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A Model in Which Well-Orderings of the Reals First Appear at a Given Projective Level, Part III—The Case of Second-Order PA

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Abstract: A model of set theory **ZFC** is defined in our recent research, in which, for a given $n \geq 3$, (A_n) there exists a good lightface Δ^1_n well-ordering of the reals, but (B_n) no well-orderings of the reals (not necessarily good) exist in the previous class Δ^1_{n-1}. Therefore, the conjunction $(\mathsf{A}_n) \wedge (\mathsf{B}_n)$ is consistent, modulo the consistency of **ZFC** itself. In this paper, we significantly clarify and strengthen this result. We prove the consistency of the conjunction $(\mathsf{A}_n) \wedge (\mathsf{B}_n)$ for any given $n \geq 3$ on the basis of the consistency of $\mathbf{PA}_2$, second-order Peano arithmetic, which is a much weaker assumption than the consistency of **ZFC** used in the earlier result. This is a new result that may lead to further progress in studies of the projective hierarchy.

Keywords: forcing; projective well-orderings; projective classes; Peano arithmetic

MSC: 03E15; 03E35

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1. Introduction

The following theorem, devoted to the problem of "effective" well-orderability of the real line $\mathbb{R}$, was established in our recent paper [1].

Theorem 1 (= Theorem 1 in [1]). *Let* $\mathtt{n} \geq 3$. *There is a generic extension of* $\mathbf{L}$, *in which*:

$(\mathsf{A}_\mathtt{n})$ *there is a* $\Delta^1_\mathtt{n}$-**good** *well-ordering of the set* $\mathbb{R}$ *of all reals, of length* ω_1;

$(\mathsf{B}_\mathtt{n})$ *there are no* $\Delta^1_{\mathtt{n}-1}$ *well-orderings of* $\mathbb{R}$, *of any kind, i.e., not necessarily good.*

Here Δ^1_n are lightface projective classes. We refer to [1] for introductory material and references to earlier research on this topic. Generally we refer to Moschovakis' monograph [2] in matters of modern notation in descriptive set theory. We recall that a Δ^1_n-*good* well-ordering is any Δ^1_n well-ordering $\preccurlyeq$ such that the class Δ^1_n is closed under $\preccurlyeq$-bounded quantification, so that if $P(y,x)$ is a binary Δ^1_n relation on the reals, then the relations

$$Q(z,x) := \exists\, y \preccurlyeq x\, P(z,y) \quad \text{and} \quad R(z,x) := \forall\, y \preccurlyeq x\, P(z,y)$$

belong to Δ^1_n as well. Gödel [3] proved that the axiom of constructibility $\mathbf{V} = \mathbf{L}$ implies the existence of a Δ^1_2-good well-ordering of $\mathbb{R}$, and such well-orderings are behind some crucial applications of constructibility in set theory, see Section 5A in [2].

The goal of this paper is to reprove Theorem 1 in the context of second-order Peano arithmetic $\mathbf{PA}_2$. This theory governs the interrelations between the natural numbers and sets of natural numbers, and is widely assumed to lay down working foundations for essential parts of modern mathematics, see, e.g., Simpson [4]. The specific version $\mathbf{PA}_2$ of second-order Peano arithmetic considered here contains Peano's axioms for numbers, the axioms of Induction and Extensionality, and the Comprehension and Countable Choice schemata resp. **CA** and $\mathbf{AC}_\omega$, see Section 8 for details.

Note that claims (A$_{\mathfrak{n}}$) and (B$_{\mathfrak{n}}$) of Theorem 1 can be adequately presented by certain formulas of the language of $\mathbf{PA}_2$, based on suitable *universal formulas* for classes $\Sigma^1_{\mathfrak{n}}$ and $\Sigma^1_{\mathfrak{n}-1}$. Therefore, for any given $\mathfrak{n} \geq 3$, the conjunction (A$_{\mathfrak{n}}$) $\wedge$ (B$_{\mathfrak{n}}$) is essentially a formula, say $\Phi_{\mathfrak{n}}$, of the language of $\mathbf{PA}_2$, and the theorem implies that $\Phi_{\mathfrak{n}}$ is consistent with **ZFC**, and, hence, with $\mathbf{PA}_2$ as well. Thus, it becomes a natural problem to establish the consistency result of Theorem 1 on the base of the $\mathbf{PA}_2$ tools rather than (much stronger) **ZFC** tools. This problem is solved by the next theorem, which is **the main result** of this paper.

Theorem 2. *Let $\mathfrak{n} \geq 3$. Assuming that $\mathbf{PA}_2$ is consistent, the conjunction of (A$_{\mathfrak{n}}$) and (B$_{\mathfrak{n}}$) of Theorem 1 is consistent with $\mathbf{PA}_2$.*

2. Outline of the Proof

Given $\mathfrak{n} \geq 3$ as in Theorem 1, a generic extension of $\mathbf{L}$ was defined in [1], in which (A$_{\mathfrak{n}}$) and (B$_{\mathfrak{n}}$) of Theorem 1 hold. This model involves a product forcing notion $\mathbb{P} \in \mathbf{L}$, defined in $\mathbf{L}$, the constructible universe, as the finite support product of $\aleph_1$-many forcing notions similar to Jensen's "minimal Π^1_2 singleton" forcing as in [5]. This forcing notion $\mathbb{P}$ depends on the index $\mathfrak{n}$ in Theorems 1 and 2, of course, but we suppress this dependence, assuming that $\mathfrak{n} \geq 3$ is fixed once and for all.

The method of finite-support products of Jensen's forcing, which we owe to Enayat [6], has brought several results similar to Theorem 1 in our recent papers. Some of them are mentioned in [1]. Some other results are as follows.

1. A model of **ZFC** in [7] with a Groszek–Laver pair (see [8]), which consists of two OD-undistinguishable E_0 classes $X \neq Y$, whose union $X \cup Y$ is a Π^1_2 set.
2. A model of **ZFC** in [9] in which, for a given $n \geq 3$, Δ^1_n reals = constructible reals.
3. Models in [10] with counterexamples to the separation theorem for both $\mathbf{\Sigma}^1_3$ and $\mathbf{\Pi}^1_3$.
4. A model of **ZFC** in [11] in which the full basis theorem holds for the effective projective hierarchy but there is no Σ^1_∞ well-ordering of the reals.

We may also mention a very recent paper [12] related to this research line.

The factors of the product forcing $\mathbb{P} \in \mathbf{L}$ in [1] are forcing notions that consist of perfect trees in $2^{<\omega}$. Therefore, $\mathbb{P} \subseteq \mathbf{L}_{\omega_1}$, and, in fact, $\mathbb{P}$ is a definable class in $\mathbf{L}_{\omega_1}$, and a CCC forcing. (We recall that a partially ordered set or class P satisfies CCC, or the countable chain condition, if every antichain $A \subseteq P$ is at most countable.) Thus, we may think about treating $\mathbb{P}$ as a class-forcing over $\mathbf{L}_{\omega_1}$. Yet, there is a serious obstacle noted in Section 31 in [1]: the construction of $\mathbb{P}$ involves a $\diamondsuit_{\omega_1}$-sequence, which goes on in $\mathbf{L}_{\omega_2}$ rather than $\mathbf{L}_{\omega_1}$. We overcome this difficulty in this paper. We make use of a recent construction of definable-$\diamondsuit$ sequences by Enayat and Hamkins [13], to obtain a definable-$\diamondsuit_{\omega_1}$ sequence inside $\mathbf{L}_{\omega_1}$ (Theorem 4), or rather by means of

$$\mathbf{ZFC}^-_{\mathbf{lc}} \;:=\; \mathbf{ZFC}^- + \text{all sets are constructible and countable}\,, \tag{1}$$

a theory known to be equiconsistent with $\mathbf{PA}_2$ and to have $\mathbf{L}_{\omega_1}$ (more exactly, $\mathbf{L}_{\omega_1^{\mathbf{L}}}$) as a natural model. The upper minus stands for the absence of the Power Sets axiom, whereas l and c in the lower index stand for the constructibility ($\mathbf{L}$) and countability.

Adapting the whole structure of the proof of Theorem 1 in [1], we introduce $\mathbb{P}$ here as a definable class forcing in $\mathbf{ZFC}^-_{\mathbf{lc}}$ satisfying the definable CCC. Then we use the class forcing theory of S. D. Friedman [14,15], developed by Antos and Gitman [16] to be applicable over $\mathbf{ZFC}^-$, and check that $\mathbb{P}$-generic extensions of a $\mathbf{ZFC}^-_{\mathbf{lc}}$ universe satisfy (A$_{\mathfrak{n}}$) $\wedge$ (B$_{\mathfrak{n}}$) above.

The final reduction to $\mathbf{PA}_2$ will be based on the equiconsistency of $\mathbf{PA}_2$, $\mathbf{ZFC}^-$, and $\mathbf{ZFC}^-_{\mathbf{lc}}$, see Section 8. This is how the proof of Theorem 2 is organized in this paper. We will freely use the notation and results of the previous article [1] in the course of the proof.

3. On Power-Less Set Theory

Recall that HC is the set of all *hereditarily countable* sets. Thus, $X \in \mathrm{HC}$ if the transitive closure $\mathrm{TC}\,(X)$ is at most countable. Note that $\mathrm{HC} = \mathbf{L}_{\omega_1}$ under $\mathbf{V} = \mathbf{L}$.

We recall that $\mathbf{ZFC}^-$ is a subtheory of **ZFC**, obtained as follows:

(a) We exclude the Power Set axiom **PS**;
(b) The *well-orderability axiom* **WA**, which claims that every set can be well-ordered, is substituted for the usual set-theoretic Axiom of Choice **AC** of **ZFC**;
(c) The Separation schema is preserved, but the Replacement schema (which happens to be not sufficiently strong in the absence of **PS**) is substituted with the *Collection* schema: $\forall\, X\, \exists\, Y\, \forall\, x \in X\, \big(\exists\, y\, \Phi(x,y) \implies \exists\, y \in Y\, \Phi(x,y)\big)$.

A comprehensive account of main features of $\mathbf{ZFC}^-$ is given in, e.g., [16,17].

We may note that if $\mathrm{HC} \subseteq \mathbf{L}$ holds, then the set $\mathbf{L}_{\omega_1} = \mathrm{HC}$ is a natural model of the theory $\mathbf{ZFC}^-_{\mathbf{lc}}$ defined by (1) above. This motivates the following definition.

Definition 1. *The ground set universe of* $\mathbf{ZFC}^-_{\mathbf{lc}}$ *(not including classes) is denoted by* $\mathbf{L}_{\omega_1}$*. Accordingly,* ω_1 *will be the collection (a proper class) of all ordinals in* $\mathbf{L}_{\omega_1}$*.*

Remark 1. *Arguing in* $\mathbf{ZFC}^-_{\mathbf{lc}}$*, we will often consider (definable) proper classes because they will play a more essential role than is common in* **ZFC***. We will also consider such things as* class-size collections of proper classes*, e.g. class-long sequences* $\langle X_\alpha \rangle_{\alpha<\omega_1}$ *of proper classes* X_α*, with the understanding that the real thing considered in this case is some (definable) class* $Y \subseteq \omega_1 \times \mathbf{L}_{\omega_1}$ *whose slices* $Y_\alpha = \{x : \langle \alpha, x\rangle \in Y\}$ *are equal to the given classes* X_α*.*

4. Some Preliminary Constructions in Power-Less Set Theory

The construction of the forcing notion $\mathbb{P}$ for the proof of Theorem 1 in [1] goes on in the universe $\mathbf{L}_{\omega_1}$ of $\mathbf{ZFC}^-_{\mathbf{lc}}$ and involves the following principal elements.

(1) The collection **PT** of all *perfect trees* $T \subseteq 2^{<\omega}$.
Arboreal forcings, i.e., those $P \subseteq \mathbf{PT}$ satisfying $s \in T \in P \implies T{\restriction}_s \in P$, in § 3 of [1].
In the context of $\mathbf{ZFC}^-_{\mathbf{lc}}$, such a P can be either a set or a (definable) proper class. Let **AF** be the class of all arboreal forcings P which are sets.
(2) By § 3 of [1], an arboreal forcing P is:
- *regular*, if, for any $S, T \in P$, the intersection $[S] \cap [T]$ of corresponding perfect sets $[S], [T]$ is clopen in $[S]$ or in $[T]$;
- *special*, if there is a *finite or countable* antichain $A \subseteq P$ such that $P = \{T{\restriction}_s : s \in T \in A\}$— the antichain A is unique and P is countable in this case.

(3) The class **MT** of all *multitrees*, i.e., maps $\boldsymbol{p} : |\boldsymbol{p}| \to \mathbf{PT}$, such that $|\boldsymbol{p}| \subseteq \omega_1$ is finite.
(4) *Multiforcings*, i.e., maps $\boldsymbol{\pi} : |\boldsymbol{\pi}| \to \{\text{arboreal forcings}\}$, such that $|\boldsymbol{\pi}| \subseteq \omega_1$, in § 6 of [1].
A multiforcing $\boldsymbol{\pi}$ is:
- *small*, in case both $|\boldsymbol{\pi}|$ and each forcing $\boldsymbol{\pi}(\xi)$, $\xi \in |\boldsymbol{\pi}|$, are (countable) sets rather than proper classes, or equivalently (in $\mathbf{ZFC}^-_{\mathbf{lc}}$), $\boldsymbol{\pi}$ itself is a set;
- *special*, in case $|\boldsymbol{\pi}|$ is countable and each $\boldsymbol{\pi}(\xi)$ is special, as in (2);
- *regular*, in case all $\boldsymbol{\pi}(\xi)$ are regular, as in (2).

All special multiforcings are small and regular—hence, sets in $\mathbf{ZFC}^-_{\mathbf{lc}}$. Non-small multiforcings are proper classes, of course. See Remark 1.
(5) If $\boldsymbol{\pi}$ is a multiforcing then $\mathbf{MT}(\boldsymbol{\pi})$ is a collection of all $\boldsymbol{\pi}$-multitrees $\boldsymbol{p}$, i.e., those satisfying $|\boldsymbol{p}| \subseteq |\boldsymbol{\pi}|$ and $\boldsymbol{p}(\xi) \in \boldsymbol{\pi}(\xi)$ for all $\xi \in |\boldsymbol{p}|$. Clearly, both $\mathbf{MT}(\boldsymbol{\pi})$ and $\boldsymbol{\pi}$ itself are just different representations of the finite-support product $\prod_{\xi \in |\boldsymbol{\pi}|} \boldsymbol{\pi}(\xi)$, see § 6 of [1]. If $\boldsymbol{\pi}$ is a set (equivalently, a small multiforcing), then $\mathbf{MT}(\boldsymbol{\pi})$ is a set as well.
(6) The collection **MFsp** of all *special (therefore small) multiforcings* $\boldsymbol{\pi}$, i.e., such that $|\boldsymbol{\pi}| \subseteq \omega_1$ is at most countable and if $\alpha \in |\boldsymbol{\pi}|$, then $\boldsymbol{\pi}(\alpha)$ is special in § 16 of [1].
(7) The operation $\bigcup^{\mathtt{cw}}$ of *component-wise union* of multiforcings, § 6 of [1].

(8) The relation $\sqsubset$ of *refinement* of multiforcings, § 7 of [1]. The relation $\pi \sqsubset_D \varphi$ (*sealing refinement*, § 8 of [1]) means that $\pi \sqsubset \varphi$ and $D \subseteq \mathbf{MT}(\pi)$ is pre-dense in $\mathbf{MT}((\pi \cup^{\mathtt{cw}} \varphi))$. Some related notions such as $\sqsubset_p$ (types of sealing refinements) are introduced in § § 9–13 of [1], and summarized in the *common sealing refinement* relation $\pi \sqsubset\!\!\sqsubset_M \varphi$ in § 14 of [1], meaning that $\pi \sqsubset_D \varphi$ holds for all D dense in $\mathbf{MT}(\pi)$, which either belong to M or are coded in some way by elements of M.

(9) For any ordinal, $\alpha \le \omega_1$— the collection $\overrightarrow{\mathbf{MF}}_\alpha$ of all $\sqsubset$-*increasing sequences* $\vec{\pi}$ of special multiforcings, of length $\mathtt{lh}(\vec{\pi}) = \alpha$, in § 16 of [1]. If $\vec{\pi} \in \overrightarrow{\mathbf{MF}}_\alpha$, then *the componentwise union* $\bigcup^{\mathtt{cw}} \vec{\pi} = \bigcup_{\gamma<\alpha} \vec{\pi}(\gamma)$ is a multiforcing, and $\bigcup^{\mathtt{cw}} \vec{\pi}$ a set in case $\alpha < \omega_1$ and a proper class in case $\alpha = \omega_1$.

(10) The collection $\overrightarrow{\mathbf{MF}} = \bigcup_{\alpha<\omega_1} \overrightarrow{\mathbf{MF}}_\alpha$ of all $\sqsubset$-*increasing sequences* of special multiforcings, of countable transfinite length. The relation $\vec{\pi} \subset \vec{\varphi}$ on $\overrightarrow{\mathbf{MF}}$ means that a sequence $\vec{\varphi}$ is a proper extension of $\vec{\pi}$. If M is any set, then $\vec{\pi} \subset_M \vec{\varphi}$ means that $\vec{\pi} \subset \vec{\varphi}$, and in addition $\pi \sqsubset\!\!\sqsubset_M \varphi(\lambda)$, where $\lambda = \mathtt{lh}(\vec{\pi})$ and $\pi = \bigcup^{\mathtt{cw}} \vec{\pi}$. See § 16 of [1].

Of those, perfect trees, special arboreal forcings, multitrees, special multiforcings, sequences in $\overrightarrow{\mathbf{MF}}$ are pretty legitimate objects (sets) in $\mathbf{ZFC}^-_{\mathbf{lc}}$. If π is a special multiforcing then $\mathbf{MT}(\pi)$ (all π-multitrees as in (5)) is a set in $\mathbf{ZFC}^-_{\mathbf{lc}}$. On the other hand, collections $\mathbf{PT}$, $\mathbf{AF}$, $\mathbf{MFsp}$, $\overrightarrow{\mathbf{MF}}$, as well as any sequence in $\overrightarrow{\mathbf{MF}}_{\omega_1}$, and similar uncountable objects are definable classes in $\mathbf{ZFC}^-_{\mathbf{lc}}$, and accordingly so that all associated results in Sections 3–14 and 16 in [1] remain true in $\mathbf{ZFC}^-_{\mathbf{lc}}$, with the understanding that some objects are proper classes, of course. In particular, the following results are true.

Theorem 3 (Theorem 4 in [1])**.** *In* $\mathbf{ZFC}^-_{\mathbf{lc}}$*, if* π *is a small regular multiforcing and* M *a countable set, then there is a special multiforcing* φ *satisfying* $|\pi| = |\varphi|$ *and* $\pi \sqsubset\!\!\sqsubset_M \varphi$*.*

Lemma 1 (Lemma 21 in [1])**.** *In* $\mathbf{ZFC}^-_{\mathbf{lc}}$*, assume that* M *is any (countable) set. Then:*

(i) *If* $\kappa < \lambda < \omega_1$ *and* $\vec{\pi} \in \overrightarrow{\mathbf{MF}}_\kappa$*, then there exists a sequence* $\vec{\varphi} \in \overrightarrow{\mathbf{MF}}_\lambda$ *such that* $\vec{\pi} \subset_M \vec{\varphi}$ *;*

(ii) *If* $\kappa < \lambda \le \omega_1$*,* $\vec{\pi} \in \overrightarrow{\mathbf{MF}}_\kappa$*,* $\vec{\varphi} \in \overrightarrow{\mathbf{MF}}_\lambda$*,* $\vec{\pi} \subset_M \vec{\varphi}$*, and a set* $D \in M$ *is open dense in* $\mathbf{MT}(\vec{\pi})$*, then* $\bigcup^{\mathtt{cw}} \vec{\pi} \sqsubset_D \varphi_{\ge\kappa} = \bigcup^{\mathtt{cw}}_{\kappa\le\alpha<\lambda} \vec{\varphi}(\alpha)$*, so that* D *is pre-dense in* $\mathbf{MT}(\vec{\varphi})$*.*

Lemma 2 (Lemma 22 in [1])**.** *In* $\mathbf{ZFC}^-_{\mathbf{lc}}$*, the following ternary relation belongs to the class* $\Delta_1 = \Delta_1^{\mathbf{L}_{\omega_1}}$*: "*$\vec{\pi}, \vec{\varphi} \in \overrightarrow{\mathbf{MF}} \wedge M \in \mathbf{L}_{\omega_1} \wedge \vec{\pi} \subset_M \vec{\varphi}$*".*

Now we approach the construction of the forcing notion $\mathbb{P}$ in § § 18–20 of [1]. The main issue with this construction is that it involves a $\diamondsuit_{\omega_1}$-sequence in [1]. The **ZFC** construction of such a sequence (as, e.g., in [18]) formally can be maintained as a proper class in $\mathbf{ZFC}^-_{\mathbf{lc}}$. However, unfortunately, the proof of the $\diamondsuit_{\omega_1}$-property does not go through in $\mathbf{ZFC}^-_{\mathbf{lc}}$ because its **ZFC** proof involves ordinals beyond ω_1, and hence, does not directly translate to the $\mathbf{ZFC}^-_{\mathbf{lc}}$ setup. The next section provides an appropriate substitution.

5. Definable $\diamondsuit_{\omega_1}$ Sequence by Means of Power-Less Set Theory

It is established in a recent paper (Theorem 4.1 in [13]) that, under $\mathbf{V} = \mathbf{L}$, there is an **Ord**-long class-sequence satisfying a $\diamondsuit$-property for all definable classes. Inspired by this result, we prove the following theorem using nearly the same method:

Theorem 4 ($\mathbf{ZFC}^-_{\mathbf{lc}}$)**.** *There is a* Δ_1 *sequence* $\langle S_\alpha \rangle_{\alpha<\omega_1}$ *of sets* $S_\alpha \subseteq \alpha$ *such that: for any definable (with parameters) classes* $S, C \subseteq \omega_1$*, if* C *is club in* ω_1*, then* $S \cap \alpha = S_\alpha$ *for some* $\alpha \in C$*.*

Any sequence of sets S_α as in the theorem can be seen as *guessing sufficiently often* initial segments $S \cap \alpha$ of any definable $S \subseteq \omega_1$ within any definable club (i.e., closed unbounded) $C \subseteq \omega_1$. This is the best possible result in $\mathbf{ZFC}^-_{\mathbf{lc}}$ of such kind. But **ZFC** with the axiom of constructibility, a stronger result is possible, in which the guessing property holds for arbitrary (not necessarily definable) sets $S, C \subseteq \omega_1$. This is basically the fundamental "diamond" theorem $\diamondsuit_{\omega_1}$ of Jensen, see, e.g., Theorem 13.21 in [18].

We use the standard notation Σ_n^X, Π_n^X, Δ_n^X (slanted lightface Σ, Π, Δ) for classes of *lightface* definability over a given transitive set X, e.g., $X = \mathbf{L}_{\omega_1}$ (no parameters allowed), and $\mathbf{\Sigma}_n^X$, $\mathbf{\Pi}_n^X$, $\mathbf{\Delta}_n^X$ for *boldface* definability in X (parameters from X allowed). Simply Σ_n means Σ_n definability in the whole universe, i.e., $\mathbf{L}_{\omega_1}$ when arguing under $\mathbf{ZFC}^-_{\mathbf{lc}}$, and Π_n, Δ_n is understood similarly. We call Σ_n, Π_n, Δ_n and the like **types** rather than classes in this paper since the word 'class' has a special technical meaning in the context of $\mathbf{ZFC}^-_{\mathbf{lc}}$, that is, a definable collection of sets.

Proof. We argue under $\mathbf{ZFC}^-_{\mathbf{lc}}$. As the axiom of constructibility is assumed by $\mathbf{ZFC}^-_{\mathbf{lc}}$, let $\leqslant_{\mathbf{L}}$ be the Gödel well-ordering of the universe, of definability type Δ_1. Define the sets S_α by transfinite recursion. Suppose that S_α has been defined for all $\alpha < \vartheta$. We put $S_\vartheta = \varnothing$ unless

(*) ϑ is a limit ordinal, and there is a set $S' \subseteq \vartheta$ and a club (closed unbounded set) $C' \subseteq \vartheta$, with both S' and C' definable (allowing parameters) in the structure $\mathbf{L}_\vartheta = \langle \mathbf{L}_\vartheta ; \in \rangle$.

If (*) holds, then pick the least such pair $\langle S', C' \rangle$, minimizing first on the Gödel codes of the defining formulas of S' and C', and then, using the well-order $\leqslant_{\mathbf{L}} \restriction \mathbf{L}_\vartheta$, on the parameters used in the definitions. For this minimal pair, let $S_\vartheta = S'$. This completes the inductive definition. The definition makes use of the truth predicate for the structure $\langle \mathbf{L}_\vartheta ; \in \rangle$.

We claim that the sequence $\langle S_\alpha \rangle_{\alpha < \omega_1}$ is as required. First of all, the truth predicate is known to be Δ_1, and so is the restricted order $\leqslant_{\mathbf{L}} \restriction \mathbf{L}_\vartheta$, with ϑ as the only parameter. It routinely follows that the class sequence $\langle S_\alpha \rangle_{\alpha < \omega_1}$ is Δ_1 in the universe of $\mathbf{ZFC}^-_{\mathbf{lc}}$.

Further, suppose towards the contrary that $S, C \subseteq \omega_1$ are counterexamples, so that C is a club in ω_1, both S, C are classes definable by specific formulas resp. $\sigma(\cdot), \kappa(\cdot)$ (with some parameters not explicitly indicated), so that

$$C = \{\gamma < \omega_1 : \kappa(\gamma)\}, \quad S = \{\gamma < \omega_1 : \sigma(\gamma)\},$$

and $S \cap \alpha \neq S_\alpha$ for all $\alpha < \omega_1$.

We may assume, without loss of generality, that these formulas are chosen so as to be minimal in the sense of the construction, so that, successively, their Gödel codes are as small as possible, and the the parameters involved form a $\leqslant_{\mathbf{L}}$-minimal tuple, respectively.

Let m be a sufficiently large natural number, larger than the Gödel codes of the formulas $\sigma(\cdot), \kappa(\cdot)$ and their subformulas, and large enough so that the minimality condition we just formulated is expressible by a Σ_m formula. Let ϑ be any limit ordinal such that all the parameters used in the definitions belong to $\mathbf{L}_\vartheta$, and such that $\mathbf{L}_\vartheta$ is an elementary submodel of the whole universe $\mathbf{L}_{\omega_1}$ of the theory $\mathbf{ZFC}^-_{\mathbf{lc}}$ considered, w.r.t. all Σ_m formulas. It follows that the restrictions $\leqslant_{\mathbf{L}} \restriction \mathbf{L}_\vartheta$, and also $S \cap \mathbf{L}_\vartheta$ and $C \cap \mathbf{L}_\vartheta$ are definable in $\mathbf{L}_\vartheta$ by the same definitions and parameters as their counterparts in $\mathbf{L}_{\omega_1}$.

Therefore, $C \cap \vartheta$ is club in $\mathbf{L}_\vartheta$, and $S \cap \vartheta, C \cap \vartheta$ form a minimal pair using those definitions $S \cap \alpha \neq S_\alpha$ for any $\alpha \in C \cap \vartheta$. Thus, by the definition of S_ϑ, it follows that $S_\vartheta = S \cap \vartheta$. Since $C \cap \vartheta$ is unbounded in ϑ and C is closed, it follows that $\vartheta \in C$, and so $S_\vartheta = S \cap \vartheta$ contradicts our assumption about S and C. So there are no such counterexample classes, and we have finished. □

Following § 18 in [1], we proceed to the next definition.

Definition 2 ($\mathbf{ZFC}^-_{\mathbf{lc}}$)**.** *We fix a sequence $\langle S_\alpha \rangle_{\alpha < \omega_1}$ given by Lemma 4.*

As usual, let $\leqslant_{\mathbf{L}}$ be the Gödel Δ_1 well-ordering of the set universe $\mathbf{L}_{\omega_1}$ under $\mathbf{ZFC}^-_{\mathbf{lc}}$.

We let $c_\alpha = \alpha$th element of $\mathbf{L}_{\omega_1}$ in the sense of $\leqslant_{\mathbf{L}}$; thus, $\mathbf{L}_{\omega_1} = \{c_\alpha : \alpha < \omega_1\}$.

If $Z \subseteq \mathbf{L}_{\omega_1}$ and $\alpha < \omega_1$, then let $(Z)_{<\alpha} = \{c_\xi \in Z : \xi < \alpha\}$.

If $\alpha < \omega_1$ then let $A_\alpha = \{c_\xi : \xi \in S_\alpha\}$. Then, $\langle A_\alpha \rangle_{\alpha < \omega_1}$ is still a Δ_1 sequence.

Let $A_\alpha^n = \{a : \langle n, a \rangle \in A_\alpha\}$.

Let $\mathfrak{M}(\alpha) = \{A_\alpha^n : n < \omega\}$. Then, $\langle \mathfrak{M}(\alpha) \rangle_{\alpha < \omega_1}$ is still a Δ_1 sequence.

6. The Key Sequence and Key Forcing Notion

The next theorem (Theorem 5) is a crucial step towards the construction of the forcing notion that will prove Theorem 2. The theorem is a close version of Theorem 7 in [1]. The construction employs some ideas related to *definable generic* transfinite constructions, and it will go on by a transfinite inductive definition of a sequence $\vec{\mathbb{\Pi}} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ in $\mathbf{L}_{\omega_1}$ from countable subsequences. The result can be viewed as a maximal branch in $\overrightarrow{\mathbf{MF}}$, *generic* with respect to all sets of a given complexity.

Definition 3 (in $\mathbf{ZFC}^-_{\mathbf{lc}}$)**.** *From now on,* ***a number*** $\mathbb{n} \geq 3$ *as in Theorem 2, is fixed.*

A sequence $\vec{\pi} \in \overrightarrow{\mathbf{MF}}$ *blocks a set or class* $W \subseteq \overrightarrow{\mathbf{MF}}$*, if either* $\vec{\pi}$ *belongs to* W *(a positive block) or no sequence* $\vec{\varphi} \in W \cap \overrightarrow{\mathbf{MF}}$ *extends* $\vec{\pi}$ *(a negative block).*

Any sequence $\vec{\mathbb{\Pi}} = \langle \mathbb{\Pi}_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$ *(that is, a definable proper class), satisfying the following four conditions (A)–(D) for this* $\mathbb{n}$*, will be called* a key sequence*:*

- *(A) The set* $|\vec{\mathbb{\Pi}}| = \bigcup_{\alpha<\kappa} |\vec{\mathbb{\Pi}}(\alpha)|$ *is equal to* ω_1 *.*
- *(B) Every* $\gamma < \omega_1$ *is a* crucial ordinal *for* $\vec{\mathbb{\Pi}}$*, so that the relation* $(\bigcup^{\mathbf{cw}}_{\alpha<\gamma} \mathbb{\Pi}_\alpha) \sqsubset_{\mathfrak{M}(\gamma)} \mathbb{\Pi}_\gamma$ *holds, where* $\mathfrak{M}(\gamma)$ *is introduced by Definition 2 and* $\sqsubset_M$ *by (8) of Section 4.*
- *(C) If in fact* $\mathbb{n} \geq 4$ *and* $W \subseteq \overrightarrow{\mathbf{MF}}$ *is a boldface* $\mathbf{\Sigma}_{\mathbb{n}-3}$ *class (a definition with parameters), then there exists an ordinal* $\gamma < \omega_1$ *such that the subsequence* $\vec{\mathbb{\Pi}} \restriction \gamma$ ***blocks*** W*—so that either* $\vec{\mathbb{\Pi}} \restriction \gamma \in W$*, or there is no sequence* $\boldsymbol{\varphi} \in W$ *extending* $\vec{\mathbb{\Pi}} \restriction \gamma$ *.*
- *(D) The sequence* $\vec{\mathbb{\Pi}}$ *belongs to the definability type* $\Delta_{\mathbb{n}-2}$ *in* $\mathbf{L}_{\omega_1}$*.*

Theorem 5 ($\mathbf{ZFC}^-_{\mathbf{lc}}$)**.** *There exists a key sequence* $\vec{\mathbb{\Pi}} = \langle \mathbb{\Pi}_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$*.*

Proof (sketch)**.** Argue under $\mathbf{ZFC}^-_{\mathbf{lc}}$, with $\mathbb{n} \geq 3$ fixed, and strictly follow the proof of Theorem 7 in [1], with Theorem 3 and Lemmas 1 and 2 as the principal references. □

Definition 4 ($\mathbf{ZFC}^-_{\mathbf{lc}}$)**.** *From now on, we fix* ***a key sequence*** $\vec{\mathbb{\Pi}} = \langle \mathbb{\Pi}_\alpha \rangle_{\alpha<\omega_1} \in \overrightarrow{\mathbf{MF}}_{\omega_1}$*, given by Theorem 5 for the number* $\mathbb{n} \geq 3$ *fixed by Definition 3. It satisfies (A)–(D) of Definition 3. We call this fixed* $\vec{\mathbb{\Pi}} \in \mathbf{L}$ ***the key sequence****.*

Based on Definition 4, we introduce some derived notions.

Definition 5 ($\mathbf{ZFC}^-_{\mathbf{lc}}$)**.** *Using the key sequence* $\vec{\mathbb{\Pi}} = \langle \mathbb{\Pi}_\alpha \rangle_{\alpha<\omega_1}$*, we define the regular multiforcing* $\mathbb{\Pi} = \bigcup^{\mathbf{cw}}_{\alpha<\omega_1} \mathbb{\Pi}_\alpha$*, and the forcing notion* $\mathbb{P} = \mathbf{MT}(\mathbb{\Pi}) = \mathbf{MT}(\vec{\mathbb{\Pi}})$*.*

We will call $\mathbb{\Pi}$ **the key multiforcing** below. Technically, $\mathbb{\Pi}$ is a class-long sequence of proper classes; see Remark 1. Accordingly, $\mathbb{P} = \mathbf{MT}(\mathbb{\Pi})$ will be our **key forcing notion**; technically, a definable proper subclass of the class $\mathbf{MT}$ of all multitrees. The following theorem presents the principal CCC property of $\mathbb{P}$ in the ground universe $\mathbf{L}_{\omega_1}$ of $\mathbf{ZFC}^-_{\mathbf{lc}}$.

Theorem 6 ($\mathbf{ZFC}^-_{\mathbf{lc}}$, Theorem 7 in [1])**.** *The forcing notion* $\mathbb{P}$ *satisfies countable chain condition CCC. Therefore,* $\mathbb{P}$*-generic extensions of* $\mathbf{L}$ *preserve cardinals.*

7. Forcing Method over Power-Less Set Theory and the Key Model

The forcing engine does not necessarily work in $\mathbf{ZFC}^-_{\mathbf{lc}}$ for an arbitrary class-size forcing notion. But there is a type of forcing notions that admits adequate treatment of forcing, similar to the standard $\mathbf{ZFC}$ case.

Definition 6 (S. D. Friedman, see [14,16])**.** *A forcing notion (a partially ordered definable class)* $P = \langle P\,; \leq \rangle$ *is* pre-tame *if for every class sequence* $\langle D_x \rangle_{x \in a}$ *of dense classes* $D_x \subseteq P$*, parametrized by elements of a set* a *(so that* $D = \{ \langle x, z \rangle : x \in a \wedge z \in D_x \}$ *is a definable class), and every condition* $p \in P$*, there is a condition* $q \leq p$ *and a sequence* $\langle d_x \rangle_{x \in a}$ *of subsets of* P *such that each* $d_x \subseteq D_x$ *is pre-dense below* q *in* P*.*

Theorem 7 (S. D. Friedman, see [14,16])**.** *In* $\mathbf{ZFC}^-$*, let* $P = \langle P\,;\leq\rangle$ *be a pre-tame class-forcing notion. Then* P *preserves* $\mathbf{ZFC}^-$ *and satisfies the main forcing principles including the truth—forcing and forcing definability theorems.*

We recall that our key forcing $\mathbb{P}$, introduced by Definition 5, is a class forcing, satisfying CCC by Theorem 6. Therefore, $\mathbb{P}$ is pre-tame under $\mathbf{ZFC}^-_{\mathbf{lc}}$, as so obviously is any CCC forcing. We conclude that Theorem 7 is applicable, and, hence, usual forcing theorems are valid for $\mathbb{P}$-generic extensions of $\mathbf{L}_{\omega_1}$, the $\mathbf{ZFC}^-_{\mathbf{lc}}$ set universe. This justifies all forcing results in Chapters III and IV of [1], including Theorems 9 and 13 there, on the basis of $\mathbf{ZFC}^-_{\mathbf{lc}}$. This argument validates the following theorem:

Theorem 8 ($\mathbf{ZFC}^-_{\mathbf{lc}}$, Theorems 9 and 13 in [1])**.** *Let* $\mathbb{n} \geq 3$ (see Definition 3 on $\mathbb{n}$). *Any* $\mathbb{P}$*-generic extension of* $\mathbf{L}_{\omega_1}$*, the* $\mathbf{ZFC}^-_{\mathbf{lc}}$ *universe of discourse, is still a model of* $\mathbf{ZFC}^-$*, in which statements* $(\mathsf{A}_{\mathbb{n}})$ *and* $(\mathsf{B}_{\mathbb{n}})$ *of Theorem 1 hold for the index* $\mathbb{n}$*.*

In other words, any $\mathbb{P}$-generic extension of $\mathbf{L}_{\omega_1}$, the $\mathbf{ZFC}^-_{\mathbf{lc}}$ universe of discourse, is still a model of $\mathbf{ZFC}^-$ and a model of statements $(\mathsf{A}_{\mathbb{n}})$ and $(\mathsf{B}_{\mathbb{n}})$ of Theorem 1. The following is a usual metamathematical corollary.

Corollary 1. *If* $\mathbf{ZFC}^-_{\mathbf{lc}}$ *is consistent, then so is the theory* $\mathbf{ZFC}^- +$ *"*$(\mathsf{A}_{\mathbb{n}}) \wedge (\mathsf{B}_{\mathbb{n}})$ *of Theorem 1".*

8. Reduction to Second-Order Peano Arithmetic

Following [4,19,20] the second order Peano arithmetic $\mathbf{PA}_2$ is a theory in the language $\mathcal{L}(\mathbf{PA}_2)$ with two sorts of variables—for natural numbers and for sets of them. We use j,k,m,n for variables over ω and x,y,z for variables over $\mathscr{P}(\omega)$, reserving capital letters for subsets of $\mathscr{P}(\omega)$ and other sets. The axioms are as follows in (1)–(5):

(1) **Peano's axioms** for numbers.
(2) **Induction** as one sentence: $\forall\, x\,\big(0 \in x \wedge \forall\, n\,(n \in x \implies n+1 \in x) \implies \forall\, n\,(n \in x)\big)$.
(3) **Extensionality** for sets of natural numbers.
(4) The **Comprehension** schema **CA**: $\exists\, x\, \forall\, k\,(k \in x \iff \Phi(k))$, for every formula Φ in which x does not occur, and in Φ, we allow parameters—free variables other than k.
(5) The schema $\mathbf{AC}_\omega$ of **Countable Choice**: $\forall\, k\, \exists\, x\, \Phi(k,x) \implies \exists\, x\, \forall\, k\, \Phi(k,(x)_k))$, for every formula Φ with parameters allowed, where $(x)_k = \{j : 2^k(2j+1) - 1 \in x\}$.

The theory $\mathbf{PA}_2$ is also known as A_2 (see, e.g., an early survey [19]), az Z_2 (in [21] or elsewhere). See also [22]. We recall that the consistency of $\mathbf{PA}_2$ is the blanket assumption in Theorem 2. Yet, we can use the following equiconsistency result:

Theorem 9. *Theories* $\mathbf{PA}_2$ *and* $\mathbf{ZFC}^-_{\mathbf{lc}}$ *are equiconsistent.*

Proof. The theorem has been a well-known fact for some time, see, e.g., Theorem 5.25 in [19]. A rather natural way of proof is as follows.

Step 1. Theory $\mathbf{ZFC}^- +$ "all sets are countable" is interpreted in $\mathbf{PA}_2$ by the *tree interpretation* described in [19], §5, especially Theorem 5.11, or in [4], Definition VII.3.10 ff. Kreisel [20], VI(a)(ii), attributed this interpretation to the type of "crude" results.
Step 2. Arguing in $\mathbf{ZFC}^- +$ "all sets are countable", we define the transitive class $\mathbf{L}$ of all constructible sets, which models $\mathbf{ZFC}^- +$ "all sets are constructible".
Step 3. We argue in $\mathbf{ZFC}^- +$ "all sets are constructible". If every ordinal is countable, then immediately all sets are countable, that is, we have $\mathbf{ZFC}^-_{\mathbf{lc}}$. If there exist uncountable ordinals, then let ω_1 be the least of them. Then, $\mathbf{L}_{\omega_1}$ is a transitive set which models $\mathbf{ZFC}^-_{\mathbf{lc}}$.

We conclude from Steps 1–3 that $\mathbf{PA}_2$ and $\mathbf{ZFC}^-_{\mathbf{LC}}$ are equiconsistent. □

Combining Theorems 9 and 8, we accomplish the proof of Theorem 2.

9. Conclusions and Problems

In this study, the method of definable $\diamondsuit_{\omega_1}$ sequences was employed to the construction of a model in which, for a given $n \geq 3$, there is a Δ^1_n-good well-ordering of the reals, but no well-orderings of the reals exist in the class Δ^1_{n-1} at the preceding level of the hierarchy, on the basis essentially of the second-order Peano arithmetic $\mathbf{PA}_2$. This considerably strengthens and extends our earlier results, in which such a model was defined on the basis of a much stronger theory **ZFC** typically assumed in for independence results by the forcing method. This is a new result and a valuable improvement upon our earlier results in [1]. The technique developed in this paper may lead to further progress in studies of different aspects of the projective hierarchy. We hope that this study will contribute to the following fundamental problem by S. D. Friedman: find a model for a given n, in which all $\mathbf{\Sigma}^1_n$ sets of reals are Lebesgue measurable and have the Baire and perfect set properties, and, at the same time, there exists a $\mathbf{\Delta}^1_{n+1}$ well-ordering of the reals, [14] (p. 209) and [15] (p. 602).

From our study, it is concluded that the technique of *definable* $\diamondsuit$*-sequences* will lead to similar consistency and independence results on the basis of the consistency of $\mathbf{PA}_2$.

Some problems that arise from our study are mentioned in § 32 in [1]. We adjoin the following problem specific to the research line of this article.

Problem 1. *Reprove the consistency result in [9], related to the equality "Δ^1_n reals = constructible reals" for any given $n \geq 3$, on the basis of the consistency of theory* $\mathbf{PA}_2$.

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