Euler's Lute and Edwards's Oud

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In a piece published in 1981, Harold M. Edwards touts the benefits of reading the masters. A quarter century later, Edwards takes seriously his own advice by publishing an encomium on Euler's *Institutiones* (1755). Although we agree with Edwards that we shall all do well by studying the masters, we argue that to derive the full benefit, one must read historically important texts without interposing a lens formed by one's own mathematical accomplishments. We show, in particular, that Edwards misses much of the style and substance of Euler's *Institutiones* by reading the text through a constructivist lens.

Euler on Ratios of Vanishing Increments

Read the Masters! is H. M. Edwards' text shortlisting Euler among the masters [Edwards 1981, p. 105]. Let us therefore read Euler. In the Preface to his 1755 *Institutiones*, Euler writes:

> In this way, we are led to a definition of *differential calculus*: It is *a method for determining the ratio of the vanishing increments that any functions take on when the variable, of which they are functions, is given a vanishing increment* [Euler 2000, p. vii] (as translated by Blanton).

Euler goes on to write that the vanishing increments involved in that ratio

> are called *differentials*, and since they are without quantity, they are also said to be *infinitely small*. Hence, by their nature they are to be so interpreted as absolutely nothing, or they are considered to be equal to nothing (ibid.).

Euler's reference to equality in his comment about infinitesimals being "equal to nothing" can be interpreted in terms of a generalized notion of equality as follows. Euler discusses two modes of comparison, arithmetic and geometric, in § 84–87 of [Euler 1755]. Euler illustrates the arithmetic mode by $ndx = 0$ and the geometric by $\frac{a \pm ndx}{a} = 1$. Euler's generalized notions of equality are similar to Leibniz's. Leibniz used a generalized notion of equality up to in the context of his transcendental law of homogeneity [Katz & Sherry 2012], showing Leibnizian calculus to have been more soundly founded than George

Berkeley's criticism thereof. Euler goes on to provide an example in terms of an infinitesimal ω:

> Thus, if the quantity x is given an increment ω, so that it becomes $x + \omega$, its square x^2 becomes $x^2 + 2x\omega + \omega^2$, and it takes the increment $2x\omega + \omega^2$. Hence, the increment of x itself, which is ω, has the ratio to the increment of the square, which is $2x\omega + \omega^2$, as 1 to $2x + \omega$. This ratio reduces to 1 to $2x$, at least when ω vanishes. Let $\omega = 0$, and the ratio of these vanishing increments, which is the main concern of differential calculus, is as 1 to $2x$. [Euler 2000, p. vii]

Euler's procedure involves an increment ω described as being infinitely small. In the first few chapters of *Institutiones*, the ratios of vanishing increments for a few elementary functions are determined via power series expansions obtained previously in the *Introductio* [Euler 1748]. However, in Chapter 4 a different picture begins to emerge, including formulas for relations among more complicated differentials. Thus Euler writes that

> if $dy = pdx$ and $dp = qdx$, then the second differential $d^2y = qdx^2$, and so it is clear, as we indicated before, that the second differential of y has a finite ratio to dx^2 (ibid., p. 68)

Here Euler is assuming that dx involves a constant progression of differentials so that $ddx = 0$. A more general situation is dealt with in § 129:

> 129. If the successive values of x, namely, $x, x^I, x^{II}, x^{III}, x^{IV}, \ldots$, do not form an arithmetic progression, but follow some other rule, then their first differentials, namely, $dx, dx^I, dx^{II}, \ldots$, will not be equal to each other, and so we do not have $d^2x = 0$ (ibid., p. 68).

Such insights are difficult to relate to from the modern viewpoint centering on the concept of derivative (rather than differential). This different nature of the infinitesimal calculus as practiced by both Leibniz and Euler was emphasized in the seminal study [Bos 1974] (on Leibniz see further in [Katz & Sherry 2013], [Sherry & Katz 2014]). Euler's § 138 on page 72 may therefore come as a surprise to someone trained in the $f \rightsquigarrow f'$ tradition:

> From the word *differential*, which denotes an infinitely small difference, we derive other names that have come into common usage. Thus we have the word *differentiate*, which means *to find a differential* (ibid., p. 72).

The term differentiate does not mean the same thing today! But the real bombshell occurs in Chapter 5, § 164. Euler writes:

> Let p/q be a given function whose differential we need to find. When we substitute $x + dx$ for x the quotient becomes

$$\frac{p + dp}{q + dq} = (p + dp)\left(\frac{1}{q} - \frac{dq}{q^2}\right) = \frac{p}{q} - \frac{p\,dq}{q^2} + \frac{dp}{q} - \frac{dp\,dq}{q^2}.$$

When p/q is subtracted, the differential remains,

$$d\left(\frac{p}{q}\right) = \frac{dp}{q} - \frac{p\,dq}{q^2},$$

and adds:

> since the term $\frac{dp\,dq}{q^2}$ vanishes [*ob evanescentem terminum* $\frac{dp\,dq}{qq}$ in the original Latin].

Euler thus obtains the formula for the differential $d(p/q)$ in terms of dp and dq, namely what we would call today the quotient rule. Note the absence of power series and the presence of the idea of discarding higher order terms, such as $\frac{dp\,dq}{q^2}$. We will see that this aspect of Euler's work is in tension with Edwards' interpretation. For more details on Euler's foundational stance, see [Reeder 2013], [Bascelli et al. 2014]. For related updates on Fermat and Cauchy, see [Katz et al. 2013] and [Borovik & Katz 2012], respectively.

How Many Strings Does a Flute Have?

Harold M. Edwards opens his piece on Euler in the *Bulletin AMS* with a musical metaphor involving the lute and the oud. These are similar instruments, but in practice are used to play very different tunes. Edwards's metaphor is meant to illustrate a claim concerning what he refers to as "Euler's definition of the derivative." Edwards claims that, although the tune of Euler's definition may sound like the ratio of a pair of infinitesimals, in reality something else is going on:

> When I understood enough of the context to realize what Euler was saying, I experienced a *shock* of recognition. It was practically the same as the definition of the derivative that I finally chose after decades of teaching calculus: 'Rewrite $\frac{\Delta y}{\Delta x}$ in a way that still makes sense when $\Delta x = 0$'[1] [Edwards 2007b, p. 576] (emphasis added).

Alas, we have read the relevant passages in Euler but neither have we experienced the epiphanous "shock" Edwards reports, nor for that matter have we detected any such similarity.

As far as listening to Euler is concerned, at the very least it needs to be pointed out that Edwards lacks a perfect pitch. The first false note is already in his title "Euler's Definition of the Derivative," for Euler did not define the derivative at all. Therefore the answer to the question "What was Euler's definition of the derivative?" is: *None*, similar to the answer to the question contained in the title of this section. Namely, the answer is that Euler doesn't *give* a definition of derivative.

Euler works with differentials throughout (for examples, see the "Euler on Ratios of Vanishing Increments" section above). The differential quotient plays an auxiliary role and always appears in a relation between differentials, such as the factor of $2x$ in $dy = 2x\,dx$ when $y = x^2$. Derivatives don't appear either in the *Introductio* or in the *Institutiones*, either under their modern name or as fluxions. Euler mentions fluxions in § 115 of the *Institutiones* as the English equivalent of differentials:

> The English mathematicians … call infinitely small differences, which we call differentials, *fluxions* and sometimes *increments*. [Euler 2000, § 115]

The noun *derivative* used in the translation [Euler 2000, § 235–238] is Blanton's and doesn't appear in the Latin original.

Euler's student Lagrange did introduce *la fonction dérivée* in his article [Lagrange 1772], but this was well after the publication of the *Institutiones* [Euler 1755], the text Edwards claims as his source. One might have thought that Edwards merely simplified the title for greater accessibility, but the same jarring note is sounded in the abstract: "Euler's method of defining the derivative of a function is not a failed effort to describe a limit." Edwards is still out of tune in his introduction: "[Euler's] definition of the derivative is misunderstood primarily because his notion of 'function' is misunderstood." [Edwards 2007b, p. 576]. A crescendo, "*Of course* Euler understood limits. Euler was Euler. But he rejected limits as the way to define derivatives" (ibid.), is followed by a coda, "Since the definition of the derivative is still two volumes in the future" (ibid., p. 579).

Where Did the Infinitesimals Go?

One of Edwards's major mathematical contributions is what is known as the Edwards curve $x^2 + y^2 = 1 + dx^2y^2$, a way of writing certain elliptic curves that, although apparently less elegant than the Weierstrass form $y^2 = x^3 + ax + b$, turns out to be more efficient, computationally and constructively speaking. His text on elliptic curves [Edwards 2007a] appeared in the same journal a few months earlier. The link to the Euler text is that, as noted in Edwards's *Essays in constructive mathematics*,

> Euler too dealt with the curve $y^2 = 1 - x^4$ …, for which explicit and beautiful formulas can be developed for the addition law, … To require that it be put in Weierstrass normal form before the group law is described loses certain symmetries that deserve to be kept. [Edwards 2005, p. 127].

This is a fascinating historical observation, but when reading Edwards on Eulerian calculus one may not even suspect that the ω appearing in Euler's discussion of the differential ratio is infinitesimal. In fact, [Edwards 2007b] is nearly an infinitesimal-free zone (although he does mention an infinitely small z on page 578). His comment on students being "taught to shrink from differentials as from an infectious disease" [Edwards 2007b, p. 579, note 5] appears to share epidemiological concerns with Cantor's vilification of infinitesimals as the *cholera bacillus of mathematics* [Meschkowski 1965, p. 505]. However, none of this is faithful to Euler, as we saw in the "Euler on Ratios of Vanishing Increments" section earlier. In fact, Edwards's comment is but the tip of the iceberg of attempted constructivist deconstructions of infinitesimals; see [Kanovei et al. 2015] for further details.

Now it is certainly possible, mathematically speaking, to redefine the derivative as the coefficient of the linear term in the Taylor series. However, this was Lagrange's approach, not

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[1] When Edwards proposes to "rewrite" the ratio, he does not mean to replace it by a different quotient but rather replace it by a different expression that is no longer a quotient. For example, $\frac{(x+dx)^2 - x^2}{dx} = \frac{(x+dx+x)(x+dx-x)}{dx} = 2x + dx$, and the last expression *still makes sense* when $dx = 0$. Now the phrase "Rewrite … in a way that still makes sense" is certainly ambiguous. Does this entail developing into a power series and taking the linear term, or perhaps a different technique? There seems to be a conflation of *definition of X* and *algorithm for computing X*, which may be deliberate given Edwards's algorithmic and constructive philosophical commitments.

Euler's; see for example [Grattan-Guinness 2000, p. 100]. Edwards, possibly because of his constructivist leanings, appears to favor Lagrange's definition more than Euler's. Edwards apparently believes that any great mathematician would agree with him on this point, and moreover he believes that Euler was a great mathematician ("Euler was Euler"). Edwards's syllogism, however, fails to persuade someone who wishes to understand Euler's actual procedures, rather than what they should "really mean" for someone who thinks he knows what the unique consistent interpretation of this must be so as to save Euler's honor. Readers of Edwards's second *Bulletin* article presumably expected to receive a fair picture of Euler's foundational stance from reading the article. This they arguably did *not* get.

Edwards's article might have been more appropriately titled *Lagrange's Definition of the Derivative*. Note, however, that in the second edition of his *Mécanique Analytique*, Lagrange fully embraced infinitesimals in the following terms:

> Once one has duly captured the spirit of this system [i.e., infinitesimal calculus], and has convinced oneself of the correctness of its results by means of the geometric method of the prime and ultimate ratios, or by means of the analytic method of derivatives, one can then exploit the infinitely small as a reliable and convenient tool so as to shorten and simplify proofs" [Lagrange 1811, p. iv] (translation ours).

Although Lagrange didn't make the short list in [Edwards 1981, p. 105], we highly recommend both Euler and Lagrange. *Read the masters!*

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A mysterious Hungarian multiplication

photo: Petőfi Literature Museum, Budapest

Szekszárd is a town some 160km south of the Hungarian capital Budapest. In its Wosinsky Mór County Museum hangs a work from 1780 by painter János Mertz, showing an (unknown) young girl pointing to a blackboard on which figures a multiplication. Art historians don't really know why this mathematical operation appears on a painting, but for mathematicians there is a secret to figure out as well.

photo: Beatrix Mecsi
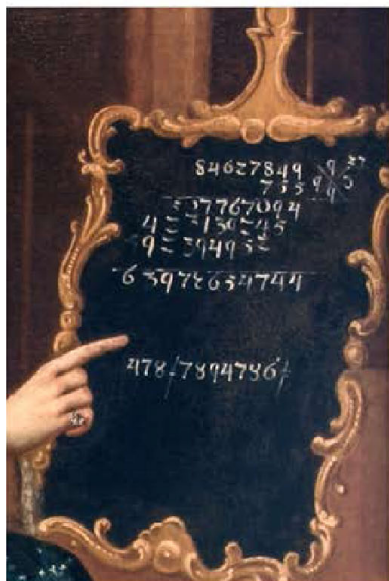
Indeed, the numbers do not seem to be correct. It is only after changing a few digits here and there that the multiplication turns out to be right. Perhaps the digits were erroneously rewritten during a restoration, or else our reading of these digits is wrong. The "casting out of nines" in the cross at the right is correct, but is there a hidden message in "4 7 8 / 7 8 9 4 7 8 6 /" at the bottom?

On the painting	What it should be
8 4 6 2 7 8 4 9	8 4 6 2 7 8 4 9
7 5 5	7 5 6
5 0 7 7 6 7 0 9 4	5 0 7 7 6 7 0 9 4
4 2 3 1 3 9 2 4 5 .	4 2 3 1 3 9 2 4 5 .
5 9 2 3 9 4 9 5 2 . .	5 9 2 3 9 4 9 4 3 . .
6 3 9 7 8 6 5 4 7 4 4	6 3 9 7 8 6 5 3 8 4 4

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