 1: for take

$$J = \{\, j \in I : \exists i \in \eta\ (j \leq i) \,\}. \qquad \dashv$$

§7. Iterated vs. product Sacks forcing. We prove Theorem 2. Recall that the cardinal $\mathfrak{l}$ was defined in Introduction.

PART 1. *Product Sacks extensions.* Let $\kappa > \mathfrak{c}^{\mathfrak{M}}$ be a cardinal in an arbitrary transitive model $\mathfrak{M}$. We prove that $\mathfrak{l} > \operatorname{card}(\mathfrak{c}^{\mathfrak{M}})$ in any countable support κ-product Sacks extension $\mathfrak{N} = \mathfrak{M}[\langle\, \mathbf{a}_\alpha : \alpha < \kappa \,\rangle]$ of $\mathfrak{M}$.

It is a standard fact that the κ-product Sacks forcing notion does not contain an antichain of cardinality bigger than $\mathfrak{c}^{\mathfrak{M}}$ in $\mathfrak{M}$ in this case.

This easily implies that for any set $\mathscr{F} \in \mathfrak{N}$ of $\mathfrak{c}^{\mathfrak{M}}$-many continuous real functions in $\mathfrak{N}$ there exists a set $K \subseteq \kappa$, $K \in \mathfrak{M}$ of cardinality $\operatorname{card} K = \mathfrak{c}^{\mathfrak{M}}$ in $\mathfrak{M}$ such that each $F \in \mathscr{F}$ is coded in

$$\mathfrak{N}' = \mathfrak{M}[\langle\, \mathbf{a}_\alpha : \alpha \in K \,\rangle].$$

Now take arbitrary ordinals $\beta \neq \gamma$ in $\kappa \setminus K$. It is also a standard fact that then $\mathbf{a}_\beta \neq F(\mathbf{a}_\gamma)$ and $\mathbf{a}_\gamma \neq F(\mathbf{a}_\beta)$ for any continuous real function F coded in $\mathfrak{N}'$, therefore the relation $<_{\mathscr{F}}$ cannot linearly order the reals in $\mathfrak{N}$.

(Note that in the model we consider in fact $\mathfrak{c} = \kappa = \mathfrak{l}$ provided the cardinal κ has uncountable cofinality in $\mathfrak{M}$. Even if $\operatorname{cof} \kappa = \aleph_0$ in $\mathfrak{M}$ then κ remains a cardinal in $\mathfrak{N}$ by the above and we still have $\mathfrak{l} \geq \kappa > \operatorname{card}(\mathfrak{c}^{\mathfrak{M}})$ in $\mathfrak{N}$.)

PART 2. *Iterated Sacks extensions.* Suppose that I is a *linear* order. (For instance I can be an ordinal, to cover the case of "ordinary" Sacks iterations, or an inverse ordinal.) Let us prove that in this case the family $\mathscr{F}$ of all continuous functions

$F: \mathscr{N} \to \mathscr{N}$ coded in $\mathfrak{M}$ witnesses that $\mathfrak{l} \le \operatorname{card}(\mathfrak{c}^{\mathfrak{M}})$ in any $\mathbb{P}_I$-generic extension $\mathfrak{N} = \mathfrak{M}[\langle \mathbf{a}_i : i \in I \rangle]$ of $\mathfrak{M}$.

Consider a pair of reals $r, r' \in \mathfrak{N}$. The sets

$$J = \{ i \in I : \mathbf{a}_i \in \mathfrak{M}[r] \} \qquad \text{and} \qquad J' = \{ i \in I : \mathbf{a}_i \in \mathfrak{M}[r'] \}$$

are initial segments of I by Theorem 1 (item 1). Therefore one of them is a part of the other one as I is a linear order. Let, e.g., $J \subseteq J'$. Then $r \le_{\mathscr{G}} r'$ by Theorem 1 (item 2), as required.

(If, in this case, $\lambda = \operatorname{cof} I > \mathfrak{c}^{\mathfrak{M}}$ in $\mathfrak{M}$ then we have $\operatorname{card}(\mathfrak{c}^{\mathfrak{M}}) < \lambda \le \mathfrak{c}$, i.e., $\mathfrak{l} < \mathfrak{c}$, in $\mathfrak{N}$. The hypothesis $\operatorname{cof} I > \mathfrak{c}^{\mathfrak{M}}$ may possibly be weakened.)

§8. Selectors for analytic equivalence relations. We prove Theorem 3 as a corollary of the following theorem.

THEOREM 27. *Suppose that I is an ordinal in a countable model $\mathfrak{M} \models \mathrm{V} = \mathrm{L}$. Then the selection principle* **SP** *holds in $\mathbb{P}_I$-generic extensions of $\mathfrak{M}$.*

PROOF OF THEOREM 3. Take a model $\mathfrak{M} \models \mathrm{V} = \mathrm{L}$, set $I = \omega_2^{\mathfrak{M}}$, and let $\mathfrak{N} = \mathfrak{M}[\langle \mathbf{a}_i : i \in I \rangle]$ be a $\mathbb{P}_I$-generic extension of $\mathfrak{M}$. Then **SP** holds in $\mathfrak{N}$ by Theorem 27. Note that $\mathfrak{N}$ preserves $\aleph_1^{\mathfrak{M}}$ by Theorem 24. It is a routine exercise to prove that (since $\mathfrak{M} \models \mathrm{V} = \mathrm{L}$) $\mathfrak{N}$ also preserves $\aleph_2^{\mathfrak{M}}$ (and any bigger cardinal, of course). Thus, as the reals $\mathbf{a}_i$ are pairwise different, $\mathfrak{N} \models 2^{\aleph_0} = \aleph_2$. Another standard argument proves that the reals do not admit a real-ordinal definable wellordering in $\mathfrak{N}$. ⊣

PROOF OF THEOREM 27. Thus let $\mathfrak{M}$ and I be as in the theorem. Consider a $\mathbb{P}_I$-generic extension $\mathfrak{N} = \mathfrak{M}[\mathbf{x}]$, where $\mathbf{x} = \langle \mathbf{a}_i : i \in I \rangle$. Fix a $\mathbf{\Sigma}^1_1$ equivalence relation E on $\mathscr{D} = 2^\omega$ in $\mathfrak{N}$. Prove that E has a $\mathbf{\Sigma}^1_2$ selector in $\mathfrak{N}$.

Fix a real $r \in \mathfrak{N}$ such that E is $\Sigma^1_1(r)$. By Theorem 25, there is a set $\rho \in \mathbf{\Xi}$ (that is, a finite or countable in $\mathfrak{M}$ subset of I) and a function $G_0 \in \mathbb{F}_\rho$ (i.e., in $\mathfrak{M}$, a continuous function $G_0 \colon \mathscr{D}^\rho \to \mathscr{N} = \omega^\omega$) such that $r = G_0^\#(\mathbf{x} \restriction \rho)$. We may assume, by Corollary 22, that, in $\mathfrak{M}$, G_0 is one-to-one on some $R_0 \in \mathtt{Perf}_\rho$ which belongs to the generic set. Then $\mathfrak{M}[r] = \mathfrak{M}[\mathbf{x} \restriction \rho]$.

On the other hand, there is a (lightface) Σ^1_1 set $P \subseteq \mathscr{D}^2 \times \mathscr{N}$ such that

$$a \mathrel{\mathsf{E}} b \iff P(a, b, r)$$

in $\mathfrak{N}$. Let us write $a \mathrel{\mathsf{E}_x} b$ instead of $P(a, b, x)$. Thus we have $\mathsf{E} = \mathsf{E}_r$ in $\mathfrak{N}$. It can be assumed that E_x is an equivalence relation on $\mathscr{D}$ for any $x \in \mathscr{N}$ in any model. (Otherwise P can be suitably changed.)

LEMMA 28. *In $\mathfrak{N}$, every E-class intersects a closed, coded in $\mathfrak{M}[r]$, set $S \subseteq \mathscr{D}$ which is a partial selector*[8] *for* E.

The lemma implies Theorem 27. Indeed, $\mathfrak{M}[r]$ is equal to $\mathrm{L}[r]$ in the sense of $\mathfrak{N}$. Define a $\Sigma^1_2(r)$ selector S for E in $\mathfrak{N}$ as follows. Consider a real $a \in \mathscr{D} = 2^\omega$ in $\mathfrak{N}$. Consider all closed, coded in $\mathfrak{M}[r]$, sets $S' \subseteq \mathscr{D}$ which are partial selectors for E and intersect the E-class of a. Let S_a be that one of them which has the least code in the sense of the Gödel $\Sigma^1_2(r)$ wellordering of $\mathfrak{M}[r]$. Put $S = \{ a : a \in S_a \}$. ⊣

[8]That is, S intersects each E-class in ≤ 1 element.

PROOF OF THE LEMMA. Let us fix an arbitrary real $a_0 \in \mathfrak{N} \cap 2^\omega$. As above there are: a set $\zeta \in \Xi$ and a function $F \in \mathbb{F}_\zeta$ such that $a_0 = F^\#(\mathbf{x} \restriction \zeta)$.

We argue in $\mathfrak{M}$. Thus $F \in \mathtt{Cont}_\zeta$, i.e., F continuously maps $\mathscr{D}^\zeta$ into $\mathscr{D}$.

We may assume that $\rho \subseteq \zeta$. Put

$$\xi = \{ j \in \zeta : \exists i \in \rho \, (j \leq i) \},$$

an initial segment in ζ. The set $R' = R_0 \restriction^{-1} \xi$ belongs to $\mathtt{Perf}_\xi$ by Lemma 10. Furthermore, as $\rho \subseteq \xi$ is cofinal in ξ and G_0 is one-to-one on R_0, there is a set $R \in \mathtt{Perf}_\xi$, $R \subseteq R'$, such that the function $G \in \mathtt{Cont}_\xi$, defined on $\mathscr{D}^\xi$ by $G(x) = G_0(x \restriction \rho)$, is one-to-one on R. (This can be easily proved using different results in Section 4.) We may assume that R belongs to the generic set.

Let us fix some $X_0 \in \mathtt{Perf}_\zeta$ such that $X_0 \restriction \xi \subseteq R$.

Define

$$H = \{ \eta \subseteq \zeta : \eta \text{ is an initial segment and } \xi \subseteq \eta \}.$$

Since I is wellordered, there exist: an initial segment $\eta_0 \in H$ and $X_1 \in \mathtt{Perf}_\zeta$, $X_1 \subseteq X_0$, such that $x \restriction \eta_0 = y \restriction \eta_0$ implies $F(x) \, \mathsf{E}_{G(x \restriction \xi)} \, F(y)$ for all $x, y \in X_1$, and none of $X' \in \mathtt{Perf}_\zeta$, $X' \subseteq X_0$, produces the same effect for some $\eta \in H$, $\eta \subsetneqq \eta_0$.

STEP 1. We assert that, for any initial segment $\eta \in H$, $\eta \subsetneqq \eta_0$, if $Y, Z \in \mathtt{Perf}_\zeta$ satisfy $Y \cup Z \subseteq X_1$ and $Y \restriction \eta = Z \restriction \eta$ then there are $Y', Z' \in \mathtt{Perf}_\zeta$ satisfying $Y' \subseteq Y$, $Z' \subseteq Z$, $Y' \restriction \eta = Z' \restriction \eta$, and $F(y) \, \not\mathsf{E}_{G(y \restriction \xi)} F(z)$ for all $y \in Y'$ and $z \in Z'$ such that $y \restriction \eta = z \restriction \eta$.

STEP 2. Suppose this has been proved. Then we can define a fusion sequence $\langle Y_u : u \in 2^{<\omega} \rangle$ of sets $Y_u \in \mathtt{Perf}_\zeta$, satisfying $Y_\Lambda \subseteq X_1$ and the following requirement. Take $m \in \omega$ and $u, v \in 2^m$. Let $\eta = \zeta_\phi[u, v]$. (A ζ-complete function ϕ is fixed.) Assume that $\xi \subseteq \eta \subsetneqq \eta_0$, so that $\eta \in H$. Finally take $y \in Y_u$ and $z \in Y_v$ satisfying $y \restriction \eta = z \restriction \eta$. The requirement for the fusion sequence is that in this case we always get $F(y) \, \not\mathsf{E}_{G(y \restriction \xi)} F(z)$.

Let $X_2 = \bigcap_n \bigcup_{u \in 2^n} Y_u$; thus $X_2 \in \mathtt{Perf}_\zeta$ and $X_2 \subseteq X_1$. Let us prove that $F(y) \, \mathsf{E}_{G(y \restriction \xi)} \, F(z)$ implies $y \restriction \eta_0 = z \restriction \eta_0$ for all $y, z \in X_2$ satisfying $y \restriction \xi = z \restriction \xi$. Indeed suppose that $y \restriction \eta_0 \neq z \restriction \eta_0$. Since ζ is wellordered, there is $i \in \eta_0$ such that $y \restriction \eta = z \restriction \eta$, where $\eta = [<i]$, but $y(i) \neq z(i)$. Note that $\xi \subseteq \eta$, because $y \restriction \xi = z \restriction \xi$, hence $\eta \in H$. Furthermore there exist $n \in \omega$ and $u, v \in 2^n$ such that $y \in Y_u$, $z \in Y_v$, and $\eta = \zeta_\phi[u, v]$. Now $F(y) \, \not\mathsf{E}_{G(y \restriction \xi)} F(z)$ by the construction, as required.

Thus we have the equivalence

$$y \restriction \eta_0 = z \restriction \eta_0 \iff F(y) \, \mathsf{E}_{G(y \restriction \xi)} \, F(z)$$

for all $y, z \in X_2$ such that $y \restriction \xi = z \restriction \xi$.

STEP 3. Let $\zeta' = \zeta \setminus \eta_0$. Note that a typical element $x \in \mathscr{D}^\zeta$ has the form $z \cup z'$, where $z \in \mathscr{D}^{\eta_0}$ while $z' \in \mathscr{D}^{\zeta'}$. Let

$$X_2(z) = \{ z' \in \mathscr{D}^{\zeta'} : z \cup z' \in X_2 \}$$

for $z \in X_2 \restriction \eta_0$. Then X_2 forces that $X_2(\mathbf{x} \restriction \eta_0)$ is nonempty in the extension. Using Theorem 25, we easily get $X_3 \in \mathtt{Perf}_\zeta$, $X_3 \subseteq X_2$, and a continuous function $G \colon (X_3 \restriction \eta_0) \to \mathscr{D}^{\zeta'}$ such that $z \cup G(z) \in X_3$ for every $z \in X_3 \restriction \eta_0$.

STEP 4. Let $X_3(y) = \{ x \in X_3 : x{\restriction}\xi = y \}$ for any $y \in X_3{\restriction}\xi$. We observe that, for any $y \in X_3{\restriction}\xi$, the set

$$S(y) = \{ F(z \cup G(z)) : z \in X_3{\restriction}\eta_0 \,\&\, z{\restriction}\xi = y \}$$

is a closed subset of

$$F\text{''}X_3(y) = \{ F(x) : x \in X_3(y) \}.$$

Moreover $S(y)$ is a partial selector for $\mathsf{E}_{G(y)}$. (Indeed suppose that $z_1 \neq z_2 \in X_3{\restriction}\eta_0$, $z_1{\restriction}\xi = z_2{\restriction}\xi = y$. Then $x_1 = F(z_1 \cup G(z_1))$ and $x_2 = F(z_2 \cup G(z_2))$ belong to $X_3(y)$ and $x_1{\restriction}\xi = x_2{\restriction}\xi = y$ but $x_1{\restriction}\eta_0 \neq x_2{\restriction}\eta_0$, hence $F(x_1) \not\mathsf{E}_{G(y)} F(x_2)$ by the above.) Finally $S(y)$ is complete in $F\text{''}X_3(y)$: for any $a \in F\text{''}X_3(y)$ there is $b \in S(y)$ satisfying $a \,\mathsf{E}_{G(y)}\, b$. (Indeed, let $a = F(x)$, where $x \in X_3(y)$, so that $x{\restriction}\xi = y$. Take $z = x{\restriction}\eta_0$ and $b = F(z \cup G(z))$.)

STEP 5. It follows that X_3 forces that $S^{\#}(\mathbf{x}{\restriction}\xi)$ is a closed set, a partial selector for $\mathsf{E}_{G(\mathbf{x}{\restriction}\xi)}$, and $F(\mathbf{x}{\restriction}\zeta)\, \mathsf{E}_{G(\mathbf{x}{\restriction}\xi)}\, b$ for some $b \in S^{\#}(\mathbf{x}{\restriction}\xi)$. We may assume that X_3 belongs to the generic set. Then $S' = S^{\#}(\mathbf{x}{\restriction}\xi)$ is, in $\mathfrak{N}$, a closed partial selector for $\mathsf{E} = \mathsf{E}_r$ since $r = G(\mathbf{x}{\restriction}\xi) = G_0(\mathbf{x}{\restriction}\rho)$. Moreover, as $a_0 = F^{\#}(\mathbf{x}{\restriction}\zeta)$, there exists $b \in S'$ such that $a_0 \,\mathsf{E}\, b$, so that S' intersects the E-class of a_0. Finally, as S' is coded in $\mathfrak{M}[\mathbf{x}{\restriction}\xi]$ by definition and G is one-to-one on $X_3{\restriction}\xi$ (recall that $X_3{\restriction}\xi \subseteq R$ by the construction), we conclude that S' is coded in $\mathfrak{M}[r]$, as required.

STEP 6. Thus it remains to prove the assertion of Step 1.

Let $\zeta^o = \{ j^o : j \in \zeta \}$ be just another copy of ζ, chosen so that $j^o = j$ for any $j \in \eta$ but $j^o \neq j$ otherwise. Put $\vartheta = \zeta \cup \zeta^o$ with the obvious order (so that $\zeta \smallsetminus \eta$ and $\zeta^o \smallsetminus \eta$ are not connected by the order). For $x \in \mathscr{D}^{\zeta}$ let $O(x) = x^o \in \mathscr{D}^{\zeta^o}$ be defined by $x^o(j^o) = x(j)$ for all $j \in \zeta$ (then $x{\restriction}\eta = O(x){\restriction}\eta$).

Let $Z^o = O\text{''}Z$ (the O-image of Z); then $Z^o \in \mathtt{Perf}_{\zeta^o}$.

The set

$$W = \{ w \in \mathtt{Perf}_{\vartheta} : w{\restriction}\zeta \in Y \,\&\, w{\restriction}\zeta^o \in Z^o \}$$

belongs to $\mathtt{Perf}_{\vartheta}$ by Lemma 9 because $Y{\restriction}\eta = Z{\restriction}\eta$. Now, by Corollary 16, there is a set $W' \in \mathtt{Perf}_{\vartheta}$, $W' \subseteq W$, such that *either* for any $w \in W'$ we have

$$F(w{\restriction}\zeta)\, \mathsf{E}_{G(w{\restriction}\xi)}\, F(O^{-1}(w{\restriction}\zeta^o)),$$

or for any $w \in W'$ we have

$$F(w{\restriction}\zeta)\, \not\mathsf{E}_{G(w{\restriction}\xi)} F(O^{-1}(w{\restriction}\zeta^o)).$$

The sets $Y' = W'{\restriction}\zeta$ and $Z' = O^{-1}\text{''}(W'{\restriction}\zeta^o)$ belong to $\mathtt{Perf}_{\zeta}$ and satisfy $Y' \subseteq Y$, $Z' \subseteq Z$, and $Y'{\restriction}\eta = Z'{\restriction}\eta$. Moreover we have

$$W' = \{ w \in \mathtt{Perf}_{\vartheta} : w{\restriction}\zeta \in Y' \,\&\, w{\restriction}\zeta^o \in (Z')^o \}$$

by assertion P-3 of Proposition 4. (Note that $(Z')^o = O\text{''}(Z') = W'{\restriction}\zeta^o$.) The dichotomy takes the form: *either* for all $y \in Y'$ and $z \in Z'$ satisfying $y{\restriction}\eta = z{\restriction}\eta$ we have $F(y) \mathsf{E}_{G(y{\restriction}\xi)} F(z)$, *or* for all $y \in Y'$ and $z \in Z'$ satisfying $y{\restriction}\eta = z{\restriction}\eta$, we have $F(y)\, \not\mathsf{E}_{G(y{\restriction}\xi)} F(z)$.

However the *either* case is impossible. (Indeed then, since $Y'{\restriction}\eta = Z'{\restriction}\eta$, we have $F(y)\, \mathsf{E}_{G(y{\restriction}\xi)}\, F(y')$ whenever $y, y' \in Y'$ satisfy $y{\restriction}\eta = y'{\restriction}\eta$, which is a contradiction with the choice of η_0 and X_1 because $\eta \subsetneqq \eta_0$.) Therefore we have the *or* case, so that the sets Y' and Z' prove the assertion of Step 1. ⊣

REMARK. We do not know whether Theorem 27 holds in the case when I is a linear order but not a wellordering. Another interesting problem (typical for the Sacks iterations) is to prove the consistency of **SP** with $2^{\aleph_0} > \aleph_2$.

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