

## A DEFINABLE NONSTANDARD MODEL OF THE REALS

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**Abstract.** We prove, in **ZFC**, the existence of a definable, countably saturated elementary extension of the reals.

**§1. Introduction.** It seems that it has been taken for granted that there is no distinguished, definable nonstandard model of the reals. (This means a countably saturated elementary extension of the reals.) Of course if  $V = L$  then there is such an extension (just take the first one in the sense of the canonical well-ordering of  $L$ ), but we mean the existence provably in **ZFC**. There were good reasons for this: without Choice we cannot prove the existence of *any* elementary extension of the reals containing an infinitely large integer.<sup>1, 2</sup> Still there is one.

**THEOREM 1.1 (ZFC).** *There exists a definable, countably saturated extension  ${}^*\mathbb{R}$  of the reals  $\mathbb{R}$ , elementary in the sense of the language containing a symbol for every finitary relation on  $\mathbb{R}$ .*

The problem of the existence of a definable proper elementary extension of  $\mathbb{R}$  was communicated to one of the authors (Kanovei) by V. A. Uspensky.

A somewhat different, but related problem of *unique existence* of a nonstandard real line  ${}^*\mathbb{R}$  has been widely discussed by specialists in nonstandard analysis.<sup>3</sup> Keisler notes in [3, § 11] that, for any cardinal  $\kappa$ , either inaccessible or satisfying  $2^\kappa = \kappa^+$ , there exists a unique, up to isomorphism,  $\kappa$ -saturated nonstandard real line  ${}^*\mathbb{R}$  of cardinality  $\kappa$ , which means that a reasonable level of uniqueness modulo isomorphism can be achieved, say, under GCH. Theorem 1.1 provides a countably saturated nonstandard real line  ${}^*\mathbb{R}$ , unique in absolute sense by virtue of a concrete definable construction in **ZFC**. A certain modification of this example also admits a reasonable model-theoretic characterization up to isomorphism (see Section 5).

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<sup>1</sup>In fact, from any nonstandard integer we can define a non-principal ultrafilter on  $\mathbb{N}$ , even a Lebesgue non-measurable set of reals [4], yet it is consistent with **ZF** (even plus Dependent Choices) that there are no such ultrafilters as well as non-measurable subsets of  $\mathbb{R}$  [5].

<sup>2</sup>It is worth mentioning that definable nonstandard elementary extensions of  $\mathbb{N}$  do exist in **ZF**. For instance, such a model can be obtained in the form of the ultrapower  $F/U$ , where  $F$  is the set of all arithmetically definable functions  $f : \mathbb{N} \rightarrow \mathbb{N}$  while  $U$  is a non-principal ultrafilter in the algebra  $A$  of all arithmetically definable sets  $X \subseteq \mathbb{N}$ .

<sup>3</sup>“What is needed is an underlying set theory which proves the unique existence of the hyperreal number system [ . . . ]” (Keisler [3, p. 229]).

The proof of Theorem 1.1 is a combination of several known arguments. First of all (and this is the key idea), arrange all non-principal ultrafilters over  $\mathbb{N}$  in a linear order  $A$ , where each ultrafilter appears repetitiously as  $D_a$ ,  $a \in A$ . Although  $A$  is not a well-ordering, we can apply the iterated ultrapower construction in the sense of [1, 6.5] (which is “a finite support iteration” in the forcing nomenclature), to obtain an ultrafilter  $D$  in the algebra of all sets  $X \subseteq \mathbb{N}^A$  concentrated on a finite number of axes  $\mathbb{N}$ . To define a  $D$ -ultrapower of  $\mathbb{R}$ , the set  $F$  of all functions  $f : \mathbb{N}^A \rightarrow \mathbb{R}$ , also concentrated on a finite number of axes  $\mathbb{N}$ , is considered. The ultrapower  $F/D$  is OD, that is, ordinal-definable, actually, definable by an explicit construction in ZFC, hence, we obtain an OD proper elementary extension of  $\mathbb{R}$ . Iterating the  $D$ -ultrapower construction  $\omega_1$  times in a more ordinary manner, i. e., with direct limits at limit steps, we obtain a definable countably saturated extension.

To make the exposition self-contained and available for a reader with only fragmentary knowledge of ultrapowers, we reproduce several well-known arguments instead of giving references to manuals.

**§2. The ultrafilter.** As usual,  $c$  is the cardinality of the continuum.

Ultrafilters on  $\mathbb{N}$  hardly admit any definable linear ordering, but maps  $a : c \rightarrow \mathcal{P}(\mathbb{N})$ , whose ranges are ultrafilters, readily do. Let  $A$  consist of all maps  $a : c \rightarrow \mathcal{P}(\mathbb{N})$  such that the set  $D_a = \text{ran } a = \{a(\xi) : \xi < c\}$  is an ultrafilter on  $\mathbb{N}$ . The set  $A$  is ordered lexicographically:  $a <_{\text{lex}} b$  means that there exists  $\xi < c$  such that  $a \upharpoonright \xi = b \upharpoonright \xi$  and  $a(\xi) < b(\xi)$  in the sense of the lexicographical linear order  $<$  on  $\mathcal{P}(\mathbb{N})$  (in the sense of the identification of any  $u \subseteq \mathbb{N}$  with its characteristic function).

For any set  $u$ ,  $\mathbb{N}^u$  denotes the set of all maps  $f : u \rightarrow \mathbb{N}$ .

Suppose that  $u \subseteq v \subseteq A$ .

If  $X \subseteq \mathbb{N}^v$  then put  $X \downarrow u = \{x \upharpoonright u : x \in X\}$ .

If  $Y \subseteq \mathbb{N}^u$  then put  $Y \uparrow v = \{x \in \mathbb{N}^v : x \upharpoonright u \in Y\}$ .

We say that a set  $X \subseteq \mathbb{N}^A$  is *concentrated* on  $u \subseteq A$ , if  $X = (X \downarrow u) \uparrow A$ ; in other words, this means the following:

$$\forall x, y \in \mathbb{N}^A \ (x \upharpoonright u = y \upharpoonright u \implies (x \in X \iff y \in X)). \tag{*}$$

We say that  $X$  is a *set of finite support*, if it is concentrated on a finite set  $u \subseteq A$ . The collection  $\mathcal{X}$  of all sets  $X \subseteq \mathbb{N}^A$  of finite support is closed under unions, intersections, complements, and differences, i. e., it is an algebra of subsets of  $\mathbb{N}^A$ . Note that if (\*) holds for finite sets  $u, v \subseteq A$  then it also holds for  $u \cap v$ . (If  $x \upharpoonright (u \cap v) = y \upharpoonright (u \cap v)$  then consider  $z \in \mathbb{N}^A$  such that  $z \upharpoonright u = x \upharpoonright u$  and  $z \upharpoonright v = y \upharpoonright v$ .) It follows that for any  $X \in \mathcal{X}$  there is a least finite  $u = \|X\| \subseteq A$  satisfying (\*).

In the remainder, if  $U$  is any subset of  $\mathcal{P}(I)$ , where  $I$  is a given set, then  $U i \Phi(i)$  (*generalized quantifier*) means that the set  $\{i \in I : \Phi(i)\}$  belongs to  $U$ .

The following definition realizes the idea of a finite iteration of ultrafilters. Suppose that  $u = a_1 < \dots < a_n \subseteq A$  is a finite set. We put

$$D_u = \{X \subseteq \mathbb{N}^u : D_{a_n} k_n \dots D_{a_2} k_2 D_{a_1} k_1 (\langle k_1, k_2, \dots, k_n \rangle \in X)\};$$

$$D = \{X \in \mathcal{X} : X \downarrow \|X\| \in D_{\|X\|}\}.$$

The following is quite clear.

- PROPOSITION 2.1. (i)  $D_u$  is an ultrafilter on  $\mathbb{N}^u$  ;  
 (ii) if  $u \subseteq v \subseteq A$ ,  $v$  finite,  $X \subseteq \mathbb{N}^u$ , then  $X \in D_u$  iff  $X \upharpoonright v \in D_v$  ;  
 (iii)  $D \subseteq \mathcal{X}$  is an ultrafilter in the algebra  $\mathcal{X}$  ;  
 (iv) if  $X \in \mathcal{X}$ ,  $u \subseteq A$  finite, and  $\|X\| \subseteq u$ , then  $X \in D \iff X \downarrow u \in D_u$ .

**§3. The ultrapower.** To match the nature of the algebra  $\mathcal{X}$  of sets  $X \subseteq \mathbb{N}^A$  of finite support, we consider the family  $F$  of all  $f : \mathbb{N}^A \rightarrow \mathbb{R}$ , concentrated on some finite set  $u \subseteq A$ , in the sense that

$$\forall x, y \in \mathbb{N}^A (x \upharpoonright u = y \upharpoonright u \implies f(x) = f(y)). \quad (\dagger)$$

As above, for any  $f \in F$  there exists a least finite  $u = \|f\| \subseteq A$  satisfying  $(\dagger)$ .

Let  $\mathcal{R}$  be the set of all finitary relations on  $\mathbb{R}$ . For any  $n$ -ary relation  $E \in \mathcal{R}$  and any  $f_1, \dots, f_n \in F$ , define

$$E^D(f_1, \dots, f_n) \iff D x \in \mathbb{N}^A E(f_1(x), \dots, f_n(x)).$$

The set  $X = \{x \in \mathbb{N}^A : E(f_1(x), \dots, f_n(x))\}$  is obviously concentrated on  $u = \|f_1\| \cup \dots \cup \|f_n\|$ , hence, it belongs to  $\mathcal{X}$ , and  $\|X\| \subseteq u = \|f_1\| \cup \dots \cup \|f_n\|$ .

In particular,  $f =^D g$  means that  $D x \in \mathbb{N}^A (f(x) = g(x))$ . The following is clear:

PROPOSITION 3.1.  $=^D$  is an equivalence relation on  $F$ , and any relation on  $F$  of the form  $E^D$  is  $=^D$ -invariant.

Put  $[f]_D = \{g \in F : f =^D g\}$ , and  ${}^*\mathbb{R} = F/D = \{[f]_D : f \in F\}$ . For any  $n$ -ary ( $n \geq 1$ ) relation  $E \in \mathcal{R}$ , let  ${}^*E$  be the relation on  ${}^*\mathbb{R}$  defined as follows:

$${}^*E([f_1]_D, \dots, [f_n]_D) \iff E^D(f_1, \dots, f_n) \iff D x \in \mathbb{N}^A E(f_1(x), \dots, f_n(x)).$$

The independence on the choice of representatives in the classes  $[f_i]_D$  follows from Proposition 3.1. Put  ${}^*\mathcal{R} = \{{}^*E : E \in \mathcal{R}\}$ . Finally, for any  $r \in \mathbb{R}$  we put  ${}^*r = [c_r]_D$ , where  $c_r \in F$  satisfies  $c_r(x) = r, \forall x$ .

Let  $\mathcal{L}$  be the first-order language containing a symbol  $E$  for any relation  $E \in \mathcal{R}$ . Then  $\langle \mathbb{R}; \mathcal{R} \rangle$  and  $\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle$  are  $\mathcal{L}$ -structures.

THEOREM 3.2. The map  $r \mapsto {}^*r$  is an elementary embedding (in the sense of the language  $\mathcal{L}$ ) of the structure  $\langle \mathbb{R}; \mathcal{R} \rangle$  into  $\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle$ .

PROOF. This is a routine modification of the ordinary argument. By  $\mathcal{L}[F]$  we denote the extension of  $\mathcal{L}$  by functions  $f \in F$  used as parameters. It does not have a direct semantics, but if  $\varphi$  is a formula of  $\mathcal{L}[F]$  and  $x \in \mathbb{N}^A$  then  $\varphi[x]$  will denote the formula obtained by the substitution of  $f(x)$  for any  $f \in F$  which occurs in  $\varphi$ . Thus,  $\varphi[x]$  is an  $\mathcal{L}$ -formula with parameters in  $\mathbb{R}$ .

LEMMA 3.3 (Łoś). For any closed  $\mathcal{L}[F]$ -formula  $\varphi(f_1, \dots, f_n)$  (all parameters  $f_i \in F$  indicated), we have:

$$\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle \models \varphi([f_1]_D, \dots, [f_n]_D) \iff D x (\langle \mathbb{R}; \mathcal{R} \rangle \models \varphi(f_1, \dots, f_n)[x]).$$

PROOF. We argue by induction on the logic complexity of  $\varphi$ . For  $\varphi$  an atomic relation  $E(f_1, \dots, f_n)$ , the result follows by the definition of  ${}^*E$ . The only notable induction step is  $\exists$  in the direction  $\Leftarrow$ . Suppose that  $\varphi$  is  $\exists y \psi(y, f_1, \dots, f_n)$ , and

$$D x (\langle \mathbb{R}; \mathcal{R} \rangle \models \varphi(f_1, \dots, f_n)[x]),$$

that is,

$$Dx (\langle \mathbb{R}; \mathcal{R} \rangle \models \exists y \psi(y, f_1, \dots, f_n)[x]).$$

Obviously there exists a function  $f \in F$ , concentrated on  $u = \|f_1\| \cup \dots \cup \|f_n\|$ , such that, for any  $x \in \mathbb{N}^A$ , if there exists a real  $y$  satisfying  $\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(y, f_1, \dots, f_n)[x]$ , then  $y = f(x)$  also satisfies this formula, i.e.,  $\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(f, f_1, \dots, f_n)[x]$ . Formally,

$$\begin{aligned} \forall x \in \mathbb{N}^A (\exists y \in \mathbb{R} (\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(y, f_1, \dots, f_n)[x]) \implies \\ \langle \mathbb{R}; \mathcal{R} \rangle \models \psi(f, f_1, \dots, f_n)[x]). \end{aligned}$$

This implies  $Dx (\langle \mathbb{R}; \mathcal{R} \rangle \models \psi(f, f_1, \dots, f_n)[x])$ . Then, by the inductive assumption,  $\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle \models \psi([f]_D, [f_1]_D, \dots, [f_n]_D)$ , hence  $\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle \models \varphi([f_1]_D, \dots, [f_n]_D)$ , as required.  $\dashv$

To accomplish the proof of Theorem 3.2, consider a closed  $\mathcal{L}$ -formula  $\varphi(r_1, \dots, r_n)$  with parameters  $r_1, \dots, r_n \in \mathbb{R}$ . We have to prove the equivalence

$$\langle \mathbb{R}; \mathcal{R} \rangle \models \varphi(r_1, \dots, r_n) \iff \langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle \models \varphi({}^*r_1, \dots, {}^*r_n).$$

Let  $f_i = c_{r_i}$ , thus,  $f_i \in F$  and  $f_i(x) = r_i, \forall x$ . Obviously  $\varphi(f_1, \dots, f_n)[x]$  coincides with  $\varphi(r_1, \dots, r_n)$  for any  $x \in \mathbb{N}^A$ , hence  $\varphi(r_1, \dots, r_n)$  is equivalent to  $Dx \varphi(f_1, \dots, f_n)[x]$ . On the other hand, by definition,  ${}^*r_i = [f_i]_D$ . Now the result follows by Lemma 3.3.  $\dashv$

**§4. The iteration.** Theorem 3.2 yields a definable proper elementary extension  $\langle {}^*\mathbb{R}; {}^*\mathcal{R} \rangle$  of the structure  $\langle \mathbb{R}; \mathcal{R} \rangle$ . Yet this extension is not countably saturated due to the fact that the ultrapower  ${}^*\mathbb{R}$  was defined with maps concentrated on finite sets  $u \subseteq A$  only. To fix this problem, we iterate the extension used above  $\omega_1$ -many times.

Suppose that  $\langle M; \mathcal{M} \rangle$  is an  $\mathcal{L}$ -structure, so that  $\mathcal{M}$  consists of finitary relations on a set  $M$ , and for any  $E \in \mathcal{R}$  there is a relation  $E^{\mathcal{M}} \in \mathcal{M}$  of the same arity, associated with  $E$ . Let  $F_M$  be the set of all maps  $f : \mathbb{N}^A \rightarrow M$  concentrated on finite sets  $u \subseteq A$ . The structure  $F_M/D = \langle {}^*M; {}^*\mathcal{M} \rangle$ , defined as in Section 3, but with the modified  $F$ , will be called *the  $D$ -ultrapower* of  $\langle M; \mathcal{M} \rangle$ . Theorem 3.2 remains true in this general setting: the map  $x \mapsto {}^*x$  ( $x \in M$ ) is an elementary embedding of  $\langle M; \mathcal{M} \rangle$  in  $\langle {}^*M; {}^*\mathcal{M} \rangle$ .

We define a sequence of  $\mathcal{L}$ -structures  $\langle M_\alpha; \mathcal{M}_\alpha \rangle$ ,  $\alpha \leq \omega_1$ , together with a system of elementary embeddings  $e_{\alpha\beta} : \langle M_\alpha; \mathcal{M}_\alpha \rangle \rightarrow \langle M_\beta; \mathcal{M}_\beta \rangle$ ,  $\alpha < \beta \leq \omega_1$ , so that

- (i)  $\langle M_0; \mathcal{M}_0 \rangle = \langle \mathbb{R}; \mathcal{R} \rangle$ ;
- (ii)  $\langle M_{\alpha+1}; \mathcal{M}_{\alpha+1} \rangle$  is the  $D$ -ultrapower of  $\langle M_\alpha; \mathcal{M}_\alpha \rangle$ , that is,  $\langle M_{\alpha+1}; \mathcal{M}_{\alpha+1} \rangle = F_\alpha/D$ , where  $F_\alpha = F_{M_\alpha}$  consists of all functions  $f : \mathbb{N}^A \rightarrow M_\alpha$  concentrated on finite sets  $u \subseteq A$ . In addition,  $e_{\alpha, \alpha+1}$  is the associated  $*$ -embedding  $\langle M_\alpha; \mathcal{M}_\alpha \rangle \rightarrow \langle M_{\alpha+1}; \mathcal{M}_{\alpha+1} \rangle$ , while  $e_{\gamma, \alpha+1} = e_{\alpha, \alpha+1} \circ e_\gamma$  for any  $\gamma < \alpha$  (in other words,  $e_{\gamma, \alpha+1}(x) = e_{\alpha, \alpha+1}(e_\gamma(x))$  for all  $x \in M_\alpha$ );
- (iii) if  $\lambda \leq \omega_1$  is a limit ordinal then  $\langle M_\lambda; \mathcal{M}_\lambda \rangle$  is the direct limit of the structures  $\langle M_\alpha; \mathcal{M}_\alpha \rangle$ ,  $\alpha < \lambda$ . This can be achieved by the following steps:

- (a)  $M_\lambda$  is defined as the set of all pairs  $\langle \alpha, x \rangle$  such that  $x \in M_\alpha$  and  $x \notin \text{ran } e_{\gamma\alpha}$  for all  $\gamma < \alpha$ .
- (b) If  $E \in \mathcal{R}$  is an  $n$ -ary relation symbol then we define an  $n$ -ary relation  $E_\lambda$  on  $M_\lambda$  as follows. Suppose that  $\mathbf{x}_i = \langle \alpha_i, x_i \rangle \in M_\lambda$  for  $i = 1, \dots, n$ . Let  $\alpha = \sup \{\alpha_1, \dots, \alpha_n\}$  and  $z_i = e_{\alpha_i, \alpha}(x_i)$  for every  $i$ , so that  $\alpha_i \leq \alpha < \lambda$  and  $z_i \in M_\alpha$ . (Note that if  $\alpha_i = \alpha$  then  $e_{\alpha_i, \alpha}$  is the identity.) Define  $E_\lambda(\mathbf{x}_1, \dots, \mathbf{x}_n)$  iff  $\langle M_\alpha; \mathcal{M}_\alpha \rangle \models E(z_1, \dots, z_n)$ .
- (c) Put  $\mathcal{M}_\lambda = \{E_\lambda : E \in \mathcal{R}\}$  – then  $\langle M_\lambda; \mathcal{M}_\lambda \rangle$  is an  $\mathcal{L}$ -structure.
- (d) Define an embedding  $e_{\alpha\lambda} : M_\alpha \rightarrow M_\lambda$  ( $\alpha < \lambda$ ) as follows. Consider any  $x \in M_\alpha$ . If there is a least  $\gamma < \alpha$  such that there exists an element  $y \in M_\gamma$  with  $x = e_{\gamma\alpha}(y)$  then let  $e_{\alpha\lambda}(x) = \langle \gamma, y \rangle$ . Otherwise put  $e_{\alpha\lambda}(x) = \langle \alpha, x \rangle$ .

A routine verification of the following is left to the reader.

**PROPOSITION 4.1.** *If  $\alpha < \beta \leq \omega_1$  then  $e_{\alpha\beta}$  is an elementary embedding of  $\langle M_\alpha; \mathcal{M}_\alpha \rangle$  to  $\langle M_\beta; \mathcal{M}_\beta \rangle$ .*

Note that the construction of the sequence of models  $\langle M_\alpha; \mathcal{M}_\alpha \rangle$  is definable, hence, so is the last member  $\langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle$  of the sequence. It remains to prove that the  $\mathcal{L}$ -structure  $\langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle$  is countably saturated.

This is also a simple argument. Suppose that, for any  $k$ ,  $\varphi_k(p_k, x)$  is an  $\mathcal{L}$ -formula with a single parameter  $p_k \in M_{\omega_1}$  (the case of many parameters does not essentially differ from the case of one parameter), and there exists an element  $x_k \in M_{\omega_1}$  such that  $\bigwedge_{i \leq k} \varphi_i(p_i, x_k)$  is true in  $\langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle$  — in other words, we have  $\langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle \models \varphi_i(p_i, x_k)$  whenever  $k \geq i$ . Fix an ordinal  $\gamma < \omega_1$  such that for any  $k, i$  there exist (then obviously unique)  $y_k, q_i \in M_\gamma$  with  $x_k = e_{\gamma\omega_1}(y_k)$  and  $p_i = e_{\gamma\omega_1}(q_i)$ . Then  $\varphi_i(q_i, y_k)$  is true in  $\langle M_\gamma; \mathcal{M}_\gamma \rangle$  whenever  $k \geq i$ .

Fix  $a \in A$  such that  $D_a$  is a non-principal ultrafilter, that is, all cofinite subsets of  $\mathbb{N}$  belong to  $D_a$ . Consider the structure  $\langle M_{\gamma+1}; \mathcal{M}_{\gamma+1} \rangle$  as the  $D$ -ultrapower of  $\langle M_\gamma; \mathcal{M}_\gamma \rangle$ . The corresponding set  $F_\gamma$  consists of all functions  $f : \mathbb{N}^A \rightarrow M_\gamma$  concentrated on finite sets  $u \subseteq A$ . In particular, the map  $f(x) = y_k$  whenever  $x(a) = k$  belongs to  $F_\gamma$ . As any set of the form  $\{k : k \geq i\}$  belongs to  $D_a$ , we have  $D_a k (\langle M_\gamma; \mathcal{M}_\gamma \rangle \models \varphi_i(q_i, y_k))$ , that is,  $D x \in \mathbb{N}^A (\langle M_\gamma; \mathcal{M}_\gamma \rangle \models \varphi_i(q_i, f)[x])$ , for any  $i \in \mathbb{N}$ . It follows, by Lemma 3.3, that  $\varphi_i(*q_i, \mathbf{y})$  holds in  $\langle M_{\gamma+1}; \mathcal{M}_{\gamma+1} \rangle$  for any  $i$ , where  $*q_i = e_{\gamma, \gamma+1}(q_i) \in M_{\gamma+1}$  while  $\mathbf{y} = [f]_D \in M_{\gamma+1}$  is the  $D$ -equivalence class of  $f$  in  $F_\gamma$ . Put  $\mathbf{x} = e_{\gamma+1, \omega_1}(\mathbf{y})$ ; then  $\varphi_i(p_i, \mathbf{x})$  is true in  $\langle M_{\omega_1}; \mathcal{M}_{\omega_1} \rangle$  for any  $i$  because obviously  $p_i = e_{\gamma+1, \omega_1}(*q_i), \forall i$ .

**§5. Varia.** By appropriate modifications of the constructions, the following can be achieved:

1. For any given infinite cardinal  $\kappa$ , a  $\kappa$ -saturated elementary extension of  $\mathbb{R}$ , definable with  $\kappa$  as the only parameter of definition.
2. A *special* elementary extension of  $\mathbb{R}$ , of as large cardinality as desired. For instance, take, in stage  $\alpha$  of the construction considered in Section 4, ultrafilters on  $\beth_\alpha$ . Then the result will be a definable special structure of cardinality  $\beth_{\omega_1}$ . Recall that special models of equal cardinality are isomorphic [1, Theorem 5.1.17]. Therefore, such a modification admits an explicit model-theoretical characterization up to isomorphism.

3. A class-size definable elementary extension of  $\mathbb{R}$ ,  $\kappa$ -saturated for any cardinal  $\kappa$ .
4. A class-size definable elementary extension of the whole set universe,  $\kappa$ -saturated for any cardinal  $\kappa$ . (Note that this cannot be strengthened to  $\text{Ord}$ -saturation, i. e., saturation with respect to all class-size families. For instance,  $\text{Ord}^M$ -saturated elementary extensions of a minimal transitive model  $M \models \text{ZFC}$ , definable in  $M$ , do not exist — see [2, Theorem 2.8].)

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