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Ulm Classification of Analytic Equivalence Relations in Generic Universes

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Abstract. We prove that if every real belongs to a set generic extension of $\boldsymbol{L}$, then every Σ^1_1 equivalence relation E on reals either admits a $\boldsymbol{\Delta}_1$ reduction to the equality on the set $2^{<\omega_1}$ of all countable binary sequences, or the Vitali equivalence E_0 continuously embeds in E. The proofs are based on a topology generated by OD sets.

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Introduction

Theorem 1.[3)] *Let E be a Σ^1_1 equivalence on reals. Assume that*

(∗) *each real belongs to a Boolean valued extension of $\boldsymbol{L}$.*

Then at least one of the following two statements holds:

(I) E *admits a $\boldsymbol{\Delta}^{\mathrm{HC}}_1$ reduction to the equality on $2^{<\omega_1}$.*

(II) $\mathsf{E}_0 \sqsubseteq_{\mathsf{c}} \mathsf{E}$, *i. e., E_0 embeds in E continuously.*

Reals may be understood either as the true reals or as points of the *Baire space* $\mathcal{N} = \omega^\omega$ or as points of the *Cantor set* $\mathcal{D} = 2^\omega$. In fact the theorem is true for all Polish spaces since all of them are Borel isomorphic to each other.

As usual $\boldsymbol{L}$ is the constructible universe. By a *Boolean valued extension* of a model M we always mean a Boolean valued extension $M^{(P)}$ via a forcing notion $P \in M$.

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3)It follows from an e-mail discussion between G. HJORTH and A. S. KECHRIS, and the author, in 1995, that G. HJORTH may have proved a similar theorem independently.

The statement that a set S belongs to a Boolean valued extension of $\boldsymbol{L}$ is adequately formalized as follows:

> *there exists a Boolean valued extension of* $\boldsymbol{L}[S]$ *in which it is true that the universe is a set generic extension of* $\boldsymbol{L}$.

The hypothesis $(*)$ follows e. g. from the assumption that the universe is a set generic extension of $\boldsymbol{L}$. (But in principle the extensions can be different for different reals.) As a matter of fact the theorem remains true in the more broad hypothesis that each real x belongs to a Boolean valued extension of $\boldsymbol{L}[z_0]$ for one and the same real z_0 which does not depend on x.

(Boldface) $\boldsymbol{\Delta}_1^{\mathrm{HC}}$ is the class of all subsets of HC (the family of all hereditarily countable sets) which are Δ_1 in HC by formulas which may contain sets in HC as parameters.

A *reduction of* E *to the equality on* $2^{<\omega_1}$ (the set of all countable binary sequences of any length $\lambda < \omega_1$) is any function U : reals $\longrightarrow 2^{<\omega_1}$ such that $x \mathsf{E} y$ if and only if $U(x) = U(y)$ holds for any pair of reals x, y. In other words such a function enumerates E-equivalence classes by elements of $2^{<\omega_1}$.

E_0 is the *Vitali equivalence relation* on $\mathcal{D} = 2^\omega$, defined by

$$x \mathsf{E}_0 y \text{ iff } x(n) = y(n) \text{ for all } n \in \omega \text{ bigger than some } n_0 = n_0(x, y).$$

Statement (II) means, by definition, the existence of a continuous $1-1$ function $\varphi : 2^\omega \longrightarrow$ reals such that

$$x \mathsf{E}_0 y \text{ iff } \varphi(x) \mathsf{E} \varphi(y) \text{ for all } x,\ y \text{ in } 2^\omega.$$

Such a function φ is called a (continuous) *embedding* of E_0 in E.

Intuitively, the Vitali relation E_0 hardly admits a reasonable enumeration of the equivalence classes, definable in ZFC: at least a ROD (real-ordinal definable) enumeration of E_0 equivalence classes by *sets of ordinals* does not exist in the Solovay model. Thus the theorem says that any $\boldsymbol{\Sigma}_1^1$ equivalence relation on reals either admits a $\boldsymbol{\Delta}_1^{\mathrm{HC}}$ enumeration of equivalence classes by elements of $2^{<\omega_1}$ or contains a homeomorphic copy of E_0, a relation which admits such an enumeration only by occasional reasons like the axiom of constructibility.

We refer the reader to [2] in matters of the early history of "Glimm-Effros" theorems – those of type: *each equivalence relation of certain kind either admits a reasonable enumeration of the equivalence classes or satisfies* $\mathsf{E}_0 \sqsubseteq_{\mathrm{c}} \mathsf{E}$ – and relevant problems in probability and measure theory.

The modern history of the topic began in HARRINGTON, KECHRIS and LOUVEAU [2], where it is proved that each Borel equivalence relation on reals either admits a Borel enumeration of the equivalence classes by reals or satisfies $\mathsf{E}_0 \sqsubseteq_{\mathrm{c}} \mathsf{E}$. The case of $\boldsymbol{\Sigma}_1^1$ relations is more complicated. HJORTH and KECHRIS [5] give examples which show that reasonable results of the "Glimm-Effros" type hardly can be obtained for $\boldsymbol{\Sigma}_1^1$ relations by simply taking a non-Borel enumeration in (I) or discontinuous embedding in (II); it seems that the enumeration of equivalence classes by *reals* does not match the nature of $\boldsymbol{\Sigma}_1^1$ relations.

To fix the problem, HJORT and KECHRIS suggested an adequate idea: enumerate the equivalence classes by elements of $2^{<\omega_1}$. (This approach is referred to as the *Ulm-type classification* in [5], in connection with an Ulm classification theorem in algebra.) They proved that the dichotomy (I) vs. (II) holds for each $\mathbf{\Sigma}^1_1$ equivalence relation on reals, assuming the existence of "sharps".[4]

Theorem 1 establishes the same result (apart of the possible compatibility of (I) and (II)) in the completely different environment of generic models. It is a principal problem to get the result in ZFC alone.[5]

Another problem is to generalize the theorem on the case of $\mathbf{\Delta}^1_2$ equivalence relations. (A generalization on $\mathbf{\Sigma}^1_2$ or $\mathbf{\Pi}^1_2$ is hardly possible [6] [7].)

Organization of the proof.

First of all, we shall consider only the case when E is a lightface Σ^1_1 relation; if in fact E is $\Sigma^1_1(z_0)$ in a real z_0, then this z_0 simply enters the reasoning in a uniform way, not influenting substantially any of the arguments.[8]

The splitting point between the statements (I) and (II) of Theorem 1 is determined in Section 1. It occurs that we have (I) in the assumption that the E equivalence class $[x]_{\mathsf{E}}$ of every real x is determined by intersections with OD (ordinal definable) sets in an appropriate collapse extension of the universe. (Case 1 in Subsection 1.2.) Otherwise (Case 2) we have (II).

Both sides of the proof depend on properties of reals in collapse extensions close to those of the Solovay model. The facts we need are reviewed in Section 2.

Section 3 proves assertion (I) of Theorem 1 in Case 1. The principal idea is based on the fact that the collapse generic models are regular enough to reduce the collection of all OD sets to essentially Souslin sets with constructible code, which yields a characterization in terms of elements of $2^{<\omega_1}$. An absoluteness argument allows to extend this fact to the universe of Theorem 1.

[4] The latter was eliminated in [5] in the case when the $\mathbf{\Sigma}^1_1$ equivalence relation of consideration occasionally has only Borel equivalence classes. The method introduced by S. FRIEDMAN and B. VELICKOVIC [1] allows to weaken the "sharps" hypothesis to the assumption that each class $\boldsymbol{L}[x]$, x being a real, contains a weakly compact cardinal.

[5] HJORTH [3] found one more theorem on $\mathbf{\Sigma}^1_1$ equivalence relations, true in both the "sharps" and the "forcing" case, but still open for ZFC.

[6] In an appropriate iterated Sacks extension of $\boldsymbol{L}$ (with "ill"founded length of iteration) there are Σ^1_2 and Π^1_2 equivalence relations which neither admit a ROD enumeration of the equivalence classes by sets of ordinals nor satisfy $\mathsf{E}_0 \sqsubseteq_c \mathsf{E}$, see KANOVEI [7].

[7] More complicated relations can be successfully investigated in strong extensions of ZFC or in special models. HJORTH [4] proved, assuming AD in $\boldsymbol{L}$[reals], that every ROD equivalence relation on reals either admits a ROD enumeration of the equivalence classes by sets of ordinals, or satisfies $\mathsf{E}_0 \sqsubseteq_c \mathsf{E}$. KANOVEI [6] proved even a stronger result (enumeration by elements of $2^{<\omega_1}$) in the Solovay model.

[8] It suffices to check that condition (*) of Theorem 1 implies its relativized form, for $\boldsymbol{L}[z_0]$ rather than $\boldsymbol{L}$. To see this, let x, z_0 be reals; we have to prove that x belongs to a Boolean valued extension of $\boldsymbol{L}[z_0]$ assuming (*). First of all, by (*), there is a Boolean valued extension V of $\boldsymbol{L}[x, z_0]$ in which it is true that the universe is a set-generic extension of $\boldsymbol{L}$. Then, by Lemma 5 below, it is also true in V that the universe is a generic extension of $\boldsymbol{L}[z_0]$. Therefore V is a Boolean valued extension of $\boldsymbol{L}[z_0]$ containing x.

Sections 4 and 5 prove (II) of Theorem 1 in Case 2. The assumption suffices to check that $\mathsf{E}_0 \sqsubseteq_{\mathsf{c}} \mathsf{E}$ in a collapse extension of the universe; moreover, E_0 embeds in E in a special sense which can be expressed by a Σ^1_2 formula. (The existence of an embedding in general needs Σ^1_3.) We conclude that E_0 embeds in E in the universe by the Shoenfield absoluteness theorem.

The construction of an embedding of E_0 into E follows the pattern given in [2], yet associated with another topology (the topology generated by OD sets), and arranged in a different way.

1 Approach to the proof of the main theorem

It will be more convenient to consider $\mathcal{D} = 2^\omega$, the Cantor set, rather than $\mathcal{N} = \omega^\omega$ as the basic Polish space for which Theorem 1 is being proved. Thus by "reals" we shall understand points of $\mathcal{D}$. (Just because points of $\mathcal{D}$ admit a very simple coding in collapse generic universes.)

We shall prove only the "lightface" version of the theorem, so that E is supposed to be a Σ^1_1 equivalence relation on reals in the course of the proof. (See footnote 8.)

The purpose of this section is to describe the factor which determines the dichotomy of Theorem 1.

1.1 Collapse extensions

Let α be an ordinal. Then $\alpha^{<\omega} = \bigcup_{n\in\omega} \alpha^n$ is the ordinary forcing notion to collapse α down to ω. We shall understand that, for conditions $p,\, q \in \alpha^{<\omega}$, $p \le q$ iff $p \subseteq q$, so that *bigger* forcing conditions are *stronger*.

If $G \subseteq \alpha^{<\omega}$ is $\alpha^{<\omega}$-generic over a transitive model M (M is a set or a class), then $f = \bigcup G$ maps ω onto α, so α is countable in $M[G] = M[f]$. Functions $f \in \alpha^\omega$ obtained this way will be called *$\alpha^{<\omega}$-generic over M*. Let $\mathsf{Clps}_\alpha(M)$ denote the set of all $\alpha^{<\omega}$-generic over M functions $f \in \alpha^\omega$.

The *Ω-collapse universe hypothesis* will be the assumption:

Ω-CUH *Ω is a limit cardinal in $\boldsymbol{L}$ and there exists a function $f_0 \in \mathsf{Clps}_\Omega(\boldsymbol{L})$ such that $\boldsymbol{V} = \boldsymbol{L}[f_0]$ in the universe.*

The notion of the *$\Omega^{<\omega}$-valued* extension of a model M is understood in the usual way, that is, as a certain inner class in M the truth in which takes values in the complete Boolean algebra over $\Omega^{<\omega}$ in M.

Definition. Let Ω be a limit $\boldsymbol{L}$-cardinal. We say that a set S *belongs to an $\Omega^{<\omega}$-valued extension of $\boldsymbol{L}$* iff there is a Boolean valued extension of $\boldsymbol{L}[S]$ where Ω-CUH holds.

The collapse models are not so nice as the Solovay model, but they contain reals which behave approximately like all reals in the Solovay model.

Definition. A set is *Ω-weak over M* (where Ω is an ordinal in a model M) iff it belongs to an $\alpha^{<\omega}$-generic extension of M for some $\alpha < \Omega$. We define

$$\mathsf{Weak}_\Omega(M) = \{x \in \mathcal{D} = 2^\omega : x \text{ is } \Omega\text{-weak over } M\}.$$

Proposition 2. *Assume* $(*)$ *of Theorem* 1. *Then for each real* z *there is a limit* $\boldsymbol{L}$*-cardinal* Ω *such that* z *belongs to an* $\Omega^{<\omega}$*-valued extension of* $\boldsymbol{L}$ *where it is true that* $z \in \mathsf{Weak}_\Omega(\boldsymbol{L})$.

Proof. By $(*)$ there are forcing notions $P, Q \in \boldsymbol{L}$ such that in $M = \boldsymbol{L}[z]^{(P)}$ it is true that the universe is a Q-generic extension of $\boldsymbol{L}$. Let $\alpha = (\operatorname{card} Q)^+$ in $\boldsymbol{L}$ and Ω be the least $\boldsymbol{L}$-cardinal bigger than α. We consider the model $M' = M^{(\Omega^{<\omega})} = \boldsymbol{L}[z]^{(P)(\Omega^{<\omega})}$, so that it is true in M' that the universe is an $\Omega^{<\omega}$-generic extension of a Q-generic extension of $\boldsymbol{L}$. It is a standard fact (and an easy corollary of Proposition 4 below) that in this case Ω-CUH is true in M'. Furthermore one easily sees that $z \in \mathsf{Weak}_\Omega(\boldsymbol{L})$ in M'. □

1.2 The dichotomy

In ZFC let $\mathcal{T}$ be the topology generated on the set $\mathcal{D} = 2^\omega$ by all OD nonempty subsets of $\mathcal{D}$. This topology plays the same role in our consideration as the Gandy-Harrington topology in the proof of the classical Glimm-Effros theorem (for Borel relations) in [2].

We define $\bar{\mathsf{E}}$ to be the $\mathcal{T}^2$-closure of E in $\mathcal{D}^2$. In other words,

$$x \,\bar{\mathsf{E}}\, y \Leftrightarrow \forall X\,[X \text{ is OD} \Rightarrow (x \in [X]_{\mathsf{E}} \Leftrightarrow y \in [X]_{\mathsf{E}})],$$

where $[X]_{\mathsf{E}} = \{y : (\exists x \in X)(x \,\mathsf{E}\, y)\}$ (the E-*saturation* of X). Thus $\bar{\mathsf{E}}$ is an OD equivalence relation on $\mathcal{D}$.

The dichotomy in [2] is determined by the equality $\mathsf{E} = \bar{\mathsf{E}}$ (where $\bar{\mathsf{E}}$ is defined via the Gandy-Harrington topology): if $\mathsf{E} = \bar{\mathsf{E}}$, then E admits a Borel enumeration of the equivalence classes by reals, otherwise E_0 embeds in E. Here the splitting condition is a bit more complicated: the essential domain of the equivalence is now a proper subset $\mathsf{Weak}_\Omega(\boldsymbol{L}) \subsetneqq \mathcal{D}$.

Case 1. For each real z there is a limit $\boldsymbol{L}$-cardinal Ω such that z belongs to an $\Omega^{<\omega}$-valued extension V of $\boldsymbol{L}$, where (in V) the following is true:

$$z \in \mathsf{Weak}_\Omega(\boldsymbol{L}) \text{ and } \mathsf{E} \text{ coincides with } \bar{\mathsf{E}} \text{ on } \mathsf{Weak}_\Omega(\boldsymbol{L}).$$

(Notice that, for a Σ^1_1 binary relation E, the assertion that E is an equivalence relation is Π^1_2, and therefore absolute for all models with the same ordinals, in particular for $\boldsymbol{L}$ and all generic extensions of $\boldsymbol{L}$.)

Case 2. Otherwise.

Theorem 3. *Suppose* $(*)$ *of Theorem* 1, *i. e., each real belongs to a Boolean valued extension of* $\boldsymbol{L}$. *Then for a given* Σ^1_1 *equivalence relation* E *we have*

- *assertion* (I) *of Theorem* 1 *in Case* 1,
- *assertion* (II) *of Theorem* 1 *in Case* 2.

This will be the form in which we prove Theorem 1. Section 3 proves the first part of Theorem 3, the second part is proved in Sections 4 and 5.

2 On collapse extensions

In this section, we fix a limit $\boldsymbol{L}$-cardinal Ω. The purpose is to establish some properties of Ω-collapse generic extensions (= the universe under the hypothesis Ω-CUH), mostly connected with weak reals.

2.1 Basic properties of collapse extensions

The hypothesis Ω-CUH will be assumed during the reasoning, but we shall not forget to specify Ω-CUH in all formulations of theorems.

Proposition 4. *Assume Ω-CUH. Let $S \subseteq \mathrm{Ord}$ be Ω-weak over $\boldsymbol{L}$. Then the universe $\boldsymbol{V}$ of all sets is an $\Omega^{<\omega}$-generic extension of $\boldsymbol{L}[S]$, and moreover we have:*

1. *If Φ is a sentence containing only sets in $\boldsymbol{L}[S]$ as parameters, then Λ decides Φ in the sense of $\Omega^{<\omega}$ as a forcing notion over $\boldsymbol{L}[S]$.*
2. *If a set $X \subseteq \boldsymbol{L}[S]$ is $\mathrm{OD}[S]$, then $X \in \boldsymbol{L}[S]$.*

(Λ is the empty function, the weakest condition in any forcing notion of the form $\Omega^{<\omega}$. $\mathrm{OD}[S]$ means *S-ordinal definable*, i.e., definable by an $\in$-formula having S and ordinals as parameters.) The proof (a copy of the proof of Theorem 4.1 in Solovay [8]) is based on the following crucial lemma:

Lemma 5. *Suppose that $P \in \boldsymbol{L}$ is a partially ordered set, and $G \subseteq P$ is a P-generic set over $\boldsymbol{L}$. Let $S \in \boldsymbol{L}[G]$, $S \subseteq \mathrm{Ord}$. Then there exists a set $\Sigma \subseteq P$, $\Sigma \in \boldsymbol{L}[S]$, such that $G \subseteq \Sigma$ and G is Σ-generic over $\boldsymbol{L}[S]$.*

Proof of the lemma (extracted from the proof of Lemma 4.4 in [8]).

We argue in $\boldsymbol{L}[S]$. Let $\underset{\sim}{S}$ be the name for S in the P-forcing language. Define a sequence of sets $A_\alpha \subseteq P$ ($\alpha \in \mathrm{Ord}$) by induction on α.

(A1) $p \in A_0$ iff there is $\sigma \in \mathrm{Ord}$ such that either $\sigma \in S$ but p P-forces $\sigma \notin \underset{\sim}{S}$ over $\boldsymbol{L}$, or $\sigma \notin S$ but p P-forces $\sigma \in \underset{\sim}{S}$ over $\boldsymbol{L}$.

(A2) $p \in A_{\alpha+1}$ iff there exists a dense over p set $D \in \boldsymbol{L}$, $D \subseteq A_\alpha$.

(A3) If α is a limit ordinal, then $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$.

One easily verifies the following (see Solovay [8]): if $p \in A_\alpha$ and $p \le q \in P$, then $q \in A_\alpha$; if $\beta < \alpha$, then $A_\beta \subseteq A_\alpha$. Evidently $A_\delta = A_{\delta+1}$ for some ordinal δ. We put $\Sigma = P \setminus A_\delta$. Thus Σ can be thought of as the set of all conditions $p \in P$ which do not force something about $\underset{\sim}{S}$ which contradicts a factual information about S.

We prove, following [8], that Σ is as required. This involves two facts.

(Σ1) $\quad G \subseteq \Sigma$.

(Assume on the contrary that $G \cap A_\gamma \neq \emptyset$ for some γ. Let γ be the least such an ordinal. Clearly γ is not limit and $\gamma \neq 0$. Let $\gamma = \alpha + 1$ and let $p \in A_\gamma \cap G$. Since G is generic, Definition (A2) implies $G \cap A_\alpha \neq \emptyset$, contradiction.)

(Σ2) $\quad$ *If $D \in \boldsymbol{L}$ is a dense subset of P, then $D \cap \Sigma$ is dense in Σ.*

(If $p \in \Sigma$, then $p \notin A_{\delta+1}$. Hence by (A2) there is $q \in D \setminus A_\delta$, $q \ge p$.)

We prove that G is Σ-generic over $\boldsymbol{L}[S]$. Let $D \in \boldsymbol{L}[S]$ be a dense subset of Σ. Suppose towards a contradiction that $D \cap G = \emptyset$. Since $D \in \boldsymbol{L}[S]$, there exists an $\in$-formula $\Phi(x, y)$ containing only ordinals as parameters and such that $\Phi(S, y)$ holds in $\boldsymbol{L}[S]$ iff $y = D$. Let $\Psi(G')$ be the conjunction of the following formulas:

(1) $S' = \underset{\sim}{S}[G']$ (the G'-interpretation of the "term" $\underset{\sim}{S}$) is a set of ordinals, and there is a unique $D' \in \boldsymbol{L}[S']$ such that $\Phi(S', D')$ holds in $\boldsymbol{L}[S']$;

(2) this D' is a dense subset of Σ', where $\Sigma' = \Sigma(S')$ is the set obtained by applying our definition of $\Sigma = \Sigma(S)$ for $S = S'$;

(3) $D' \cap G' = \emptyset$.

Then $\Psi(G)$ is true in $\boldsymbol{L}[G]$ by our assumptions. Let $p \in G$ P-force $\Psi(G)$ over $\boldsymbol{L}$. Then $p \in \Sigma$ by (Σ1). By the density there exists a condition $q \in D$, $q \geq p$. Consider a Σ-generic over $\boldsymbol{L}[S]$ set $G' \subseteq \Sigma$ containing q. Then G' is also P-generic over $\boldsymbol{L}$ by (Σ2). We observe that $\underline{S}[G'] = S$ because $G' \subseteq \Sigma$. Therefore D' and Σ' (as in the description of Ψ) coincide with resp. D and Σ. In particular $q \in D' \cap G'$, a contradiction, because p forces (3). □

Proof of Proposition 4. By Lemma 5 (for $P = \Omega^{<\omega}$) we have that the universe is a Σ-generic extension of $\boldsymbol{L}[S]$ for a tree $\Sigma \subseteq \Omega^{<\omega}$, $\Sigma \in \boldsymbol{L}[S]$. Note that Ω is a cardinal in $\boldsymbol{L}[S]$ by the choice S. On the other hand, Ω is countable in the universe by Ω-CUH, therefore the collapse of Ω is Σ-forced by some $u \in G$. Now obviously the set of all Ω-branching points of Σ is cofinal over u in Σ. It follows that the set $\{v \in \Sigma : u \subseteq v\}$ includes in $\boldsymbol{L}[S]$ a cofinal subset order isomorphic to $\Omega^{<\omega}$.

For the items 1. and 2. argue as in the proofs of Lemma 3.5 and Corollary 3.5 in [8] for $\boldsymbol{L}[S]$ as the initial model. □

2.2 Coding reals and sets of reals in collapse extensions

The following definitions intend to introduce a useful coding system for reals (i.e., points of $\mathcal{D} = 2^\omega$ in this paper) and sets of reals in the collapse extensions.

Let $\alpha \in \mathrm{Ord}$. By $\mathbb{T}_\alpha$ we denote the set of all indexed sets t of the form $\langle \alpha, \langle t_n : n \in \omega \rangle \rangle$ – the "terms" – such that $t_n \subseteq \alpha^{<\omega}$ for each n.

We put $\mathbb{T}_{<\Omega} = \bigcup_{\alpha<\Omega} \mathbb{T}_\alpha$ for any ordinal Ω.

"Terms" $t \in \mathbb{T}_\alpha$ are used to code functions $C : \alpha^\omega \longrightarrow \mathcal{D}$. Given $f \in \alpha^\omega$, we define $x = \mathrm{C}_t(f) \in \mathcal{D}$ by $x(n) = 1$ iff $f \restriction m \in t_n$ for some m.

Assume that $t \in \mathbb{T}_\alpha$, $u \in \alpha^{<\omega}$, and M is an arbitrary model. We introduce the sets $\mathsf{X}_{t,u}(M) = \{\mathrm{C}_t(f) : u \subset f \in \mathsf{Clps}_\alpha(M)\}$ and $\mathsf{X}_t(M) = \mathsf{X}_{t,\Lambda}(M)$= the C_t-image of $\mathsf{Clps}_\alpha(M)$, where, we recall, $\mathsf{Clps}_\alpha(M)$ is the set of all functions $f \in \Omega^\omega$ which are $\Omega^{<\omega}$-generic over M. (Λ is the empty sequence.)

Proposition 6. *Assume Ω-CUH. Let $S \subseteq \mathrm{Ord}$ be Ω-weak over $\boldsymbol{L}$. Then every $\mathrm{OD}[S]$ set $X \subseteq \mathsf{Weak}_\Omega(\boldsymbol{L}[S])$ is a union of sets of the form $\mathsf{X}_t(\boldsymbol{L}[S])$, where $t \in \mathbb{T}_{<\Omega} \cap \boldsymbol{L}[S]$. Moreover, if $t \in \mathbb{T}_\alpha \cap \boldsymbol{L}[S]$, $\alpha < \Omega$, and $u \in \alpha^{<\omega}$, then every $\mathrm{OD}[S]$ set $X \subseteq \mathsf{X}_{t,u}(\boldsymbol{L}[S])$ is a union of sets of the form $\mathsf{X}_{t,v}(\boldsymbol{L}[S])$, where $u \subseteq v \in \alpha^{<\omega}$.*

Proof. Every $x_0 \in X$ belongs to an $\alpha^{<\omega}$-generic extension of $\boldsymbol{L}[S]$ for some $\alpha < \Omega$. Thus $x_0 \in \boldsymbol{L}[S, f_0]$, where $f_0 \in \mathsf{Clps}_\alpha(\boldsymbol{L}[S])$. Let $\boldsymbol{x}$ be a name of x_0 in the $\alpha^{<\omega}$-forcing language. Put $t_n = \{u \in \alpha^{<\omega} : u \text{ forces } \boldsymbol{x}(n) = 1 \text{ in } \boldsymbol{L}[S]\}$ for all n, and $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle$, so that $t \in \mathbb{T}_\alpha \cap \boldsymbol{L}[S]$ and $x_0 = \mathrm{C}_t(f_0)$.

Suppose that $X = \{x : \Phi(S, x)\}$ in the universe, where the formula Φ contains only S and ordinals as parameters. Let $\Psi(S, f)$ denote the formula: "Λ $\Omega^{<\omega}$-forces $\Phi(S, \mathrm{C}_t(f))$ over the universe", so $\Psi(S, f_0)$ holds in $\boldsymbol{L}[S, f_0]$ by Proposition 4. Since f_0 is generic, there exists $u \in \Omega^{<\omega}$, $u \subset f_0$, such that $\Psi(S, f)$ holds in $\boldsymbol{L}[S, f]$ whenever $f \in \mathsf{Clps}_\alpha(\boldsymbol{L}[S])$ satisfies $u \subset f$. It follows that $\mathsf{X}_{t,u}(\boldsymbol{L}[S]) \subseteq X$ by Proposition 4, and, we recall, $x_0 = \mathrm{C}_t(f_0) \in \mathsf{X}_{t,u}(\boldsymbol{L}[S])$. Finally the set $\mathsf{X}_{t,u}(\boldsymbol{L}[S])$ is equal to $\mathsf{X}_{t'}(\boldsymbol{L}[S])$ for some other $t' \in \mathbb{T}_\alpha \cap \boldsymbol{L}[S]$.

The "moreover" part is proved similarly. □

3 The case of closed relations: classifiable reals

In this section, we prove the "Case 1" of Theorem 3. Thus E continues to be a Σ^1_1 equivalence relation on reals.

3.1 Classifiable reals

Assuming Ω-CUH, there is a constructible Δ^{HC}_1 enumeration $\{\tau[\xi] : \xi < \omega_1\}$ of $\mathbb{T}_{<\omega_1} \cap \boldsymbol{L}$ such that each "term" $t \in \mathbb{T}_{<\Omega} \cap \boldsymbol{L}$ has uncountably many numbers $\xi < \Omega$ satisfying $t = \tau[\xi]$. The following lemma gives a special characterization for $\bar{\mathsf{E}}$, the $\mathcal{T}^2$-closure of E, based on this enumeration.

Lemma 7. *Assume Ω-CUH. Let x, y be reals in $\mathsf{Weak}_\Omega(\boldsymbol{L})$. Then $x \,\bar{\mathsf{E}}\, y$ if and only if for each $\xi < \Omega$ we have*

$$x \in [\mathsf{X}_{\tau[\xi]}(\boldsymbol{L}_\xi)]_{\mathsf{E}} \Leftrightarrow y \in [\mathsf{X}_{\tau[\xi]}(\boldsymbol{L}_\xi)]_{\mathsf{E}}.$$

Proof. The "only if" part is clear, since the sets $\mathsf{X}_{\tau[\xi]}(\boldsymbol{L}_\gamma)$ are OD. Let us prove the "if" direction. Assume that not $x \,\bar{\mathsf{E}}\, y$. Then there exists an OD set X such that $x \in [X]_{\mathsf{E}}$ but $y \notin [X]_{\mathsf{E}}$. By Proposition 6 we have $x \in \mathsf{X}_t(\boldsymbol{L}) \subseteq [X]_{\mathsf{E}}$, where $t = \langle \alpha, \langle t_n : n \in \omega \rangle \rangle \in \mathbb{T}_\alpha \cap \boldsymbol{L}$, $\alpha < \Omega$. Since Ω is a limit cardinal in $\boldsymbol{L}$, there is an $\boldsymbol{L}$-cardinal γ, $\alpha < \gamma < \Omega$, such that $\mathsf{Clps}_\alpha(\boldsymbol{L}) = \mathsf{Clps}_\alpha(\boldsymbol{L}_\gamma)$. As $t = \tau[\xi]$ for some ξ, $\gamma \le \xi < \Omega$, we have $\mathsf{X}_t(\boldsymbol{L}) = \mathsf{X}_{\tau[\xi]}(\boldsymbol{L}_\xi)$. □

Let $x \in \mathfrak{D}$. We define, for all $\xi < \Omega$, $\varphi_x(\xi) = 1$ iff $x \in [\mathsf{X}_{\tau[\xi]}(\boldsymbol{L}_\xi)]_{\mathsf{E}}$. Thus $\varphi_x \in 2^\Omega$, and $x \,\bar{\mathsf{E}}\, y$ iff $\varphi_x = \varphi_y$ for all x, $y \in \mathsf{Weak}_\Omega(\boldsymbol{L})$ by the lemma.

This is a nice point: we have defined a very straightforward enumeration of E-equivalence classes of "weak" reals, essentially by reals, under the assumption Ω-CUH. However the enumeration is too complicated to be reproduced in the original universe. Another idea enters the reasoning.

Definition (extracted from HJORTH and KECHRIS [5]). Let $\mathrm{Def}_{\mathsf{E}}$ be the set of all triples $\langle x, \psi, t \rangle$ such that $x \in \mathfrak{D}$, $\psi \in 2^\gamma$ and $t \in \mathbb{T}_\alpha \cap \boldsymbol{L}_\gamma[\psi]$ [9], where $\alpha < \gamma < \omega_1$ and the following conditions are satisfied:

(a) $\boldsymbol{L}_\gamma[\psi]$ models ZFC^- (ZFC minus the Power Set Axiom) so that ψ can occur as an extra class parameter in Replacement and Separation.

(b) In $\boldsymbol{L}_\gamma[\psi]$, $\langle \Lambda, \Lambda \rangle$ forces $\mathrm{C}_t(\boldsymbol{f}) \,\mathsf{E}\, \mathrm{C}_t(\boldsymbol{g})$ in the sense of $\alpha^{<\omega} \times \alpha^{<\omega}$ as the forcing, where $\boldsymbol{f}$ and $\boldsymbol{g}$ are the names for the generic functions in α^ω in the $(\alpha^{<\omega} \times \alpha^{<\omega})$-forcing language.

(c) For each $\xi < \gamma$, $\psi(\xi) = 1$ iff $x \in [\mathsf{X}_{\tau[\xi]}(\boldsymbol{L}_\xi)]_{\mathsf{E}}$ (i.e., $\psi = \varphi_x \restriction \gamma$, provided $\gamma \le \Omega$).

(d) x belongs to $[\mathsf{X}_t(\boldsymbol{L}_\gamma[\psi])]_{\mathsf{E}}$.

A real x is E-*classifiable* iff there are ψ and t such that $\langle x, \psi, t \rangle \in \mathrm{Def}_{\mathsf{E}}$.

[9] By $\boldsymbol{L}_\gamma[\psi]$ we understand the result of the Gödel construction of length γ arranged so that only the restriction $\psi \restriction \gamma'$ is available at each step $\gamma' < \gamma$. Note that $\psi \notin \boldsymbol{L}_\gamma[\psi]$.

Lemma 8. $\mathsf{Def}_{\mathsf{E}}$ *is a* Δ_1^{HC} *set (provided* E *is* Σ_1^1*).*

Proof. The requirements (a) and (b) are Δ_1^{HC} because they are relativized to $\boldsymbol{L}_\gamma[\psi]$ and the enumeration $\tau[\xi]$ was chosen in Δ_1^{HC}.

Condition (d) is obviously Σ_1^{HC}, so it remains to convert it to Π_1^{HC} form. Notice that in the assumption of (a) and (b) the set $X = \mathsf{X}_t(\boldsymbol{L}_\gamma[\psi])$ consists of mutually E-equivalent reals. (Consider a pair of $\alpha^{<\omega}$-generic over $\boldsymbol{L}_\gamma[\psi]$ functions $f,\, g \in \alpha^\omega$, not necessarily a *generic pair*. Let $h \in \alpha^\omega$ be an $\alpha^{<\omega}$-generic over both $\boldsymbol{L}_\gamma[\psi, f]$ and $\boldsymbol{L}_\gamma[\psi, g]$ function. Then by (b) the formula $\mathrm{C}_t(h)\,\mathsf{E}\,\mathrm{C}_t(f)$ holds in $\boldsymbol{L}_\gamma[\psi, f, h]$, hence in the universe by the Shoenfield absoluteness. Similarly, $\mathrm{C}_t(h)\,\mathsf{E}\,\mathrm{C}_t(g)$. It follows that $\mathrm{C}_t(f)\,\mathsf{E}\,\mathrm{C}_t(g)$, as required.) Therefore (d) is equivalent to $(\forall y \in \mathsf{X}_t(\boldsymbol{L}_\gamma[\psi]))\,(x\,\mathsf{E}\,y)$, and this is clearly Π_1^{HC}.

Consider (c). The right-hand side of the equivalence "iff" in (c) is Σ_1^1 with inserted Δ_1^{HC} functions, therefore Δ_1^{HC}. It follows that (c) itself is Δ_1^{HC}. □

3.2 Getting enumeration of the equivalence classes

The following lemma will allow to define a Δ_1^{HC} enumeration of the equivalence classes for the given Σ_1^1 equivalence relation E by elements of $2^{<\omega_1}$.

Lemma 9. *Assuming Case* 1 *of Subsection* 1.2*, all reals* x *are* E*-classifiable.*

Proof. Let $x \in \mathfrak{D}$. By the assumption of Case 1, there is a limit $\boldsymbol{L}$-cardinal Ω such that x belongs to an $\Omega^{<\omega}$-valued extension V of $\boldsymbol{L}$ where it is true that E coincides with $\bar{\mathsf{E}}$ on $\mathsf{Weak}_\Omega(\boldsymbol{L})$ and $x \in \mathsf{Weak}_\Omega(\boldsymbol{L})$. By Lemma 8 and the Shoenfield absoluteness, it suffices to prove that x is E-classifiable in V.

We argue in the "auxiliary" universe V. Thus Ω-CUH will be assumed. We observe that φ_x is Ω-weak over $\boldsymbol{L}$: indeed, $\varphi_x \in \boldsymbol{L}[x]$ by Proposition 4 as φ_x is $\mathrm{OD}[x]$. On the other hand $[x]_{\mathsf{E}}$ is $\mathrm{OD}[\varphi_x]$. (Clearly $[x]_{\mathsf{E}}$ is the E-saturation of the set $Y = [x]_{\mathsf{E}} \cap \mathsf{Weak}_\Omega(\boldsymbol{L}) = [x]_{\bar{\mathsf{E}}} \cap \mathsf{Weak}_\Omega(\boldsymbol{L})$. However Y is $\mathrm{OD}[\varphi_x]$ by Lemma 7.) Therefore by Proposition 6, $x \in \mathsf{X}_t(\boldsymbol{L}[\varphi_x]) \subseteq [x]_{\mathsf{E}}$ for some $t \in \mathbb{T}_\alpha \cap \boldsymbol{L}[\varphi]$, where $\alpha < \Omega$. There is an ordinal γ, $\alpha < \gamma < \Omega$, such that the model $\boldsymbol{L}_\gamma[\varphi_x \restriction \gamma]$ contains t and satisfies (a). We put $\psi = \varphi_x \restriction \gamma$ and prove that $\langle x, \psi, t\rangle \in \mathsf{Def}_{\mathsf{E}}$. We have (c) by definition, while (d) holds because $x \in \mathsf{X}_t(\boldsymbol{L}[\varphi_x]) \subseteq [\mathsf{X}_t(\boldsymbol{L}_\gamma[\psi])]_{\mathsf{E}}$.

Now we check (b). Otherwise there exist conditions $u,\, v \in \alpha^{<\omega}$ such that $\langle u, v\rangle$ $(\alpha^{<\omega}\times\alpha^{<\omega})$-forces that not $\mathrm{C}_t(f)\,\mathsf{E}\,\mathrm{C}_t(g)$ in $\boldsymbol{L}_\gamma[\psi]$. Consider an $\alpha^{<\omega}\times\alpha^{<\omega}$-generic over $\boldsymbol{L}[\varphi_x]$ pair $\langle f, g\rangle \in \alpha^\omega \times \alpha^\omega$ such that $u \subset f$ and $v \subset g$. Then both $y = \mathrm{C}_t(f)$ and $z = \mathrm{C}_t(g)$ belong to $\mathsf{X}_t(\boldsymbol{L}[\varphi_x])$, so $y\,\mathsf{E}\,z$ as $\mathsf{X}_t(\boldsymbol{L}[\varphi_x]) \subseteq [x]_{\mathsf{E}}$. On the other hand, $\langle f, g\rangle$ also is generic over $\boldsymbol{L}_\gamma[\psi]$, therefore $y\,\mathsf{E}\,z$ is false in $\boldsymbol{L}_\gamma[\psi, f, g]$ (this is forced by $\langle u, v\rangle$), hence in the universe as E is Σ_1^1, which is a contradiction with the above. This ends the proof of (b). □

Definition. Suppose that $x \in \mathfrak{D}$ is E-classifiable. Let γ_x denote the least ordinal $\gamma < \omega_1$ such that $\mathsf{Def}_{\mathsf{E}}(x, \varphi_x \restriction \gamma, t)$ for some t. We put $\psi_x = \varphi_x \restriction \gamma_x$ and define ν_x as the least ordinal $\nu < \gamma_x$ such that the νth, in the sense of the Gödel $\mathrm{OD}[\psi_x]$ wellordering, element of $\boldsymbol{L}_{\gamma_x}[\psi_x]$ is a "term" $t = t_x \in \mathbb{T}_{<\gamma_x} \cap \boldsymbol{L}_{\gamma_x}[\psi_x]$ which satisfies $\mathsf{Def}_{\mathsf{E}}(x, \psi_x, t)$. Finally we set $U_{\mathsf{E}}(x) = \langle \psi_x, \nu_x\rangle$.

Lemma 10. *If every* $x \in \mathfrak{D}$ *is* E*-classifiable, then the map* U_{E} *is a* Δ_1^{HC} *enumeration of the* E*-eqivalence classes.*

Proof. First of all, $U = U_{\mathsf{E}}$ is Δ_1^{HC} by Lemma 8. If $x \mathsf{E} y$, then $U(x) = U(y)$ since the definition is E-invariant for x. We prove the converse. Assume that $U(x) = U(y)$, so $\psi_x = \psi_y = \psi \in 2^{<\omega_1}$ and $t_x = t_y = t \in \mathbb{T}_\alpha \cap \boldsymbol{L}_\gamma[\psi]$, where $\alpha < \gamma = \operatorname{dom}\psi < \omega_1$. By (d) we have $\mathrm{C}_t(f) \mathsf{E} x$ and $\mathrm{C}_t(g) \mathsf{E} y$ for some $\alpha^{<\omega}$-generic over $\boldsymbol{L}_\gamma[\psi]$ functions $f, g \in \alpha^\omega$. However $\mathrm{C}_t(f) \mathsf{E} \mathrm{C}_t(g)$ (see the proof of Lemma 8). □

Corollary 11. *In the assumption of Case* 1 *of Subsection* 1.2, E *admits a* Δ_1^{HC} *enumeration of the equivalence classes by elements of* $2^{<\omega_1}$.

Proof. The range of the function U is covered by a subset $R \subseteq \mathrm{HC}$ (all pairs $\langle\psi, \nu\rangle$ such that ...) which admits a $1-1$ Δ_1^{HC} correspondence with $2^{<\omega_1}$. □

This completes the proof of the "Case 1" part of Theorem 3.

4 OD forcing

This section starts the proof of the "Case 2" part of Theorem 3. At the beginning, we reduce the problem to a more elementary form.

4.1 Explanation

Suppose that each real x belongs to a Boolean valued extension of $\boldsymbol{L}$ ((*) of Theorem 1), but the assumption of Case 1 in Subsection 1.2 fails.

Let $z_0 \in \mathfrak{D}$ witness that the assumption of Case 1 fails. By Proposition 2, there is a limit $\boldsymbol{L}$-cardinal Ω such that z_0 belongs to an $\Omega^{<\omega}$-valued extension V of $\boldsymbol{L}$ (so that Ω-CUH holds in V) and $z_0 \in \mathsf{Weak}_\Omega(\boldsymbol{L})$ in V. By the choice of z_0 we have $\mathsf{E} \subsetneqq \bar{\mathsf{E}}$ on the set $\mathsf{Weak}_\Omega(\boldsymbol{L})$ in V.

This is our starting position in the proof of the "Case 2" part of Theorem 3. The general plan will be first to prove that E_0 continuously embeds in E in *the auxiliary Boolean valued universe* V, and second, to get the result in the universe of Theorem 3 by the Shoenfield absoluteness theorem.

The second part does not seem easy: the existence of a continuous embedding of E_0 into E is a Σ_3^1 statement. To fix the problem, we introduce a special type of embeddings the existence of which is expressed by a Σ_2^1 formula. Recall that any Σ_1^1 set E has the form $\mathsf{E} = \bigcup_{\alpha<\omega_1} \mathsf{E}^\alpha$, where E^α are Borel sets – the *approximations*, satisfying $\mathsf{E}^\alpha \subseteq \mathsf{E}^\gamma$ whenever $\alpha < \gamma < \omega_1$, and uniquely defined as soon as a Π_1^0 set P which projects onto E is fixed.

Definition. A function $\varphi : \mathfrak{D} \xrightarrow{1-1} \mathfrak{D}$ is a *special embedding* of E_0 into E iff

(1) for all $x, y \in \mathfrak{D}$, not $x \mathsf{E}_0 y$ implies not $\varphi(x) \mathsf{E} \varphi(y)$;

(2) there exists an ordinal $\alpha < \omega_1$ such that $\langle\varphi(0^k{}^\frown 0^\frown z), \varphi(0^k{}^\frown 1^\frown z)\rangle \in \mathsf{E}^\alpha$ for all $z \in \mathfrak{D}$ and $k \in \omega$. (0^k is the sequence of k zeros.)

We prove that this is an embedding, i.e., $x \mathsf{E}_0 y$ implies $\varphi(x) \mathsf{E} \varphi(y)$. Suppose $x \mathsf{E}_0 y$. One easily proves that then x is connected with y by a finite chain of pairs of the form $x' = 0^k{}^\frown 0^\frown z$, $y' = 0^k{}^\frown 1^\frown z$. We have $\varphi(x) \mathsf{E} \varphi(y)$ by (2).

The existence of a continuous special embedding of E_0 into E is obviously a Σ_2^1 property. Thus, by the Shoenfield absoluteness, the following theorem (applied in V) suffices to complete the proof of the "Case 2" part of Theorem 3.

Theorem 12. *Assume* Ω-CUH. *Suppose that the set* $\mathsf{Weak}_\Omega(\boldsymbol{L})$ *is nonempty,* E *is a* Σ^1_1 *equivalence relation, and* $\mathsf{E} \subsetneqq \bar{\mathsf{E}}$ *on* $\mathsf{Weak}_\Omega(\boldsymbol{L})$. *Then* E_0 *admits a special continuous embedding into* E.

The proof of this theorem takes this and the next section. We assume Ω-CUH and fix a Σ^1_1 equivalence E satisfying $\mathsf{E} \subsetneqq \bar{\mathsf{E}}$ on the set $\mathsf{Weak}_\Omega(\boldsymbol{L}) \neq \emptyset$.

4.2 Three forcing notions

In the course of the proof we shall make use of the following three forcing notions associated with the topology generated by OD sets:

$$\mathbb{X} = \{X \subseteq \mathsf{Weak}_\Omega(\boldsymbol{L}) \,:\, X \text{ is OD and nonempty}\};$$

$$\mathbb{X}_2 = \{P \subseteq \mathsf{Weak}^{(2)}_\Omega(\boldsymbol{L}) \,:\, P \text{ is OD and nonempty}\},$$

where $\mathsf{Weak}^{(2)}_\Omega(\boldsymbol{L}) = \{\langle x, y\rangle \in \mathcal{D}^2 \,:\, \langle x, y\rangle \text{ is } \Omega\text{-weak over } \boldsymbol{L}\}$;

$$\mathbb{P} = \{P \in \mathbb{X}_2 \,:\, P = (\mathfrak{pr}_1 P \times \mathfrak{pr}_2 P) \cap \bar{\mathsf{E}}\},$$

where $\mathfrak{pr}_1 P = \{x \,:\, \exists y\, P(x,y)\}$ and $\mathfrak{pr}_2 P = \{y \,:\, \exists x\, P(x,y)\}$ for any $P \subseteq \mathcal{D}^2$.

All three sets are nonempty as the set $\mathsf{Weak}_\Omega(\boldsymbol{L})$ is OD and nonempty.

Smaller sets will all the time be *stronger* forcing conditions.

It occurs that, assuming Ω-CUH, the forcing notions include dense subsets of remarkably simple nature.

Definition. A set X is Ω-*small* iff there exist an ordinal $\lambda < \Omega$ and an OD function $f : \lambda \xrightarrow{\text{onto}} X$. A forcing condition $X \in \mathbb{X}$ is $\mathbb{X}$-*primitive* if and only if the set $\mathbb{X}_{\subseteq X} = \{Y \in \mathbb{X} \,:\, Y \subseteq X\}$ is Ω-small, and analogously for $\mathbb{X}_2$ and $\mathbb{P}$.

Lemma 13. *Assume* Ω-CUH. *Let* X *be* Ω-*small. Then* X *is countable,* $X \subseteq \mathrm{OD}$, *and the "*OD *power set"* $\mathcal{P}^{\mathrm{OD}}(X) = \mathrm{OD} \cap \mathcal{P}(X)$ *is also* Ω-*small.*

Proof. To see that $\mathcal{P}^{\mathrm{OD}}(X)$ is Ω-small recall that Ω is a limit $\boldsymbol{L}$-ordinal in the assumption Ω-CUH, and use Proposition 4. $\square$

Lemma 14. *Assume* Ω-CUH. *The set of all* $\mathbb{X}$-*primitive conditions* $X \in \mathbb{X}$ *is dense in* $\mathbb{X}$, *and analogously for* $\mathbb{X}_2$ *and* $\mathbb{P}$.

Proof. Note that every set of the form $X = \mathsf{X}_t(\boldsymbol{L})$, where $t \in \mathbb{T}_\alpha \cap \boldsymbol{L}$ and $\alpha < \Omega$, is $\mathbb{X}$-primitive. (Indeed, by Proposition 6 every OD subset of X is uniquely determined by an OD subset of $\alpha^{<\omega}$. Now use Lemma 13.) This implies the result for $\mathbb{X}$, since the set of all sets of the form $\mathsf{X}_t(\boldsymbol{L})$ is dense in $\mathbb{X}$ by Proposition 6. $\mathbb{X}_2$ is simply a two-dimentional copy of $\mathbb{X}$.

As for $\mathbb{P}$, one easily proves that any condition $P \in \mathbb{P}$ is $\mathbb{P}$-primitive whenever both $\mathfrak{pr}_1 P$ and $\mathfrak{pr}_2 P$ are $\mathbb{X}$-primitive. $\square$

Let us consider $\mathbb{X}$ as a forcing notion over OD. We say that a set $G \subseteq \mathbb{X}$ is $\mathbb{X}$-*generic* iff it nonempty intersects each dense OD subset of $\mathbb{X}$. The notions of $\mathbb{X}_2$-*generic* and $\mathbb{P}$-*generic* sets have the analogous meaning.

Lemma 15. *Assume* Ω-CUH. *If* G *is* $\mathbb{X}$-*generic, then* $\bigcap G$ *is a singleton* $\{a\}$. *If* G *is* $\mathbb{X}_2$-*generic or* $\mathbb{P}$-*generic, then* $\bigcap G$ *is a singleton* $\{\langle a, b\rangle\}$.

P r o o f. We prove the result for $\mathbb{X}$; the results for $\mathbb{X}_2$ and $\mathbb{P}$ can be obtained by an analogous argument. Assume, towards contradiction, that $\bigcap G = \emptyset$. (Clearly $\bigcap G$ cannot contain more than one real.) Note that $\mathbb{X}$ is OD order isomorphic to a p.o. set in $\boldsymbol{L}$. (Indeed, it is known that there is an OD map δ of Ord onto the class of all OD sets. Since $\mathbb{X}$ is OD, $\mathbb{X}$ is a 1–1 OD image of an OD set $\mathbb{X}'$ of ordinals via δ. By Proposition 4 both $\mathbb{X}'$ and the δ-preimage of the order on $\mathbb{X}$ belong to $\boldsymbol{L}$.) Now, using Proposition 4, one easily proves that the assumption $\bigcap G = \emptyset$ is forced, so that there is $X \in \mathbb{X}$ such that $\bigcap G = \emptyset$ for *every* $\mathbb{X}$-generic set $G \subseteq \mathbb{X}$ containing X. We can assume that $X = \mathrm{X}_t(\boldsymbol{L})$, where $t \in \mathbb{T}_\alpha \cap \boldsymbol{L}$ and $\alpha < \Omega$, in particular X is $\mathbb{X}$-primitive. Let $\{\mathfrak{X}_n : n \in \omega\}$ be an enumeration of all OD dense subsets of $\mathbb{X}_{\subseteq X}$ (Lemma 13 is applied). Proposition 6 yields an increasing $\alpha^{<\omega}$-generic over $\boldsymbol{L}$ sequence $u_0 \subset u_1 \subset u_2 \subset \cdots$ with $u_n \in \alpha^{<\omega}$ such that $X_n = \mathrm{X}_{t,u_n}(\boldsymbol{L}) \in \mathfrak{X}_n$ for every n. We obtain an $\mathbb{X}$-generic set $G \subseteq \mathbb{X}$ containing X and all sets X_n. Now let $f = \bigcup_{n \in \omega} u_n$, so that $f \in \alpha^\omega$ is $\alpha^{<\omega}$-generic over $\boldsymbol{L}$. Then $x = \mathrm{C}_t(f)$ belongs to X_n for all n, so $x \in \bigcap G$, which is a contradiction. □

R e m a r k 16. Surprisingly enough *every* real $x \in \mathsf{Weak}_\Omega(\boldsymbol{L})$ is $\mathbb{X}$-generic in the sense that the associated set $G_x = \{X \in \mathbb{X} : x \in X\}$ is $\mathbb{X}$-generic. (Otherwise take the nonempty OD set X of all x which witness the opposite. Then $X \in \mathbb{X}$. Take an $\mathbb{X}$-primitive $Y \subseteq X$. By the primitivity there exists an $\mathbb{X}$-generic set G containing Y. To get a contradiction apply Lemma 15.)

Similarly $\mathbb{X}_2$-generic pairs are simply all pairs $\langle x, y\rangle \in \mathsf{Weak}^{(2)}_\Omega(\boldsymbol{L})$.

The question is not so clear for $\mathbb{P}$ which is a very interesting product-like forcing. If $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic set so that $\bigcap G$ is a singleton, let $\bigcap G = \{\langle a_G, b_G\rangle\}$. The pairs $\langle a_G, b_G\rangle$ of this form will be called *$\mathbb{P}$-generic*.

4.3 The key set

We recall that, by the assumption of Theorem 12, $\mathsf{E} \subsetneqq \bar{\mathsf{E}}$ on $\mathsf{Weak}_\Omega(\boldsymbol{L})$. This means that there exist $\bar{\mathsf{E}}$-classes of reals in $\mathsf{Weak}_\Omega(\boldsymbol{L})$ which include more than one E-class. We call the union of all those $\bar{\mathsf{E}}$-classes,

$$H = \{x \in \mathsf{Weak}_\Omega(\boldsymbol{L}) : (\exists y \in \mathsf{Weak}_\Omega(\boldsymbol{L}))\,(x \mathrel{\bar{\mathsf{E}}} y \text{ but not } x \mathrel{\mathsf{E}} y)\},$$

the *key set* from the title. Clearly H is OD, nonempty, and E-invariant *inside* $\mathsf{Weak}_\Omega(\boldsymbol{L})$, and moreover $H^2 \cap \bar{\mathsf{E}} \neq \emptyset$, so that in particular $H^2 \cap \bar{\mathsf{E}} \in \mathbb{P}$.

The following theorem is a counterpart of the proposition in HARRINGTON e. a. [2] that $\mathsf{E}\restriction H$ is meager in $\bar{\mathsf{E}}\restriction H$.

T h e o r e m 17. *Assume Ω-CUH. If $\langle a, b\rangle$ is a $\mathbb{P}$-generic pair, then $a \mathrel{\bar{\mathsf{E}}} b$. If in addition $a, b \in H$, then not $a \mathrel{\mathsf{E}} b$.*

P r o o f.

Part 1. Suppose on the contrary that not $a \mathrel{\bar{\mathsf{E}}} b$. Then there exists an OD set C such that $x \in A = [C]_{\mathsf{E}}$ and $y \in B = \mathcal{D} \setminus A$. By the genericity of $\langle a, b\rangle$ there exists a condition $P \in G$ (where $G \subseteq \mathbb{P}$ is a $\mathbb{P}$-generic set which defines the pair $\langle a, b\rangle$ in the sense that $\{\langle a, b\rangle\} = \bigcap G$) such that $\mathrm{pr}_1\, P \subseteq A$ and $\mathrm{pr}_2\, B \subseteq B$, therefore $P \subseteq (A \times B) \cap \bar{\mathsf{E}} = \emptyset$, which is impossible.

Part 2. The proof that not $a \mathrel{\mathsf{E}} b$ is not so easy. Assume otherwise. As in the proof of Lemma 15 there is a set $P_0 \in \mathbb{P}$, $P_0 \subseteq H \times H$, such that $a \mathrel{\mathsf{E}} b$ holds for *each* $\mathbb{P}$-generic pair $\langle a, b\rangle \in P_0$. We observe that $\mathsf{E} \subsetneq \bar{\mathsf{E}}$ on the OD nonempty set $X_0 = \mathfrak{pr}_1 P_0 \subseteq H$. (Otherwise $\mathsf{E} = \bar{\mathsf{E}}$ even on $[X_0]_{\mathsf{E}}$. This implies $[X_0]_{\mathsf{E}} \cap H = \emptyset$, which contradicts the above.) Let us fix reals $a, a' \in X_0$ such that $a \,\bar{\mathsf{E}}\, a'$ but not $a \,\mathsf{E}\, a'$.

Claim. *There is a real b such that both $\langle a, b\rangle$ and $\langle a', b\rangle$ belong to P_0 and are $\mathbb{P}$-generic pairs.*

This ends the proof of Theorem 17 (Part 2): Indeed, we have $a \,\mathsf{E}\, b$ and $a' \,\mathsf{E}\, b$ by the choice of P_0, contradicting not $a \,\mathsf{E}\, a'$. □

Proof of the Claim. For sets X and Y and a binary relation R let us write $X \,\mathsf{R}\, Y$ iff $(\forall x \in X)(\exists y \in Y)(x \,\mathsf{R}\, y)$ and $(\forall y \in Y)(\exists x \in X)(x \,\mathsf{R}\, y)$. It follows from Remark 16 and Lemma 14 that there exists a $\mathbb{P}$-primitive condition $P_1 \in \mathbb{P}$, $P_1 \subseteq P_0$, such that $a \in X_1 = \mathfrak{pr}_1 P_1$. Define $Y_1 = \mathfrak{pr}_2 P_1$. Then $X_1 \,\bar{\mathsf{E}}\, Y_1$ and $P_1 = (X_1 \times Y_1) \cap \bar{\mathsf{E}}$. We let $P' = \{\langle x, y\rangle \in P_0 : y \in Y_1\}$. Then $P_1 \subseteq P' \subseteq P_0$ and $P' \in \mathbb{P}$. Furthermore $a' \in X' = \mathfrak{pr}_1 P'$. (Indeed, since $a \in X_1$ and $X_1 \,\bar{\mathsf{E}}\, Y_1$, there exists $y \in Y_1$ such that $a \,\bar{\mathsf{E}}\, y$; then $a' \,\bar{\mathsf{E}}\, y$ as well because $a \,\bar{\mathsf{E}}\, a'$, therefore $\langle a', y\rangle \in P'$.) As above there exists a $\mathbb{P}$-primitive set $P'_1 \in \mathbb{P}$, $P'_1 \subseteq P'$, such that $a' \in X'_1 = \mathfrak{pr}_1 P'_1$. Then $Y'_1 = \mathfrak{pr}_2 P'_1 \subseteq Y_1$. By Lemma 15 $\mathbb{P}$ admits only countably many dense OD sets below P_1 and below P'_1. Let $\{\mathfrak{P}_n : n \geq 2\}$ and $\{\mathfrak{P}'_n : n \geq 2\}$ be enumerations of both families of dense sets. We define sets P_n, $P'_n \in \mathbb{P}$ $(n \geq 2)$, satisfying

(i) $a \in X_n = \mathfrak{pr}_1 P_n$ and $a' \in X'_n = \mathfrak{pr}_1 P'_n$;

(ii) $Y'_n = \mathfrak{pr}_2 P'_n \subseteq Y_n = \mathfrak{pr}_2 P_n$ and $Y_{n+1} \subseteq Y'_n$;

(iii) $P_{n+1} \subseteq P_n$, $P'_{n+1} \subseteq P'_n$, $P_n \in \mathfrak{P}_n$, and $P'_n \in \mathfrak{P}'_n$.

By (iii) both sequences $\{P_n : n \geq 1\}$ and $\{P'_n : n \geq 1\}$ are $\mathbb{P}$-generic, so by Lemma 15 they result in two generic pairs, $\langle a, b\rangle \in P_0$ and $\langle a', b\rangle \in P_0$, having the first terms equal to a and a' by (i) and second terms equal to each other by (ii). Thus it suffices to execute the construction of P_n and P'_n.

The construction goes on by induction on n. Assume that P_n and P'_n have been defined. We define P_{n+1}. By (ii), the set $P^* = (X_n \times Y'_n) \cap \bar{\mathsf{E}} \subseteq P_n$ belongs to $\mathbb{P}$ and satisfies $a \in X^* = \mathfrak{pr}_1 P^*$. (Indeed, $\langle a, y\rangle \in P^*$, where y satisfies $\langle a', y\rangle \in P'_n$, because $a \,\bar{\mathsf{E}}\, a'$.) However $\mathfrak{P}_{n+1}$ is dense in $\mathbb{P}$ below $P^* \subseteq P_0$, so

$$\mathfrak{pr}_1 \mathfrak{P}_{n+1} = \{\mathfrak{pr}_1 P' : P' \in \mathfrak{P}_{n+1}\}$$

is a dense OD set in $\mathbb{X}$ below $X^* = \mathfrak{pr}_1 P^*$. Accordingly to Remark 16, we have $a \in \mathfrak{pr}_1 P'$ for some $P' \in \mathfrak{P}_{n+1}$, $P' \subseteq P^*$. It remains to put $P_{n+1} = P'$, and then $X_{n+1} = \mathfrak{pr}_1 P_{n+1}$ and $Y_{n+1} = \mathfrak{pr}_2 P_{n+1}$. To define P'_{n+1} we set $P^* = (X'_n \times Y_{n+1}) \cap \bar{\mathsf{E}}$, etc. □ (Claim)

We end the section with one more property related to the key set H.

Lemma 18. *Assume Ω-CUH. Suppose that $X, Y \in \mathbb{X}$ and $X \,\bar{\mathsf{E}}\, Y$. Then we have:*

(A) *$X \times Y$ contains a pair $\langle x, y\rangle \in \mathsf{Weak}^{(2)}_{\Omega}(\boldsymbol{L})$ such that $x \,\mathsf{E}\, y$.*

(B) *If $X \cup Y \subseteq H$, then there exist sets X', $Y' \in \mathbb{X}$ such that $X' \subseteq X$, $Y' \subseteq Y$, still $X' \,\bar{\mathsf{E}}\, Y'$ and $X' \cap Y' = \emptyset$.*

Proof.

(A) By Proposition 6 we have sets $X' = \mathsf{X}_t(\boldsymbol{L}) \subseteq X$ and $Y' = \mathsf{X}_{t'}(\boldsymbol{L}) \subseteq Y$, where $t, t' \in \mathbb{T}_\alpha \cap \boldsymbol{L}$ for some $\alpha < \Omega$, such that $P = \bar{\mathsf{E}} \cap (X' \times Y')$ is still nonempty. Then

$Q = P \cap \mathsf{E} \neq \emptyset$ as well by the definition of $\tilde{\mathsf{E}}$. Since Ω is a limit $\boldsymbol{L}$-cardinal, we have $X = \mathsf{X}_t(\boldsymbol{L}_\beta)$ and $Y = \mathsf{X}_{t'}(\boldsymbol{L}_\beta)$ for a suitable β, $\alpha \leq \beta < \Omega$. Take an arbitrary $\beta^{<\omega}$-generic over $\boldsymbol{L}$ function $f \in \beta^\omega$. The statement $Q \neq \emptyset$ becomes a Σ^1_1 formula with reals in $\boldsymbol{L}[f]$ (those coding f, t, t') as parameters. Since all sets in $\boldsymbol{L}[f]$ are Ω-weak over $\boldsymbol{L}$, it remains to apply the Shoenfield absoluteness theorem.

(B) There are reals $x \in X$ and $y \in Y$ such that $x \mathrel{\tilde{\mathsf{E}}} y$ and $x \neq y$. (Otherwise $\tilde{\mathsf{E}}$ is the equality on $X = Y$, hence $\mathsf{E} = \tilde{\mathsf{E}}$ on X, which is impossible, see the proof of Theorem 17.) Let e. g. $x(k) = 0$ and $y(k) = 1$. Define

$$X' = \{x' \in X \,:\, x'(k) = 0 \,\&\, (\exists y' \in Y)\,(x' \mathrel{\tilde{\mathsf{E}}} y' \,\&\, y'(k) = 1)\},$$

and Y' accordingly. □

5 Embedding E_0 into E

In this section we complete the proof of Theorem 12. We prove, assuming Ω-CUH and $\mathsf{E} \subsetneqq \tilde{\mathsf{E}}$ on $\mathsf{Weak}_\Omega(\boldsymbol{L}) \neq \emptyset$, that E_0 continuously specially embeds in E.

5.1 Generic splitting systems

By the assumption the set H of Subsection 4.3 is nonempty; obviously H is OD. By Lemma 14 there exists an $\mathbb{X}$-primitive set $X_0 \in \mathbb{X}$, $X_0 \subseteq H$. Then the set $P_0 = (X_0 \times X_0) \cap \tilde{\mathsf{E}}$ belongs to $\mathbb{P}$ and is $\mathbb{P}$-primitive. We observe that

$$\mathfrak{pr}_1\, P_0 \;=\; \mathfrak{pr}_2\, P_0 \;=\; X_0 \subseteq H \subseteq \mathsf{Weak}_\Omega(\boldsymbol{L}).$$

We shall define a family of sets X_u ($u \in 2^{<\omega}$) satisfying

(a) $X_\Lambda = X_0$, $X_u \in \mathbb{X}$ and $X_{u^\frown i} \subseteq X_u$ for all u and $i = 0, 1$.

In addition to the sets X_u we shall define relations $\mathsf{R}_{u,v} \subseteq X_u \times X_v$ for *some* pairs $\langle u, v\rangle$, to provide important connections between branches in $2^{<\omega}$.

Let u, $v \in 2^n$. We say that $\langle u, v\rangle$ is a *crucial pair* iff $u = 0^{k\frown}0^\frown w$ and $v = 0^{k\frown}1^\frown w$ for some $k < n$ (0^k is the sequence of k terms equal to 0) and some $w \in 2^{n-k-1}$ (possibly $k = n-1$, that is, $w = \Lambda$, the empty sequence).

Thus we define binary relations $\mathsf{R}_{u,v} \subseteq X_u \times X_v$ for all crucial pairs $\langle u, v\rangle$, so that the following requirements will be satisfied:

(b) $\mathsf{R}_{u,v} \in \mathbb{X}_2$, $\mathfrak{pr}_1\, \mathsf{R}_{u,v} = X_u$, $\mathfrak{pr}_2\, \mathsf{R}_{u,v} = X_v$, and $\mathsf{R}_{u^\frown i, v^\frown i} \subseteq \mathsf{R}_{u,v}$ for every crucial pair $\langle u, v\rangle$ and each $i = 0, 1$.

(c) For any k, $\mathsf{R}_{0^{k\frown}0,\, 0^{k\frown}1}$ is an $\mathbb{X}_2$-primitive set satisfying $\mathsf{R}_{0^{k\frown}0,\, 0^{k\frown}1} \subseteq \mathsf{E}^\alpha$ for some ordinal $\alpha = \alpha(k) < \omega_1$. (Recall that E^α denotes the αth approximation of E, a Borel subset of E, see Subsection 4.1.)

Take notice that if $\langle u, v\rangle$ is a crucial pair, then $\langle u^\frown i, v^\frown i\rangle$ is crucial as well, but $\langle u^\frown i, v^\frown j\rangle$ is not crucial for $i \neq j$, unless $u = v = 0^k$ for some k.

Remark 19. *Conditions* (b) *and* (c) *imply* $X_u \mathrel{\mathsf{R}_{u,v}} X_v$, *hence* $X_u \mathrel{\mathsf{E}} X_v$, *for all crucial pairs* u, v. *Moreover, then we have* $X_u \mathrel{\mathsf{E}} X_v$ *and* $X_u \mathrel{\tilde{\mathsf{E}}} X_v$ *for all pairs* $u, v \in 2^n$ *as each pair in* 2^n *is tied by a chain of crucial pairs.* □

Recall that $X \mathrel{\mathsf{R}} Y$ means $(\forall x \in X)(\exists y \in Y)\,(x \mathrel{\mathsf{R}} y)$ and $(\forall y \in Y)(\exists x \in X)\,(x \mathrel{\mathsf{R}} y)$.

Three more requirements will concern genericity.

In order to guarantee that the sequence $\{X_{a\restriction m} : m \in \omega\}$ is $\mathbb{X}$-generic for any branch $a \in 2^\omega$, we require

(g1) $X_u \in \mathfrak{X}_n$ whenever $u \in 2^{n+1}$,

where $\{\mathfrak{X}_n : n \in \omega\}$ is a fixed (maybe not OD) enumeration of all OD dense subsets of $\mathbb{X}_{\subseteq X_0}$. Then for any $a \in 2^\omega$ the intersection $\bigcap_{n\in\omega} X_{a\restriction n}$ contains a single real $\varphi(a) \in H$ by Lemma 15 and the map φ is Polish-continuous.

We now want to arrange matters so that $\langle \varphi(a), \varphi(b)\rangle$ is $\mathbb{P}$-generic whenever not $a \, \mathsf{E}_0 \, b$. Let $\{\mathfrak{P}_n : n \in \omega\}$ be a fixed enumeration of all OD dense sets in $\mathbb{P}_{\subseteq P_0}$. It may be assumed that $\mathfrak{P}_{n+1} \subseteq \mathfrak{P}_n$. We require that

(g2) If $u,\, v \in 2^{n+1}$ and $u(n) \neq v(n)$ (that is, the last terms of u, v are different), then $X_u \cap X_v = \emptyset$ and $P_{u,v} = (X_u \times X_v) \cap \bar{\mathsf{E}} \in \mathfrak{P}_n$.

If this holds and not $a \, \mathsf{E}_0 \, b$ (so that $a(n) \neq b(n)$ for infinitely many numbers n), then $\langle \varphi(a), \varphi(b)\rangle$ is $\mathbb{P}$-generic, therefore not $\varphi(a) \, \mathsf{E} \, \varphi(b)$ by Theorem 17. Moreover if simply $a \neq b$, then $\varphi(a) \neq \varphi(b)$, so φ is a bijection.

On the other hand, we need some $\alpha < \omega_1$ to witness item (2) in the Definition in Subsection 4.1. Let $\{\mathfrak{Q}_n(Q) : n \in \omega\}$ be a fixed enumeration of all OD dense subsets of $(\mathbb{X}_2)_{\subseteq Q}$, for any $\mathbb{X}_2$-primitive set $Q \in \mathbb{X}_2$. We shall assume that $\mathfrak{Q}_{n+1}(Q) \subseteq \mathfrak{Q}_n(Q)$ for all Q and n. The last genericity requirement is

(g3) If $k \leq n$ and $w \in 2^{n-k}$, then $\mathsf{R}_{0^k{}^\frown 0^\frown w,\, 0^k{}^\frown 1^\frown w} \in \mathfrak{Q}_n(\mathsf{R}_{0^k{}^\frown 0,\, 0^k{}^\frown 1})$.

Assume this holds and consider a pair of reals $a = 0^k{}^\frown 0^\frown c$ and $b = 0^k{}^\frown 1^\frown c$ for some k and $c \in 2^\omega$. The sequence of sets $R_m = \mathsf{R}_{0^k{}^\frown 0^\frown c\restriction m,\, 0^k{}^\frown 1^\frown c\restriction m}$ $(m \in \omega)$ is then $\mathbb{X}_2$-generic, so that by Lemma 15 the intersection $\bigcap_m R_m$ is a singleton – which can be only equal to $\langle \varphi(a), \varphi(b)\rangle$. Therefore we have $\varphi(0^k{}^\frown 0^\frown c) \, \mathsf{E}^{\alpha(k)} \, \varphi(0^k{}^\frown 1^\frown c)$ by (c). Now set $\alpha = \sup_k \alpha(k)$.

Thus, assuming λ-CUH, requirements (a), (b), (c), (g1), (g2), (g3) suffice for φ to be a special continuous 1–1 embedding E_0 in E. Therefore Theorem 12 is reduced to the construction of sets X_u and $\mathsf{R}_{u,v}$ satisfying (a), (b), (c), (g1), (g2), (g3). Before the construction starts, we prove

Lemma 20. *Assume Ω-CUH. Let $n \in \omega$ and let X_u be a nonempty* OD *set for each $u \in 2^n$ while, for every crucial pair u, $v \in 2^n$, $\mathsf{R}_{u,v} \subseteq \mathfrak{D}^2$ is an* OD *set satisfying $X_u \, \mathsf{R}_{u,v} \, X_v$. Then we have:*

1. *If $u_0 \in 2^n$ and $X' \subseteq X_{u_0}$ is* OD *and nonempty, then there exists a system of* OD *nonempty sets $Y_u \subseteq X_u$ $(u \in 2^n)$ such that still $Y_u \, \mathsf{R}_{u,v} \, Y_v$ holds for all crucial pairs u, v, and in addition $Y_{u_0} = X'$.*

2. *If u_0, $v_0 \in 2^n$ is a crucial pair and nonempty* OD *sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$ satisfy $X' \, \mathsf{R}_{u_0,v_0} \, X''$, then there exists a system of* OD *nonempty sets $Y_u \subseteq X_u$ $(u \in 2^n)$ such that still $Y_u \, \mathsf{R}_{u,v} \, Y_v$ holds for all crucial pairs u, v, and in addition $Y_{u_0} = X'$, $Y_{v_0} = X''$.*

Proof. Item 1 easily follows from item 2. To prove item 2, we use induction on n. We prove the lemma for $n+1$ provided it is proved for some $n \geq 1$. The principal idea is to divide 2^{n+1} on two copies of 2^n, $U_0 = \{s^\frown 0 : s \in 2^n\}$ and $U_1 = \{s^\frown 1 : s \in 2^n\}$, connected by the only crucial pair of $\hat{u} = 0^n{}^\frown 0$ and $\hat{v} = 0^n{}^\frown 1$, and handle them separately using the induction hypothesis.

If $u_0 = \hat{u}$ and $v_0 = \hat{v}$, then we apply the induction hypothesis (item 1) independently for the families $\{X_u : u \in U_0\}$ and $\{X_u : u \in U_1\}$ and the given sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$. Assembling the results, we get nonempty OD sets $Y_u \subseteq X_u$ $(u \in 2^{n+1})$ such that $Y_u \mathsf{R}_{u,v} Y_v$ for all crucial pairs u, v, possibly with the exception of the pair of $u = u_0 = \hat{u}$ and $v = v_0 = \hat{v}$, and $Y_{u_0} = X'$, $Y_{v_0} = X''$. However $Y_{\hat{u}} \mathsf{R}_{\hat{u},\hat{v}} Y_{\hat{v}}$ by the choice of X' and Y'.

Otherwise u_0 and v_0 belong to one and the same domain, say to U_0. First apply the induction hypothesis (item 2) to the family $\{X_u : u \in U_0\}$ and the sets $X' \subseteq X_{u_0}$ and $X'' \subseteq X_{v_0}$, getting a system of nonempty OD sets $Y_u \subseteq X_u$ $(u \in U_0)$, in particular an OD nonempty set $Y_{\hat{u}} \subseteq X_{\hat{u}}$. Now put $Y_{\hat{v}} = \{y \in X_{\hat{v}} : (\exists x \in Y_{\hat{u}})\,(x \mathsf{R}_{\hat{u},\hat{v}}\, y)\}$, so that $Y_{\hat{u}} \mathsf{R}_{\hat{u},\hat{v}} Y_{\hat{v}}$ holds, and apply the hypothesis (item 1) to the family $\{X_u : u \in U_1\}$ and the set $Y_{\hat{v}} \subseteq X_{\hat{v}}$. □

5.2 The construction

To begin with, we put $X_\Lambda = X_0$.

Assume that the sets X_s $(s \in 2^n)$ and relations $\mathsf{R}_{s,t}$ for all crucial pairs of s, $t \in 2^k$ $(k \le n)$ are defined, and expand the construction at level $n+1$.

We first put $A_{s^\frown i} = X_s$ for all $s \in 2^n$ and $i = 0, 1$. We also define $\mathsf{Q}_{u,v} = \mathsf{R}_{s,t}$ for any crucial pair of $u = s^\frown i$, $v = t^\frown i$ in 2^{n+1} other than the pair $\hat{u} = 0^n{}^\frown 0$, $\hat{v} = 0^n{}^\frown 1$. For the latter one we put $\mathsf{Q}_{\hat{u},\hat{v}} = \bar{\mathsf{E}}$, so that $A_u \mathsf{Q}_{u,v} A_v$ holds for all crucial pairs $\langle u, v\rangle$ in 2^{n+1}.

The sets A_u and relations $\mathsf{Q}_{u,v}$ will be reduced in several steps to meet requirements (a), (b), (c) and (g1), (g2), (g3) of Subsection 5.1.

P a r t 1. After 2^{n+1} steps of the procedure of Lemma 20 (item 1) we obtain a system of nonempty OD sets $B_u \subseteq A_u$ $(u \in 2^{n+1})$ such that $B_u \in \mathfrak{X}_n$ for all u and $B_u \mathsf{Q}_{u,v} B_v$ for all crucial pairs u, v in 2^{n+1}. Thus (g1) is fixed.

P a r t 2. To fix (g2), consider a pair of $u_0 = s_0{}^\frown 0$, $v_0 = t_0{}^\frown 1$ $(s_0, t_0 \in 2^n)$. Then $B_{u_0} \bar{\mathsf{E}} B_{v_0}$ (see Remark 19), therefore $Q = (B_{u_0} \times B_{v_0}) \cap \bar{\mathsf{E}} \in \mathbb{P}$. We observe that by the density of $\mathfrak{P}_n$ there is a set $P \in \mathfrak{P}_n$, $P \subseteq Q$. Then $B' = \mathfrak{pr}_1 P \subseteq B_{u_0}$ and $B'' = \mathfrak{pr}_2 P \subseteq B_{v_0}$ are nonempty OD sets satisfying $B' \bar{\mathsf{E}} B''$. We may assume, by Lemma 18, that $B' \cap B'' = \emptyset$. We apply Lemma 20 (item 1) for the two systems of sets $\{B_{s^\frown 0} : s \in 2^n\}$ and $\{B_{t^\frown 1} : t \in 2^n\}$ separately (compare with the proof of Lemma 20!), and the sets $B' \subseteq B_{s_0{}^\frown 0}$, $B'' \subseteq B_{t_0{}^\frown 1}$, respectively. This results in a system of nonempty OD sets $B'_u \subseteq B_u$ $(u \in 2^{n+1})$ with $B'_{u_0} = B'$, $B'_{v_0} = B''$, so that $(B'_{u_0} \times B'_{v_0}) \cap \bar{\mathsf{E}} = P \in \mathfrak{P}_n$, and still $B'_u \mathsf{Q}_{u,v} B'_v$ for all crucial pairs u, $v \in 2^{n+1}$, perhaps with the exception of the pair of $\hat{u} = 0^n{}^\frown 0$, $\hat{v} = 0^n{}^\frown 1$, which is the only one connecting the domains. To handle this pair, note that $B'_{\hat{u}} \bar{\mathsf{E}} B'_{u_0}$ and $B'_{\hat{v}} \bar{\mathsf{E}} B'_{v_0}$ (Remark 19 is applied in each of the two domains), so $B'_{\hat{u}} \bar{\mathsf{E}} B'_{\hat{v}}$, since $B' \bar{\mathsf{E}} B''$. Finally we observe that $\mathsf{Q}_{\hat{u},\hat{v}}$ is so far equal to $\bar{\mathsf{E}}$. After 4^n steps (the number of pairs u_0, v_0 to be considered) we obtain a system of nonempty OD sets $C_u \subseteq B_u$ $(u \in 2^{n+1})$ such that the set $(C_u \times C_v) \cap \bar{\mathsf{E}}$ belongs to $\mathfrak{P}_n$ whenever $u(n) \ne v(n)$, and still $C_u \mathsf{Q}_{u,v} C_v$ for all crucial pairs u, $v \in 2^{n+1}$. Thus (g2) is fixed.

Part 3. We fix (c) for the exceptional crucial pair of $\hat{u} = 0^n{}^\frown 0$, $\hat{v} = 0^n{}^\frown 1$. Since we have $C_{\hat{u}} \bar{\mathsf{E}} C_{\hat{v}}$, the set $\mathsf{R} = (C_{\hat{u}} \times C_{\hat{v}}) \cap \mathsf{E} \cap \mathsf{Weak}^{(2)}_{\Omega}(\boldsymbol{L})$ is nonempty by Lemma 18. Then, as $\mathsf{R} \subseteq \mathsf{E}$, the set $\mathsf{R} \cap \mathsf{E}^\alpha$ is nonempty for some $\alpha < \omega_1$. (E^α is the αth approximation of the Σ^1_1-set E.) There is an $\mathbb{X}_2$-primitive set $\mathsf{Q} \in \mathbb{X}_2$, $\mathsf{Q} \subseteq \mathsf{R} \cap \mathsf{E}^\alpha$, by Lemma 14. We consider the OD sets $C' = \mathfrak{pr}_1 \mathsf{Q}$ $(\subseteq C_{\hat{u}})$, $C'' = \mathfrak{pr}_2 \mathsf{Q}$ $(\subseteq C_{\hat{v}})$; obviously $C' \mathsf{Q} C''$, so that $C' \mathsf{Q}_{\hat{u},\hat{v}} C''$. (Recall that at the moment $\mathsf{Q}_{\hat{u},\hat{v}} = \bar{\mathsf{E}}$.) Lemma 20 yields nonempty OD sets $Y_u \subseteq C_u$ $(u \in 2^{n+1})$ still satisfying $Y_u \mathsf{Q}_{u,v} Y_v$ for all crucial pairs u, v in 2^{n+1}, and $Y_{\hat{u}} = C'$, $Y_{\hat{v}} = C''$. We re-define $\mathsf{Q}_{\hat{u},\hat{v}}$ by $\mathsf{Q}_{\hat{u},\hat{v}} = \mathsf{Q}$ (then $\mathsf{Q}_{\hat{u},\hat{v}} \subseteq \mathsf{E}^\alpha$), but this keeps $Y_{\hat{u}} \mathsf{Q}_{\hat{u},\hat{v}} Y_{\hat{v}}$.

Part 4. To fix (g3) consider a crucial pair u_0, v_0 in 2^{n+1}. Then $u_0 = 0^k{}^\frown 0^\frown w$, $v_0 = 0^k{}^\frown 1^\frown w$ for some $k \leq n$ and $w \in 2^{n-k}$. It follows that $\mathsf{Q}' = \mathsf{Q}_{u_0,v_0} \cap (Y_{u_0} \times Y_{v_0})$ is a nonempty (since $Y_{u_0} \mathsf{Q}_{u_0,v_0} Y_{v_0}$) OD subset of $\mathsf{R}_{0^k{}^\frown 0, 0^k{}^\frown 1}$ by the construction. Pick a set $\mathsf{Q} \in \mathfrak{Q}_n(\mathsf{R}_{0^k{}^\frown 0, 0^k{}^\frown 1})$, $\mathsf{Q} \subseteq \mathsf{Q}'$, put $Y' = \mathfrak{pr}_1 \mathsf{Q}$ and $Y'' = \mathfrak{pr}_2 \mathsf{Q}$ (then $Y' \mathsf{Q} Y''$ and $Y' \mathsf{Q}_{u_0,v_0} Y''$), and apply Lemma 20 (item 2) for the system of sets Y_u $(u \in 2^{n+1})$ and the sets $Y' \subseteq Y_{u_0}$, $Y'' \subseteq Y_{v_0}$. After this we define the "new" Q_{u_0,v_0} by $\mathsf{Q}_{u_0,v_0} = \mathsf{Q}$.

Repeat this consecutively for all crucial pairs; the finally obtained sets – let them be X_u $(u \in 2^{n+1})$ – and the final relations $\mathsf{R}_{u,v}$ $(u, v \in 2^{n+1})$ defined as the restrictions of the relations $\mathsf{Q}_{u,v}$ to $X_u \times X_v$ are as required.

This ends the construction.

This also ends the proof of Theorems 12 and 3, and Theorem 1. □

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