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DEFINABLE MINIMAL COLLAPSE FUNCTIONS AT ARBITRARY PROJECTIVE LEVELS

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Abstract. Using a nonLaver modification of Uri Abraham's minimal Δ^1_3 collapse function, we define a generic extension $L[a]$ by a real a, in which, for a given $n \geq 3$, $\{a\}$ is a lightface Π^1_n singleton, a effectively codes a cofinal map $\omega \to \omega_1^L$ minimal over L, while every Σ^1_n set $X \subseteq \omega$ is still constructible.

§**1. Introduction.** It is well-known that all sets $x \subseteq \omega$ of the lightface class Σ^1_2 or Π^1_2 are Goedel-constructible. In fact this is an immediate corollary of the Shoenfield absoluteness theorem. But one gets models with nonconstructible sets which belong to the analytic hierarchy just above the mentioned threshold. In particular it is consistent with **ZFC** that there exists a Δ^1_3 real and a Π^1_2 real singleton, see [12], and such a real can be of minimal $\mathbf{L}$-degree, [13].

Many more results on definable sets of different kind have been obtained on the base of forcing methods invented in the abovementioned papers. Most of them employ versions of the *almost disjoint* coding method of [12]. A recent article [7] contains several powerful applications of almost disjoint coding, in particular, to the construction of models with Δ^1_3 well-orderings of the reals, in which the reals have some very special properties. The paper also contains further references.

Yet the almost disjoint coding technique is pretty useless in the case of models containing definable generic objects and *minimal* over the ground model with respect to this or another property. The first example of such a model was presented by Jensen [13]. Namely, Jensen's forcing notion $\boldsymbol{J} \in \mathbf{L}$ consists of perfect trees in $\omega^{<\omega}$ (a subset of the Sacks forcing), and if a real $a \in \omega^\omega$ is $\boldsymbol{J}$-generic over $\mathbf{L}$ then 1) it is true in $\mathbf{L}[a]$ that $\{a\}$ is a nonconstructible Π^1_2 singleton, and 2) a is *minimal over* $\mathbf{L}$, in the sense that if $b \in \mathbf{L}[a] \cap \omega^\omega$ then either $b \in \mathbf{L}$ or $a \in \mathbf{L}[b]$. (See also 28A in [11] on this forcing.)

Several variations of Jensen's forcing are known. In particular, a model in [15] in which, for a given $n \geq 3$ there exists a minimal nonconstructible Π^1_n singleton but all Σ^1_n sets $x \subseteq \omega$ are constructible, an ω_2-long iteration of Jensen's forcing in [1], a model in [17] in which there is an equivalence class of the equivalence relation E_0 [1] (a E_0-*class*, for brevity), which is a lightface Π^1_2 set in ω^ω, not containing OD

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[1] E_0 is defined on the Baire space ω^ω so that $x \;\mathsf{E}_0\; y$ iff the set $\{n : x(n) \neq y(n)\}$ is finite.

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elements, an extension of this result to arbitrary projective classes in [19], related models in [9] (Π^1_2 Groszek–Laver pairs of E_0-classes), [18,20,21] (nonuniformizable projective sets with countable vertical cross-sections), and a very recent [8] (a model in which the axiom of dependent choices Π^1_2-DC fails, but the countable AC holds for sets of reals).

The research of this paper was inspired by another minimal-style forcing construction, a generic extension $\mathbf{L}[a]$ by Uri Abraham [2] such that 1) $\{a\}$ is a nonconstructible Π^1_2 singleton in $\mathbf{L}[a]$, 2) $\omega_1^{\mathbf{L}[a]} = \omega_2^{\mathbf{L}}$ (so a codes a collapse of $\omega_1^{\mathbf{L}}$), and 3) a is *a minimal collapse over* $\mathbf{L}$, in the sense that if $b \in \mathbf{L}[a]$, $b : \omega \to \omega_1^{\mathbf{L}}$, and b is cofinal in $\omega_1^{\mathbf{L}}$, then $a \in \mathbf{L}[b]$. Abraham's forcing in [2] consists of Laver-style trees in ${\omega_1}^{<\omega}$, and its complicated construction in $\mathbf{L}$, while having a certain semblance of Jensen's method in [13], involves some crucial novel ideas related to the Laver forcing.

Our main result extends this research line. The next theorem asserts the existence of a model of **ZFC**, in which, for a given number $\mathbb{n} \geq 2$, there exists a $\Pi^1_{\mathbb{n}}$ real singleton which codes the collapse of the ordinal $\omega_1^{\mathbf{L}}$ in a minimal way, and in the same time reals that belong to $\Sigma^1_{\mathbb{n}}$ do not code the collapse of $\omega_1^{\mathbf{L}}$. The abovementioned result of [2] corresponds to the case $\mathbb{n} = 2$ in this theorem, of course. We use the blackboard $\mathbb{n}$ to distinguish the fixed number $\mathbb{n}$ in the theorem from other numbers n in the text.

THEOREM 1.1. *Let* $\mathbb{n} \geq 2$. *There is a generic extension* $\mathbf{L}[a]$ *of* $\mathbf{L}$, *by a real* $a \in \omega^\omega$, *such that the following is true in* $\mathbf{L}[a]$:

(i) $\omega_1^{\mathbf{L}[a]} = \omega_2^{\mathbf{L}}$;
(ii) (minimality) *if* $b \in \mathbf{L}[a]$, $b : \omega \to \omega_1^{\mathbf{L}}$, *b is cofinal in* $\omega_1^{\mathbf{L}}$, *then* $a \in \mathbf{L}[b]$;
(iii) *the singleton* $\{a\}$ *is a (lightface)* $\Pi^1_{\mathbb{n}}$ *set*;
(iv) (vacuous for $\mathbb{n} = 2$) *every* $\Sigma^1_{\mathbb{n}}$ *set* $x \subseteq \omega$ *belongs to* $\mathbf{L}$.

§2. Structure of the proof. The proof of Theorem 1.1 is organized as follows: Basic notions, related to ω_1-branching trees in ${\omega_1}^{<\omega}$ (*wide trees*), are introduced in Section 3. Unlike [2], we'll not focus on Laver-style trees, which makes basic constructions somewhat simpler. Every set $\mathbf{P}$ of wide trees T, closed under restrictions, is considered as *a forcing by wide trees*, a *WT-forcing* in brief, Section 4. Every WT-forcing adjoins a $\mathbf{P}$-generic element $a \in {\omega_1}^\omega$.

Section 6 presents a nonLaver modification of Uri Abraham's method in [2, 2.14], designed to define uncountable decreasing sequences of wide trees. Basically, any collection F of wide trees, satisfying rather natural conditions of Definition 6.1, yields a wide tree $\mathbf{wr}(F)$, so that if $F \subseteq F'$ then $\mathbf{wr}(F') \subseteq \mathbf{wr}(F)$. We apply the method to prove Theorem 5.3 in Sections 7, 8, which allows, given a wide tree S and a family of continuous functions $f_\alpha : {\omega_1}^\omega \to {\omega_1}^\omega$, $\alpha < \omega_1^{\mathbf{L}}$, to define a smaller wide tree $T \subseteq S$, regular in some sense with respect to each f_α.

Another technical device, also having its roots in [2], is introduced in Section 9. It allows to shrink a given wide tree S to a smaller wide tree T such that any predense set $U \subseteq S$ in a given family of $\aleph_1$-many such sets meets every infinite branch in T except for a bounded set of them (Corollary 9.5).

Then, arguing in the constructible universe $\mathbf{L}$, we define a forcing notion to prove Theorem 1.1 in Section 11 in the form $\mathbb{P} = \bigcup_{\alpha<\omega_2} \mathbb{P}_\alpha$. The summands $\mathbb{P}_\alpha$ are

$\aleph_1$-large WT-forcings defined by induction. Any $\mathbb{P}$-generic extension of $\mathbf{L}$ happens to be a model for Theorem 1.1, which we prove in the remainder.

The inductive construction of $\mathbb{P}_\alpha$ involves two key genericity ideas. The first idea, essentially by Jensen [13], is to make every level $\mathbb{P}_\alpha$ of the construction *generic* in some sense over the union of lower levels $\mathbb{P}_\xi$, $\xi < \alpha$. This is based on a construction developed in Sections 10, 11, which includes the abovementioned modification of Abraham's method. The iterated genericity of the levels $\mathbb{P}_\alpha$ implies that the two sets are equal in any $\mathbb{P}$-generic extension of $\mathbf{L}$:

1) the singleton $\{\mathbf{a}[G]\}$ of the principal generic element $\mathbf{a}[G] \in {\omega_1}^\omega$,

2) the intersection $\bigcap_{\alpha<\omega_2} \bigcup_{T\in\mathbb{P}_\alpha}$.

This equality is established in Sections 12, 13, on the base of studies of continuous functions in Sections 7, 8. It will eventually lead to (ii) of Theorem 1.1.

The second idea goes back to old papers [10], [15]. In $\mathbf{L}$, let $\overline{\mathbb{WTF}}$ be the set of all countable sequences $\overline{\mathbf{P}} = \langle \mathbf{P}_\xi \rangle_{\xi<\alpha}$ $(\alpha < \omega_1)$, compatible with the first genericity idea at each step $\xi < \alpha$. Then a whole sequence $\langle \mathbf{P}_\alpha \rangle_{\alpha<\omega_1}$ can be interpreted as a maximal chain in $\overline{\mathbb{WTF}}$. It happens that if such a chain is *generic*, in some sense precisely defined in Section 11, (ii) of Theorem 11.4, with respect to all $\mathbf{\Sigma}^1_{\mathfrak{n}-1}$ subsets of $\overline{\mathbb{WTF}}$, then the ensuing forcing notion $\mathbb{P} = \bigcup_{\alpha<\omega_1} \mathbb{P}_\alpha$ inherits some basic forcing properties of the whole forcing by (all) wide trees, up to the $\mathfrak{n}$-th level of projective hierarchy. This includes, in particular, the invariance of the forcing relation with respect to some natural transformations of wide trees (the forcing conditions), leading eventually to the proof of (iv) of Theorem 1.1 in Sections 15–18.

§3. Wide trees. Let ${\omega_1}^{<\omega}$ be the set of all *strings* (finite sequences) of ordinals $\xi < \omega_1$—including *the empty string* Λ. If $s \in {\omega_1}^{<\omega}$ then $\mathtt{lh}(s) < \omega$ is the *length* of a string s, and $\mathtt{max}\, s < \omega_1$ is the largest term in s. Let $\omega_1^n = \{s \in {\omega_1}^{<\omega} : \mathtt{lh}(s) = n\}$ (strings of length n). If $t \in {\omega_1}^{<\omega}$ and $\xi < \omega_1$, then $t^\frown\xi$ is the extension of t by ξ as the rightmost term. If $s, t \in {\omega_1}^{<\omega}$ then $s \subseteq t$ means that the string t extends s, while $s \subset t$ means a proper extension. A set $T \subseteq {\omega_1}^{<\omega}$ is a *tree* iff $s \in T \implies t \in T$ whenever $s, t \in {\omega_1}^{<\omega}$ and $t \subset s$. Then:

- if $s \in T$ then $\mathbf{succ}_T(s) = \{t \in T : s \subset t \wedge \mathtt{lh}(t) = \mathtt{lh}(s) + 1\}$, the set of all *successors* of s in T. If $\mathbf{succ}_T(s) = \varnothing$ then s is an *endnode* of T;
- $\mathbf{BN}(T) = \{s \in T : \mathtt{card}\,(\mathbf{succ}_T(s)) \geq 2\}$, all *branching nodes* of T, and $\mathbf{BN}_n(T) = \{s \in \mathbf{BN}(T) : \mathtt{card}\,(\{u \in \mathbf{BN}(T) : u \subset s\}) = n\}$;
- if $u \in T$ then define $T \restriction_u = \{t \in T : u \subseteq t \vee t \subseteq u\}$, a *restricted tree*;
- if T is not pairwise $\subseteq$-compatible then there is a largest string $u \in T$ such that $T \restriction_u = T$, denoted by $u = \mathtt{stem}(T)$, then $\{\mathtt{stem}(T)\} = \mathbf{BN}_0(T)$;
- define $[T] = \{x \in {\omega_1}^\omega : \forall m\, (x \restriction m \in T)\}$, a closed set in ${\omega_1}^\omega$.

Definition 3.1. A set $U \subseteq T$ is *dense* in a tree T if $\forall s \in T\, \exists u \in U\, (s \subseteq u)$, *open dense*, if in addition $s \in U$ holds whenever $s \in T$, $u \in U$, $u \subseteq s$, and *predense*, if the set $U' = \{s \in T : \exists u \in U (u \subseteq s)\}$ is dene.

DEFINITION 3.2. A tree $\varnothing \neq T \subseteq \omega_1{}^{<\omega}$ is a *wide tree*, in symbol $T \in \mathbf{WT}$, if any $s \in T$ can be extended to a branching node $t \in \mathbf{BN}(T)$, $s \subseteq t$, and if $t \in \mathbf{BN}(T)$ then $\mathtt{card}\,(\mathbf{succ}_T(s)) = \aleph_1$—that is, all branching nodes are ω_1-branching.

A bigger set $\mathbf{WT}'$ consists of all trees $\varnothing \neq T \subseteq \omega_1{}^{<\omega}$ such that each subtree of the form $T\restriction_s$, $s \in T$, is uncountable. Clearly $\mathbf{WT} \subsetneqq \mathbf{WT}'$, but $\mathbf{WT}$ is still dense in $\mathbf{WT}'$, so that every tree $T \in \mathbf{WT}'$ contains a subtree $S \in \mathbf{WT}$, $S \subseteq T$.

Generally, $\mathbf{WT}$ and $\mathbf{WT}'$ belong to the category of *uncountably splitting* versions of the perfect set forcing. Similar forcing notions, as well as their Laver-style versions (which require every node above the stem to be a wide-splitting node), have been thoroughly studied in set theoretic papers, see *e.g.,* Abraham [2], Bukovsky [4], Bukovsky and Copláková-Hartová [5], Jech [11, Chapter 28], Kanamori [14], Kurilić [22], Namba [23], to mention a few.

LEMMA 3.3. *Suppose that* $T \in \mathbf{WT}$. *If* $s \in T$ *then* $T\restriction_s \in \mathbf{WT}$. *If* $x \in X \subseteq [T]$, X *is open in* $[T]$, *then there is* $s \in T$ *such that* $s \subset x$ *and* $T\restriction_s \subseteq X$.

DEFINITION 3.4. We introduce two notions of inclusion between trees which partially honor the branching structure. If $S, T \subseteq \omega_1{}^{<\omega}$ are trees then define:

– $S \subseteq_n T$ iff $\mathbf{BN}_n(T) \subseteq S \subseteq T$;
– $S \subseteq'_n T$ iff $S \subseteq T$ and $\mathbf{BN}_{n-1}(S) = \mathbf{BN}_{n-1}(T)$.

LEMMA 3.5. (i) *The relations* $\subseteq_0$ *and* $\subseteq'_0$ *coincide with just* $\subseteq$;
(ii) $S \subseteq'_{n+1} T \implies S \subseteq_n T \implies S \subseteq'_n T$;
(iii) *if* $S \subseteq_n T$ *then* $\forall\, u \in \mathbf{BN}_n(T)$(*there is a unique* $v \in \mathbf{BN}_n(S)$ *with* $u \subseteq v$);
(iv) *if* $S \subseteq'_n T$ *then* $S \subseteq_n T$ *iff* $\forall\, u \in \mathbf{BN}_{n-1}(T)\,(\mathbf{succ}_T(u) = \mathbf{succ}_S(u))$.

LEMMA 3.6. *Let* $T \in \mathbf{WT}$, $n < \omega$. *Assume that if* $u \in \mathbf{BN}_n(T)$ *then* $T_u \in \mathbf{WT}$, $T_u \subseteq T\restriction_u$. *Then the tree* $S = \bigcup_{u \in \mathbf{BN}_n(T)} T_u$ *belongs to* $\mathbf{WT}$ *and satisfies* $S \subseteq_n T$ *and* $S\restriction_u = T_u$ *for all* $u \in \mathbf{BN}_n(T)$.

Note that under the conditions of the lemma, if $u \in \mathbf{BN}_n(T)$ then $u \subseteq \mathtt{stem}(T_u)$, and in addition $\mathbf{BN}_n(S) = \{\mathtt{stem}(T_u) : u \in \mathbf{BN}_n(T)\}$.

LEMMA 3.7. *Assume that* $\cdots \subseteq_4 T_4 \subseteq_3 T_3 \subseteq_2 T_2 \subseteq_1 T_1 \subseteq_0 T_0$ *is an infinite decreasing sequence of trees in* $\mathbf{WT}$. *Then the tree* $T = \bigcap_n T_n$ *belongs to* $\mathbf{WT}$, *and we have* $T \subseteq_n T_n$, *and hence* $\mathbf{BN}_n(T) = \mathbf{BN}_n(T_{n+1})$, *for all* n.

§4. Wide tree forcing notions and dense sets.

A nonempty set $\mathbb{P} \subseteq \mathbf{WT}$ is a *wide tree forcing*, WT-forcing in brief, if we have $T\restriction_u \in \mathbb{P}$ whenever $u \in T \in \mathbb{P}$. Thus $\mathbf{WT}$ itself is a WT-forcing, and if $S \in \mathbf{WT}$ then the set $\{S\restriction_t : t \in S\}$ is a WT-forcing.

REMARK 4.1. Any WT-forcing $\mathbf{P}$ can be considered as a forcing notion ordered so that if $T \subseteq T'$, then T is a stronger condition. The forcing $\mathbf{P}$ adjoins a cofinal element $x \in \omega_1{}^{\omega}$. More exactly if a set $G \subseteq \mathbf{P}$ is $\mathbf{P}$-generic over a given set universe $\mathbf{V}$ (and $\mathbf{P} \in \mathbf{V}$ is assumed) then the intersection $\bigcap_{T \in G}[T]$ contains a unique element $\mathbf{a}[G] \in (\omega_1^{\mathbf{V}})^\omega$, and $\mathbf{a}[G]$ satisfies $G = \{T \in \mathbf{P} : \mathbf{a}[G] \in [T]\}$, $\mathbf{V}[G] = \mathbf{V}[\mathbf{a}[G]]$, and $\sup \mathbf{a}[G] = \omega_1^{\mathbf{V}}$ (cardinality collapse).

Elements $\mathbf{a}[G]$ of this kind are called $\mathbf{P}$-*generic*.

To prove Theorem 1.1 we'll make use of a certain WT-forcing $\mathbb{P} \subseteq \mathbf{WT}$.

Definition 4.2. A set $D \subseteq \mathbf{P}$ is *dense* in $\mathbf{P}$ if for any $S \in \mathbf{P}$ there is a tree $T \in D$, $T \subseteq S$, *open dense*, if in addition $S \in D$ holds whenever $S \in \mathbf{P}$, $T \in D$, $S \subseteq T$, and *predense*, if the set $D' = \{S \in \mathbf{P} : \exists T \in D (S \subseteq T)\}$ is dense.

If $T \in \mathbf{WT}$ and $D \subseteq \mathbf{WT}$ then let $D \Uparrow T = \{s \in T : \exists S \in D\,(T \restriction_s \subseteq S)\}$.

Lemma 4.3. *Assume that* $\mathbf{P}$ *is a WT-forcing, and* $D_n \subseteq \mathbf{P}$ *is predense in* $\mathbf{P}$ *for all* n. *Let* $S_0 \in \mathbf{P}$. *Then there is a tree* $T \in \mathbf{WT}$ (not necessarily in $\mathbf{P}$!) *such that* $T \subseteq S_0$ *and if* $n < \omega$ *then* $\mathbf{BN}_n(T) \subseteq D_n \Uparrow T$.

Proof. We wlog assume that each D_n is open dense; otherwise replace it by $D_n' = \{S' \in \mathbf{P} : \exists S \in D_n\,(S' \subseteq S)\}$. Using Lemma 3.6 and the open density, define a sequence $\cdots \subseteq_4 T_4 \subseteq_3 T_3 \subseteq_2 T_2 \subseteq_1 T_1 \subseteq_0 T_0 \subseteq S_0$, such that if $n < \omega$ and $s \in \mathbf{BN}_n(T_{n+1})$ then $T_{n+1} \restriction_s \in D_n$. By Lemma 3.7, the tree $T = \bigcap_n T_n$ is as required: if $s \in \mathbf{BN}_n(T)$ then $s \in \mathbf{BN}_n(T_{n+1})$, so that $T \restriction_u \subseteq T_{n+1} \restriction_u \in D_n$. ⊣

There is no way to directly extend Lemma 4.3 to the case of ω_1-sequences of dense sets. But a somewhat weaker result of Lemma 9.4 will be possible.

§**5. Bounded sets and continuous maps.** It is known from descriptive set theory that if a continuous map $f : P \to \omega^\omega$ is defined on a perfect set $P \subseteq \omega^\omega$ then f is a bijection or a constant on a suitable perfect subset $P' \subseteq P$. A similar but somewhat more complicated dichotomy holds for wide trees. Say that a set $X \subseteq {\omega_1}^\omega$ is *bounded*, if there is an ordinal $\beta < \omega_1$ such that $X \subseteq \beta^\omega$. Note that if $T \in \mathbf{WT}$ then the set $[T]$ is **un**bounded.

Lemma 5.1. *Let* $S \in \mathbf{WT}$ *and* $f : [S] \to {\omega_1}^\omega$ *be continuous. There is a tree* $T \subseteq S$, $T \in \mathbf{WT}$, *such that either* $f\,$"$[T]$ *is bounded or* $f \restriction [T]$ *is a bijection*.

Proof. Suppose that for no $T \in \mathbf{WT}$, $T \subseteq S$, $f \restriction [T]$ is bounded. Then, as the set $\mathbf{BN}_1(S)$ is uncountable, by a simple cardinality argument there exist: an uncountable set $U \subseteq \mathbf{BN}_1(S)$, a number k, and for each $t \in U$—an ordinal $\xi_t < \omega_1$ and a tree $U_t \in \mathbf{WT}$ satisfying $U_t \subseteq S \restriction_t$, $f(x)(k) = \xi_t$ for all $x \in [U_t]$ (same k for all $t \in U$!), and if $t \neq t'$ belong to U then $\xi_t \neq \xi_{t'}$.

Then the tree $S_1 = \bigcup_{t \in U} U_t$ belongs to $\mathbf{WT}$ and satisfies $S_1 \subseteq_1' S$. In addition, there is a number $k = k_1$ such that if $u \neq u'$ belong to $\mathbf{BN}_1(S_1)$ and $x, x' \in [S_1]$, $u \subset x$, $u' \subset x'$, then $f(x)(k_1) \neq f(x')(k_1)$.

Similarly, there is a tree $S_2 \in \mathbf{WT}$, $S_2 \subseteq_2' S_1$, and a number k_2, such that if $u \neq u'$ belong to $\mathbf{BN}_2(S_2)$ and $x, x' \in [S_1]$, $u \subset x$, $u' \subset x'$, then $\langle f(x)(k_1), f(x)(k_2)\rangle \neq \langle f(x')(k_1), f(x')(k_2)\rangle$.

Iterating this construction appropriately by induction, we get a required tree $T = \bigcap_n S_n \in \mathbf{WT}$ by Lemma 3.6. ⊣

The next Theorem 5.3 presents a dichotomy somewhat different than the one considered by Lemma 5.1, and related to the case of $\aleph_1$-many maps.

Definition 5.2. If $U, V \in \mathbf{WT}$ and $f : [U] \to {\omega_1}^\omega$ is a continuous map, then $\mathbb{H}(U, f, V)$ is the set of all strings $s \in U$ such that

- either (1) $[V] \cap (f\,"[U \restriction_s])$ is bounded,
- or (2) $f \restriction [U \restriction_s]$ is a *total identity*, that is, $f(x) = x$ for all $x \in [U \restriction_s]$.

Note that (1) and (2) are incompatible provided $U \subseteq V$.

THEOREM 5.3 (the proof will be given in Section 8). *Assume that* $S \in \mathbf{WT}$, *and, for each* $\alpha < \omega_1$, $f_\alpha : [S] \to {\omega_1}^\omega$ *is a continuous function. Then there exists a tree* $T \in \mathbf{WT}$, $T \subseteq_1 S$, *such that, for every* $\alpha < \omega_1$, *the set* $\mathbb{H}(T, f_\alpha, T)$ *is dense in T.*

§6. Iteration of wide trees. Here we develop another method of construction of trees in $\mathbf{WT}$, similar to a construction introduced in [2, 2.14], and designed for the proof of Theorem 5.3.

DEFINITION 6.1. A function J is an *iteration* (*of wide trees*), in symbol $J \in \mathbf{IWT}$, if its *core* $C = \mathtt{dom}\, J$ is a subtree of ${\omega_1}^{<\omega}$ (possibly with endnodes and/or isolated branches), all values $J(u)$ are trees in $\mathbf{WT}$, and in addition

(1) if $u \subseteq v$ belong to C then $v \in J(u)$ and $J(v) \subseteq J(u) \restriction_v$;

(2) if $u \subset v$ belong to C, $\mathtt{lh}(v) = \mathtt{lh}(u) + 1$, $u \notin \mathbf{BN}(J(u))$, then $J(v) = J(u)$.

In this case we define the *wrap* of J,

(3) $\mathbf{wr}(J) = \{s \in {\omega_1}^{<\omega} : \forall u \in \mathtt{dom}\, J\ (u \subset s \implies s \in J(u)\}$.

If $\mathbf{P} \subseteq \mathbf{WT}$ then let $\mathbf{IWT}(\mathbf{P})$ consist of all iterations $J \in \mathbf{IWT}$ with $\mathtt{ran}\, J \subseteq \mathbf{P}$. An iteration $J \in \mathbf{IWT}$ is *small* if the core $C = \mathtt{dom}\, J$ is at most countable.

In particular $\varnothing \in \mathbf{IWT}$ and $\mathbf{wr}(\varnothing) = {\omega_1}^{<\omega}$.

If $C \subseteq {\omega_1}^{<\omega}$ is a tree and $s \in {\omega_1}^{<\omega}$ then let $\mathbf{proj}_C(s)$ (the *projection*) be the largest string in C with $u \subseteq s$; in particular $\mathbf{proj}_C(s) = s$ provided $s \in C$.

LEMMA 6.2. *If* $J \in \mathbf{IWT}$ *then* $T = \mathbf{wr}(J) \in \mathbf{WT}$, $C = \mathtt{dom}\, J \subseteq T$, *and*

(i) *if* $s \in C$ *then* $s \subseteq \mathtt{stem}(J(s))$, $T \restriction_s \subseteq J(s)$, *and* $\mathbf{succ}_T(s) = \mathbf{succ}_{J(s)}(s)$;

(ii) *if* $s \in T \smallsetminus C$ *and* $u = \mathbf{proj}_C(s)$ *then* $s \in J(u)$ *and* $T \restriction_s = J(u) \restriction_s$;

(iii) *if* $s \in C$ *is an endnode in* C *then we have* $T \restriction_s = J(s)$.

PROOF. If $u \in C$ then $u \in T$ by 6.1(1), so we have $C \subseteq T$.

(i) If $s \in C$ then $J(s) = J(s) \restriction_s$ by 6.1(1) with $u = v = s$, so that obviously $s \subseteq \mathtt{stem}(J(s))$. If now $t \in T$ and $s \subseteq t$ then $t \in J(s)$ by 6.1(3), therefore $T \restriction_s \subseteq J(s)$. This implies $\mathbf{succ}_T(s) \subseteq \mathbf{succ}_{J(s)}(s)$. To get the equality, let $t = s^\frown \xi \in \mathbf{succ}_{J(s)}(s)$. Then $t \in T$ by 6.1(1),(3), so $t \in \mathbf{succ}_T(s)$, as required.

(ii) If $s \notin C$ then by 6.1(1), (3) the criterion of $s \in T = \mathbf{wr}(J)$ is just $s \in J(u)$, where $u = \mathbf{proj}_C(s)$. This easily implies the result. And (iii) is similar to (ii).

To prove $T \in \mathbf{WT}$, let $s \in T$. We have to prove that (a) if $s \in \mathbf{BN}(T)$ then $\mathbf{succ}_T(s)$ is uncountable, and (b) there is a string $s' \in \mathbf{BN}(T)$ with $s \subseteq s'$. By (i), (ii) we have (a) immediately, so it remains to check (b).

CASE 1. $s \in T \smallsetminus C$. Then $T \restriction_s = J(u) \restriction_s$ by (ii), where $u = \mathbf{proj}_C(s)$. But $J(u) \restriction_s \in \mathbf{WT}$ by Lemma 3.3, which easily implies (b).

CASE 2. s is an endnode in C, so $T \restriction_s = J(s) \in \mathbf{WT}$ by (iii); follow Case 1.

CASE 3. there is an endnode s' in C with $s \subseteq s'$—apply Case 2 for s'.

CASE 4. if all the above fails then there is an infinite branch in C containing s, that is, $b \in {\omega_1}^\omega$ such that $b \restriction n \in C$, $\forall n$, and $s = b \restriction n_0$, where $n_0 = \mathtt{lh}(s)$. Then

$b \restriction n \in J(s)$ for all n by (i). Therefore, as $J(s) \in \mathbf{WT}$, there is a least number $k \geq n_0$ with $t = b \restriction k \in \mathbf{BN}(J(s))$. Then by the way $J(t) = J(s)$ by 6.1(2), hence $t \in \mathbf{BN}(J(t))$, and finally $t \in \mathbf{BN}(T)$ by (i), as required. ⊣

Lemma 6.2 allows to maintain infinite, even uncountable $\subseteq$-decreasing sequences of trees in $\mathbf{WT}$, with the help of the following two rather obvious results.

LEMMA 6.3. *If $J \subseteq J'$ are iterations in* $\mathbf{IWT}$ *then* $\mathbf{wr}(J') \subseteq \mathbf{wr}(J)$.

If $\langle J_\xi \rangle_{\xi<\lambda}$ is a $\subset$-increasing sequence of iterations $J_\xi \in \mathbf{IWT}$ then $J = \bigcup_{\xi<\lambda} J_\xi \in$ $\mathbf{IWT}$ *and* $\mathbf{wr}(J) = \bigcap_\xi \mathbf{wr}(J_\xi)$.

LEMMA 6.4. *Let $J \in \mathbf{IWT}$, $\mathtt{dom}\, J = C \subseteq C' \subseteq T = \mathbf{wr}(J)$, C' be a tree.*

- Define a *natural extension* J' of J to C' by $\mathtt{dom}\, J' = C'$, $J'(s) = J(s)$ for $s \in C$, and if $s \in C' \smallsetminus C$ and $u = \mathbf{proj}_C(s)$ then $J'(s) = J(u)\restriction_s$.

Then $J' \in \mathbf{IWT}$, $J \subseteq J'$, $\mathbf{wr}(J') = \mathbf{wr}(J)$.

Condition (2) of Definition 6.1 imposes important restrictions on the construction of iterations, basically justifying proper reduction only at successors of branching nodes. Nevertheless it leaves us enough freedom.

LEMMA 6.5. *Assume that* $\mathbf{P}$ *is a WT-forcing, $J \in \mathbf{IWT}(\mathbf{P})$, $C = \mathtt{dom}\, J$, $s \in T =$ $\mathbf{wr}(J)$, and $s \notin C$ or s is an endnode in C. Let $U \in \mathbf{P}$, $U \subseteq T\restriction_s$. Then there exists an iteration $J' \in \mathbf{IWT}(\mathbf{P})$ and a string $s' \in \mathtt{dom}\, J'$ such that $J \subseteq J'$, $s \subseteq s'$, and $J'(s') \subseteq U$.*

PROOF. Let $t = \mathtt{stem}(U)$, thus $s \subseteq t \in \mathbf{BN}(U)$ and all shorter strings $v \subset t$ do not belong to $\mathbf{BN}(U)$. Pick any $s' \in U$ with $\mathtt{lh}(s') = \mathtt{lh}(t)+1$; then $t \subset s' \notin C$. Let $u = \mathbf{proj}_C(s)$. Let $J' \in \mathbf{IWT}(\mathbf{P})$ be the extension of J to the domain $C' = C \cup \{v : u \subset v \subseteq t\} \cup \{s'\}$ by $J'(u) = J(s) = J(s)\restriction_u$ whenever $s \subset u \subseteq t$, and finally $J'(s') = U\restriction_{s'}$. To see that 6.1(2) is satisfied for J' at $u = t$ and $v = s'$, recall that $t \in \mathbf{BN}(U)$, hence $t \in \mathbf{BN}(J(s)) = \mathbf{BN}(J(t))$ as well. ⊣

§7. Key dichotomy lemma.

LEMMA 7.1. *Assume that* $\mathbf{P} \subseteq \mathbf{WT}$ *is a WT-forcing, $J \in \mathbf{IWT}(\mathbf{P})$ is a small iteration, $S = \mathbf{wr}(J)$, $g_0 \in C = \mathtt{dom}\, J$, and $f : [S] \to {\omega_1}^\omega$ is continuous. There is a small iteration $J' \in \mathbf{IWT}(\mathbf{P})$ and a string $g \in C' = \mathtt{dom}\, J'$, such that $g_0 \subseteq g$, $J \subseteq J'$, and $g \in \mathbb{H}(T, f, T)$, where $T = \mathbf{wr}(J')$, that is,*

(1) *$[T] \cap (f\,"[T\restriction_g])$ is bounded,* **or**

(2) *f is a total identity on $[T\restriction_g]$.*

The lemma will be crucial in the proof of Theorem 5.3 in Section 8.

PROOF. Pick any $g_1 \in S \smallsetminus C$ satisfying $g_0 \subseteq g_1$. Let $u = \mathbf{proj}_C(g_1)$. If $f\,"[S\restriction_{g_1}] \subseteq [C]$ (a bounded set) then let $J_1 \in \mathbf{IWT}(\mathbf{P})$ be the natural extension of J to the domain $C_1 = C \cup \{s : u \subset s \subseteq g_1\}$ by Lemma 6.4. Thus $J \subseteq J_1$, $\mathtt{dom}\, J_1 = C_1$, $J_1(g_1) = J(u)\restriction_{g_1}$, and $\mathbf{wr}(J_1) = S$. Therefore $J' = J_1$ and $g = g_1$ satisfy (1).

Thus suppose that $x_1 \in [S\restriction_{g_1}]$, and $y_1 = f(x_1) \in [S] \smallsetminus [C]$. As f is continuous while ${\omega_1}^\omega \smallsetminus [C]$ open, there is a longer string $g_2 \in S \smallsetminus C$, $g_1 \subset g_2$, such that $f(x) \notin [C]$ for all $x \in S\restriction_{g_2}$. If f is a total identity on $[T\restriction_{g_2}]$ then let $J_2 \in \mathbf{IWT}(\mathbf{P})$ be the natural extension of J to the domain $C_2 = C \cup \{s : u \subset s \subseteq g_2\}$ by Lemma 6.4; now $J' = J_2$ and $g = g_2$ satisfy (2).

Thus suppose that $x_2 \in [S\restriction_{g_2}]$, and $y_2 = f(x_2) \neq x_2$. There is a yet longer string $g_3 \in S \smallsetminus C$, $g_2 \subset g_3$, such that $f(x) \neq x$ and $f(x) \notin [C]$ for all $x \in S\restriction_{g_3}$. If $(f\,"[S\restriction_{g_3}]) \cap [S] = \varnothing$ then let $J_3 \in \mathbf{IWT}(\mathbf{P})$ be the natural extension of J to the domain $C_3 = C \cup \{s : u \subset s \subseteq g_3\}$; now $J' = J_3$, $g = g_3$ satisfy (1).

Thus suppose that $x_3 \in S\restriction_{g_3}$ and $y_3 = f(g_3) \in [S]$. In addition, $x_3 \neq y_3 \notin [C]$ holds as $g_2 \subseteq g_3$, hence there is $m \geq \mathtt{lh}(g_3)$ such that $t = y_3\restriction m \in \mathbf{BN}(S) \smallsetminus C$ and $t \neq s = x_3\restriction m$. Let $t' = y_3\restriction(m+1)$ (a successor of t in S). There is a string $h \in S$ such that $t' \subset h$ but $h \neq t'' = y_3\restriction \ell$, where $\ell = \mathtt{lh}(h)$. As f is continuous, pick a number $n \geq n_3 = \mathtt{lh}(g_3)$ such that $t'' \subset f(x)$ holds for all $x \in [S\restriction_g]$, where $g = x_3\restriction n$. Recall that $u = \mathbf{proj}_C(g)$. Let $v = \mathbf{proj}_C(t)$,

$$C' = C \cup \{w \in {\omega_1}^{<\omega} : u \subset w \subseteq g\} \cup \{w \in {\omega_1}^{<\omega} : v \subset w \subseteq t'\},$$

and extend the iteration J to the domain C' by $J'(w) = J(u)\restriction_w$ whenever $u \subset w \subseteq g$, $J'(w) = J(v)\restriction_w$ whenever $v \subset w \subset t$, and $J'(t') = J(v)\restriction_h$.

Now it suffices to prove (1) in the form $[T] \cap (f\,"[T\restriction_g]) = \varnothing$. Let $g \subset x \in [S]$. Then $y = f(x)$ satisfies $t'' \subset y$, hence $h \not\subset y$. Let's show that $y \notin [T]$. It suffices to check $t'' \notin T$. Suppose otherwise. Then, as $t' \in C'$, we have $t'' \in J'(t')$ by 6.1(3). However $J'(t') = J(v)\restriction_h$, so it follows that t'' and h are compatible, which contradicts to the construction, as required. ⊣

§8. The proof of the restriction theorem.

Theorem 5.3. We argue under the assumptions of Theorem 5.3. The set $\mathbf{P} = \{S\restriction_t : t \in S\}$ is a WT-forcing and $S \in \mathbf{P}$. By Lemmas 7.1 and 6.3, 6.4, there is a $\subseteq$-increasing sequence of small iterations $J_\gamma \in \mathbf{IWT}(\mathbf{P})$, $\gamma < \omega_1$, with domains $C_\gamma = \mathtt{dom}\,J_\gamma$ and trees $S_\gamma = \mathbf{wr}(J_\gamma)$, such that $C_0 = \{u : u \subseteq \sigma\}$, where $\sigma = \mathtt{stem}(S)$, and $J_0(u) = S$ for all $u \in C_0$, the sets $C = \bigcup_{\gamma<\omega_1} C_\gamma$ and $T = \bigcap_{\gamma<\omega_1} T_\gamma$ coincide (Lemma 6.4 is applied), and in addition (Lemma 7.1 is applied), if $s_0 \in C = T$ and $\alpha < \omega_1$ then there is an index $\gamma = \gamma(s_0, \alpha) < \omega_1$ and a string $s \in C_\gamma$ such that $s_0 \subseteq s$ and $s \in \mathbb{H}(T_\gamma, f_\alpha, T_\gamma)$. Then $J = \bigcup_\alpha J_\alpha \in \mathbf{IWT}(\mathbf{P})$, $C = \mathtt{dom}\,J$, and $T = \mathbf{wr}(J)$, by Lemma 6.3. Moreover, as $T \subseteq T_\gamma$, we have $s \in \mathbb{H}(T, f_\alpha, T)$ as well. It follows that the set $\mathbb{H}(T, f_\alpha, T)$ is dense in T, and obviously *open* dense. And finally $T \subseteq_1 S$ by Lemma 6.2(i) with $s = \sigma = \mathtt{stem}(S)$. (Recall that $J_0(\sigma) = S$.) ⊣

§9. Belts and covering. Here we introduce the last major tool employed in the definition of the forcing notion for Theorem 1.1. It is based on the following definition.

Definition 9.1. A set $H \subseteq {\omega_1}^{<\omega}$:

– *meets* $x \in {\omega_1}^{\omega}$ iff $\exists\, m\,(x\restriction m \in H)$;

– is a *belt* for a tree $T \in \mathbf{WT}$, if it meets every $x \in [T]$;

– *weakly covers* T, in symbol $T \subseteq^{\mathtt{w}} B$, if there is an ordinal $\beta < \omega_1$ such that H is a belt for each subtree $T\restriction_s$, where $s \in T$ and $\max s \geq \beta$—in other words, we require H to meet every $x \in [T]$ with $\sup x \geq \beta$.

For instance, if $n < \omega$ then $\mathbf{BN}_n(T)$ is a belt for $T \in \mathbf{WT}$.

LEMMA 9.2. *Let $H \subseteq T$ weakly cover $T \in \mathbf{WT}$ with a parameter $\beta < \omega_1$. Then*

(i) *H is predense in T;*
(ii) *H weakly covers any tree $S \in \mathbf{WT}$, $S \subseteq T$, with the same β;*
(iii) *the set $X = \{x \in [T] : H$ does not meet $x\}$ satisfies $X \subseteq \beta^\omega$.*

PROOF. (iii) Let $x \in [T] \smallsetminus \beta^\omega$, $x(j) \geq \beta$ for some j. Let $s = x \restriction (j+1)$. Then H is a belt for $T \restriction_s$, hence H meets x. ⊣

REMARK 9.3. Being a belt is equivalent to the well-foundedness of the subtree $T' = \{s \in T : \neg \exists t \in H\, (t \subseteq s)\}$, hence it is an *absolute* notion, in spite of an explicit reference to the nonabsolute notion of $[T]$. It follows that to weakly cover with a parameter β is an *absolute* notion, too.

Now assume that $H \subseteq T$ weakly covers $T \in \mathbf{WT}$ with a parameter $\beta < \omega_1$. Let $x \in [T]$ be an element cofinal in ω_1 $(=\omega_1^{\mathbf{V}}$ of the given set universe $\mathbf{V})$, which may exist in an extension of $\mathbf{V}$, Remark 4.1. *We claim that H meets x.* Indeed, $x \notin \beta^\omega$ by the cofinality, and on the other hand, the absoluteness of the weak covering allows to apply Lemma 9.2(iii) in the extension containing x.

LEMMA 9.4. *Assume that $\mathbf{P}$ is a WT-forcing, $T \in \mathbf{P}$, and $D_\xi \subseteq \mathbf{P}$ is open dense in $\mathbf{P}$ for all $\xi < \omega_1$. Then there is a tree $S \in \mathbf{WT}$ such that $S \subseteq_1 T$, and each set $D_\xi {\Uparrow} S = \{t \in S : \exists\, U \in D_\xi\, (S \restriction_t \subseteq U)\}$ weakly covers S.*

PROOF. If $\alpha < \omega_1$ then fix an enumeration of the countable set $\{D_\xi : \xi \leq \alpha\} = \{D_k^\alpha : k < \omega\}$. Using Lemma 3.6 and the open-density of each D_ξ in $\mathbf{P}$, define a sequence $\dots \subseteq_5 T_4 \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0 = T$ of trees in $\mathbf{WT}$, such that if $n \geq 1$ and $u \in \mathbf{BN}_n(T_n)$ then $T_n \restriction_u \in \bigcap_{j,k \leq n} D_k^{u(j)}$. The tree $S = \bigcap_n T_n$ belongs to $\mathbf{WT}$ and satisfies $S \subseteq_{n+1} T_n$ and $\mathbf{BN}_n(S) = \mathbf{BN}_n(T_n)$ for all n, by Lemma 3.7. In particular $S \subseteq_1 T$. Now suppose that $\xi < \omega_1$.

We claim that ξ itself witnesses $D_\xi {\Uparrow} S$ to weakly cover S. Let $x \in [S]$ and $x(j) = \alpha \geq \xi$ for some j. Then $D_\xi = D_k^\alpha$ for some k. Let $n = 1 + \max\{j, k\}$. There is a number $m \geq n$ such that $u = x \restriction m$ belongs to $\mathbf{BN}_n(S) = \mathbf{BN}_n(T_n)$. Then $S \restriction_u \subseteq T_n \restriction_u \in D_k^{u(j)} = D_k^\alpha = D_\xi$ by construction, and we are done. ⊣

COROLLARY 9.5. *If $T \in \mathbf{WT}$ and $H_\xi \subseteq T$ is open dense in T for all $\xi < \omega_1$ then there is $S \in \mathbf{WT}$ such that $S \subseteq_1 T$ and each $H_\xi \cap S$ weakly covers S.*

PROOF. Apply the lemma for the sets $\mathbf{P} = \{T \restriction_s : s \in T\}$ and $D_\xi = \{T \restriction_s : s \in H_\xi\}$. ⊣

§10. Refining wide tree forcing notions. The forcing notion to prove Theorem 1.1 will be defined in the form of an ω_1-union of its parts, that is, WT-forcings of cardinality $\leq \aleph_1$.

DEFINITION 10.1. Let $\mathfrak{M}$ be any set and $\mathbf{P}$ be a WT-forcing. Another WT-forcing $\mathbf{Q}$ is an *$\mathfrak{M}$-refinement* of $\mathbf{P}$, in symbol $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$, if the following holds:

(A) $\mathbf{Q}$ is dense in $\mathbf{P} \cup \mathbf{Q}$;
(B) $\mathbf{Q}$ *refines* $\mathbf{P}$: if $Q \in \mathbf{Q}$ then there exists $T \in \mathbf{P}$ satisfying $Q \subseteq T$;
(C) if $U \in \mathbf{Q}$, and a set $D \in \mathfrak{M}$, $D \subseteq \mathbf{P}$ is predense in $\mathbf{P}$, then the set $D {\Uparrow} U = \{s \in U : \exists\, S \in D\, (U \restriction_s \subseteq S)\}$ weakly covers U;

(D) if $T_0 \in \mathbf{P}$ and $\langle D_n \rangle_{n<\omega} \in \mathfrak{M}$ is a sequence of predense sets $D_n \subseteq \mathbf{P}$ then there is a tree $T \in \mathbf{Q}$ such that $T \subseteq T_0$, and $\mathbf{BN}_n(T) \subseteq D_n \Uparrow T$ for all n;
(E) if $T_0 \in \mathbf{P}$ and $f : {\omega_1}^\omega \to {\omega_1}^\omega$, $f \in \mathfrak{M}$, is continuous, then there is $T \in \mathbf{Q}$ such that $T \subseteq T_0$, and either f"$[T]$ is bounded or $f \restriction [T]$ is a bijection;
(F) if $f \in \mathfrak{M}$, $f : {\omega_1}^\omega \to {\omega_1}^\omega$ is a continuous map, and $U, V \in \mathbf{Q}$, then the set $\mathbb{H}(U, f, V)$, of all strings $s \in U$ such that $[V] \cap (f$"$[U \restriction_s])$ is bounded or $f \restriction [U \restriction_s]$ is a total identity, weakly covers U.

If $\mathfrak{M} = \varnothing$ then we write $\mathbf{P} \sqsubset \mathbf{Q}$ instead of $\mathbf{P} \sqsubset_\varnothing \mathbf{Q}$; in this case (C)–(F) are trivial. Generally, in the role of $\mathfrak{M}$, we'll consider transitive models of the theory $\mathbf{ZFC}'$ which includes all $\mathbf{ZFC}$ axioms except for the Power Set axiom, but an axiom is adjoined, which claims the existence of ω_1 and $\mathscr{P}(\omega_1)$. (Then the existence of sets like ${\omega_1}^{<\omega}$ and $\mathbf{WT}$ easily follows.)

Lemma 10.2. *Let* $\mathbf{P}, \mathbf{Q}, \mathbf{R}$ *be WT-forcings satisfying* $\mathbf{P} \sqsubset \mathbf{Q} \wedge \mathbf{Q} \sqsubset \mathbf{R}$. *Then* $\mathbf{P} \sqsubset \mathbf{R}$, *and if* (K) *is one of* (C), (D), (E), (F) *and the pair* $\mathbf{P} \sqsubset \mathbf{Q}$ *satisfies* (K) *with some* $\mathfrak{M}$, *then the pair* $\mathbf{P} \sqsubset \mathbf{R}$ *satisfies* (K) *with the same* $\mathfrak{M}$.

Proof. (C) Let $R \in \mathbf{R}$. As $\mathbf{Q} \sqsubset \mathbf{R}$, there is a tree $Q \in \mathbf{Q}$ with $R \subseteq Q$. Then $D \Uparrow Q$ weakly covers Q by (C) for $\mathbf{P}, \mathbf{Q}$. Then easily $D \Uparrow R$ weakly covers the tree R.

(D) If $T' \subseteq T$ and $t \in \mathbf{BN}_n(T')$ then there is a string $s \in \mathbf{BN}_n(T)$ with $s \subseteq s'$.
(F) If $U' \subseteq U$ and $V' \subseteq V$ then $\mathbb{H}(U, f, V) \cap U' \subseteq \mathbb{H}(U', f, V')$. ⊣

Lemma 10.3. *Assume that* $\mathfrak{M} \models \mathbf{ZFC}'$ *is a transitive model, and* $\mathbf{P} \in \mathfrak{M}$ *and* $\mathbf{Q}$ *are WT-forcings satisfying* $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$. *Then*

(i) *if a set* $D \in \mathfrak{M}$, $D \subseteq \mathbf{P}$ *is predense in* $\mathbf{P}$ *then* D *is predense in* $\mathbf{P} \cup \mathbf{Q}$;
(ii) *if* $T, T' \in \mathbf{P}$ *are incompatible in* $\mathbf{P}$ *then* T, T' *are incompatible in* $\mathbf{P} \cup \mathbf{Q}$.

Proof. (i) Let $U \in \mathbf{Q}$. Then $D \Uparrow U$ weakly covers U by 10.1(C). Let $s \in D \Uparrow U$. Then $U' = U \restriction_s \in \mathbf{Q}$, $U' \subseteq U$, and $U' \subseteq S$ for some $S \in D$.

(ii) The sets $D(T) = \{S \in \mathbf{P} : S \subseteq T \vee [S] \cap [T] = \varnothing\}$ and $D(T')$ belong to $\mathfrak{M}$ and are open dense in $\mathbf{P}$ by Lemma 3.3. Therefore $D = D(T) \cap D(T')$ is open dense either, and in fact $S \in D \Longrightarrow [S] \cap [T] = \varnothing \vee [S] \cap [T'] = \varnothing$ by the incompatibility. It follows that if $U \in \mathbf{Q}$ and, by (i), $S \in D$ and $U' \in \mathbf{Q}$, $U' \subseteq U$, $U' \subseteq S \cap U$, then $[U'] \cap [T] = \varnothing$ or $[U'] \cap [T'] = \varnothing$, hence U cannot witness the compatibility of T, T'. ⊣

We now establish the existence of refinements.

Theorem 10.4. *Assume that* $\mathfrak{M} \models \mathbf{ZFC}'$ *is a transitive model of cardinality* $\leq \aleph_1$, *and* $\mathbf{P} \in \mathfrak{M}$ *is a WT-forcing,* $\mathtt{card}\,\mathbf{P} \leq \aleph_1$ *in* $\mathfrak{M}$. *Then there exists a WT-forcing* $\mathbf{Q}$ *of cardinality* $\aleph_1$, *satisfying* $\mathbf{P} \sqsubset_{\mathfrak{M}} \mathbf{Q}$.

Proof. *Step 1*. If $P \in \mathbf{P}$ then by Lemma 9.4 there is a tree $T(P) \in \mathbf{WT}$, $T(P) \subseteq P$, such that $D \Uparrow T(P)$ weakly covers $T(P)$ for each $D \in \mathfrak{M}$, $D \subseteq \mathbf{P}$, predense in $\mathbf{P}$. The set $\mathbf{P}' = \{T(P) \restriction_s : P \in \mathbf{P} \wedge s \in T(P)\}$ is a WT-forcing of cardinality $\aleph_1$ and 10.1(A),(B),(C) hold for $\mathbf{Q} = \mathbf{P}'$.

Step 2. To fulfill 10.1(E), if $P' \in \mathbf{P}'$ and $f : {\omega_1}^\omega \to {\omega_1}^\omega$, $f \in \mathfrak{M}$ is continuous, then by Lemma 5.1 there is a tree $T(P', f) \in \mathbf{WT}$, such that $T(P', f) \subseteq T$, and either $f\,``[T(P',f)]$ is bounded or $f\restriction[T(P',f)]$ is a bijection. We let $\mathbf{P}''=\{T(P',f)\restriction_s : P'\in\mathbf{P}'\wedge s\in T(P',f)\}$. Now 10.1(A),(B),(C),(E) hold for the forcing $\mathbf{Q}=\mathbf{P}''$.

Step 3. To fulfill 10.1(D), note first of all that each set $D\in\mathfrak{M}$, $D\subseteq\mathbf{P}$, predense in $\mathbf{P}$, remains predense in $\mathbf{P}\cup\mathbf{P}''$ by Lemma 10.3(i). If $P''\in\mathbf{P}''$ and $\mathbf{d}=\langle D_n\rangle_{n<\omega}\in\mathfrak{M}$ is a sequence of predense sets $D_n\subseteq\mathbf{P}$, then by Lemma 4.3 there is a tree $T(P'',\mathbf{d})\in\mathbf{WT}$ such that $T(P'',\mathbf{d})\subseteq P''$, and if $n<\omega$ and $s\in\mathbf{BN}_n(T(P'',\mathbf{d}))$ then $\exists S\in D_n\,(T(P'',\mathbf{d})\restriction_s\subseteq S)$. We let

$$\mathbf{P}'''=\{T(P'',\mathbf{d})\restriction_s : P''\in\mathbf{P}''\wedge\mathbf{d}\in\mathfrak{M}\}.$$

Now 10.1(A),(B),(C),(D),(E) hold for $\mathbf{Q}=\mathbf{P}'''$.

To fulfill 10.1(F), we begin with some notation. If $S\in\mathbf{WT}$ and $\alpha<\omega_1$ then let $\alpha^\frown S=\{\alpha^\frown s : s\in S\}$; then $\alpha^\frown S\in\mathbf{WT}$ and $\langle\alpha\rangle\subseteq\mathtt{stem}(\alpha^\frown S)$. Conversely, if $W\in\mathbf{WT}$ and $\langle\alpha\rangle\subseteq\mathtt{stem}(W)$ then let $W{\downarrow}=\{s\in{\omega_1}^{<\omega} : \alpha^\frown s\in W\}$; then $W{\downarrow}\in\mathbf{WT}$ and $W=\alpha^\frown(W{\downarrow})$. We have $(\alpha^\frown S){\downarrow}=S$, of course.

Step 4. Let $\mathbf{P}'''=\{R_\alpha : \alpha<\omega_1\}$. We convert $\mathbf{P}'''$ into a single tree

$$R=\{\Lambda\}\cup\bigcup\nolimits_{\alpha<\omega_1}(\alpha^\frown R_\alpha)\in\mathbf{WT};\quad\text{ then }\ R_\alpha=(R\restriction_{\langle\alpha\rangle}){\downarrow},\ \forall\,\alpha.$$

If $f:{\omega_1}^\omega\to{\omega_1}^\omega$ is continuous and $\alpha,\beta<\omega_1$ then define $f_{\alpha\beta}:{\omega_1}^\omega\to{\omega_1}^\omega$ so that $f_{\alpha\beta}(\alpha^\frown x)=\beta^\frown f(x)$, $f_{\alpha\beta}(\beta^\frown x)=\alpha^\frown f(x)$, and $f_{\alpha\beta}(y)=y$ whenever $y(0)\neq\alpha,\beta$. The set of continuous functions $F=\{f_{\alpha\beta} : f\in\mathfrak{M}\wedge\alpha,\beta<\omega_1\}$ is still of cardinality $\aleph_1$. By Theorem 5.3 there exists a tree $T\in\mathbf{WT}$, $T\subseteq_1 R$, such that if $h\in F$ then the set $\mathbb{H}(T,h,T)$ is open dense in T. Therefore by Corollary 9.5 there is a tree $Q\in\mathbf{WT}$ such that $Q\subseteq_1 T$ (hence $Q\subseteq_1 R$ as well) and if $h\in F$ then $\mathbb{H}(T,h,T)$ weakly covers Q. Then $\mathbb{H}(Q,h,Q)$ weakly covers Q as well by Lemma 9.2(ii) since $\mathbb{H}(T,h,T)\cap Q\subseteq\mathbb{H}(Q,h,Q)$.

Step 5. Note that if $\alpha<\omega_1$ then the one-term string $\langle\alpha\rangle$ belongs to Q since $Q\subseteq_1 R$. Now let $Q_\alpha=(Q\restriction_{\langle\alpha\rangle}){\downarrow}=\{q\in{\omega_1}^{<\omega} : \alpha^\frown q\in Q\}$. We claim that *the WT-forcing* $\mathbf{Q}=\{Q_\alpha\restriction_q : \alpha<\omega_1\wedge q\in Q_\alpha\}$ *satisfies* $\mathbf{P}\sqsubset_{\mathfrak{M}}\mathbf{Q}$.

First of all, $\mathbf{P}\sqsubset\mathbf{P}'\sqsubset\mathbf{P}''\sqsubset\mathbf{P}'''\sqsubset\mathbf{Q}$ by construction, and hence $\mathbf{P}\sqsubset\mathbf{Q}$ holds, and we have 10.1(C),(D),(E) for the pair $\mathbf{P}\sqsubset\mathbf{Q}$ by Lemma 10.2.

To check 10.1(F), let $f\in\mathfrak{M}$, $f:{\omega_1}^\omega\to{\omega_1}^\omega$ be continuous, and $U=Q_\alpha$, $V=Q_\beta$ be trees in $\mathbf{Q}$. To prove that $\mathbb{H}(U,f,V)$ weakly covers U, let $h=f_{\alpha\beta}$. Then $\mathbb{H}(Q,h,Q)$ weakly covers Q by Step 4. Thus there is an ordinal $\xi<\omega_1$ such that if $x\in[Q]$ and $\mathtt{sup}\,x\geq\xi$ then $\mathbb{H}(Q,h,Q)$ meets x, so $x\restriction m\in\mathbb{H}(Q,h,Q)$ for some m. We claim that ξ witnesses that $\mathbb{H}(U,f,V)$ weakly covers U.

Assume that $y\in[U]=[Q_\alpha]$, $\mathtt{max}\,y\geq\xi$. Then $x=\alpha^\frown y\in[Q]$, so $s=x\restriction(m+1)\in\mathbb{H}(Q,h,Q)$ for some m, by the above. Then $s=\alpha^\frown t$, where $t=y\restriction m$. It remains to prove that $t\in\mathbb{H}(U,f,V)$.

Case 1. $[Q]\cap(h\,``[Q\restriction_s])$ is bounded. However $h=f_{\alpha\beta}$ and $U=Q_\alpha$, $V=Q_\beta$, hence $[Q]\cap(h\,``[Q\restriction_s])=\beta^\frown([V]\cap(f\,``[U\restriction_t]))$. Thus the set $[V]\cap(f\,``[U\restriction_t])$ is bounded, therefore $t\in\mathbb{H}(U,f,V)$.

Case 2. $h\restriction[Q\restriction_s]$ is a total identity, $h(x)=x$ whenever $x\in Q\restriction_s$. Then $\beta=\alpha$, $U=V$, and $f\restriction[U\restriction_t]$ is a total identity, thus still $t\in\mathbb{H}(U,f,V)$. $\dashv$

§11. Blocking sequences and the forcing. *We argue in the constructible universe* $\mathbf{L}$ in this section.

The forcing to prove Theorem 1.1 will be defined as the union of a ω_1-sequence of WT-forcings of size $\aleph_1$, increasing in the sense of a relation $\sqsubset$ (Definition 10.1). We here introduce the notational system to be used in this construction.

DEFINITION 11.1. Let $\mathbb{WTF}$ be the set of all WT-forcings of cardinality $\leq\aleph_1$.

If $\overline{\mathbf{P}} = \langle \mathbf{P}_\alpha \rangle_{\alpha<\lambda}$ is a sequence of forcings $\mathbf{P}_\alpha \in \mathbb{WTF}$, then let $\mathfrak{M}(\overline{\mathbf{P}})$ be the least transitive model of $\mathbf{ZFC}^-$ of the form $\mathbf{L}_\vartheta$, containing $\overline{\mathbf{P}}$, in which both λ and the set $\bigcup \overline{\mathbf{P}} = \bigcup_{\alpha<\lambda} \mathbf{P}_\alpha$ are of cardinality $\leq\aleph_1$.

If $\lambda \leq \omega_2$ then let $\overline{\mathbb{WTF}}_\lambda$ be the set of all λ-sequences $\overline{\mathbf{P}} = \langle \mathbf{P}_\alpha \rangle_{\alpha<\lambda}$ of forcings $\mathbf{P}_\alpha \in \mathbb{WTF}$, satisfying the following:

($*$) if $\gamma < \lambda$ then $\bigcup(\overline{\mathbf{P}}\restriction\gamma) \sqsubset_{\mathfrak{M}(\overline{\mathbf{P}}\restriction\gamma)} \mathbf{P}_\gamma$.

Let $\overline{\mathbb{WTF}} = \bigcup_{\lambda<\omega_2} \overline{\mathbb{WTF}}_\lambda$.

The set $\overline{\mathbb{WTF}} \cup \overline{\mathbb{WTF}}_{\omega_2}$ is ordered by the end-extension relations $\subset$, $\subseteq$.

LEMMA 11.2. *Assume that* $\kappa < \lambda < \omega_2$, *and* $\overline{\mathbf{P}} = \langle \mathbf{P}_\alpha \rangle_{\alpha<\kappa} \in \overline{\mathbb{WTF}}$. *Then*:

(i) *the union* $\mathbf{P} = \bigcup \overline{\mathbf{P}}$ *belongs to* $\mathbb{WTF}$;

(ii) *there is a sequence* $\overline{\mathbf{Q}} \in \overline{\mathbb{WTF}}$ *such that* $\mathtt{dom}(\overline{\mathbf{Q}}) = \lambda$ *and* $\overline{\mathbf{P}} \subset \overline{\mathbf{Q}}$.

PROOF. To prove (ii) apply Theorem 10.4 by induction on λ. ⊣

DEFINITION 11.3 (Key definition). A sequence $\overline{\mathbf{P}} \in \overline{\mathbb{WTF}}$ *blocks* a set $W \subseteq \overline{\mathbb{WTF}}$ if either $\overline{\mathbf{P}} \in W$ or there is no sequence $\overline{\mathbf{Q}} \in W$ satisfying $\overline{\mathbf{P}} \subseteq \overline{\mathbf{Q}}$.

11.1. Sets $\mathbf{H}_\kappa$ and definability classes. Recall that $\mathbf{H}_\kappa$ is the set of all sets *hereditarily of cardinality* $<\kappa$. Thus $x \in \mathbf{H}_\kappa$ if $\mathtt{card}(\mathrm{TC}(x)) < \kappa$. In particular $\mathrm{HC} = \mathbf{H}_{\omega_1}$ is the set of all *hereditarily countable* sets, while $\mathbf{H}\omega_2$ is the set of all sets *hereditarily of cardinality* $\leq\aleph_1$; $\mathrm{HC} = \mathbf{L}_{\omega_1}$ and $\mathbf{H}\omega_2 = \mathbf{L}_{\omega_2}$ in $\mathbf{L}$.

$\varSigma_n(\mathbf{H}_\kappa)$, resp., $\varSigma_n^{\mathbf{H}_\kappa}$ is the class of all sets $X \subseteq \mathbf{H}_\kappa$, definable in $\mathbf{H}_\kappa$ by a $\varSigma_n$ formula with parameters in $\mathbf{H}_\kappa$, resp., with no parameters. The classes $\varPi_n(\mathbf{H}_\kappa)$, $\varPi_n^{\mathbf{H}_\kappa}$ have the same meaning (with $\varPi_n$ formulas), and $\varDelta_n(\mathbf{H}_\kappa) = \varSigma_n(\mathbf{H}_\kappa) \cap \varPi_n(\mathbf{H}_\kappa)$, $\varDelta_n^{\mathbf{H}_\kappa} = \varSigma_n^{\mathbf{H}_\kappa} \cap \varPi_n^{\mathbf{H}_\kappa}$, as usual. In particular, $\varDelta_0(\mathbf{H}_\kappa) = \varSigma_0(\mathbf{H}_\kappa) = \varPi_0(\mathbf{H}_\kappa)$ and $\varDelta_0^{\mathbf{H}_\kappa} = \varSigma_0^{\mathbf{H}_\kappa} = \varPi_0^{\mathbf{H}_\kappa}$ (definability by bounded formulas, with/without parameters). See more on $\in$-definability in [3, Part B, Chapter 5, Section 4] or elsewhere.

In particular, we consider the classes $\varSigma_n^{\mathbf{H}\omega_2}$, $\varPi_n^{\mathbf{H}\omega_2}$, $\varDelta_n^{\mathbf{H}\omega_2}$ of definability in $\mathbf{H}\omega_2$ (parameters not allowed) and $\varSigma_n(\mathbf{H}\omega_2)$, $\varPi_n(\mathbf{H}\omega_2)$, $\varDelta_n(\mathbf{H}\omega_2)$ (all parameters in $\mathbf{H}\omega_2$ allowed)—this is the case $\kappa = \aleph_2$ in the above definitions.

THEOREM 11.4 (The blocking sequence theorem, in $\mathbf{L}$). *Let* $\mathfrak{n} \geq 2$. *There exists a sequence* $\overline{\mathbb{P}} = \langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_2} \in \overline{\mathbb{WTF}}_{\omega_2}$ *satisfying the following two conditions*:

(i) $\overline{\mathbb{P}}$, *as the set of pairs* $\langle \alpha, \mathbb{P}_\alpha \rangle$, *belongs to the definability class* $\varDelta_{\mathfrak{n}-1}^{\mathbf{H}\omega_2}$;

(ii) *if* $\mathfrak{n} \geq 3$ *and* $W \subseteq \overline{\mathbb{WTF}}$ *is a* $\varSigma_{\mathfrak{n}-2}(\mathbf{H}\omega_2)$ *set then there is an ordinal* $\gamma < \omega_2$ *such that the restricted sequence* $\overline{\mathbb{P}}\restriction\gamma = \langle \mathbb{P}_\alpha \rangle_{\alpha<\gamma} \in \overline{\mathbb{WTF}}$ *blocks* W.

PROOF. Let $\leq_{\mathbf{L}}$ be the canonical $\varDelta_1$ wellordering of $\mathbf{L}$; thus its restriction to $\mathbf{H}\omega_2 = \mathbf{L}_{\omega_2}$ is $\varDelta_1^{\mathbf{H}\omega_2}$. As $\mathfrak{n} \geq 3$, there exists a universal $\varSigma_{\mathfrak{n}-2}^{\mathbf{H}\omega_2}$ set $\mathfrak{U}^{\mathfrak{n}} \subseteq \omega_2 \times \mathbf{H}\omega_2$. That is, $\mathfrak{U}^{\mathfrak{n}}$ is $\varSigma_{\mathfrak{n}-2}^{\mathbf{H}\omega_2}$ (parameter-free $\varSigma_{\mathfrak{n}-2}$ definable in $\mathbf{H}\omega_2$), and for every set $X \subseteq \mathbf{H}\omega_2$ of type $\varSigma_{\mathfrak{n}-2}(\mathbf{H}\omega_2)$ ($\varSigma_{\mathfrak{n}-2}$ definable in $\mathbf{H}\omega_2$ with arbitrary parameters) there is an ordinal $\delta < \omega_1$ such that $X = \mathfrak{U}^{\mathtt{n}}_{\delta} = \{x : \langle \delta, x \rangle \in \mathfrak{U}^{\mathtt{n}}\}$. The choice of ω_2 as the domain of parameters is validated by the assumption $\mathbf{V} = \mathbf{L}$, which implies the existence of a $\Delta_1^{\mathbf{H}\omega_2}$ surjection $\omega_2 \xrightarrow{\text{onto}} \mathbf{H}\omega_2$.

Coming back to Definition 11.3, note that for any sequence $\overline{\mathbf{P}} \in \overline{\mathbb{WTF}}$ and any set $W \subseteq \overline{\mathbb{WTF}}$ there is a sequence $\overline{\mathbf{Q}} \in \overline{\mathbb{WTF}}$ which satisfies $\overline{\mathbf{P}} \subset \overline{\mathbf{Q}}$ and blocks W. This allows us to define $\overline{\mathbf{Q}}_\alpha \in \overline{\mathbb{WTF}}$ by induction on $\alpha < \omega_1$ so that $\overline{\mathbf{Q}}_0 = \varnothing$, $\overline{\mathbf{Q}}_\lambda = \bigcup_{\alpha<\lambda} \overline{\mathbf{Q}}_\alpha$, and each $\overline{\mathbf{Q}}_{\alpha+1}$ is equal to the $\leq_{\mathbf{L}}$-least sequence $\overline{\mathbf{Q}} \in \overline{\mathbb{WTF}}$ which satisfies 1) $\overline{\mathbf{Q}}_\alpha \subset \overline{\mathbf{Q}}$ and 2) if $\mathtt{n} \geq 3$ then $\overline{\mathbf{Q}}$ blocks $\mathfrak{U}^{\mathtt{n}}_\alpha$.

Then $\overline{\mathbb{P}} = \bigcup_{\alpha<\omega_2} \overline{\mathbf{Q}}_\alpha \in \overline{\mathbb{WTF}}_{\omega_2}$. Condition (ii) holds by construction, while (i) follows by a routine verification, based on the fact that $\overline{\mathbb{WTF}} \in \Delta_1^{\mathbf{H}\omega_2}$ and $\mathfrak{U}^{\mathtt{n}} \in \Sigma_{\mathtt{n}-2}^{\mathbf{H}\omega_2}$ (provided $\mathtt{n} \geq 3$). $\dashv$

DEFINITION 11.5 (In $\mathbf{L}$). We fix a sequence $\overline{\mathbb{P}} = \langle \mathbb{P}_\alpha \rangle_{\alpha<\omega_2} \in \overline{\mathbb{WTF}}_{\omega_2}$, given by Theorem 11.4 for the number $\mathtt{n} \geq 2$, for which Theorem 1.1 is to be proved.

In particular $\overline{\mathbb{P}}$ satisfies (i) and (ii) of Theorem 11.4.

If $\gamma < \omega_2$ then let $\mathfrak{M}_\gamma = \mathfrak{M}(\overline{\mathbb{P}} \restriction \gamma)$, and $\mathbb{P}_{<\gamma} = \bigcup_{\alpha<\gamma} \mathbb{P}_\alpha$, $\mathbb{P} = \bigcup_{\alpha<\omega_2} \mathbb{P}_\alpha$.

§12. Some forcing properties. The WT-forcing $\mathbb{P} \in \mathbf{L}$ defined by 11.5 will be the forcing notion for the proof of Theorem 1.1. The next lemma presents some properties of $\mathbb{P}$. *We continue to argue in* $\mathbf{L}$ *under the conditions and following notation of Definition 11.5.*

LEMMA 12.1. *$\mathbb{P}$ is a WT-forcing, all sets $\mathbb{P}_\alpha$, $\mathbb{P}_{<\gamma}$ belong to $\mathbb{WTF}$.*
In addition:

(i) *if $\alpha < \omega_2$ then $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbb{P}_\gamma$;*
(ii) *if $\alpha < \omega_2$ and the set $D \in \mathfrak{M}_\alpha$, $D \subseteq \mathbb{P}_{<\alpha}$ is predense in $\mathbb{P}_{<\alpha}$ then it is predense in $\mathbb{P}$, too;*
(iii) *every set $\mathbb{P}_\alpha$ is predense in $\mathbb{P}$;*
(iv) *if $\alpha < \omega_2$ and trees $T, T' \in \mathbb{P}_{<\alpha}$ are incompatible in $\mathbb{P}_{<\alpha}$ then T, T' are incompatible in $\mathbb{P}$, too;*
(v) *if $f : {\omega_1}^\omega \to {\omega_1}^\omega$ is continuous then the set of all trees $T \in \mathbb{P}$ such that $f\,``[T]$ is bounded or $f \restriction [T]$ is a bijection, is dense in $\mathbb{P}$;*
(vi) *if $f : {\omega_1}^\omega \to {\omega_1}^\omega$ is continuous then the set of all trees $T \in \mathbb{P}$ such that*:
  (1) *$f \restriction [T]$ is a total identity,* **or**,
  (2) *for some $\gamma < \omega_2$, $f \restriction [T]$* **avoids** *$\mathbb{P}_\gamma$ in the sense that if $V \in \mathbb{P}_\gamma$ then the subset $\{s \in T : [V] \cap (f\,``[T \restriction_s]$ is bounded$\}$ weakly covers T,*

  is dense in $\mathbb{P}$;
(vii) (very important!) *if $\mathtt{n} \geq 3$ and a set $Q \subseteq \mathbf{WT}$ belongs to $\Sigma_{\mathtt{n}-2}(\mathbf{H}\omega_2)$, then $\mathbb{P} \cap (Q \cup Q^-)$ is dense in $\mathbb{P}$, where $Q^- = \{T \in \mathbf{WT} : \neg \exists S \in Q\,(S \subseteq T)\}$.*

PROOF. (i) holds by $(*)$ of Definition 11.1.

(ii) We use induction on γ, $\alpha \leq \gamma < \omega_2$, to check that if D is predense in $\mathbb{P}_{<\gamma}$ then it remains predense in $\mathbb{P}_{<\gamma} \cup \mathbb{P}_\gamma = \mathbb{P}_{<\gamma+1}$ by (i) and Lemma 10.3(i). Limit steps, including the final step to $\mathbb{P}$ ($\gamma = \omega_2$) are routine.

(iii) $\mathbb{P}_\alpha$ is dense in $\mathbb{P}_{<\alpha+1} = \mathbb{P}_{<\alpha} \cup \mathbb{P}_\alpha$ by 10.1(A). It remains to refer to (ii).

(vi) Prove by induction on γ that if $\alpha < \gamma \leq \omega_1$ then T, T' are incompatible in $\mathbb{P}_{<\gamma}$, using (i) and Lemma 10.3(ii).

To prove (v) and (vi) let $T_0 \in \mathbb{P}$. There is an ordinal $\gamma < \omega_2$ such that $T_0 \in \mathbb{P}_{<\gamma}$ and $f \in \mathfrak{M}_\gamma$. We have $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbb{P}_\gamma$ by (i). Therefore by (E) of Definition 10.1 there is a tree $T \in \mathbb{P}_\gamma$ such that $T \subseteq T_0$ and $f\,"[T]$ is bounded or $f \restriction [T]$ is a bijection, so we get (v). Further by (F) of Definition 10.1 if $V \in \mathbb{P}_\gamma$ then the set $\mathbb{H}(T, f, V)$, of all strings $s \in T$ such that $[V] \cap (f\,"[T \restriction_s])$ is bounded or $f \restriction [T \restriction_s]$ is a total identity, weakly covers T. We have two cases.

Case 1. $f \restriction [T \restriction_s]$ is a total identity for at least one $s \in T$. Then the corresponding subtree $T' = T \restriction_s$ satisfies (vi)(1).

Case 2. for each $V \in \mathbb{P}_\gamma$, the set $H(V)$ of all strings $s \in T$ such that $[V] \cap (f\,"[T \restriction_s])$ is bounded, weakly covers T, thus T itself satisfies (vi)(2).

(vii) Suppose that $\mathfrak{n} \geq 3$. Let $T_0 \in \mathbb{P}$, that is, $T_0 \in \mathbb{P}_{<\alpha_0}$, $\alpha_0 < \omega_2$. The set W of all sequences $\overline{\mathbf{P}} \in \overline{\mathbb{WTF}}$, such that $\overline{\mathbb{P}} \restriction \alpha_0 \subseteq \overline{\mathbf{P}}$ and $\exists\, T \in Q \cap (\bigcup \overline{\mathbf{P}})\,(T \subseteq T_0)$, belongs to $\Sigma_{\mathfrak{n}-2}(\mathbf{H}\omega_2)$ along with Q. Therefore there is an ordinal $\alpha < \omega_2$ such that $\overline{\mathbb{P}} \restriction \alpha$ blocks W. We have two cases.

Case 1. $\overline{\mathbb{P}} \restriction \alpha \in W$. Then the related tree $T \subseteq T_0$ belongs to $Q \cap \mathbb{P}$.

Case 2. there is no sequence in W which extends $\overline{\mathbb{P}} \restriction \alpha$. Let $\gamma = \mathtt{max}\{\alpha, \alpha_0\}$. Then $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbb{P}_\gamma$ by (i). As $\alpha_0 \leq \gamma$, there is a tree $T \in \mathbb{P}_\gamma$, $T \subseteq T_0$. We claim that $T \in Q^-$, which completes the proof in Case 2.

Suppose to the contrary that $T \notin Q^-$, thus there is a tree $S \in Q$, $S \subseteq T$. The set $\mathbf{R} = \mathbb{P}_\gamma \cup \{S \restriction_t : t \in S\}$ is a WT-forcing and obviously $\mathbb{P}_\gamma \sqsubset \mathbf{R}$, hence still $\mathbb{P}_{<\gamma} \sqsubset_{\mathfrak{M}_\gamma} \mathbf{R}$ holds by Lemma 10.2. It follows that the sequence $\overline{\mathbf{R}}$ defined by $\mathtt{dom}\,\overline{\mathbf{R}} = \gamma + 1$, $\overline{\mathbf{R}} \restriction \gamma = \overline{\mathbb{P}} \restriction \gamma$, and $\overline{\mathbf{R}}(\gamma) = \mathbf{R}$, belongs to $\overline{\mathbb{WTF}}$, and even $\overline{\mathbf{R}} \in W$ since $S \in Q \cap \mathbf{R}$. Yet $\overline{\mathbb{P}} \restriction \alpha \subset \overline{\mathbf{R}}$, which contradicts to the Case 2 hypothesis. ⊣

To prove a chain condition for $\mathbb{P}$, we'll need the following general lemma. See Definition 11.5 on models $\mathfrak{M}_\alpha$.

Lemma 12.2 (In $\mathbf{L}$). *If $X \subseteq \mathbf{H}\omega_2 = \mathbf{L}_{\omega_2}$ then the set $\mathscr{O}_X$ of all ordinals $\alpha < \omega_2$ such that the model $\langle \mathbf{L}_\alpha ; X \cap \mathbf{L}_\alpha \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_2} ; X \rangle$ and $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$, is unbounded in ω_2.*

Proof. Let $\alpha_0 < \omega_2$. There is an elementary submodel M of $\langle \mathbf{L}_{\omega_3} ; \in \rangle$, of cardinality $\mathtt{card}\, M = \aleph_1$, which contains α_0, ω_2, X and is such that the set $M \cap \mathbf{L}_{\omega_2}$ is transitive. Consider the Mostowski collapse $\phi : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$. Let $\alpha = \phi(\omega_2)$. Then $\alpha_0 < \alpha < \lambda < \omega_2$ and $\phi(X) = X \cap \mathbf{L}_\alpha$ by the choice of M. We conclude that $\langle \mathbf{L}_\alpha ; X \cap \mathbf{L}_\alpha \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_2} ; X \rangle$. And $\mathtt{card}\, \alpha > \aleph_1$ in $\mathbf{L}_\lambda$, hence $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\alpha$. Then $X \cap \mathbf{L}_\alpha \in \mathfrak{M}_\alpha$, as $X \cap \mathbf{L}_\alpha \in \mathbf{L}_\lambda$ by construction. ⊣

Corollary 12.3 (In $\mathbf{L}$). (i) *If $A \subseteq \mathbb{P}$ is an antichain then $\mathtt{card}\, A \leq \aleph_1$.*

(ii) *Let $D_n \subseteq \mathbb{P}$ be predense in $\mathbb{P}$, for each n. Then the set of all trees $T \in \mathbb{P}$, satisfying $\forall\, n\, (\mathbf{BN}_n(T) \subseteq D_n {\Uparrow} T)$, is dense in $\mathbb{P}$.*

Proof. (i) Let $A \subseteq \mathbb{P}$ be a maximal antichain. By Lemma 12.2 there is an ordinal α such that $\langle \mathbf{L}_\alpha ; \mathbb{P}', A' \rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_2} ; \mathbb{P}, A \rangle$, where $\mathbb{P}' = \mathbb{P} \cap \mathbf{L}_\alpha$ and $A' = A \cap \mathbb{P}_{<\alpha}$, and in addition $\mathbb{P}', A' \in \mathfrak{M}_\alpha$. By the elementarity, we have $\mathbb{P}' = \mathbb{P}_{<\alpha}$ and $A' = A \cap \mathbb{P}_{<\alpha} \in \mathfrak{M}_\alpha$, and A' is a maximal antichain, hence a predense set, in $\mathbb{P}_{<\alpha}$. But then A' is a predense set, hence, a maximal antichain, in the whole set $\mathbb{P}$ by Lemma 12.1(ii). Thus $A = A'$, and $\mathtt{card}\, A = \mathtt{card}\, A' \leq \aleph_1$.

(ii) We wlog assume that all D_n are open dense, for if not then replace D_n by the set $\{T \in \mathbb{P} : \exists S \in D_n\,(T \subseteq S)\}$. Let $T_0 \in \mathbb{P}$. Pick a maximal antichain $A_n \subseteq D_n$ in each D_n. Then all sets A_n are maximal antichains in $\mathbb{P}$ by the open density, and $\mathtt{card}\, A_n \le \aleph_1$ by (i). Therefore there is an ordinal $\alpha < \omega_2$ such that the set $A = \bigcup_n A_n$ satisfies $A \subseteq \mathbb{P}_{<\alpha}$ and A, T_0, and the sequence $\langle A_n \rangle_{n<\omega}$ belong to $\mathfrak{M}_\alpha$. By the maximality of D_n and Lemma 12.1(iv), each $D'_n = D_n \cap \mathbb{P}_{<\alpha}$ is dense in $\mathbb{P}_{<\alpha}$. It follows by Lemma 12.1(i) and (D) of Definition 10.1 that there is a tree $T \in \mathbb{P}_\alpha$ such that $T \subseteq T_0$ and $\mathbf{BN}_n(T) \subseteq D_n \Uparrow T$ for all n. ⊣

§13. **The model.** This section presents some key properties of $\mathbb{P}$-generic extensions $\mathbf{L}[G]$ of $\mathbf{L}$ obtained by adjoining a $\mathbb{P}$-generic set $G \subseteq \mathbb{P}$ to $\mathbf{L}$. Recall that the forcing notion $\mathbb{P} \in \mathbf{L}$ was introduced by Definition 11.5, along with some related notation.

Corollary 13.1. *If a set $G \subseteq \mathbb{P}$ is $\mathbb{P}$-generic over $\mathbf{L}$ then $\omega_1^{\mathbf{L}} < \omega_1^{\mathbf{L}[G]} = \omega_2^{\mathbf{L}}$.*

Proof. That $\omega_1^{\mathbf{L}} < \omega_1^{\mathbf{L}[G]}$ follows from the fact that $\mathbf{a}[G]$ is a cofinal map $\omega \to \omega_1^{\mathbf{L}}$. To prove $\omega_1^{\mathbf{L}[G]} = \omega_2^{\mathbf{L}}$ use Corollary 12.3. ⊣

Blanket agreement 13.2. Arguing in generic extensions of $\mathbf{L}$, we'll use standard notation like $\omega_\xi^{\mathbf{L}}$ to denote $\mathbf{L}$-cardinals. We also use $(\mathbf{WT})^{\mathbf{L}}$ to denote "the set $\mathbf{WT}$ defined in $\mathbf{L}$". Thus for instance $\mathbb{P} \subseteq (\mathbf{WT})^{\mathbf{L}}$.

We'll make use of a *coding system for continuous maps*, helpful whenever "the same" continuous $f : {\omega_1}^\omega \to \mathbf{Ord}^\omega$ is considered in different models.

Definition 13.3. Let $\vartheta \in \mathbf{Ord}$. A *code of continuous function* from $(\omega_1^{\mathbf{L}})^\omega$ to ϑ^ω is any map $\mathbf{c} : \mathtt{dom}\,\mathbf{c} \to \vartheta$ with $\mathtt{dom}\,\mathbf{c} \subseteq (\omega_1^{\mathbf{L}})^{<\omega} \times \omega$, such that the sets $S^{\mathbf{c}}_{n\xi} = \{s \in (\omega_1^{\mathbf{L}})^{<\omega} : \langle s, n\rangle \in \mathtt{dom}\,\mathbf{c} \wedge \mathbf{c}(s,n) = \xi\}$ satisfy the following for any n:

(1) if $\xi \ne \eta$, $u \in S^{\mathbf{c}}_{n\xi}$, $v \in S^{\mathbf{c}}_{n\eta}$, then u, v are incompatible, and

(2) $S^{\mathbf{c}}_n = \bigcup_\xi S^{\mathbf{c}}_{n\xi}$ is a belt for $(\omega_1^{\mathbf{L}})^{<\omega}$, *i.e.*, $\forall\, x \in (\omega_1^{\mathbf{L}})^\omega\, \exists\, m\,(x {\restriction} m \in S^{\mathbf{c}}_n)$.

Let $\mathbf{CCF}_\vartheta$ be the set of all such codes. If $\mathbf{c} \in \mathbf{CCF}_\vartheta$ then a continuous $f_{\mathbf{c}} : (\omega_1^{\mathbf{L}})^\omega \to \vartheta^\omega$ is defined as follows. If $x \in (\omega_1^{\mathbf{L}})^\omega$ and $n < \omega$, then by definition there is a unique $\xi < \vartheta$ such that $x {\restriction} k \in S^{\mathbf{c}}_{n\xi}$ for some k. Let $f_{\mathbf{c}}(x)(n) = \xi$.

If $f : (\omega_1^{\mathbf{L}})^\omega \to \vartheta^\omega$ is continuous then its code $\mathbf{c} = \mathbf{code}(f) \in \mathbf{CCF}_\vartheta$ is defined by $S^{\mathbf{c}}_{n\xi} = \{s \in (\omega_1^{\mathbf{L}})^{<\omega} : \forall\, x \in (\omega_1^{\mathbf{L}})^\omega\,(s \subset x \implies f(x)(n) = \xi)\}$; then $f_{\mathbf{c}} = f$.

Remark 13.4 (Absoluteness). Being a code in $\mathbf{CCF}_\vartheta$ is absolute since so is the condition of being a belt, see Remark 9.3.

Lemma 13.5. *If $G \subseteq \mathbb{P}$ is generic over $\mathbf{L}$, $\vartheta \in \mathbf{Ord}$, $y \in \vartheta^\omega \cap \mathbf{L}[G]$, then*

(i) *there is a code $\mathbf{c} \in \mathbf{CCF}_\vartheta \cap \mathbf{L}$ such that $y = f_{\mathbf{c}}(\mathbf{a}[G])$;*

(ii) *if $\vartheta = \omega_1^{\mathbf{L}}$ then y is bounded in $\omega_1^{\mathbf{L}}$ or $G \in \mathbf{L}[y]$;*

(iii) *if $\vartheta = \omega_1^{\mathbf{L}}$ and y is unbounded in $\omega_1^{\mathbf{L}}$ then $y = \mathbf{a}[G]$ or there is an ordinal $\gamma < \omega_2^{\mathbf{L}}$ such that $y \notin \bigcup_{V \in \mathbb{P}_\gamma} [V]$.*

Proof. (i) There is a $\mathbb{P}$-name $t \in \mathbf{L}$ satisfying $y = t[G]$ (the G-valuation of t). It can be assumed that $\mathbb{P}$ forces that t is valuated as an element of ϑ^ω.

Arguing in L, let $\tau_{n\xi} = \{T \in \mathbb{P} : T \text{ forces } t(n) = \xi\}$ ($n < \omega$ and $\xi < \vartheta$). The sets $\tau_n = \bigcup_\xi \tau_{n\xi}$ are open dense in $\mathbb{P}$. It follows by Corollary 12.3(ii) that there is a tree $T \in G$ such that $T\restriction_s \in \tau_n$ whenever $n < \omega$ and $s \in \mathbf{BN}_n(T)$. This allows us to define, **still in L**, a continuous $f' : [T] \to \vartheta^\omega$ by $f'(x)(n) = \xi$ iff the only string $s \in \mathbf{BN}_n(T)$ with $s \subset x$ belongs to $\tau_{n\xi}$. Let $f : \omega_1{}^\omega \to \vartheta^\omega$ be a continuous extension of f'. Then $\mathbf{c} = \mathbf{code}(f) \in \mathbf{CCF}_\vartheta \cap \mathbf{L}$, and easily $y = f_\mathbf{c}(\mathbf{a}[G])$.

(ii) Let, by (i), $\mathbf{c} \in \mathbf{CCF}_{\omega_1^\mathbf{L}} \cap \mathbf{L}$ and $y = f_\mathbf{c}(\mathbf{a}[G])$. By Lemma 12.1(v), there is a tree $T \in G$ such that, in $\mathbf{L}$, $f_\mathbf{c}``[T]$ is bounded or $f_\mathbf{c}\restriction[T]$ is a bijection.

Case 1. in $\mathbf{L}$, $f_\mathbf{c}``[T]$ is bounded, that is, there is an ordinal $\beta < \omega_1^\mathbf{L}$ satisfying $f_\mathbf{c}(x) \in \beta^\omega$ for all $x \in [T] \cap \mathbf{L}$. But $f_\mathbf{c}$ is continuous while $[T] \cap \mathbf{L}$ is dense in $[T]$ in $\mathbf{L}[G]$. It follows that $f_\mathbf{c}(x) \in \beta^\omega$ for all $x \in [T] \cap \mathbf{L}[G]$. In particular $y = f_\mathbf{c}(\mathbf{a}[G]) \in \beta^\omega$ since $\mathbf{a}[G] \in [T]$ (because $T \in G$), so y is bounded.

Case 2. in $\mathbf{L}$, $f_\mathbf{c}\restriction[T]$ is a bijection. The bijectivity is equivalent to the well-foundedness of the tree $W_\mathbf{c}$ of all pairs $\langle s, t\rangle$ of strings $s, t \in T$ such that $\mathtt{lh}(s) = \mathtt{lh}(t)$ and there exist no strings u, v satisfying: $u \subseteq s$, $v \subseteq t$, and $u \in S^\mathbf{c}_{n\xi}$, $v \in S^\mathbf{c}_{n\eta}$ for some n and $\xi \neq \eta$. Therefore the bijectivity of $f_\mathbf{c}\restriction[T]$ is an absolute property of $\mathbf{c}, T$. Thus $f_\mathbf{c}\restriction[T]$ is a bijection in $\mathbf{L}[G]$, and we have $\mathbf{a}[G] = f_\mathbf{c}^{-1}(y) \in \mathbf{L}[y]$, as required.

(iii) We still assume that, by (i), $y = f_\mathbf{c}(\mathbf{a}[G])$, where $\mathbf{c} \in \mathbf{CCF}_{\omega_1^\mathbf{L}} \cap \mathbf{L}$. By Lemma 12.1(vi), there is a tree $T \in G$ such that, in $\mathbf{L}$, $f_\mathbf{c}\restriction[T]$ is a total identity or, for some $\gamma < \omega_2^\mathbf{L}$, $f_\mathbf{c}\restriction[T]$ avoids $\mathbb{P}_\gamma$ in the sense of 12.1(vi).

Case 1. in $\mathbf{L}$, $f_\mathbf{c}\restriction[T]$ is a total identity, that is, $f_\mathbf{c}(x) = x$ for all $x \in [T] \cap \mathbf{L}$. By the same simple continuity/density argument, we have $f_\mathbf{c}(x) = x$ for all $x \in [T] \cap \mathbf{L}[G]$, in particular $y = f_\mathbf{c}(\mathbf{a}[G]) = \mathbf{a}[G]$.

Case 2. $\gamma < \omega_2^\mathbf{L}$ and, in $\mathbf{L}$: $f_\mathbf{c}\restriction[T]$ avoids $\mathbb{P}_\gamma$, that is, if $V \in \mathbb{P}_\gamma$ then the subset $T(V) = \{s \in T : [V] \cap (f``[T\restriction_s]) \text{ is bounded}\}$ (defined in $\mathbf{L}$) weakly covers T. Now let $V \in \mathbb{P}_\gamma$ and check that $y \notin [V]$. By the Case 2 assumption, $T(V)$ weakly covers T. Therefore, as $\mathbf{a}[G] \in [T]$ is definitely unbounded, there is a string $s \in T(V)$ satisfying $s \subseteq \mathbf{a}[G]$. Then $S = T\restriction_s \in G$ and $[V] \cap (f``[S])$ is bounded, so that there is an ordinal $\beta < \omega_1^\mathbf{L}$ satisfying: if $x \in [S] \cap \mathbf{L}$ and $f_\mathbf{c}(x) \in [V]$ then $f_\mathbf{c}(x) \in \beta^\omega$. We claim that *the implication $f_\mathbf{c}(x) \in [V] \implies f_\mathbf{c}(x) \in \beta^\omega$ also holds for all $x \in [S] \cap \mathbf{L}[G]$*. Assume that this is established. As $x = \mathbf{a}[G] \in [S]$ (because $S \in G$), we then have $y \in [V] \implies y \in \beta^\omega$. (Recall that $y = f_\mathbf{c}(\mathbf{a}[G])$.) But y is unbounded, hence $y \notin [V]$, as required.

To prove the claim, let $x_0 \in [S] \cap \mathbf{L}[G]$ be a counterexample, so $y_0 = f_\mathbf{c}(x_0) \in [V]$ but $y_0(n_0) = \xi$ for some n_0 and $\xi \geq \beta$. The existence of such x_0 is equivalent to the non–well-foundedness of the tree W of all strings $s \in S$ such that $s \in S^\mathbf{c}_{n_0\eta}$ for all $\eta \neq \xi$, and there is no string $u \notin V$ satisfying: $\forall\, j < \mathtt{lh}(u)\, (s \in S^\mathbf{c}_{ju(j)})$. Therefore the existence of x_0 is an absolute property of $\mathbf{c}, S, V$. Thus such an $x_0 \in [S]$ exists already in $\mathbf{L}$, contrary to the Case 2 assumption. ⊣

Corollary 13.6. *Let $G \subseteq \mathbb{P}$ be generic over* $\mathbf{L}$. *Then it holds in* $\mathbf{L}[G]$ *that*

(i) $\mathbf{a}[G]$ *is the only member of the intersection* $\bigcap_{\gamma<\omega_1} \bigcup_{T\in\mathbb{P}_\gamma} [T]$;
(ii) *the set* $\{\mathbf{a}[G]\}$ *is a* $\Pi^{\mathrm{HC}}_{\mathfrak{n}-1}$ *singleton*;
(iii) *there is a* $\Pi^1_\mathfrak{n}$ *real singleton* $\{r\}$, $r \in \omega^\omega$, *such that* $\mathbf{L}[r] = \mathbf{L}[\mathbf{a}[G]]$.

PROOF. (i) Every set $\mathbb{P}_\gamma$ is predense in $\mathbb{P}$ by Lemma 12.1(iii). It is implied by the genericity that $\mathbf{a}[G] \in \bigcup_{T\in\mathbb{P}_\gamma}[T]$. The uniqueness follows from Lemma 13.5(iii).

(ii) The sequence $\overline{\mathbb{P}} = \{\langle\gamma, \mathbb{P}_\gamma\rangle : \gamma < \omega_2^{\mathbf{L}}\}$ is of type $\Delta^{\mathbf{H}\omega_2}_{\mathfrak{n}-1}$ in $\mathbf{L}$ by Definition 11.5. However $\mathbf{H}\omega_2$ in the sense of $\mathbf{L}$ coincides with the constructible part of HC (=hereditarily countable sets) in the sense of $\mathbf{L}[G]$, because $\omega_1^{\mathbf{L}[G]} = \omega_2^{\mathbf{L}}$ by Corollary 13.1. It easily follows that $\overline{\mathbb{P}}$ is $\Delta^{\mathrm{HC}}_{\mathfrak{n}-1}$ in $\mathbf{L}[G]$. On the other hand,

$$\{\mathbf{a}[G]\} = \{x : \forall\,\gamma\,\forall\,\mathbb{Q}\,(\langle\gamma,\mathbb{Q}\rangle \in \overline{\mathbb{P}} \Longrightarrow \exists\, T \in \mathbb{Q}\,(x \in [T])\}$$

by (i). This yields the result since $\exists\, T \in \mathbb{Q}$ is a bounded quantifier.

(iii) If $r \in \omega^\omega$ then let $(r)_n(k) = r(2^n(2k+1)-1)$, thus $(r)_n \in \omega^\omega$. Let W be the Π^1_1 set of all reals which code an ordinal, and let $|w| < \omega_1$ be the ordinal coded by $w \in W$. Let $r \in \omega^\omega$ be defined so that each $(r)_n$ belongs to $W \cap \mathbf{L}$ and is $\le_{\mathbf{L}}$-minimal of all $w \in W \cap \mathbf{L}$ satisfying $|w| = \mathbf{a}[G](n)$. Thus r is a real in $\mathbf{L}[G]$. The singleton $\{r\}$ is defined in HC of $\mathbf{L}[G]$ by the following formula:

$\forall\, n$, $(r)_n \in W \cap \mathbf{L}$ and $(r)_n$ is $\le_{\mathbf{L}}$-minimal of all $w \in W \cap \mathbf{L}$ with $|w| = |(r)_n|$, and $\forall\, x \in \mathbf{Ord}^\omega\,(\forall\, n\,(x(n) = |(r)_n|) \Longrightarrow x = \mathbf{a}[G])$.

It easily follows by the result of (ii) that $\{r\}$ is a $\Pi^{\mathrm{HC}}_{\mathfrak{n}-1}$ singleton as well, hence a $\Pi^1_{\mathfrak{n}}$ singleton. ⊣

Corollary 13.1 and Corollary 13.6(ii), (iii) account for items (i), (ii), (iii) of Theorem 1.1. Item (iv) of the theorem is based on different ideas related to claim (vii) of Lemma 12.1. From now on we work towards this goal.

§14. Shoenfield's transformation of Σ^1_2 formulas. The following known transformation of Σ^1_2 formulas involves an idea in the proof of the Shoenfield absoluteness theorem. We present it here in a form useful for our purposes.

BLANKET AGREEMENT 14.1. From now on p, q, r denote reals in ω^ω.

THEOREM 14.2. *Let $\varphi(p_1,\dots,p_n)$ be a Σ^1_2 formula of the form*

$$\left.\begin{array}{l}\varphi(p_1,\dots,p_n) := \exists\, q\,\forall\, r\,\exists\, m\, R(q\restriction m, r\restriction m, p_1\restriction m,\dots,p_n\restriction m)\,,\\ \text{where } R \subseteq (\omega^{<\omega})^{n+2},\ R \in \mathbf{L}, \text{ and } q, r, p_i \text{ are variables over } \omega^\omega\end{array}\right\}, \tag{$*$}$$

and $\vartheta \ge \aleph_1^{\mathbf{L}}$ a cardinal in $\mathbf{L}$. Then there is a relation $Q = Q_\vartheta(R) \subseteq \vartheta^{<\omega} \times (\omega^{<\omega})^{n+1}$, $Q \in \mathbf{L}$, such that Q is $\Delta^{\mathbf{H}_\vartheta}_0(R)$ as a subset of $\mathbf{H}_\vartheta$ in $\mathbf{L}$[2]*, and it holds in any generic extension M of $\mathbf{L}$ with $\vartheta \ge \omega_1^M$ that: if $p_1,\dots,p_n \in \omega^\omega$ then*

$$\varphi(p_1,\dots,p_n) \iff \exists\,\chi \in \vartheta^\omega\,\exists\, q \in \omega^\omega\,\forall\, m\, Q(\chi\restriction m, q\restriction m, p_1\restriction m,\dots,p_n\restriction m)\,.$$

PROOF. $\varphi(p_1,\dots,p_n)$ is equivalent to $\exists\, q\,(W_{q,p_1,\dots,p_n}$ is wellfounded$)$, where

$$W_{q,p_1,\dots,p_n} = \{u \in \omega^{<\omega} : \forall\, j \le \mathtt{lh}(u)\,\neg\, R(q\restriction j, u\restriction j, p_1\restriction j,\dots,p_n\restriction j)\}\,,$$

hence—in any universe M as in the theorem—to the formula:

$$\exists\, q \in \omega^\omega\,\exists\, f : W_{q,p_1,\dots,p_n} \to \vartheta\,(f \text{ is order-preserving})\,.$$

[2] Meaning that the equality $Q = \{w \in \mathbf{H}_\vartheta : \psi(w)\}$ holds in $\mathbf{L}$, where ψ is a bounded formula with R as the only parameter.

By "order-preserving" we mean: if $u, v \in W_{q,p_1,\dots,p_n}$ then $u \leq_{\mathrm{LS}} v \Longleftrightarrow f(u) \leq f(v)$, where $\leq_{\mathrm{LS}}$ is the Lusin–Sierpinski (= Kleene–Brouwer) order on strings.

Fix a recursive bijection $k \mapsto s_k : \omega \xrightarrow{\text{onto}} \omega^{<\omega}$, with the inverse bijection $\mathtt{num} : \omega^{<\omega} \to \omega$, so that $s = s_{\mathtt{num}(s)}$. We assume that $\mathtt{lh}(s) \leq \mathtt{num}(s)$, $\forall s$. Let

$$W^m_{q,p_1,\dots,p_n} = \{s \in W_{q,p_1,\dots,p_n} : \mathtt{num}(s) < m\},$$

a finite set. Then $\varphi(p_1, \dots, p_n)$ is equivalent to the formula

$$\exists q \in \omega^\omega \, \exists \chi \in \vartheta^\omega \, \forall m \, (\chi \circ \mathtt{num} \text{ is order-preserving on } W^m_{q,p_1,\dots,p_n}).$$

($\chi \circ \mathtt{num}$ is the superposition.) The subformula in brackets depends on $\chi \restriction m$ and $q \restriction m, p_1 \restriction m, \dots, p_n \restriction m$ only. In other words, we have a relation $Q = Q_\vartheta(R) \subseteq \vartheta^{<\omega} \times (\omega^{<\omega})^{n+1}$, still $Q \in \mathbf{L}$, such that $\varphi(p_1, \dots, p_n)$ is equivalent to the formula

$$\exists \chi \in \vartheta^\omega \, \exists q \in \omega^\omega \, \forall m \, Q(\chi \restriction m, q \restriction m, p_1 \restriction m, \dots, p_n \restriction m). \qquad (\dagger)$$

Namely Q contains all tuples $\langle \sigma, v, u_1, \dots, u_n \rangle$ of strings $\sigma \in \vartheta^{<\omega}$ and $v, u_i \in \omega^{<\omega}$ of same length $\mathtt{lh}(\sigma) = \mathtt{lh}(v) = \mathtt{lh}(u_i) =$ some m, such that the superposition $\sigma \circ \mathtt{num}$ (defined on the set $S_m = \{s_j : j < m\}$) is order-preserving on the set

$$W^m_{v,u_1,\dots,u_n} = \{u \in S_m : \forall j \leq \mathtt{lh}(u) \, \neg R(v \restriction j, u \restriction j, u_1 \restriction j, \dots, u_n \restriction j)\}.$$

To see that Q is $\Delta_0^{\mathbf{H}_\vartheta}(R)$, note first that $\vartheta = \mathbf{Ord} \cap \mathbf{H}_\vartheta$, which eliminates ϑ and $\vartheta^{<\omega}$ from the list of parameters. In the rest, we skip a routine verification of all elements of the definition of Q being expressible by bounded formulas. ⊣

§15. Auxiliary forcing relation. Here we introduce a key tool for the proof of claim (iv) of Theorem 1.1. This is a forcing-like relation $\mathtt{forc}$. It is not explicitly connected with the forcing notion $\mathbb{P}$ (but rather connected with the full wide tree forcing **WT**), however it will be compatible with $\mathbb{P}$ for formulas of certain quantifier complexity (Theorem 17.1). The crucial advantage of $\mathtt{forc}$ will be its invariance under a certain group of transformations (Lemma 16.3), a property that cannot be expected for $\mathbb{P}$. This will be the key argument in the proof of Theorem 1.1 below in Section 18.

Blanket agreement 15.1. From now on, we let $\Theta = \omega_2^{\mathbf{L}}$, so $\Theta = \omega_2$ in $\mathbf{L}$ but $\Theta = \omega_1$ in $\mathbb{P}$-generic extensions of $\mathbf{L}$.

We argue in L. We consider a language $\mathscr{L}$ whose elementary formulas, called $\mathscr{L}\Sigma^1_2$ (in spite that they are looking more like Σ^1_1), are those of the form

$$\left.\begin{array}{l} \varphi(p_1, \dots, p_n) := \exists \chi \in \Theta^\omega \, \exists q \in \omega^\omega \, \forall m \, Q(\chi \restriction m, q \restriction m, p_1 \restriction m, \dots, p_n \restriction m), \\ \text{where } Q \in \mathbf{L},\ Q \subseteq \Theta^{<\omega} \times (\omega^{<\omega})^{n+1},\ Q \text{ is a } \Delta_0(\mathbf{H}\omega_2) \text{ set}, \\ \text{and } q, p_i \text{ are variables over } \omega^\omega. \end{array}\right\} \quad (1)$$

The dual class $\mathscr{L}\Pi^1_2$ consists of formulas

$$\left.\begin{array}{l} \varphi(p_1, \dots, p_n) := \forall \chi \in \Theta^\omega \, \forall q \in \omega^\omega \, \exists m \, Q(\chi \restriction m, q \restriction m, p_1 \restriction m, \dots, p_n \restriction m), \\ \text{with the same specifications.} \end{array}\right\} \quad (2)$$

Higher classes $\mathscr{L}\Sigma^1_k$ and $\mathscr{L}\Pi^1_k$ are defined naturally, *e.g.*, $\mathscr{L}\Sigma^1_5$ contains formulas of the form $\exists q_1 \, \forall q_2 \, \exists q_3 \, \Phi(q_1, q_2, q_3)$, where Φ is $\mathscr{L}\Pi^1_2$ and q_i vary over ω^ω.

We allow codes $\mathbf{c} \in \mathbf{CCF}_\omega$ to substitute free variables over ω^ω. If $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$ is an $\mathscr{L}$-formula, and $x \in \omega_1{}^\omega$, then $\varphi[x]$ denotes the formula $\varphi(f_{\mathbf{c}_1}(x), \dots, f_{\mathbf{c}_n}(x))$, where all $f_{\mathbf{c}_i}(x)$ are reals in ω^ω, of course.

Definition 15.2 (In $\mathbf{L}$). We define a relation $T \,\mathtt{forc}\, \varphi$ between trees $T \in \mathbf{WT}$ and closed $\mathscr{L}$-formulas in $\bigcup_{k\ge 2}(\mathscr{L}\Sigma^1_k \cup \mathscr{L}\Pi^1_k)$. Recall that $\Theta = \omega_2$ (in $\mathbf{L}$).

(A) Let $\varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$ be a $\mathscr{L}\Sigma^1_2$ formula as in (1), and $\mathbf{c}_1, \dots, \mathbf{c}_n \in \mathbf{CCF}_\omega$. Let finally $T \in \mathbf{WT}$. We define $T \,\mathtt{forc}\, \varphi$ iff there exist codes $\mathbf{c} \in \mathbf{CCF}_\omega$ and $\mathbf{d} \in \mathbf{CCF}_\Theta$ such that the following holds for all $x \in [T]$:
$$\forall m\, Q(f_{\mathbf{d}}(x)\restriction m, f_{\mathbf{c}}(x)\restriction m, f_{\mathbf{c}_1}(x)\restriction m, \dots, f_{\mathbf{c}_n}(x)\restriction m).$$

(B) If φ is a closed $\mathscr{L}\Pi^1_k$ formula, $k \ge 2$, then $T \,\mathtt{forc}\, \varphi$ iff there is no tree $S \in \mathbf{WT}$ such that $S \subseteq T$ and $S \,\mathtt{forc}\, \varphi^-$, where φ^- is the result of canonical transformation of $\neg\varphi$ to $\mathscr{L}\Sigma^1_k$ form.

(C) If $\varphi := \exists x\, \psi(x)$ is a closed $\mathscr{L}\Sigma^1_{k+1}$ formula, $k \ge 2$ (ψ being of type $\mathscr{L}\Pi^1_k$), then $T \,\mathtt{forc}\, \varphi$ iff there is a code $\mathbf{c} \in \mathbf{CCF}_\omega$ such that $T \,\mathtt{forc}\, \psi(\mathbf{c})$.

If $\varphi(p_1, \dots, p_n)$ is an $\mathscr{L}$-formula then let
$$\mathbf{Forc}(\varphi) = \{\langle T, \mathbf{c}_1, \dots, \mathbf{c}_n\rangle : T \in \mathbf{WT} \wedge \mathbf{c}_i \in \mathbf{CCF}_\omega \wedge T \,\mathtt{forc}\, \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)\}.$$
In particular if φ is closed then $\mathbf{Forc}(\varphi) = \{T \in \mathbf{WT} : T \,\mathtt{forc}\, \varphi\}$. We also define $\mathbf{Des}(\varphi) = \mathbf{Forc}(\varphi) \cup \mathbf{Forc}(\varphi^-)$ in this case.

Theorem 15.3 (In $\mathbf{L}$). *If $k \ge 2$ and φ is a formula in $\mathscr{L}\Sigma^1_k$, resp., $\mathscr{L}\Pi^1_k$, then the set* $\mathbf{Forc}(\varphi)$ *belongs to $\Sigma_{k-1}(\mathbf{H}\omega_2)$, resp., $\Pi_{k-1}(\mathbf{H}\omega_2)$.*

Proof. The proof goes on by induction on k. We begin with $\mathscr{L}\Sigma^1_2$ formulas. We argue under the assumptions and following notation of (1) above. According to definition 15.2(A), the existence quantifiers over $\mathbf{c}$ and $\mathbf{d}$ comply with the Σ_1 definability, but we have to prove that the set
$$W = \{\langle \mathbf{d}, \mathbf{c}, \mathbf{c}_1, \dots, \mathbf{c}_n, T, m\rangle \in \mathbf{CCF}_\Theta \times (\mathbf{CCF}_\omega)^{n+1} \times \mathbf{WT} \times \omega : \\ \forall x \in [T]\, Q(f_{\mathbf{d}}(x)\restriction m, f_{\mathbf{c}}(x)\restriction m, f_{\mathbf{c}_1}(x)\restriction m, \dots, f_{\mathbf{c}_n}(x)\restriction m)\}$$
belongs to $\Sigma_1(\mathbf{H}\omega_2)$. Recall that Q is $\Delta_0(\mathbf{H}\omega_2)$ by (1). It can also be mentioned that $\mathbf{CCF}_\Theta \cup \mathbf{CCF}_\omega \cup \mathbf{WT} \subseteq \mathbf{H}\omega_2$, so that $W \subseteq \mathbf{H}\omega_2$ anyway.

The hostile elements in the definition of W, which do not allow it to be $\Sigma_1(\mathbf{H}\omega_2)$ straightaway, are the quantifier $\forall x \in [T]$ in the second line, and the quantifier $\forall x \in \omega_1{}^\omega$ in (2) of Definition 13.3. (As we argue in $\mathbf{L}$, the upper index $\mathbf{L}$ as in 13.3 is removed.) But, $\omega_1{}^\omega \in \mathbf{H}\omega_2$ (under $\mathbf{V} = \mathbf{L}$), hence, as we don't care here about the choice of parameters in $\mathbf{H}\omega_2$,[3] we can pick up $\omega_1{}^\omega$ as the extra parameter. The quantifier $\forall x \in \omega_1{}^\omega$ in (2) of 13.3 then immediately becomes bounded, while the quantifier $\forall x \in [T]\,(\dots x \dots)$ in the definition of W changes to $\forall x \in \omega_1{}^\omega\,(x \in [T] \Longrightarrow \dots x \dots)$, hence becomes bounded as well, and overall we get even $W \in \Delta_0(\mathbf{H}\omega_2)$, as required.

The induction steps are easy applications of 15.2(B),(C). ⊣

Recall that a number $\mathtt{n} \ge 2$ is fixed by Definition 11.5.

[3] If we do care then the result holds too but by means of more thoroughful arguments.

Lemma 15.4 (in $\mathbf{L}$). *Let φ be a closed formula in $\mathscr{L}\Sigma^1_k \cup \mathscr{L}\Pi^1_k$, $k \geq 2$. Then the set $\mathbf{Des}(\varphi)$ is dense in $\mathbf{WT}$. If $k < \mathfrak{n}$, then $\mathbf{Des}(\varphi) \cap \mathbb{P}$ is dense in $\mathbb{P}$.*

Proof. The first claim is a simple application of Definition 15.2(B). The second claim follows from the first one by Lemmas 15.3 and 12.1(vii). ⊣

§16. Invariance. It happens that the relation $\mathtt{forc}$ is invariant under some natural transformations of wide trees. Here we prove the invariance. *We still argue in* $\mathbf{L}$.

Let $S \in \mathbf{WT}$. To define a canonical homeomorphism $h_S : [S] \xrightarrow{\text{onto}} {\omega_1}^\omega$, assume that $x \in [S]$. Let $k < \omega$. Then $x \restriction m_k \in \mathbf{BN}_k(T)$ for some (unique) number m_k. The set $\Xi(x,k) = \{\xi < \omega_1 : (x \restriction m_k)^\frown \xi \in S\}$ has cardinality $\mathtt{card}(\Xi(x,k)) = \aleph_1$; let $\Xi(x,k) = \{\xi_\gamma : \gamma < \omega_1\}$ be the enumeration in the increasing order. In particular, $x(m_k) = \xi_\gamma$ for some (unique) $\gamma = \gamma(x,k)$. Define $y = h_S(x) \in {\omega_1}^\omega$ by $y(k) = \gamma(x,k)$, $\forall k$. The map h_S is a required homeomorphism.

It follows that if $T \in \mathbf{WT}$ is another tree then $h_{ST} = {h_T}^{-1} \circ h_S$ (the superposition) is a homeomorphism of $[S]$ onto $[T]$. Moreover, in this case, if $U \subseteq S$ is a subtree then the according subtree $h_{ST} \cdot U = \{h_{ST}(x) \restriction m : x \in [U] \wedge m < \omega\} \subseteq T$ satisfies $U \in \mathbf{WT}$ iff $h_{ST} \cdot U \in \mathbf{WT}$, and $[h_{ST} \cdot U] = \{h_{ST}(x) : x \in [U]\}$.

Lemma 16.1 (In $\mathbf{L}$). *If $S, T \in \mathbf{WT}$ and $U \in \mathbf{WT}$, $U \subseteq S$, then $V = h_{ST} \cdot U \in \mathbf{WT}$, $V \subseteq T$, and $h_{UV} = h_{ST} \restriction [U]$.* ⊣

If $\lambda \in \mathbf{Ord}$ and $f : [S] \to \lambda^\omega$ then a function $h_{ST} \cdot f = f \circ h_{ST}^{-1} : [T] \to \lambda^\omega$ is defined by $(h_{ST} \cdot f)(x) = f(h_{ST}(x))$, equivalently, $(h_{ST} \cdot f)(h_{ST}(x)) = f(x)$. If $\mathbf{c}, \mathbf{c}' \in \mathbf{CCF}_\lambda$ then we symbolically write $\mathbf{c}' \restriction T = h_{ST} \cdot (\mathbf{c} \restriction S)$, in case the associated functions $f_{\mathbf{c}}$ and $f_{\mathbf{c}'}$ satisfy: $f_{\mathbf{c}'} \restriction [T] = h_{ST} \cdot (f_{\mathbf{c}} \restriction [S])$.

Lemma 16.2 (In $\mathbf{L}$). *If $S, T \in \mathbf{WT}$, $\lambda \in \mathbf{Ord}$, and $\mathbf{c} \in \mathbf{CCF}_\lambda$ then there is a code $\mathbf{c}' \in \mathbf{CCF}_\lambda$ satisfying $\mathbf{c}' \restriction T = h_{ST} \cdot (\mathbf{c} \restriction S)$.*

Proof. The map $f = f_{\mathbf{c}} \restriction [S] : [S] \to \lambda^\omega$ is continuous, hence so is the transformed map $f' = h_{ST} \cdot f : [T] \to \lambda^\omega$. Let $g : {\omega_1}^\omega \to \lambda^\omega$ be any continuous extension of f', and let $\mathbf{c}' = \mathbf{code}(g)$. ⊣

Finally if $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$ is a $\mathscr{L}$-formula, and $\varphi' := \varphi(\mathbf{c}'_1, \dots, \mathbf{c}'_n)$, where $\mathbf{c}'_1, \dots, \mathbf{c}'_n$ is another set of codes $\mathbf{c}'_i \in \mathbf{CCF}_\omega$, then we symbolically write $\varphi' \restriction T = h_{ST} \cdot (\varphi \restriction S)$, in case $\mathbf{c}'_i \restriction T = h_{ST} \cdot (\mathbf{c}_i \restriction S)$ holds for each $i = 1, \dots, n$.

Lemma 16.3 (In $\mathbf{L}$). *Let $S, T \in \mathbf{WT}$ and let φ, φ' be closed formulas in $\mathscr{L}\Sigma^1_k \cup \mathscr{L}\Pi^1_k$, $k \geq 2$, and finally $\varphi' \restriction T = h_{ST} \cdot (\varphi \restriction S)$. Then $S \,\mathtt{forc}\, \varphi$ iff $T \,\mathtt{forc}\, \varphi'$.*

Proof. We argue by induction. Let φ, φ' be $\mathscr{L}\Sigma^1_2$, so that $\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n)$ and $\varphi' := \varphi(\mathbf{c}'_1, \dots, \mathbf{c}'_n)$, where $\mathbf{c}_1, \dots, \mathbf{c}'_n, \mathbf{c}'_1, \dots, \mathbf{c}'_n$ are codes in $\mathbf{CCF}_\omega$, and

$$\varphi(p_1, \dots, p_n) := \exists \chi \in \Theta^\omega \, \exists q \in \omega^\omega \, \forall m \, Q(\chi \restriction m, q \restriction m, p_1 \restriction m, \dots, p_n \restriction m)$$

is a formula as in (1) of Section 15, and $\mathbf{c}'_i \restriction T = h_{ST} \cdot (\mathbf{c}_i \restriction S)$ holds for each i.

Assume that $S \,\mathtt{forc}\, \varphi$. Then by definition (Definition 15.2(A)) there are codes $\mathbf{c} \in \mathbf{CCF}_\omega$ and $\mathbf{d} \in \mathbf{CCF}_\Theta$ such that

$$\forall x \in [S] \, \forall m \, Q(f_{\mathbf{d}}(x) \restriction m, f_{\mathbf{c}}(x) \restriction m, f_{\mathbf{c}_1}(x) \restriction m, \dots, f_{\mathbf{c}_n}(x) \restriction m).$$

Pick, by Lemma 16.2, codes $\mathbf{c}' \in \mathbf{CCF}_\omega$ and $\mathbf{d}' \in \mathbf{CCF}_\Theta$ with $\mathbf{c}' \restriction T = h_{ST} \cdot (\mathbf{c} \restriction S)$ and $\mathbf{d}' \restriction T = h_{ST} \cdot (\mathbf{d} \restriction S)$. Then we obtain

$$\forall\, y \in [T]\,\forall\, m\, Q(f_{\mathbf{d}'}(y)\restriction m, f_{\mathbf{c}'}(y)\restriction m, f_{\mathbf{c}'_1}(y)\restriction m, \dots, f_{\mathbf{c}'_n}(y)\restriction m),$$

and hence the codes $\mathbf{c}'$ and $\mathbf{d}'$ witness $T\,\mathtt{forc}\,\varphi'$.

Step $\mathscr{L}\Sigma^1_k \to \mathscr{L}\Pi^1_k$. Let φ be a closed formula in $\mathscr{L}\Pi^1_k$, so that φ is ψ^-, where ψ is $\mathscr{L}\Sigma^1_k$, and accordingly φ' is $(\psi')^-$, $\psi'\restriction T = h_{ST}\cdot(\psi\restriction S)$. Assuming that $S\,\mathtt{forc}\,\varphi$, prove that $T\,\mathtt{forc}\,\varphi'$. Suppose to the contrary that $T\,\mathtt{forc}\,\varphi'$ fails. Then, by Definition 15.2(B), there is a tree $V \in \mathbf{WT}$, $V \subseteq T$, $V\,\mathtt{forc}\,\psi'$. We let $U = h_{ST}\cdot V$, so that $U \in \mathbf{WT}$, $U \subseteq S$, $V = h_{ST}\cdot U$. And, by the way, $h_{UV} = h_{ST}\restriction[U]$ by Lemma 16.1, thus still $\psi'\restriction V = h_{UV}\cdot(\psi\restriction[U])$. It follows that $U\,\mathtt{forc}\,\psi$, by the inductive hypothesis, which contradicts to $S\,\mathtt{forc}\,\varphi$.

Step $\mathscr{L}\Pi^1_k \to \mathscr{L}\Sigma^1_{k+1}$. Let φ be a closed formula in $\mathscr{L}\Sigma^1_{k+1}$, so that φ is $\exists\, q\, \psi(q)$, where $\psi(q)$ is $\mathscr{L}\Pi^1_k$, and accordingly φ' is $\exists\, q\, \psi'(q)$, $\psi'\restriction T = h_{ST}\cdot(\psi\restriction S)$. Assuming that $S\,\mathtt{forc}\,\varphi$, prove that $T\,\mathtt{forc}\,\varphi'$. By Definition 15.2(C), there is a code $\mathbf{c} \in \mathbf{CCF}_\omega$ satisfying $S\,\mathtt{forc}\,\psi(\mathbf{c})$. By Lemma 16.2, there exists a code $\mathbf{c}' \in \mathbf{CCF}_\omega$ such that $\mathbf{c}'\restriction T = h_{ST}\cdot(\mathbf{c}\restriction S)$. Then $\psi'(\mathbf{c}')\restriction T = h_{ST}\cdot(\psi(\mathbf{c})\restriction S)$. It follows that $T\,\mathtt{forc}\,\psi'(\mathbf{c}')$, by the inductive hypothesis, hence $T\,\mathtt{forc}\,\varphi'$. ⊣

Corollary 16.4. *Let $S, T \in \mathbf{WT}$ and let φ be a closed formula in $\mathscr{L}\Sigma^1_k \cup \mathscr{L}\Pi^1_k$, $k \ge 2$, with no codes in $\mathbf{CCF}_\omega$ as parameters. Then $S\,\mathtt{forc}\,\varphi$ if and only if $T\,\mathtt{forc}\,\varphi$.*

§17. Forcing and truth. Recall that $\mathbb{n} \ge 2$ is fixed by Definition 11.5.

Moreover we'll assume that $\mathbb{n} \ge 3$, because we now focus on the proof of claim (iv) of Theorem 1.1, vacuous in the case $\mathbb{n} = 2$.

The last part of the proof of Theorem 1.1 will be the next theorem which connects the forcing relation $\mathtt{forc}$ with the truth in $\mathbb{P}$-generic extensions. This will be the key ingredient of the proof of Theorem 1.1(iv): we use the invariant relation $\mathtt{forc}$ to surprisingly approximate the forcing $\mathbb{P}$, definitely noninvariant under the transformations considered in Section 16.

Theorem 17.1. *Assume that $2 \le k < \mathbb{n}$, $\varphi \in \mathbf{L}$ is a closed formula in $\mathscr{L}\Pi^1_k \cup \mathscr{L}\Sigma^1_{k+1}$, and a set $G \subseteq \mathbb{P}$ is generic over $\mathbf{L}$. Then the sentence $\varphi[\mathbf{a}[G]]$ is true in $\mathbf{L}[G]$ if and only if $\exists\, T \in G\,(T\,\mathtt{forc}\,\varphi)$.*

Proof. *We argue in $\mathbf{L}[G]$. Base of induction*: φ is a closed $\mathscr{L}\Sigma^1_2$ formula,

$$\varphi := \varphi(\mathbf{c}_1, \dots, \mathbf{c}_n) := \exists\, \chi \in \mathbb{O}^\omega\, \exists\, q \in \omega^\omega\, \forall\, m\, Q(\chi\restriction m, q\restriction m, \mathbf{c}_1\restriction m, \dots, \mathbf{c}_n\restriction m),$$

as in 15.2(A) and (1) of Section 15. Assume that $T \in G$ and $T\,\mathtt{forc}\,\varphi$. Then by Definition 15.2(A) there are codes $\mathbf{c} \in \mathbf{CCF}_\omega \cap \mathbf{L}$ and $\mathbf{d} \in \mathbf{CCF}_{\mathbb{O}} \cap \mathbf{L}$ such that

$$\forall\, m\, \forall\, x \in [T] \cap \mathbf{L}\, Q(f_{\mathbf{d}}(x)\restriction m, f_{\mathbf{c}}(x)\restriction m, f_{\mathbf{c}_1}(x)\restriction m, \dots, f_{\mathbf{c}_n}(x)\restriction m).$$

(Recall Remark 13.4 on the absoluteness of being a code in any $\mathbf{CCF}_\lambda$.) However all functions $f_{\mathbf{d}}, f_{\mathbf{c}}, f_{\mathbf{c}_i}$ are continuous. It follows that the last displayed formula can be strengthened to

$$\forall\, x \in [T]\, \forall\, m\, Q(f_{\mathbf{d}}(x)\restriction m, f_{\mathbf{c}}(x)\restriction m, f_{\mathbf{c}_1}(x)\restriction m, \dots, f_{\mathbf{c}_n}(x)\restriction m).$$

Therefore, as $\mathbf{a}[G] \in [T]$ (because $T \in G$), we obtain

$$\forall\, m\, Q(f_{\mathbf{d}}(\mathbf{a}[G])\restriction m, f_{\mathbf{c}}(\mathbf{a}[G])\restriction m, f_{\mathbf{c}_1}(\mathbf{a}[G])\restriction m, \dots, f_{\mathbf{c}_n}(\mathbf{a}[G])\restriction m).$$

Thus elements $\chi = f_{\mathbf{d}}(\mathbf{a}[G])$ and $q = f_{\mathbf{c}}(\mathbf{a}[G])$ witness $\varphi[\mathbf{a}[G]]$ to be true.

To establish the inverse, suppose that $\varphi[\mathbf{a}[G]]$ is true in $\mathbf{L}[G]$, that is,

$$\forall m\, Q(\chi \restriction m, q \restriction m, f_{\mathbf{c}_1}(\mathbf{a}[G]) \restriction m, \dots, f_{\mathbf{c}_n}(\mathbf{a}[G]) \restriction m)$$

true for some $\chi \in \circledcirc^\omega$ and $q \in \omega^\omega$ in $\mathbf{L}[G]$. By Lemma 13.5 there are codes $\mathbf{d} \in \mathbf{CCF}_\circledcirc \cap \mathbf{L}$ and $\mathbf{c} \in \mathbf{CCF}_\omega \cap \mathbf{L}$ such that $\chi = f_\mathbf{d}(\mathbf{a}[G])$ and $q = f_\mathbf{c}(\mathbf{a}[G])$. Thus there is a tree $T \in G$ which $\mathbb{P}$-forces the formula

$$\forall m\, Q(f_\mathbf{d}(\mathbf{a}[\underline{G}]) \restriction m, f_\mathbf{c}(\mathbf{a}[\underline{G}]) \restriction m, f_{\mathbf{c}_1}(\mathbf{a}[\underline{G}]) \restriction m, \dots, f_{\mathbf{c}_n}(\mathbf{a}[\underline{G}]) \restriction m) \qquad (*)$$

over $\mathbf{L}$. We claim that the codes $\mathbf{c}$ and $\mathbf{d}$ witness $T \,\mathtt{forc}\, \varphi$ as in 15.2(A). Indeed otherwise there are $x \in [T]$ and m such that

$$\neg\, Q(f_\mathbf{d}(x) \restriction m, f_\mathbf{c}(x) \restriction m, f_{\mathbf{c}_1}(x) \restriction m, \dots, f_{\mathbf{c}_n}(x) \restriction m). \qquad (\dagger)$$

But, the maps $f_\mathbf{d}, f_\mathbf{c}, f_{\mathbf{c}_i}$ are continuous. It follows that there is a string $u = x \restriction j$ for some j such that ($\dagger$) holds for **all** $x \in [S]$, where $S = T\restriction_u \in \mathbb{P}$. But then clearly T cannot $\mathbb{P}$-force $(*)$ as S forces the opposite.

Step $\mathscr{L}\Sigma^1_k \Longrightarrow \mathscr{L}\Pi^1_k$, $k < \mathfrak{n}$. Let φ be a $\mathscr{L}\Pi^1_k$ formula. By Lemma 15.4, there is a tree $T \in G$ such that either $T \,\mathtt{forc}\, \varphi$ or $T \,\mathtt{forc}\, \varphi^-$. Assume that $T \,\mathtt{forc}\, \varphi$; we have to prove that $\varphi[\mathbf{a}[G]]$ is true. Suppose otherwise. Then $\varphi^-[\mathbf{a}[G]]$ is true. By the inductive hypothesis, there is a tree $S \in G$ such that $S \,\mathtt{forc}\, \varphi^-$. But the trees S, T belong to the same generic set G, hence they are compatible, which leads to a contradiction with the assumption $T \,\mathtt{forc}\, \varphi$, according to Definition 15.2(B). Now assume that $T \,\mathtt{forc}\, \varphi^-$. Then $\varphi^-[\mathbf{a}[G]]$ is true by the inductive hypothesis, hence $\varphi[\mathbf{a}[G]]$ is false. On the other hand, there is no tree $S \in G$ such that $S \,\mathtt{forc}\, \varphi^-$, just as above.

Step $\mathscr{L}\Pi^1_k \Longrightarrow \mathscr{L}\Sigma^1_{k+1}$, $k < \mathfrak{n}$. Let φ be $\exists\, x\, \psi(x)$ where ψ is $\mathscr{L}\Pi^1_k$. Assume that $T \in G$ and $T \,\mathtt{forc}\, \varphi$. Then by Definition 15.2(C) there is a code $\mathbf{c} \in \mathbf{CCF} \cap \mathbf{L}$ such that $T \,\mathtt{forc}\, \psi(\mathbf{c})$. By the inductive hypothesis, the formula $\psi(\mathbf{c})[\mathbf{a}[G]]$, that is, $\psi[\mathbf{a}[G]](f_\mathbf{c}(\mathbf{a}[G]))$, is true in $\mathbf{L}[G]$. But then $\varphi[\mathbf{a}[G]]$ is true as well.

Conversely assume that $\varphi[\mathbf{a}[G]]$ is true. Then there is a real $y \in \mathbf{L}[G] \cap \omega^\omega$ such that $\psi[\mathbf{a}[G]](y)$ is true. By Lemma 13.5(i), $y = f_\mathbf{c}(\mathbf{a}[G])$ for a code $\mathbf{c} \in \mathbf{CCF}_\omega \cap \mathbf{L}$. But then $\psi(\mathbf{c})[\mathbf{a}[G]]$ is true in $\mathbf{L}[G]$. By the inductive hypothesis, there is a tree $T \in G$ satisfying $T \,\mathtt{forc}\, \psi(\mathbf{c})$. Then $T \,\mathtt{forc}\, \varphi$ as well. $\dashv$

§18. The final argument.

Theorem 1.1, the main theorem. We assert that any $\mathbb{P}$-generic extension $\mathbf{L}[G] = \mathbf{L}[\mathbf{a}[G]]$ satisfies conditions (i), (ii), (iii), (iv) of the theorem. Regarding (i), (ii), (iii) see a summary in the very end of Section 13. Let's concentrate on (iv). Let $\Phi(j)$ be a parameter-free $\Sigma^1_\mathfrak{n}$ formula. (The case when Φ has real parameters in $\mathbf{L}$ can also be handled with some extra care.) Thus

$$\Phi(j) \;:=\; \exists\, r_1\, \forall\, r_2 \cdots \forall(\exists)\, r_n\, \exists(\forall)\, m\, R_j(r_1 \restriction m, r_2 \restriction m, \dots, r_\mathfrak{n} \restriction m)\,,$$

where r_i are variables over ω^ω, $R_j \subseteq (\omega^{<\omega})^\mathfrak{n}$, $R_j \in \mathbf{L}$, and the map $j \mapsto R_j$ is arithmetically definable in $\mathbf{L}$. Applying Theorem 14.2 in $\mathbf{L}$ with $\circledcirc = \omega_2^\mathbf{L} = \omega_1^{\mathbf{L}[G]}$ and $M = \mathbf{L}[G]$, we get relations $Q_j = Q_\circledcirc(R_j)$, and closed $\mathscr{L}\Sigma^1_\mathfrak{n}$ formulas

$$\varphi_j \;:=\; \exists\, r_1\, \forall\, r_2 \cdots \exists(\forall)\, r_{n-2}\, \forall\,(\exists)\, \chi \in \circledcirc^\omega\, \forall\,(\exists)\, q\, \exists\,(\forall)\, m$$
$$Q_j(\chi \restriction m, q \restriction m, r_1 \restriction m, r_2 \restriction m, \dots, r_{n-2} \restriction m)\,,$$

satisfying $\Phi(j) \iff \varphi_j$, $\forall\, j$, both in $\mathbf{L}$ and in any $\mathbb{P}$-generic extension $\mathbf{L}[G]$ of $\mathbf{L}$. It follows, by Theorem 17.1, that the set $X = \{j : \Phi(j)^{\mathbf{L}[G]}\}$ (defined in $\mathbf{L}[G]$)

satisfies $X = \{j : \exists T \in G\,(T\,\mathtt{forc}\,\varphi_j)\}$. Furthermore, as the formulas φ_j do not contain codes in $\mathbf{CCF}_\omega$, it follows, by Corollary 16.4, that $X = \{j : T\,\mathtt{forc}\,\varphi_j\}$, where T is any particular tree in $(\mathbf{WT})^{\mathbf{L}}$, one and he same for all j. We conclude that $X \in \mathbf{L}$, as required. ⊣

§19. A problem. It is a challenge to figure out what kind of models the method of the proof of Theorem 1.1 gives for forcing notions with trees with the splitting parameter bigger than $\aleph_1$. For instance, let $\mathbf{WT}_{\omega_2}$ be the set of all trees $T \subseteq \omega_2{}^{<\omega}$ with no isolated branches, whose all branching nodes are ω_2-branching nodes. This is a non-Laver version of the Namba forcing; the Namba forcing *per se* requires that in addition every node above the stem is a branching node. The forcing $\mathbf{WT}_{\omega_2}$ (or an equivalent forcing) is considered *e.g.,* in [4], [11, Section 28], or [6, 18.4].

Clearly $\mathbf{WT}_{\omega_2}$ adds a cofinal infinite sequence, say $\vec{a} = \langle \alpha_n \rangle_{n<\omega}$, in $\omega_2^{\mathbf{V}}$. Moreover, if CH holds in the ground universe then, essentially by Namba–Bukovsky, $\mathbf{WT}_{\omega_2}$ does not add new reals, hence, does not collapse $\omega_1^{\mathbf{V}}$. (See [11, Section 28] for a simple proof.) Thus $\vec{a} \in \mathbf{H}_\lambda$ in the extension $\mathbf{V}[\vec{a}]$, where $\lambda = \omega_2^{\mathbf{V}[\vec{a}]} > \omega_2^{\mathbf{V}}$. (Where $\mathbf{V}$ is the ground set universe, as usual.)

It is then an interesting problem to check whether there are results for the definability of $\vec{a}$ in $\mathbf{H}_\lambda$ similar to the results in [2] and those of this paper, and first of all whether there is a subforcing $\mathbb{P} \subseteq \mathbf{WT}_{\omega_2}$ which retains the above properties of $\mathbf{WT}_{\omega_2}$ and such that in addition any $\mathbb{P}$-generic sequence $\vec{a}$ is definable in $\mathbf{V}[\vec{a}]$. The argument presented in this paper fails in this setting at Lemma 9.4 though.

It will also be interesting to accomodate the methods used in this paper to the (more complex) case of *iterated* extensions as *e.g.,* in [16].

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