# ON SOME APPLICATIONS OF HEYTING-VALUED ANALYSIS II

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This paper contains three sections. In Section 1 the model of intellectual system (IS) based on sheaves and generic models is discussed. The author realizes all conditionality and scantiness of this model. but probably, the very idea of this type of models is new for the artificial intelligence. Sections 2 and 3 are purely mathematical. The material of these two sections can be used for the proof of convergency of an iterative scheme mentioned in the end of Section 1. Also Sections 2 and 3 contain some results concerning well known problems: in Section 2 we study the problem of "natural translation" of classical logic into intuitionistic one for various theories such as arithmetic. algebra. analysis etc. (see [9,p.127. 7]), and in Section 3 we study Macintyre problem about the transfer of model completeness of a theory from stalks of a sheave to the structure of all global sections of the sheave ([1,p.88. 2,p.175]). It is interesting to note that this method - the method of Heyting-valued analysis ([3-5]) - can be successfully used in such different questions.

We assume the reader is rather familiar with the Heyting-valued analysis [3]. but this is not necessary. The sign $\leftrightharpoons$ means "is equal by definition " or "is equivalent by definition".

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Section 1. On one possible model of an intellectual system

In connection with forming the concept of the decision-making intellectual system (abbreviation: IS) one can designate a number of composing notions of this concept which we give below. First, the motive (= the resulting goal of a compound action or quest), second, the goal (= intermediate. current goal in realization of the motive), third, internal, "deeply structured" model of the external world. The latter model, having leading part in the intelligence we shall call the internal model (or, simply, the model). Let us continue the list of composing notions of this concept: first, the external world (= the situation or the state of affair), the description of which the IS sensomotorically receives at each given moment of time. Second, an action (of IS), changing the sensomotoric information on conditions of own achieving of the external world. Functioning of IS takes place on two scales of time: on the larger scale of time it has the form of learning, development, which first of all consists of changing the internal model of the world as well as the means of access to this model, and on the lesser scale of time there are coping with problems generated by the system of motivation. The role of the internal model on the larger scale of time consists of maintaining the IS's stability (homeostasis), while the role of the model on the lesser scale of time consists the possibility to activate the set of partial situations in response to each sensomotoric input. A particular situation describes such representations of objects of the extreme world and, in addition, such relationships between these representations that correspond to actual state of affairs.

In accordance with one of the known possible definitions ([13, p.3]) an intellectual system (IS) can be defined as a system which generates chains of goal-directed actions under general pressing of some motives and on the basis of some internal model of external world via constructing a situative model of the current state of affairs.

It would be desirable to study this concept of IS (the philosophical concept in its essency) in purely mathematical and computationally realizable terms. This would permit on mathematical and experimental levels to testify the validity of any given set of properties being intuitively ascribed to an IS. We give below such description of the above notions in purely mathematical terms (certainly, preliminary).

The time is understood descretely as steps enumerated by natural numbers. A motive is understood as an increasing sequence $\{Th_n\}$ of theories (= of lists of axioms): $Th_n \subseteq Th_{n+1}$. Intuitively, the sequence $\{Th_n\}$ describes one of such sequences of pictures of the external world that their realization will discharge the tension caused by given motive (= the potentially realizable motive). The degree of this discharging is characterized by the notion of "satisfiability" of IS. The latter notion is important mostly for the scale proper to the process of learning. Now we consider the work of IS in the lesser scale of time.

We define the internal model as follows. Fix a Stone topological space X. To each point p from X we correlate some mathematical structure (= model K) having the same signature for all p. Let E be disjoint union of all $K_p$ where p ranges over X. In E we fix a topology such that the function $\Pi:E\to X$ defined by the equivalence $\Pi(s)=p \Leftrightarrow s\in K_p$ is a continuous function and also is a local homomorphism. Call by a section a function of the form: $k:\mathcal{U}\to E$, where $\mathcal{U}$ is open set in X and $\Pi(k(p))=p$ for all p from $\mathcal{U}$. Denote by $\mathcal{T}(X)$ the topology in X, i.e. the lattice of all open sets in X. We also denote by $\Gamma(E,X)$ the set of all sections. Intuitively, $K_p$ is a particular situation (a particular experience) at point p while one section $k(\bullet)$ is the representation of one fixed object in various situations (in which this object is taken into account). Therefore, $\Gamma(E,X)$ intuitively corresponds to the reflection of objects of the external world in the internal model of IS. Informally speaking, the set $\{\Gamma(E,\mathcal{U}) \mid \mathcal{U}\in\mathcal{T}(X)\}$ corresponds to the reflection of situative aspects of the external world within the internal model of IS.

For every list of axioms (= of theories) $T_1,\dots,T_m$ we denote

$$X(T_1,\dots,T_m) \rightleftharpoons X(\bigcup_{i=1}^{m} T_i) \rightleftharpoons \{p\in X \mid K_p \models T_1 \& \dots \& T_m\}.$$ Intuitively, sensomotoric

information (= the description of the external world) T excites (= activizes) the domain X(T) of the internal model. If IS has performed (within the frame of reasonable chain of actions $P_1,\dots,P_{n-2},\dots$) the next action $P_{n-1}$ (where $P_1\subseteq\dots\subseteq P_{n-2}\subseteq P_{n-1}$) and IS has obtained the description $T_n$ of the external world on n-th step, then the domain $X(T_n,P_{n-1})$ of the internal model is activized. We do not make it precise whether the covering space <E,X> is the whole internal model or only its part being actualizable by the motive (and by other circumstances).

And so, one-moment description of the external world is represented by some list of axiom (= by some theory) T. It is possible that the language of description of the motive (i.e. the language in which the theory $Th_n$ is written) is more abstract in comparison with the language of description of the external world (i.e. with the language in which the theory T is written). The latter language we denote by Я. We also denote by Я(C) the extension of language Я by countable list of new constants. We call by an action a finite consistent set of atomic formulas (in language Я(C)) and their negations. We could call by an action also a finite consistent set of quantifier free (= basic) formulas (in language Я(C)) and would come to some results. An action compatible with a theory T will be called an action for theory T. Intuitively, an action means a finite number of "pushing and non-pushing some potentially infinite number of buttons (= of constants from C)".

We define inductively some iterative procedure to obtain the satisfying of the motive. Let the actions $P_1 \subseteq \ldots \subseteq P_{n-1}$ have been already chosen and performed. Then IS is (sensomotorically) given a theory $T_n$ being the next description of the external world. It can happen that the theory $T_n$ contradicts the descriptions which are analogous to the previous descriptions. Further, from the two conditions below we simultaneously find the goal $\psi_n$ of n-th iteration and the action $P_n$ being performed at the n-th iteration. Now we formulate the two above mentioned conditions. The first condition says that the maximum of the function of argument $\psi$ of the form

$$\max\{m \mid \mu(X(\psi \Rightarrow Th_m)) - \mu(X(T_n, P_{n-1}))\mid < \varepsilon\}$$

is accessed at the value $\psi_n$ for the argument $\psi$. Here $\mu(\cdot)$ is a real-valued measure on X and $\varepsilon$ is a fixed real number (parameter). The second condition says that $P_{n-1} \subseteq P_n$ and $P_n \Vdash_{T_n} \psi_n$, where $\Vdash_{T_n}$ is the finite forcing in theory $T_n$.
Here $P_n \Vdash$ means that in all generic models of theories $T_n$ and $P_n$ (where $P_n$ is an action for the theory $T_n$) the formula $\psi_n$ is valid. In other words, the valuation of the formula $\psi_n$ for the theory $T_n$ (in A. Robinson's sense) is minored with $P_n$.

We could give sufficient conditions and close to them necessary conditions of the convergence of iterative procedures of the type described. One could attempt to realize procedures of this type algorithmically. Some additional considerations are given in [6].

Section 2. The natural translation of a classical theory of algebras with metrics into an intuitionistic theory of algebras with metrics.

An algebra with metrics is defined as $\langle K,+,-,\cdot,0,1,\|\cdot\|\rangle$, where $+:K^2\to K$, $-:K\to K$, $\cdot:K^2\to K$; $0,1\in K$, $\|\cdot\|:K\to \underline{R}$; $\underline{R}$ is defined as the set of all Dedekind sections $\lambda=\langle\lambda,\lambda_1\rangle$ on $Q$, and $\cdot<\cdot$ is predicate on $\underline{R}$, defined by $(\lambda<\mu)\leftrightharpoons(Er)\in Q(r\in\lambda_1\&r\in\mu)$. Usual ZF-formulas describing $Q$, $Q\geq 0$, $\underline{R}$, $\underline{C}$ (as the set of all pairs of Dedekind sections) we denote by $\mathscr{X}_1,\mathscr{X}_2,\mathscr{X}_3,\mathscr{X}_4$.

We call a formula $\mathscr{X}(\cdot,\ldots,\cdot)$ (in the language of ZF) a Dedekind formula iff

1) $HZF\vdash\forall f,+,-,\cdot,0,1,\|\cdot\|(\mathscr{X}(f,+,-,\cdot,0,1,\|\cdot\|)\Rightarrow\langle f,+,-,\cdot,\|\cdot\|\rangle$ is an algebra with metrics (here the formula $\mathscr{X}_3$ is used);

2) $ZFC\vdash\forall f,+,-,\cdot,0,1,\|\cdot\|(\mathscr{X}(f,\ldots,\|\cdot\|)\Rightarrow\forall k,t\in f(k=t\Rightarrow\|k-t\|=0)$;

3) $\forall\Omega(\llbracket\mathscr{X}(f,\ldots,\|\cdot\|)\rrbracket_\Omega\leq$
$[(f(k)\to\llbracket\|k\|_\Omega=\|k\|_B\rrbracket_B)\&(f(k)\&f(t)\to\llbracket k+_\Omega t=k+_B t\rrbracket_B)\&\ldots])$,
where ... means the same conditions for $-,\cdot,0,1$;

4) $\forall\Omega(\llbracket\mathscr{X}(f,\ldots,\|\cdot\|)\rrbracket_\Omega\leq\llbracket\mathscr{X}(f,\ldots,\|\cdot\|)\rrbracket_B)$,

where $\underline{B}$ is canonically constructed from $\Omega$. Namely, $\underline{B}$ is the (unique) complete Boolean algebra such that can be injected (as a complete Heyting algebra) into $\underline{B}$. The evaluation $\llbracket\cdot\rrbracket_B$ is computed in the Boolean-valued universum $V^{\underline{B}}$ and $\llbracket\cdot\rrbracket_\Omega$ is computed in the Heyting-valued universum $V^\Omega$. Here HZF devotes the intuitionistic set theory introduced by R. Greyson.

Example 1. 1) A set of Dedekind sections being closed under the usual ring operations (defined in $\underline{R}$) considered together with these operations can be described by the formula which we denote by $\mathscr{X}$. Let also $\|\lambda\|\leftrightharpoons|\lambda|\leftrightharpoons\max\{\lambda,-\lambda\}$, i.e.

$\mathscr{X}(f,\ldots,\|\cdot\|\leftrightharpoons\forall x\in f(\mathscr{X}_3(x)\&\forall x_1,x_2\in f(x_1+x_2,-x_1,x_1\cdot x_2,0,1\in f)\&\ldots$

(Here we again use the formula $\mathscr{X}_1$). It is easy to see that $\mathscr{X}$ is a Dedekind formula. It describes in $V^\Omega$ and $V^{\underline{B}}$ the families of all subrings of $\underline{R}$. Let $\underline{R}_\Omega$ be an object described in $V^\Omega$ by formula $\mathscr{X}_3$ and $\underline{R}_{\underline{B}}$ be an object described in $V^{\underline{B}}$ by the same formula. Of course,

$\underline{R}_\Omega\neq\underline{R}_{\underline{B}}$, but $\llbracket\underline{R}_\Omega$ is a subring of $\underline{R}_{\underline{B}}\rrbracket_{\underline{B}}=1$.

2) Let a set of complex numbers be closed under the usual ring operations (defined in C) considered together with these operations can be described by the formula which we denote by $\mathscr{X}$. We remind that complex number is a pair of Dedekind sections. It is easy to see that $\mathscr{X}$ is a

Dedekind formula. It describes in $V^{\Omega}$ and $V^{\underline{B}}$ the families of all subrings of C.

3) The families of the subrings of the hypercomplex systems over $\underline{Q}$, $\underline{R}$ or $\underline{C}$ can be described by Dedekind formulas in the same manner.

4) In all cases mentioned above we can add the formula $\forall x\in f(Ey)\in f(x=0 \vee x\cdot y=1)$ to $\mathfrak{X}$ and then we can obtain the corresponding family of fields.

5) In all cases mentioned above we can add to formula $\mathfrak{X}$ the formula stating that the field is real closed (or algebraic closed).

6) Thus, all usual classes of number rings and fields (groups and algebras) can be described by Dedekind formulas. Because of this we can apply Theorem 1 (below) to all these classes.

7) Each implication $\varphi \Rightarrow \psi$ of the type described in Theorem 1, which is true in $\underline{R}$ (or $\underline{C}$) will be a consequence of the axioms of a real (or algebraic) closed field (for the corresponding theories are complete). So, Theorem 1 can be applied to this implication, too.

Let $\varphi$, $\psi$ be formulas in the language of the theory of rings. By $\varphi^{+}$ we denote the formula obtained from $\varphi$ by substitution of $k\#t$ instead of $k\neq t$. Here # denotes the apartness predicate. In the simplest case (when metric is $\|\cdot\|:K\to\underline{R}$) the apartness predicate is defined by $(k\#t)\leftrightharpoons\|k-t\|^{2}>0$. Later we give other examples of ranges of values of metrics in which the apartness predicate can also be defined. Let $k\doteq t$ abbreviate $\|k-t\|=0$. By $\psi^{+}$ we denote the formula obtained from $\psi$ by the two simultaneous substitutions: the substitution of $k\#t$ instead of $k\neq t$ and the substitution of $k\doteq t$ instead of $k=t$. By $cHa\models(x_1,\dots x_n)$ we mean $\forall_{\Omega}(\llbracket\varphi(\check{x}_1,\dots,\check{x}_n)\rrbracket_{\Omega}=1)$, where $x_1,\dots,x_n\in V$ and $\Omega$ ranges over the class of all complete Heyting algebras. The predicate $cHa\models(\cdot)$ can be considered as a new intuitionistic semantics for the set theory. In main features this predicate was defined in [10], and this translation was defined in [7,8].

<u>Theorem 1</u>. Let $\mathfrak{X}$ be a Dedekind formula in the language of ZF. $\varphi$ and $\psi$ are formulas in the language of the ring theory, $\psi$ is a AE-formula. If

$$ZFC\vdash\forall f,\dots,\|\cdot\|(\mathfrak{X}(f,\dots,\|\cdot\|)\Rightarrow\forall k_1,\dots,k_n\in f[\varphi_f(\bar{k})\Rightarrow\psi_f(\bar{k})]),$$

then

$$cHa\models\forall f,\dots,\|\cdot\|(\mathfrak{X}(f,\dots,\|\cdot\|)\Rightarrow\forall k_1,\dots,k_n\in f[\varphi^{+}_f(\bar{k})\Rightarrow\psi^{+}_f(\bar{k})]).$$

<u>Proof</u>. Assume that $f,\dots,\|\cdot\|\in V^{\Omega}$, $k_1,\dots,k_n\in D(f)$, $u\leftrightharpoons\llbracket\mathfrak{X}(f,\dots,\|\cdot\|)\rrbracket\,\&\,f(k_1)\,\&\dots\&\,f(k_n)\,\&\,\llbracket\varphi^{+}_f(\bar{k})\rrbracket_{\Omega}$. Then $u\leq\llbracket\varphi^{+}_f(\bar{k})\rrbracket_{\underline{B}}$. The formula $\mathfrak{X}$ is a Dedekind one, hence

$u \leq [\![\mathfrak{X}(f,\dots,\|\cdot\|)]\!]_{\underline{B}} \& f(k_1)\&\dots\&f(k_n)\&[\![\varphi_f^+(k)]\!]_{\underline{B}}$. By our assumption $u \leq [\![\psi_f(\bar{k})]\!]_{\underline{B}}$, therefore $u \leq [\![\psi_f^+(\bar{k})]\!]_{\underline{B}}$ (because $\mathfrak{X}$ is a Dedekind formula).

Let $\psi_f^+(k)$ be the formula $\forall t_1,\dots,t_m \in f \& \forall \varepsilon_1,\dots,\varepsilon_s \in Q_{>0}(El_1,\dots,l_r) \in f\chi(k)$.

In this case $\forall t_1,\dots,t_m \in D(f) \forall \varepsilon_1,\dots,\varepsilon_s \in Q_{>0}(u \leq f(t_1)\&\dots\&f(t_m) \text{->}$
$[\![(El_1,\dots,l_r) \in f\chi(\bar{k},\bar{t},\bar{\bar{\varepsilon}})]\!]_{\underline{B}})$.

i.e. $u\&f(t_1)\&\dots\&f(t_m) \leq [\![(E\bar{l}) \in f\chi(\bar{k},\bar{t},\bar{\bar{\varepsilon}})]\!]_{\underline{B}})$.

Therefore $u \leq [\![(E\bar{l}) \in f(X(\bar{k},\bar{t},\bar{\bar{\varepsilon}}))]\!]_{\Omega}$, i.e. $u \leq [\![\forall \bar{t} \in f \forall \bar{\varepsilon} \in Q_{>0}(E\bar{l}) \in fX(\bar{k},\bar{t},\bar{\varepsilon})]\!]_{\Omega}$.

Remark. An analogous theorem can also be stated for each particular complete Heyting algebra $\Omega$ and the corresponding Boolean algebra $\underline{B}$. In this case we fix an arbitrary algebra with metric $<f,+,-,\cdot,0,1,\|\cdot\|>$ in $V^{\Omega}$ and consider the same algebra in $V^{\underline{B}}$. (Here "metric" may also be a mapping $\|\cdot\|: f \text{-->} \tilde{Y}$, where $Y$ is a locally campact ring, and $\tilde{Y}_{\Omega}$ is a completion of the uniform space $\check{Y}$ in $V^{\Omega}$ by Cauchy filters, and $\tilde{Y}_{\underline{B}}$ is the completion in $V^{\underline{B}}$ constructed in the same way. In this case $[\![\tilde{Y}_{\Omega}$ is a subring of $\tilde{Y}_{\underline{B}}]\!]_{\underline{B}}=1$.) We can state that if

$$f(k_1)\&\dots\&f(k_n) \leq [\![f \models (\varphi(\bar{k}) \Rightarrow \psi(\bar{k}))]\!]_{\underline{B}},$$

then

$$f(k_1)\&\dots\&f(k_n) \leq [\![f \models (\varphi^+(\bar{k}) \Rightarrow \psi^+(\bar{k}))]\!]_{\Omega}.$$

This construction can be successfully applied, for example, to an arbitrary ring K in the following way. Let $\Omega$ be the topology of the Stone space constructed for the Boolean algebra of all central idemponents of ring K. It is possible to define mapping $K \mapsto K'$, where $K'$ is an element from $V^{\Omega}$ and also a translation $\varphi \mapsto \varphi'$, such that $([\![K' \models \varphi]\!]_{\Omega}=1) \Leftrightarrow (K \models \varphi')$ for all formulas $\varphi$ and for all rings K. Therefore, if $K \models ((\varphi^+)'\&i)$ and $ZFC \vdash ((\varphi\&i') \Rightarrow \psi)$, then $K \models (\psi^+)'$. For example one can take i="is an abelian regular ring" and i'="is a skew field". Details see in [3]. Even the simplest case $K'=\tilde{Y}_{\Omega}$ is interesting: in this case $K=C_1(X,Y)$, where X is Stone space for the algebra $\Omega$. If $\Omega$ is a topology of a topological space Z and, for example, Y is equal to $\underline{R}$, then $C_1(X,Y) \cong C(Z,Y)$. We can substitute (in Theorem 1) an arbitrary formula $\varphi_1 \text{-->} \varphi_2$ instead of premise $\varphi$, where $\varphi_1$ is a B-decidable AE-formula and $\varphi_2$ is in the weak A-normal form.

Main results of this section were reported by the author to the 7th Congress of Logic (Salzburg, July 1983).

## Section 3. Localizations and evaluations

The results of this section were reported by the author to 8th Congress of Logic (Moscow, August 1987).

In the following K is an associative ring with 1 (however our results are still valid for much wider class of algebras). This ring is identified with the Pierce sheave. Its stalk coincides with $K_p \leftrightharpoons K/\bar{p}$ where p is a point in a Stone space X(K) of the Boolean B(K) of all central idemponents of K. Let us fix a theory T, having some T* as its model companion. Let us also fix some class of rings $\underline{K} \leftrightharpoons \{K | \{K_p\} |= T\}$. By $\underline{K}^*$ we denote $\{K | (\{K_p\} |= T^*) \& K |= \Phi_1 \& \Phi_2\}$, where $(\{Kp\} |= T) \leftrightharpoons \forall p \in X(K)(K_p |= T)$ and $\Phi_1$ is the property "to be normal" and $\Phi_2$ is the property " to be atomless" for the ring K. Here $\Phi_1 = \forall k \in K(Ee_0) \in B(K) \forall e \in B(K)(e \cdot k = 0 <=> e \leq e_0)$. In an ordinary way we define the evaluation $[\![\cdot]\!]_{\mathfrak{T}(K)}$ for the formulas (with parameters from K) in the language of the ring theory. Sometimes we shall write $\mathfrak{T}$ instead of $\mathfrak{T}(K)$, where $\mathfrak{T}(K)$ is the topology of the space X(K).

Macintyre's problem is: when the class $\underline{K}$ has a model companion $\underline{K}'$ and what is an axiomatization of this $\underline{K}'$. We will give sufficient conditions for the fact that exactly the class $\underline{K}^*$ (defined as above) is the model companion for $\underline{K}$ and will give an explicit axiomatization for this class $\underline{K}^*$. This can be considered as a possible answer to the Macintyre problem.

<u>Proposition 1</u>. Let K be a normal ring and $\{K_p\} |= T^*$, where theory T* is model complete. Then for each formula $\varphi$ we have two properties: $[\![\varphi(k_1, \ldots, k_n)]\!]_{\mathfrak{T}} = \{p \in X(K) | K_p |= \varphi(k_1(p), \ldots, k_n(p))\}$ and $[\![\varphi(k_1 \ldots k_n)]\!]_{\mathfrak{T}}$ is a closed-open set, where $k_1, \ldots, k_n \in K$.

<u>Proof</u>. For atomic formula (k=t) we have $[\![k=t]\!] = e_0$, where $e_0$ comes owing to the definition of normality for the element k-t. For the case of connectivness v, &, $\neg$ all is obvious. For the case (E) note that $[\![(Ex)\varphi]\!] = \{p-X | K_p |= (Ex)\varphi\}$. Using model completeness for the formula

$(\neg(Ex)\varphi)(x_1, \ldots, x_n)$ we come to reducing E-formula $\psi(x_1, \ldots, x_n)$ and by normality of the ring K we obtain that the set $[\![(Ex)\varphi]\!]$ is open-closed. In the case of $\forall$ note that $[\![\forall x\varphi]\!]\subseteq\{p\in X\mid K_p\models\forall x\varphi\}\subseteq\bigcap\{[\![\varphi(k)]\!]\mid k\in K\}$ and take into account that the set $\{p\in X\mid K_p\models(Ex)\neg\varphi\}$ is open-closed.

<u>Proposition 2</u>. Let K be a normal ring, $\varphi$ be an AE-formula. If $\forall p\in e(K_p\models(k_1(p),\dots,k_n(p))$. then $[\![\varphi(\bar{k})]\!]_{\mathcal{T}}\geq e$ for each $e\in B(K)$.

<u>Proposition 3</u>. Let K be a normal ring, $\varphi$ be a formula having no quantifiers within the scope of implication. If $p\in[\![\varphi(\bar{k})]\!]_{\mathcal{T}}$, then $K_p\models\varphi(k_1(p),\dots,k_n(p))$.

Class $\underline{K}$ is called Boolean-regular, if $B(K)\subseteq B(L)$ for every $K,L\in\underline{K}$ such that $K\subseteq L$ (this is true, for example, if $(K,L\in\underline{K})\&(K\subseteq L)\Rightarrow Z(K)\subseteq Z(L)$, where Z(K) is the centre of the ring K).

<u>Proposition 4</u>. a) The A-model completeness of a class $\underline{K}$ implies Boolean-regularity of $\underline{K}$.

b) Boolean-regularity of $\underline{K}$ is equivalent to the condition $K\subseteq L\Rightarrow\forall p_1\in X(L)((p_1\cap K)\in X(K))$ and also is equivalent to the condition

$$K\subseteq L\Rightarrow(Ep_1)\in X(L)((p_1\cap K)\in X(K)).$$

<u>Proof.</u> For each $e\in B(K)$ transfer the formula $\forall x(e\cdot x=x\cdot e)$ from K onto L.

b) Let us verify successively three implications between the above three conditions on $\underline{K}$. Certainly, $q\neq p_1$ $K=p_1\cap B(K)\subseteq B(K)$ and q contains 0 but does not contain 1 and, moreover, q is closed w.r.p. operation v. And, because of the Boolean-regularity of the class $\underline{K}$ we have that the set q is transitive downward set and is a prime ideal. If $e\in B(K)\setminus B(L)$, then $(p_1\cap K)$ can not be prime. $\square$

A class $\underline{K}$ is called Boolean-simple, iff

$(K,L\in K)\&(K\subseteq L)\Rightarrow\forall e_1\in B(L)(e_1\neq 0\Rightarrow(Ep_1)\in e_1(Ep)\in X(K)$

$[p\supseteq p_1\cap K\&\forall k\in K((Ee)\in p_1(e\cdot k=k)\Rightarrow(Ee)\in p(e\cdot k=k)]$.

If $\underline{K}$ is a Boolean-regular class and $\forall K\in\underline{K}(K\models\Phi_1)$, we can reduce this condition to the simplier one:

$(K,L\in\underline{K})(K\subseteq L)\Rightarrow\forall p_1\in X(L)[(p_1\cdot L)\cap K\subseteq(p_1\cap K)\cdot K]$.

<u>Proposition 5</u>. Let $\underline{K}$ be a Boolean-regular class and $\forall K\in\underline{K}(K\models\Phi_1)$. Then

a) If $\underline{K}$ is AE-model complete, then $\underline{K}$ is Boolean-simple.

b) Boolean-simplicity of $\underline{K}$ is equivalent to following condition:

$K\subseteq L \Rightarrow \forall k\in K\forall e\in B(K)((K|=\psi(k,e))\Rightarrow(L|=\psi(k,e)))$. where

$\psi \leftrightharpoons \forall e_0(Et)(e_0^2=e_0 \& e_0\cdot t=t\cdot e_0 \& e_0\cdot k=0 \Rightarrow e_0\cdot e=e_0)$.

<u>Proof.</u> Clause a) immediately follows from clause b).

b) Let the respective condition on $\underline{K}$ holds good. And let $e\leftrightharpoons[\![k=0]\!]_K$. Then $L|=\psi(k,e)$, i.e. $[\![k=0]\!]_L\leq[\![k=0]\!]_K$. From this it follows successively $[\![k\neq 0]\!]_K\leq[k\neq 0]_L$, $\forall p_1\in X(L)\forall k\in K[((Ee)\in p_1(e\cdot k=k))\Rightarrow [\![k\neq 0]\!]_K\leq e_1$, $[\![k\neq 0]\!]_K\in p_1\cap K$, $[\![k\neq 0]\!]_K\cdot k=k]$, i.e. $(Ee)\in(p_1\cap K)(e\cdot k=k)$. The latter proved condition is even much stronger than the Boolean-simplicity of class $\underline{K}$. On the contrary, assume $e_0\in B(L)$ and $e_0\cdot k=0$. Suppose $e_0\cap(1-e)\neq 0$. By the assumption we choose a point $p_1$ in $X(L)$ such that $p_1\in e_0\cap(1-e)$ and $\bar{p}_1\cap K\subseteq\bar{p}$, where $p$ is equal to $p_1\cap K$ and also $p$ is a prime ideal in $B(K)$. There exists $e_1\in p$ for which $e_1\cdot k=k$ as $(1-e_0)\cdot k=k$. Hence, $(1-e_1)\in p_1$, $(1-e_1)\cdot k=0$, $(1-e_1)\leq e$, $e\notin p_1$. But, on the other hand, $e\in p_1$. Contradiction. $\square$

A class $\underline{K}$ is called Boolean-absolute iff it is Boolean-regular and Boolean-simple.

Theory T is called autonomous (respectively, normally autonomus) iff every model K of this theory can be embedded into a ring F such that $\{F_p\}|=T$ (respectively. into a normal ring F, such that $\{F_p\}|=T$).

Let $X_1(K)$ denote the set of all proper ideals of $B(K)$. If $q\in X_1(K)$, then $\bar{q}\leftrightharpoons q\cdot K$ is an ideal in K. By $\{K_q\}$ we denote $\{K/\bar{q}\,|\,q\in X_1(K)\}$. Theory T is called totally autonomus iff $\{F_q\}|=T$ for each model F of theory T.

Let T be a theory in the disjunctive normal form. By T' we denote $\{\varphi'|\varphi\in T\}$ ($\varphi'$ was defined in the Remark in Section 2; details see in [3]). Let us mention that $\varphi'$ is a Horn formula.

<u>Theorem 2.</u> a) The class $\underline{K}^*$ is axiomatizable (and even Horn axiomatizable). If $\underline{K}_1$ is defined as $\{K|(\{K_p\}|=T)\&(K|=\Phi_1)\}$, where T is an AE-theory (or $\underline{K}_1$ is defined as $\{K|\{K_p\}|=T\}$, where T is an AE-positive theory), then $\underline{K}_1$ is Horn axiomatizable. Namely,

$$\underline{K}_1=\{K|(K|=T')\&(K|=\Phi_1)\}.$$

b) If $\underline{K}^*$ is a Boolean absolute class then $\underline{K}^*$ is modelly complete.

c) If T* is a normally autonomous theory, then $\underline{K}$ can be embedded into $\underline{K}^*$.

d) Let $T \subseteq T^*$. If $T^*$ is a normally autonomus theory and $\underline{K}^*$ is a Boolean-absolute class, then $\underline{K}^*$ is a Horn model companion for $\underline{K}$ (it is true that $\underline{K}^* \subseteq \mathrm{Mod}T' \subseteq \underline{K}$). Under the same conditions $(T^*)'+\Phi_1+\Phi_2$ is the model companion for $T'$.

e) If $T^*$ is a normally autonomus theory, $T$ is a totally autonomus theory, and $\underline{K}^*$ is a Boolean-absolute class, then $\underline{K}^*$ is the Horn model companion for $\underline{K}$ (here we suppose that the possibility of embedding of $T^*$ into $T$ is provable in ZFC).

<u>Proof.</u> a) The class $\underline{K}_1 \rightleftharpoons \{K \mid \{K_p\} \models T^*, K \models \Phi_1\}$ is Horn axiomatizable, namely $\underline{K}_1 = \{K \mid K \models (T^*)', K \models \Phi_1\}$. In fact, if $K \in \underline{K}_1$, then by Proposition 1 and the Remark from Section 1 we can obtain $K \models \varphi'$, where $\varphi \in T^*$. Conversely, by the same Remark and Proposition 3 we have $\{K_p\} \models \varphi T^*$. All formulas $\varphi'$ and the formula $\Phi_1$ are Horn formulas. The class $\underline{K}^*$ is designed within the class $\underline{K}_1$ by Horn axiom $\Phi_2$. The second proposition of this clause generates the first proposition of that clause as any modelly complete theory is AE-axiomatizable. If $K \in \underline{K}_1$, then on the basis of Proposition 2 and the Remark we can obtain $K \models T'$. If $K \models T'\Phi_1$, then with the help of the Proposition 3 and the Remark we can obtain $K \in \underline{K}_1$.

b) Let $\psi$ be a primitive formula with parameters from ring $K$, where $K \in \underline{K}^*$. Let in addition ring $L$ be any extension of the ring $K$ and $L \in \underline{K}^*$. Let us denote by $\psi_l$ the statement expressing the existence of a solution of a subsystem of the original system. This subsystem is assumed to consist of all equalities and at most one inequality of the original system. An index $l$ enumerates all such subsystems. It was proved in [2], that if a ring $L$ is normal and atomless, then $(L \models \psi) \Leftrightarrow [(\forall l(\{p \in X(L) \mid L_p \models \psi_l\} \neq \emptyset)) \& (\bigcup_{e}\{p \mid L_p \models \psi_l\} = X(L))]$. We shall demonstrate this fact, too. By Proposition 1 the set in figured brackets coincides with the set $[\![\psi_l]\!]_{\mathcal{T}}$ and also is open-closed. From the left-handside to the right-handside the above equivalence is obvious. Conversely, denote $u_l \rightleftharpoons [\![\psi_l]\!]_{\mathcal{T}}$, where $[\![\psi_l]\!] \in B(L)$. Let us form the Boolean subalgebra in $B(L)$ generated by the finite set $\{u_l\}$. This subalgebra is finite and therefore is atomic. In $u_1$ we further choose an atom $v_1$. If in $u_2$ there is no other atom than $v_1$, then (otherwise we choose an atom $v_2$ in $u_2$, different from $v_1$) we split the atom $v_1$ into two disjoint parts $v_1$ and $v_2$ and after that we again obtain

$v_1 \leq u_1$, $v_2 \leq u_2$, $v_1 \neq v_2$. Here is essentially used the atomlessness of the

ring $L$. Continuing this process over all $e$, we shall obtain the disjunctive set $\{v_l\} \subseteq B(L)$, where $0 \neq v_l \leq u_l$. Finally, we accomplish this set by elements $W_1 \in B(L)$, such that $W_1 \leq u_1$ and $\{\ldots v_1 \ldots\ldots\ldots w_1 \ldots\}$ is a decomposition of unit 1. Using the accessibility of the evaluation, we get $\{k_1\}$ for which $u_1 \leq [\![\psi_1(k_1)]\!]$. Cluing all $\bar{k}_1$ on $v_1$ and all $\bar{k}_1$ on $w_1$, we obtain $\bar{k} \in L$. And so, $\bar{k}$ is a sollution of the original system $\psi$ in L.

By the agency of this equivalence we have $(L|=)\neg\psi) <=> [(\underset{e}{V}([\![\psi 1]\!]=0)) \vee (\underset{e}{U}[\![\psi_1]\!]<1)]$, where in place of 0 and X(L) we write 0 and 1, respectively.

We successively rewrite the first disjunctive member in the following way: $\underset{e}{V}([\![\neg\psi 1]\!]=1)$; using model completeness of the theory T* we find an E-formula $\psi'1$ (having $\underset{s}{V}\psi_{1s}$ as the quantifiers part), being equivalent to $\neg\psi_1$ in all models for T*, i.e. $\underset{e}{V}([\![\psi'_1]\!]=1)$; next, using accessibility of the evaluation we further get $\underset{e}{V}(E\bar{k})(U[\![\psi_{1s}]\!]=1)$, i.e. $\underset{e}{V}(E\bar{k})(E\bar{e}_{1s}) \in B(L)(e_{1s} \leq [\![\psi_{1s}]\!] \& \Pi(1-e_{1s})=0$. The latter can be written in the form $(E\bar{k})(E\bar{e}_{1s}) V \forall_{e}\, e_{0}\, s\, [\Pi(1-e_{1s})=0 \& e_{11} \cdot k_1 = e_{11} \cdot t_1 \& \ldots \& ((e_0 \cdot k_2 = e_0 \cdot t_2) => e_0 \leq 1-e_{11}) \& \ldots]$, where $k_1 = t_1$ is one of the equalities, while $k_2 \neq t_2$ is one of the inequalities from $\psi_{11}$. Here $e_{1s}$, $e_0$ are special variables ranging over B(L).

Similarly, for the second disjunctive member we obtain the chain of equivalences: $(Ep)EX(L)\underset{e}{\&}(p \notin [\![\psi_1]\!])$, $(Ep) \in X(L)(p \notin [V\psi_1])$, $[\![\neg \underset{e}{V}\psi_1]\!]>0$; next, using model completeness of the theory T* we find E-formula with the quantifierless part $V\psi_s$ such that $[\![\psi]\!]>0$; further, via accessibility of the evaluation we get $(Ek)([\![\underset{s}{V}\psi_s]\!]>0)$, $(Ek)(Ee)\underset{s}{V}((e \neq 0) \& ([\![V\psi_s]\!] \geq e))$. The latter can be written in the form

$(Ek)(Ee)\underset{s}{V}\forall e_0 (e_0 \neq 0 \& e \cdot k_1 = e \cdot t_1 \& \ldots \& ((e_0 \cdot k_2 = e_0 \cdot t_2) => e_0 \leq 1-e) \& \ldots)$, where $k_1 \neq t_1$ is one of the equalities, while $k_2 \neq t_2$ is one of the inequalities from the $\psi_s$. Here $e, e_0$ are special variables ranging over B(L).

By the agency of the absoluteness of the class $\underline{K}$* and Proposition 5b we obtain that $\neg\psi$ can be transferred from K onto L, where $K \subseteq L, K, L \in \underline{K}$*.

Finally, the class $\underline{K}$* is model complete for the axiomatizable class $\underline{K}$* the criterion of A. Robinson holds good.

c) If $K \in \underline{K}$*, then K is embeddable into $\underset{P}{\Pi} K_p$, where $K_p|=T$. By the definition of model companion a ring $K_p$ is embeddible into $F^p$, where $F^p$ is a model for $T^*$. By the condition of normal autonomous of the theory $T^*$ we may consider $F^p$ to be a normal ring whose all localizations are models for $T^*$. We define in $F^p$ the discrete topology. Let $X_0$ be the Cantor discontinum. Denote by $\overline{F^p}$ the algebra $C(X_0, F^p)$ of all continuous mappings of $X_0$ into $F^p$. The ring $\overline{F^p}$ consists of locally constant $F^p$-valued mappings of $X_0$. having finite set of values. The ring $F^p$ is embeddible into the ring $\overline{F^p}$. Furthermore, we shall show that $\overline{F^p}\in\underline{K}^*$. The class $\underline{K}^*$ is closed under arbitrary products because it is a Horn class. Therefore, $(\prod_p F^p)\in K$ and, hence, $K \rightarrowtail \prod_p K_p \rightarrowtail \prod_p F^p \rightarrowtail \prod_p \overline{F^p} \in \underline{K}^*$.

Now, consider the ring $\overline{F} \leftrightharpoons C(X_0, K)$, where $F$ is a normal ring such that all its localizations are models for $T^*$. It is true that $B(\overline{F})=C(X_0, B(F))$. The set $\langle x_0, p_0\rangle$ by definition being equal to

$\{f\in B(\overline{F}) \mid f(x_0)\in p_0\}$ is a prime ideal in $B(\overline{F})$ for every $x_0\in X$ and $p_0\in X(F)$. Any point from $X(F)$ has such a form (what symbolically can be written by equality $X(F)=X_0 * X(F)$, because for every $p\in X(\overline{F})$ there exists $x_0\in X_0$ for which $p_0=\{f(x_0) \mid f\in p\}$ does not contain the unit of $B(F)$. Otherwise, $\{\{x_0\in X_0 \mid f(x_0)=1\} \mid f\in p\}$ would be an open cover of $X_0$

and its subcover would give us such $f_1, \ldots, f_n\in p$, that $f_1 \vee \ldots \vee f_n \in p$ and

$f_1 \vee \ldots \vee f_n = 1$, that is a contradiction. The above $p_0$ is a prime ideal in

$B(F)$. Therefore, $p \subseteq \langle x_0, p_0\rangle$. By maximality of $p$ (all this takes place

in Boolean algebras) the latter is possible only if $p=\langle x_0, p_0\rangle$.

Furthermore, $\overline{\langle x_0, p_0\rangle}=\{f\in\overline{F} \mid f(x_0)\in p_0\}$, where $\overline{p}_0 \leftrightharpoons p_0 F$ and $\overline{\langle x_0, p_0\rangle} \leftrightharpoons$

$\langle x_0, p_0\rangle \cdot \overline{F}$. Thus, $(\overline{F})_{\langle x_0, p_0\rangle}=\overline{F}/\overline{\langle x_0, p_0\rangle}\cong\overline{F}/\overline{p}_0=F_{p_0}$, where $p_0$ ranges over

$X(F)$. By the assumption for all $F_{p_0}$ it is true that $F_{p_0} \models T^*$. From this

it follows $\{(\overline{F})_{\langle x_0, p_0\rangle}\} \models T^*$. Let us verify the normality of $F$. If $f\in\overline{F}$,

then we put $e_0(x) \leftrightharpoons e_i$, where $e_i$ is an element from $F$, corresponding to the element $f(x)$ from $F$ under the normality property of $F$. Such an $e_0$ satisfies the definition of normality for $f$ in $\bar{F}$.

Assume that $f$ is an atom in $B(\bar{F})$. At least one "step" of the function $f$, for example, $f(x_0)$ is different from 0. This step is a constant function on an open-closed set, containing at least two distinct points. Eliminating one of these two points together with some its open-closed neighbourhood we could come to the conclusion that $f$ is not an atom. This would contradict to the assumption.

d) From clauses b) and c) it follows immediately the first proposition. If $K \in \underline{K}^*$, then $K \models (\Phi_1 \& (T^*)')$ and moreover $K \models T'$. If $K \models T'$, then by Remark in Section 1 and Proposition 3 we shall obtain $K \in \underline{K}$. From this it follows the last proposition.

e) To clause d) one should add only the embeddability of the class $\underline{K}^*$ into $\underline{K}$. Besides this in clause d) there is nowhere usage of the condition which we want to eliminate now.

This follows from general Theorem 3 which will be proved below.

<u>Theorem 3</u>. Let a theory $T$ be model embeddable into a theory $T_1$ (and this fact be provable in ZFC). If $T$ is an AE-theory, $T_1$ is a totally autonomous theory and $K$ is a ring such that $K \models \Phi_1$ and $\{K_p\} \models T$, then $K$ can be embedded into the class $\underline{K}_1 = \{L \mid \{L_p\} \models T_1\}$.

To apply this theorem to the proof of Theorem 2, let $T$ be equal to $T^*$, $T_1$ be equal to $T$. Theory $T^*$ can be modelly embedded into $T$ by the definition of the model companion of a theory. It is known that $T^*$ is an AE-theory and that each $K$ belonging to $\underline{K}^*$ is normal.

<u>Remark</u>. 1) We have used only the following two properties in the proof of clause c) of Theorem 2: the closedness of $\underline{K}^*$ with respect to the products and the existence of model embedding of $T$ into $T^*$. Therefore, Theorems 2c) and 3 can be considered as "embedding theorem for locally axiomatizable classes".

2) Actually, in Theorem 2d) we have proved the formula $(T^*)' = (T')^*$ (assuming that the term $(T^*)'$ contains axioms $\Phi_1 \& \Phi_2$). $\square$

<u>Proof</u> of Theorem 3. Let $K$ satisfy the assumption of the theorem. By Proposition 2 we shall get $[\![T]\!]_{\mathcal{T}} = 1$. From here we further get $[\![T]\!]_{B(K)} = 1$. By Theorem 16e [3] the latter means $[\![T]\!]_{\underline{B},K'} = 1$, or $[\![K' \models T]\!]_{\underline{B}} = 1$, where $\underline{B} \leftrightharpoons \underline{B}(K)$. By the assumption the formula $\forall x(Ef)(x \models T \Rightarrow \bar{x} \subseteq f \& f \models T_1)$ is deducible in ZFC. From here by the accessibility in $V^{\underline{B}}$ we get $[\![K' \subseteq f \& f \models T_1]\!]_{\underline{B}} = 1$, where $f \in V^{\underline{B}}$. Denote $(f)\hat{\ }\underline{B} = L$. Again by Theorem 16e we have $K \subseteq L$ (in the sense that $k \mapsto p_k$) and having verified the inclusion $L \in \underline{K}_1$, we shall obtain the proof of the theorem.

In fact, $\underline{B}$ is imbeddeble into $B(L)$ by the rule $b \mapsto b \cdot 1 + \neg b \cdot 0$ (the right handside is the result of glueing together 1 and 0 in $f$). This imbedding will be denoted by h. Denote $L_{(p_0)} \leftrightharpoons L/\overline{h(p_0)} = L/(h(p_0)\cdot L)$, where $p_0$ is a point of Stone space $S(\underline{B})$ of the Boolean algebra $\underline{B}$, while $\overline{h(p_0)}$ is an ideal in L. We shall show that $L_{(p_0)}$ coincides with the "stalk" f in the point p, i.e. with the factorization of L by the equivalence relation $(k \sim_{p_0} t) \leftrightharpoons ([\![k=t]\!]_{\underline{B}} \notin p_0)$ (see [3]). We shall also show that $([\![f \models \varphi(k_1,\dots,k_n)]\!]_{\underline{B}} = 1) \Leftrightarrow \{L_{(p_0)}\} \models \varphi([k_1]_{p_0},\dots,[k_n]_{p_0})$, where $p_0$ ranges over $S(\underline{B})$ and $k_1,\dots,k_n \in L$. The first statement means: $(Eb)\in p_0 (h(b)\cdot(k-t)=k-t) \Leftrightarrow [\![k=t]\!]_{\underline{B}} \notin p_0$. From left to right we have $h(\neg b)=1-h(b)$, $(1-h(b))\cdot(k-t)=0$ in L, and further $\neg b \notin p_0$ and $[\![(1-h(b))\cdot(k-t)=0]\!]_{\underline{B}}=1$, $\neg b \le [\![k-t=0]\!]_{\underline{B}}$. Conversely, let $b \leftrightharpoons [\![k=t]\!]_{\underline{B}} \notin p_0$, $b \le [\![h(b)\cdot(k-t)=0]\!]_{\underline{B}}$ and $\neg b \le [\![h(b)\ (k-t)=0]\!]_{\underline{B}}$, $h(b)\cdot(k-t)=0$ in L and $\neg b \in p_0$, $h(\neg b)\cdot(k-t)=k-t$. The second statement can be verified in the form $p_0 \in [\![f \models \varphi(\bar{k})]\!]_{\underline{B}} \Leftrightarrow L_{(p_0)} \models \varphi([k]_{p_0})$ for connectives $\&, \neg, (E)$ by induction on the length of the formula $\varphi$. And so, $\{L_{(p_0)}\} \models T_1$.

For any $p \in X(L)$ we form $p_0 \leftrightharpoons h^{-1}(p) \subseteq \underline{B}$. We have $p_0 \in S(\underline{B})$. Denote the ideal $\overline{h(p_0)}$ in L by a. Then $a \subseteq \bar{p}$ and $L_p = L/\bar{p} \cong (L/a)(\bar{p}/a) = L_{(p_0)}/\overline{p/a}$. Note that $q \leftrightharpoons p/a$ has the properties: $q \subseteq B(L/a)$ and q is closed w.r.p. operation v and does not contain $[1]_a$. In fact, if $[1]_a = [e]_a$ for $e \in p$, then $1-e=e_0\cdot r$, $e_0 \in h(p_0)$, $1=e+e_0\cdot r=(e \vee e_0)\cdot 1$, $e \vee e_0 \in p$. So, we have a contradiction. We add q those elements from $(L/a)$, which can be majored by some $[e]_a$ from q. The $q_1$ obtained in such a way will be a proper ideal in $B(L/a)$ and, in addition, $\bar{q}_1 = \bar{q}$. So, $L_p = L_{(p_0)}/\bar{q}_1$, where $q_1 \in X_1(L_{(p_0)})$ and by the condition we finally obtain $L_p \models T_1$. □

Let us now consider the converse problem: how to transfer model completeness of $\underline{K}^*$ and $\underline{K}$ to their local theories. The following generalization of the notion of Boolean-simplicity of a class is convenient for this purpose. Let $\underline{K}_1$, $\underline{K}$ be two classes of rings. We call $\underline{K}_1$ a Boolean-simple class for $\underline{K}$ iff $\forall K\in\underline{K}\forall L\in\underline{K}_1(K\subseteq L\Rightarrow(Ep_1)\in X(L)(Ep)\in X(K)$ $[p\supseteq p_1\cap K\&\forall k\in K((Ee)\in p_1(e\cdot k=k))\Rightarrow((Ee)\in p(e\cdot k=k))]$.

The condition "a class $\underline{K}_1$ is a model companion for a class $\underline{K}$ and $\underline{K}_1$, $\underline{K}$ are mutually Boolean-simple" would serve as the assumption on the classes $\underline{K}_1$, $\underline{K}$ in Theorem 4 below. We denote the above condition by (*). But, in Theorem 4 we actually use the weaker assumption on the classes $\underline{K}_1$, $\underline{K}$. This weaker assumption uses a special type of the classes $\underline{K}_1$, $\underline{K}$ in Theorem 4. Let us remember that the ring $L=C(X_0,L)$ was defined in the proof of Theorem 2. So, we call the class $\underline{K}_1$ a weak model companion for the class $\underline{K}$ iff:

1) for two rings $\bar{L}_1$, $\bar{L}_2$ belonging to $\underline{K}_1$, where $L_1, L_2\models T_1$, $L_1\subseteq L_2$ and for each Horn AEAE-formula $\varphi$ with constants from $L_1$, such that $\bar{L}_1\models\varphi$ there must be true that $\bar{L}_2\models\varphi$;
2) each $K\in\underline{K}$ such that $K\models T$ can be embedded into some $L\in\underline{K}_1$; and each $L\in\underline{K}_1$ such that $L\models T_1$ can be embedded into some $K\in\underline{K}$;
3) for each $K\in\underline{K}$ such that $K\models T$ and each $L\in\underline{K}_1$, such that $K\subseteq L$ there exists $p\in X(K)$ and $p_1\in X(L)$ such that $p\supseteq p_1\cap L$ and $\forall k\in(K\cap\bar{p}_1)(k\in\bar{p})$;
4) for each $\bar{L}\in\underline{K}_1$ such that $L\models T_1$ and each $K\in\underline{K}$ such that $\bar{L}\subseteq K$ there exists $p\in X(\bar{L})$ and $p_1\in X(K)$ such that $p\supseteq p_1\cap\bar{L}$ and $\forall l\in L(l\in\bar{p}_1\Rightarrow l\in\bar{p})$. It is clear that condition (*) implies that $\underline{K}_1$ is a weak companion for $\underline{K}$.

<u>Theorem 4</u>. Assume that $\{K|\{K_p\}\models T_1,\ K\models(T'_1+\Phi_1+\Phi_2)\}\subseteq\underline{K}_1\subseteq\{K|\{K_p\}\models T_1\}$, $\underline{K}\neq\{K|\{K_p\}\models T\}$ and $\underline{K}_1$ is a weak model companion for the class $\underline{K}$. If $T\vdash\Phi_S$ and $T_1\vdash\Phi_S$, then the theory $T_1$ is a model companion for T. (If $T\subseteq T_1$, then the condition 4 in the definition of a weak model companion can be omitted.)

<u>Proof.</u> Here we shall show the completeness of the theory $T_1$. Let $F_1$ and $F_2$ (where $F_1\subseteq F_2$) be two models for $T_1$ and $\psi$ be a formula which is primitive over $F_1$, $F_1\models\neg\psi$. From the two rings $\bar{F}_1\neq C(X_0,F_1)$ and $\bar{F}_2\neq C(X_0,F_2)$, where $\bar{F}_1\subseteq\bar{F}_2$ (see proof of Theorem 2). The localizations of $\bar{F}_1$ and $\bar{F}_2$ coincides with $F_1$ and $F_2$, respectively. Since $T_1\vdash\Phi_S$ we

have $\{(\bar F_1)_p\}\models T_1$, $\bar F_1\models\Phi_1\&\Phi_2$ and moreover all localizations $\bar F_1$ (and $\bar F_2$) coincide with $F_1$ (and $F_2$, resp.). In this situation by induction on the length of an arbitrary formula $\varphi$ we verify the following generalization of Proposition 2:

$$\{(\bar F_1)_p\}\models\varphi(\lambda_1^p,\dots,\lambda_n^p)\Rightarrow[\![\varphi(\lambda_1,\dots,\lambda_n)]\!]_{\mathfrak{T}(\bar F_1)}=1,\quad \text{where } \lambda_1,\dots,\lambda_n$$

are constants from $F_1$. The proof of Proposition 2 needs only to be accomplished by observing that $(\bar F_1)_p$ is isomorphic to $F_1$ by the formula $[f]_{\bar p}\mapsto f(x_0)$, where $x_0$ corresponds to p. And so, $\bar F_1$, $\bar F_2\in\underline{K}_1$. For any A-formula $\neg\psi$ (as well as for any formula with constant parameters) there must be valid in a normal ring the condition $[\![\neg\psi]\!]_{\mathfrak{T}(\bar F_1)}=1$. By the Remark $\bar F_1\models(\neg\psi)'$. Then by the assumption $\bar F_2\models(\neg\psi)'$ and by the Remark $[\![\neg\psi]\!]_{\mathfrak{T}(\bar F_2)}=1$, i.e. $F_2\cong(F_2)_p\models\neg\psi$.

Let K be a model for T. As $T\vdash\Phi_5$, then $K\in\underline{K}$. By the condition $K\subseteq L\in\underline{K}_1$. By the condition of Boolean-simplicity there can be found such $p_1\in X(L)$ and $p\in X(K)$ that $p\supseteq p_1\cap K$, i.e. $p=p_1\cap K=\{0\}$. From here we get $K\to L_{p_1}$, $k\mapsto[k]_{\bar p_1}$ is an imbedding and by the assumption of Theorem we also get $L_{p_1}\models T_1$. If $T\subseteq T_1$, then the proof is done.

It rests to verify that $T_1$ is model imbeddable into T. let $L\models T_1$. Form $\bar L\in\underline{K}_1$. By the assumption of Theorem we get $\bar L\subseteq K\in\underline{K}$ and we also find $p_1\in X(K)$ and $p\in X(\bar L)$, for which $p\supseteq q\varsubsetneq(p_1\cap\bar L)$ and p has the corresponding properties. As we saw in the proof of Theorem 2c this p has the form $\langle x_0,p_0\rangle$. where $p_0\in X(L)$, i.e. $p_0=\{0\}$. Therefore $\bar p=\{f\in\bar L\mid f(x_0)=0\}$. Let us put $L\rightarrowtail K/\bar p_1$, $e\mapsto[e]_{\bar p_1}$. By the assumption this is really an imbedding. From here we obtain the required imbedding of L into $K_{p_1}$, where $K_{p_1}\models T$. $\square$

<u>Corollary.</u> The class $\underline{K}$ of all abelian regular rings has no weak (ordinary) model companion within classes having the form $\underline{K}_1$ (from Theorem 4), where $T_1\vdash\Phi_5$ ($T_1\vdash$ "is simple").

<u>Proof</u>. If such a $\underline{K}_1$ is a model companion for $\underline{K}$, then $T_1$ must be a model companion for the solid theory, that is impossible.$\square$

<u>Remark</u>. The conditions $T|-\Phi_5$, $T_1|-\Phi_5$ (Theorem 4 and its Corollary) can be weakened. Formula $\Phi_5$ is defined on page 22. □

A primitive formula is called 1-primitive iff it contains no more than one inequality. We shall say that "theory T decides a class of formulas $\Sigma$" iff $T|-\varphi$ or $T|-\neg\varphi$ for each formula $\varphi\in\Sigma$.

<u>Theorem 5</u>. a) Assume that local theory T decides all 1-primitive statements and that all sets B(K) for each K belonging to the class $\underline{K}\leftrightharpoons\{K|\{K_p\}|=T\}$ are infinite. Then the theory $Th\underline{K}$ decides all E-statements.

b) if $\underline{K}^*$ is a Boolean-absolute class and the model complete theory T* decides all 1-primitive statements then the theory

$(Th\underline{K}^*)<=>(\Phi_1+\Phi_2+ +(T^*)')$

is a model complete Horn theory.

<u>Proof</u>. a) Let $K\in\underline{K}$ and $\psi$ be a primitive sentence. (The satisfiability of the assumption of K is guaranteed, for example, by the atomlessness of K). Then all $\psi_l$, $1\leq l\leq L$, which are formed by $\psi$ as in the proof of theorem 2b will be 1-primitive sentences. By the assumption either $T|-\psi_l$ for all l or $T|-\neg\psi l_0$ for some $l_0$. In the first case $\forall p(K_p|=\&\ \psi_l)$. Let us choose in X(K) exactly L distinct points $p_1,\dots,p_L$. By separability they will have neighbourhoods $u_1,\dots,u_L$ beeing disjunctive to each other. In $p_1$ there will be valid $\psi_1$ for some $k_1(p_1),\dots,k_n(p_1)$, while the equalities from $\psi_1$ are valid also on some open-closed neighbourhood $u_1'$ inside $u_1$. Therefore $k_1,\dots,k_n$ on $u_1$ satisfies $\psi_1$. In the same way we shall find $t_1,\dots,t_n$ on $u_L'$, satisfying $\psi_L$. Further, let us glue all $k_1,\dots,t_1$ and so on. Then we get

$k_1,\dots,k_n$ on $u_1'\cup\dots\cup u_L'$, satisfying $\psi$. We extend these $k_1,\dots,k_n$

onto the complement to $u_1'\cup\dots\cup u_L'$ in such a way that all equalities from $\psi l_0$ will be satisfied. And so, $K|=\psi$. In the second case $\forall p(K_p|=\psi e_0)$. From here (even without normality and atomlessness) we get $K|=\neg\psi$.

Now we easily obtain the decidability of all E-formulas.

b) We infere this clause from clause a) and Theorem 2a,b by taking into account the fact that in modelly complete theory any formula is equivalent to some E-formula.

<u>Example 2</u>. The part of our article up to Example 3 constitutes the content of the current Example 2 which includes some propositions and theorems. Here we restrict ourselves to the associative rings with

unit (however, this restriction is not necessary: we can consider also rings without unit and non-associative rings studied in [11]). Let $L$ be the class of all primary PI-rings A (over the commutative ring R with unit), which have fixed degree S. The centre Z(A) of the ring A (which will be denoted later by F) is an integral ring (i.e. a commutative ring without divisors of zero). Let us remember that algebra A is called a PI-ring iff there exists a noncommutative polynome over R with at least one senior coefficient 1 such that this polynome is equal to 0 for each element of A. The degree of such an algebra A defined as the least degree of such polynomes. The algebra $A_R$ (considered as a central algebra) can be embedded into its (classical) quotient ring $S_A \cong A \otimes_F F_{cl}$, where $F_{cl}$ is the quotient field for F. The field $F_{cl}$ can also be embedded into $S_A$. These embeddings can be defined by the formulas $a \mapsto a \otimes 1$ and $f \cdot g^{-1} \mapsto 1 \otimes f \cdot g^{-1}$ (because modules $A_F$ and $F_{F_{cl}}$ are forsion-free). Each maximal linearly independent system $\{a_i\}$ in A over F generates a basis $\{a_i \otimes 1\}$ in $S_A$ over $F_d$; and each basis in $S_A$ can be converted into such a basis (details see in [12]). So, some properties of A are connected with the corresponding properties of $S_A$ and vice versa. The algebra $S_A$ is a m-dimensional algebra over its centre $F_d$ and is simple by Pozner theorem. For each simple algebra A having dimension m over its centre, we have $m=n^2$ for some integer n. We call these algebras "n-algebras" [12]. In our case $n=[\frac{S}{2}]$. It is possible to axiomatize within the class of primary rings the property "ring A is a PI-algebra of s-degree" by the formula $A \models S_{2 \cdot n} \& \neg S_{2 \cdot n-2}$, where $S_k$ is a standard identity of the degree k ([12,p.498]). Another axiomatization can be given by the formula "there exists a maximal linearly independent (over F) system with n elements". So the class $\underline{L}$ can be axiomatized by these two axioms. We denote them by $T_{\underline{L}}$.

Let us consider the class $\underline{L}_{T_0}$ of all n-algebras A, such that their center F satisfies the fixed theory $T_0$. We have $\underline{L}_{T_0} \subseteq \underline{L}$, because $A_F$ satisfies the same identities as the matrix ring $M_n(F)$ ([12,p.497]), and, consequently, it satisfies the standard identity $S_{2 \cdot n}$.

<u>Proposition 6</u>. The class $\underline{L}_{T_0}$ can be axiomatized (by some theory T formulated below).

<u>Proof</u>. Artin-Weddernburn theorem implies that $A_F \cong M_k(D)$, where D is a solid and also a s-dimensional algebra over its centre $F \cong Z(D)$ ([12, p.283]). It is clear that $n^2 = k^2$ s. So, we can axiomatize $\underline{L}_{T_0}$ by

the theory T which contains the following axiom

$$\bigvee_{k^2\cdot s=n^2}(E\{e_{ij}\}^{k})\forall x(Ey)\{\sum_{i=1}^{k} e_{ii}=1\& e_{ij}\cdot e_{pq}=\delta_{jp}\cdot e_{iq}\ \&$$

$$[x\cdot e_{ij}=e_{ij}\cdot x \Rightarrow y\cdot e_{ij}=e_{ij}\cdot y\&(x=0 \vee x\cdot y=y\cdot x=1)])\&$$

$$(Ex_1,\dots,x_s)[\forall x(E)y_1,\dots,y_s\forall u\ \ (x_1\cdot e_{ij}=e_{ij}\cdot x\&\dots\& y_1\cdot u=u\cdot y_1\&\dots\&$$

$$x=x_1\cdot y_1+\dots+x_s\cdot y_s)\&(x_1,\dots,x_s)$$

are linearly independent over the centre)]}, where $\delta_{jp}$ is the Kroneker symbol and also axioms of $T_0$ relativized to the centre F. Let A be the model of T. It is aesy to check that $A\cong M_k(Z(\{e_{ij}\}))$, where $Z(\{e_{ij}\})$ is a centralizator of the system of "matrix units" $\{e_{ij}\}$. This centralizator is always a ring, in our case it is a s-dimensional solid over the centre of the ring A. □

<u>Proposition 7</u> a) The class $\underline{K}_{T_0}\leftrightharpoons\{K\mid\{K_p\}\models T\}$ is a Boolean-regular class (as well as all its subclasses).

b) Moreover, $K\subseteq L\Rightarrow Z(K)\subseteq Z(L)$ for all $K,L\in\underline{K}_{T_0}$.

<u>Proof.</u> Let us mention that $\underline{L}_{T_0}\subseteq\underline{K}_{T_0}$. It suffieces to verify only clause b). We call by a marked polynome a multi-linear non-commutative polynome with the only coefficients $\pm 1$ (containing at least one variable and being not equal to zero) such that all values of the polynome in every central algebra $M_n(F)$ belong to its centre. Here F is an arbitrary simple field ($\underline{Q}$ or residue field). There exists a marked polynome (for example, polynome $\psi$ constructed by Yu. Rozmyslov is marked). It is clear that every marked polynome has the same list of properties for all $n\cdot n$ - matrix rings over all fields and even for all n-algebras (because each n-algebra $A_F$ has the same identities as $M_n(F)$, where $F\leftrightharpoons Z(A)$. The image $\psi(A)$ of $\psi$ on A coincides with the whole centre F for each algebra A. Assume that $f\in Z(K)$ and $K\subseteq L$, where $K,L\in\underline{K}_{T_0}$.

For each point $p_0\in X(K)$ we have $k(p_0)\in Z(K_{p_0})$. So, there exists $t_1^{p_0},\dots,t_m^{p_0}\leftrightharpoons\bar{t}^{p_0}$, such that $k(p_0)=\psi(\bar{t}^{p_0}(p_0))$. This equality is also true in some closed-open neighbourhood $e_{p_0}$ of the point $p_0$, i.e.

$\forall p\in e_{p_0}$ $(k(p)=\psi(\bar{t}^{p_0}(p)))$. Choose a disjoint covering $e_{p_1},\dots,e_{p_e}$ of the whole $X(K)$. Let us combine $t_1^{p_1},\dots,t_1^{p_e}$ on $e_{p_1},\dots,e_{p_e}$ into a section $t_1 \leftrightharpoons \sum_{i=1}^{e} t_i^{p_i}, e_{p_i}$ and construct also $t_2,\dots,t_m$. Then $k=\psi(\bar{t})$. Because of $\bar{t}\in L$ we have $k=\psi(\bar{t})$ "in L" and $k(p)=\psi(\bar{t}(p))$ for each $p\in X(L)$. Therefore, $k(p)\in Z(L_p)$ and $k\in Z(L)$.

Theorem 6. If $T_0$ is a modelly complete theory then $T$ is a modelly complete theory.

Proof. Let $A,B$ be in $\underline{L}_{T_0}$, $A\subseteq B$ and $F\leftrightharpoons Z(A)$, $G\leftrightharpoons Z(B)$. By Proposition 7 we have $F\subseteq G$. It is easy to show that each basis $\{x_1,\dots,x_n\}$ in $A$ over $F$ is also a basis in $B$ over $G$. Indeed. let us construct a subalgebra $A\cdot G$ in $B$ over $G$, generated by a subring $A$. Then $Z\leftrightharpoons Z_{A\cdot G}(A)\supseteq F,G$: $F,G\subseteq Z(A\cdot G)$ and $A\cdot G$ is an algebra over $F$. According to [12, p.289], the mapping $a\otimes c\mapsto a\cdot c$ is an isomorphism of F-algebras $A\otimes_F Z_{A\cdot G}$ and $A\cdot G$. This mapping is also an isomorphism of G-algebras.

Consequently, $n=\dim_G B\geq \dim_G A\cdot G=\dim_G (A\otimes_F Z)_G=(\dim_F A)\cdot(\dim_G Z)=n\cdot[Z:G]$ and, therefore, $Z=G$. So, $\dim_G A\cdot G=n$ and $A\cdot G=B$ and, therefore, $B_G\cong A\otimes_F G)_G$. Now each basis $\{a_i\}$ in $A_F$ can be converted into a basis $\{a_i\otimes 1\}$ in $(A\otimes_F G)$, and, consequently, to a basis in $B_G$. Now the proof can be finished as in [11]. q.e.d.

For example, all following theories are model complete: the theory of the solid quaternions, the theory of n-algebras with real (or algebraical) closed centre, the theory of the class of rings elementary equivalent to the solid of quaternions or the ring $M_2(\underline{R})$, etc.

Corollary. Let $T_0{}^*$ be a model companion for $T_0$. Then the theory $T^*$ (corresponding to the class $\underline{L}_{T_0^*}$) is a model companion for the theory $T$.

Theorem 7. Let $T_0$ be a model complete theory. Then the class $\underline{K}_{T_0} \leftrightharpoons \{K \mid (\{K_p\} \models T)\ \&\ K\models\Phi_2\}$ is model complete and has a Horn axiomatization. It is also a Boolean-absolute class.

Proof. We shall establish that $\underline{K}_{T_0}$ is Boolean-absolute and apply Theorem 6 using Theorem 2B). □

For example, all following classes are model complete:
$\{K \mid \forall p((K_p \equiv \underline{H})\ \&\ (K \models \Phi_2))\}$ and $\{K \mid (\forall p(K_p \equiv \underline{H} \vee K_p \equiv M_2(\underline{R}))\ \&\ K \models \Phi_2\}$.

Theorem 8. Let $T_0^*$ be a model companion for $T_0$. Then the class $\underline{K}_{T_0^*} \rightleftharpoons \{K \mid (K \models \Phi_2)\ \&\ (\{K_p\} \models T^*)\}$ has a Horn axiomatization and is a model companion for the class $\underline{K}_{T_0}$.

Theorem 9. Let $T_0^*$ be the theory of algebraical closed fields. Then the class $\underline{K}_{T_0}{}^*$ is Horn axiomatizable and is a model companion for the class $\underline{K}_{\underline{L}} \rightleftharpoons \{K \mid \{K_p\} \models T_{\underline{L}}\}$.

Example 3. Let us present Example 2 in the more simple situation of matrix rings. Let T be the set of all statements (in the language of the ring theory) which are true in the matrix ring $M_n(F)$, where F is a fixed field, i.e. $T \rightleftharpoons Th(M_n(F))$. Similarly, $T^* \rightleftharpoons Th(M_n(F_1))$, where $F_1$ is an algebraically closed field. It is mentioned in [11] that the theory $T^*$ is modelly complete. We assume also that Th(F) and $Th(F_1)$ can be mutually model embedded into each other. In this case theories T and $T^*$ can be also mutually model embedded, i.e. $T^*$ is a model companion for T. Theories T and $T^*$ are normal and total autonomus, because they contain the statement $\Phi_5$, where

$$\Phi_5 \rightleftharpoons \forall e (Et)(e^2 = e \ \&\ e \cdot t = t \cdot e \Rightarrow e = 0 \vee e = 1).$$

Let us form the classes

$\underline{K} \rightleftharpoons \{K \mid \forall p(K_p \equiv M_n(F))\}$ and $\underline{K}^* \rightleftharpoons \{K \mid (K \models \Phi_2)\ \&\ \forall p(K_p \equiv M_n(F_1))\}$.

They correspond to local theories T and $T^*$ as usual. The class $\underline{K}^*$ is Boolean-regular, moreover, $(K \subseteq L) \Rightarrow Z(K) \subseteq Z(L)$ for all $K, L \in \underline{K}^*$. Let us check the latter fact. There exists a system $\{e_{ij}\}$, $1 \le i,j \le n$ of elements of $M_n(F_1)$ ("matrix units"), such that

$$\sum_{i=1}^{n} e_{ii} = 1 \text{ and } e_{ij} \cdot e_{pq} = \delta_{jp} \cdot e_{iq} \quad (\text{where } \delta_{jp} \text{ is the Kroneker symbol}).$$

This fact can be expressed by an E-formula. (Here and later we may substitute any commutative ring for $F_1$). By Proposition 5 and Remark in Section 2 this E-formula is true in K; the corresponding system of "matrix units" we denote by $\{e_{ij}\}$. It is a system of matrix units also for L. It is easy to see that $L \cong M_n(G)$, where $G \leftrightharpoons Z_L(\{e_{ij}\})$ is a centralizator for the system $\{e_{ij}\}$ (i.e. $Z_L(X) \leftrightharpoons \{l \in L \mid \forall x \in X(l \cdot x = x \cdot l)\}$). Here G is a ring and elements l of G are mapped into $l \cdot E$ by this isomorphism. The standard identity of the order $2 \cdot n$ holds for L by the Amizur-Levisciy theorem and by Proposition 5 and Remark in Section 2. Then Leron-Vopne theorem implies that G is a commutative ring. Now we see that if $k \in Z(K)$, then $k \in G$ and $kE \cdot m = m \cdot kE$ for matrix m corresponding to each $e \in L$. So, we have proved the mentioned fact.

All localizations of the ring K belonging to the class $\underline{K}^*$ are simple rings. Indeed, if $K \equiv M_n(F)$, where F is a field, then the formula $(E\bar{e}_{ij})\forall x,y(\sum_{i=1}^{n} e_{ii} = 1 \& e_{ij} \cdot e_{pq} = \delta_{jp} \cdot e_{iq} \& e_{ij} \cdot x = x \cdot e_{ij})$ holds for $M_n(F)$, and consequently, for K. As above, $K \cong M_n(G)$, where $G \leftrightharpoons Z(\{e_{ij}\})$ and G is a center of K. It is clear that $G \equiv F$ and, sequently, G is a field. All both-sided ideals of $M_n(G)$ are equal to $M_n(a)$ for some both-sided ideals a in G. Therefore K is a simple ring. So, the class $\underline{K}^*$ is a Boolean-regular and Boolean-simple. The elements of the class $\underline{K}^*$ are biregular rings, so $\forall K \in \underline{K}^*(K \models \Phi_1)$, i.e. $\underline{K}^*$ has a required form. Consequently, $Th\underline{K}^*$ is a model complete Horn theory and a model companion for the class $\underline{K}$. In a usual way we obtain the completeness and decidability for the theory $Th\underline{K}^*$. For example, if the theory of the field $F_1$ is decidable, so is the theory of $M_n(F_1)$ ([11,p.36]). By Theorem 5 we see that the theory $Th\underline{K}^*$ is also complete and decidable (if $ThF_1$ is recursively axiomatizable).

Analogously, we get that if T is a theory of the ring of $n*n$-matrix over an arbitrary commutative regular ring, then the theory $T^*$ of the ring of $n*n$-matrises over some commutative regular algebraically closed atomless rings is a model companion for T. The same is true for the corresponding classes $\underline{K}^* \leftrightharpoons \{K \mid (K \models \Phi_2) \& \{K_p\} \models T^*\}$ and $\underline{K} \leftrightharpoons \{K \mid \{K_p\} \models T\}$.

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