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On coding uncountable sets by reals

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If $A \subseteq \omega_1$, then there exists a cardinal preserving generic extension $\mathbb{L}[A][x]$ of $\mathbb{L}[A]$ by a real x such that
1) $A \in \mathbb{L}[x]$ and A is $\Delta_1^{\mathrm{HC}}(x)$ in $\mathbb{L}[x]$;
2) x is minimal over $\mathbb{L}[A]$, that is, if a set Y belongs to $\mathbb{L}[x]$, then either $x \in \mathbb{L}[A, Y]$ or $Y \in \mathbb{L}[A]$.
The forcing we use implicitly provides reshaping of the given set A.

1 Introduction

The following is our main result.

Theorem 1.1 *Suppose that $A \subseteq \omega_1$ and $\mathbb{V} = \mathbb{L}[A]$. Then there exists a cardinal preserving generic extension $\mathbb{V}[x]$ of the ground universe $\mathbb{V}$ by a generic real x such that*
(i) *$A \in \mathbb{L}[x]$ – this implies $\mathbb{V}[x] = \mathbb{L}[x]$, and $A \in \Delta_1^{\mathrm{HC}}(x)$ in $\mathbb{V}[x]$,*
(ii) *x is a minimal real over $\mathbb{V}$, that is, $x \notin \mathbb{V}$, and if a set Y belongs to $\mathbb{V}[x]$, then $x \in \mathbb{V}[Y]$ or $Y \in \mathbb{V}$,* [1)]
(iii) *there is a club $C \in \mathbb{V}[x]$, $C \subseteq \omega_1$, that reshapes A, i.e., if $\alpha \in C$, then $\alpha < \omega_1^{\mathbb{L}[A \cap \alpha]}$.*

We may compress the properties (i) and (ii) of a real x in the theorem by saying that x *minimally codes* the set $A \subseteq \omega_1$. Jensen and Solovay [12] proposed a method of coding of uncountable sets by reals by means of almost disjoint forcing. In the context of Theorem 1.1, this coding method consists of two parts. The first part is the *reshaping* of A by means of a generic club (closed and unbounded set) $C \subseteq \omega_1$ with the properties that
1) C does not add new reals to $\mathbb{L}[A]$, and
2) if $\xi \in C$ is a limit ordinal, then $\xi < \omega_1^{\mathbb{L}[A \cap \xi]}$ (see Theorem 14.1 below).

After this is done, a type of almost-disjoint ccc forcing is employed to produce a generic real x over $\mathbb{L}[A][C]$ such that A and C belong to $\mathbb{L}[x]$, that is, x codes those two sets. These methods were expanded to a greatly more complicated technique of *coding the universe* [5].

The almost-disjoint forcing technique does not provide minimal reals. The first result on *minimal* coding was published in [13]: a generic minimal upper bound of the constructibility degrees of any model satisfying CH, with the minimality understood only in the sense of reals (*weak* minimality in discussions below). The coding technique in [13] involves a subforcing of the Sacks forcing close to a forcing notion introduced in [11]. [2)]

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1) One may want to strengthen the minimality requirement in (ii) of Theorem 1.1 as follows: $x \in \mathbb{L}[Y]$ or $Y \in \mathbb{V}$. But this is, generally speaking, impossible, see Section 16.

2) Sy Friedman informed us that similar results were independently obtained by M. Groszek; but her relevant studies are unpublished. According to the anonymous referee, similar results (a generic minimal real, but only in the sense of reals) were also obtained by P. Welch, also unpublished.

Leaving aside coding results of less relevance, let us mention that Sy Friedman developed the minimal coding technique to a very powerful method of *minimal coding of the universe*, so that basically any universe $\mathbb{V}$ of ZFC + GCH can be extended by adding a class generic cardinal-preserving and cofinalities-preserving real x so that $\mathbb{V}[x] = \mathbb{L}[x]$ and x is minimal over $\mathbb{V}$ in the sense of (ii) of Theorem 1.1. See [6, Section 1] or, with a sketchy proof and further references, [7, Theorem 8.21]. As we were informed by Sy Friedman, a certain (still rather complicated) reduction of methods involved in the proof of [7, Theorem 8.21] also yields the proof of Theorem 1.1, (i) and (ii), of this paper. (See also [8] for another application of this version of the minimal coding technique.)

The goal of this paper is to present a self-contained proof of Theorem 1.1 by means of a more elementary coding that involves a technique going back to [11, 13]. The forcing we employ will consist of perfect trees $T \subseteq 2^{<\omega}$, that is, it will be a subforcing of the Sacks forcing. That the Sacks forcing itself does not suffice, generally speaking, to prove the theorem, will be shown in Section 12.

Before the proof starts, let us briefly discuss the reshaping problem already mentioned above. (The importance of this issue was revealed by Sy Friedman in a broad context, see for example [7, Ch. 8].) The reshaping in this context means a reduction of the general case of Theorem 1.1 to the case when

$$(*) \qquad \xi < \omega_1^{\mathbb{L}[A\cap\xi]} \text{ for all limit } \xi < \omega_1.$$

If this is the case, then Theorem 1.1 can be proved by means of a ccc forcing, as in Section 13 below. On the other hand, if the true $\omega_1 = \omega_1^{\mathbb{L}[A]}$ is *not* a Mahlo cardinal in $\mathbb{L}$, then there is a set $A' \in \mathbb{L}[A]$, $A' \subseteq \omega_1$, with $\mathbb{L}[A] = \mathbb{L}[A']$, and satisfying $(*)$, so that the reshaping can be done internally in the non-Mahlo case. However if $\mathbb{V} = \mathbb{L}[A]$, with $A \subseteq \omega_1$ coding a generic filter over $\mathbb{L}$ for the Levy collapse of a weakly compact cardinal, then by some results of Kunen, any ccc-generic real x over $\mathbb{L}[A]$ satisfies $\omega_1^{\mathbb{L}[x]} < \omega_1$, and hence x cannot code A. Thus the reshaping is not always doable internally.

But if (ii) of Theorem 1.1 is weakened to the form when $Y \subseteq \omega$ (essentially, the minimality among reals), then the reshaping by a generic set C satisfying 1) and 2) above, as in [12], is innocuous in the sense that a real x minimal in the weak sense over $\mathbb{L}[A][C]$ will remain minimal in the weak sense over $\mathbb{L}[A]$ as well. But the general minimality fails as C is a set of intermediate degree.

2 On perfect trees

Consider the set 2^ω of all $x : \omega \longrightarrow 2 = \{0, 1\}$, the Cantor space.

If $s \in 2^{<\omega}$ (a finite binary string), then put $[s] = \{x \in 2^\omega : s \subset x\}$ and let $\operatorname{lh} s$ be the length of s.

If $X \subseteq 2^\omega$ is a perfect set, then $S = \operatorname{tree} X = \{s \in 2^{<\omega} : X \cap [s] \neq \emptyset\}$ is a perfect tree in $2^{<\omega}$ and $X = [S] = \{a \in 2^\omega : \forall m\, (a \restriction m \in S)\}$. Let Perf be the set of all perfect trees $S \subseteq 2^{<\omega}$. Define, for $S \in \mathrm{Perf}$, $\operatorname{diam} S = \dfrac{1}{m_0 + 1}$, where m_0 is the largest such m that $a \restriction m = b \restriction m$ for all $a, b \in [S]$.

A set $E \subseteq \mathrm{Perf}$ is an *antichain* if and only if $[S] \cap [T] = \emptyset$ for all $S \neq T$ in E. [3)]

A set $D \subseteq \mathbb{X}$ is *dense in* $\mathbb{X} \subseteq \mathrm{Perf}$ if and only if for each $S \in \mathbb{X}$ there exists $T \in D$ such that $T \subseteq S$. Note that if $S, T \in \mathrm{Perf}$, then $S \subseteq T$ is equivalent to $[S] \subseteq [T]$.

If $t \in S \in \mathrm{Perf}$, then put $S \restriction t = \{s \in S : s \subseteq t \vee t \subset s\}$; this is still a tree in Perf and $[S \restriction t] = [S] \cap [t]$. Say that a set $\mathbb{X} \subseteq \mathrm{Perf}$ is *CO-dense* if and only if for any $t \in S \in \mathbb{X}$ there exists a tree $T \in \mathbb{X}$ such that $[T] \subseteq [S \restriction t]$. In this case, we have even more: if $S \in \mathbb{X}$ and a set $\emptyset \neq U \subseteq [S]$ is open in $[S]$, then there exists a tree $T \in \mathbb{X}$ such that $[T] \subseteq U$.

Lemma 2.1 *If $\mathbb{X} \subseteq \mathrm{Perf}$ is CO-dense and $S_1, \ldots, S_n \in \mathbb{X}$, then there are trees $T_1, \ldots, T_n \in \mathbb{X}$ satisfying $T_i \subseteq S_i$ for all i and $[T_i] \cap [T_j] = \emptyset$ for $i \neq j$.*

P r o o f. Let, say, $n = 2$. Pick $x_1 \in [S_1]$, $x_2 \in [S_2]$, $x_1 \neq x_2$. There is $m \in \omega$ with $x_1 \restriction m = u \neq v = x_2 \restriction m$. The trees $T'_1 = S_1 \restriction u$ and $T'_2 = S_2 \restriction v$ satisfy $[T'_1] \cap [T'_2] = \emptyset$. By the CO-density there exist trees $T_1, T_2 \in \mathbb{X}$ with $T_i \subseteq T'_i$, $i = 1, 2$. □

[3)] This is somewhat stronger than the usual notion of antichain in a poset.

If $S \in \mathrm{Perf}$ and $\mathbb{X} \subseteq \mathrm{Perf}$, then $S \subseteq^{\mathrm{fin}} \bigcup \mathbb{X}$ means that there is a *finite* set $\mathbb{X}' \subseteq \mathbb{X}$ such that $[S] \subseteq \bigcup_{T \in \mathbb{X}'}[T]$, or, which is equivalent, simply $S \subseteq \bigcup \mathbb{X}'$.

Lemma 2.2 *Let $\mathbb{X} \subseteq \mathrm{Perf}$ be a countable CO-dense set and $\{D_n : n \in \omega\}$ a family of sets $D_n \subseteq \mathbb{X}$ dense in $\mathbb{X}$. Then there is a countable antichain $E \subseteq \mathrm{Perf} \setminus \mathbb{X}$ such that*

1) *$T \subseteq^{\mathrm{fin}} \bigcup D_n$ for all n and all $T \in E$, and*
2) *for each $S \in \mathbb{X}$ there is $T \in E$ with $T \subseteq S$.*

Proof. Let $\mathbb{X} = \{S_n : n \in \omega\}$. It suffices to define a family of trees $T^n_s \in \mathbb{X}$, where $n \in \omega$ and $s \in 2^{<\omega}$, such that

(i) $T^n_\Lambda = S_n$ for all n, where Λ is the empty string;
(ii) $T^n_{s^\wedge 0} \cup T^n_{s^\wedge 1} \subseteq T^n_s$, but $T^n_{s^\wedge 0} \cap T^n_{s^\wedge 1} = \emptyset$, where $s^\wedge i$ is the extension of s by i as the rightmost term;
(iii) $\operatorname{diam} T^n_s \leq \dfrac{1}{\operatorname{lh} s}$;
(iv) the sets $X^n_k = \bigcup_{\operatorname{lh} s = k}[T^n_s]$ satisfy $X^n_n \cap X^m_n = \emptyset$ for $m < n$;
(v) if $n \in \omega$, $k \geq 1$, $\operatorname{lh} s = k$, then $T^n_s \subseteq T$ for some $T \in D_{k-1}$;
(vi) if $n, k \in \omega$, then either there is $s \in 2^{<\omega}$ such that $\operatorname{lh} s = k$ and $[T^n_s] \cap [S_k] = \emptyset$, or $X^n_k \subsetneqq [S_k]$.

Details (rather elementary) are left to the reader; in particular Lemma 2.1 is applied to fix (iv). After the construction is accomplished define the trees $T_n = \bigcap_{k \in \omega} \bigcup_{\operatorname{lh} s = k} T^n_s$ and $E = \{T_n : n \in \omega\}$. Note that (vi) implies $E \cap \mathbb{X} = \emptyset$. □

3 Coding

The goal is to uniformly define, for any perfect set $X \subseteq 2^\omega$ and any real b in a rather large set $B \subseteq 2^\omega$, a countable family of perfect sets $X_{nb} \subseteq X$, CO-dense in X, and also a uniform decoding map f such that $f(X, x) = b$ for all $x \in \bigcup_n X_{nb}$. (The notation will be changed.)

If $x \in 2^\omega$ and $n \in \omega$, then let $(x)_n(k) = x(2^n(2k+1) - 1)$, so that $(x)_n \in 2^\omega$.

Define $x_{\mathrm{even}} \in 2^\omega$ so that $x_{\mathrm{even}}(n) = x(2n)$ for all $n \in \omega$. Consider the sets

$$\boldsymbol{R}^0 = \{b \in 2^\omega : \forall n\, ((b)_n = (b)_0)\} \quad \text{and} \quad \boldsymbol{R} = \{x \in 2^\omega : (\exists\, b \in \boldsymbol{R}^0)\, (b \mathrel{\mathsf{E}_0} x)\},$$

where, for $x, y \in 2^\omega$, $x \mathrel{\mathsf{E}_0} y$ if and only if $x(n) = y(n)$ for all but finitely many n. The set $\boldsymbol{R}$ will be the B in the explanation above. For every $x \in \boldsymbol{R}$ there is a unique $b = \boldsymbol{b}(x) \in \boldsymbol{R}^0$ with $x \mathrel{\mathsf{E}_0} b$. Let $\operatorname{dif}(x)$ be the least natural n such that $x(i) = b(i)$ for all $i \geq n$. If $x \in \boldsymbol{R}^{\mathrm{ev}} = \{x \in 2^\omega : x_{\mathrm{even}} \in \boldsymbol{R}\}$, then let $\boldsymbol{b}^{\mathrm{ev}}(x) = \boldsymbol{b}(x_{\mathrm{even}})$ and $\boldsymbol{u}(x) = x \restriction (2 \operatorname{dif}(x_{\mathrm{even}}))$.

If $b \in 2^\omega$ and $n \geq 1$, then let $U(b, n)$ be the set of all strings $s \in 2^{<\omega}$ of length $\operatorname{lh} s = 2n$ such that $s(2n-2) \neq b(n-1)$. Separately let $U(b, 0) = \{\Lambda\}$. Put $U(b) = \bigcup_{n \in \omega} U(b, n)$. If $u \in U(b, n)$, then define

$$\widehat{Y}(b, u) = \{x \in [u] : (\forall\, k \geq n)\, (x(2k) = b(k))\} \quad \text{and} \quad \boldsymbol{T}(b, u) = \operatorname{tree} \widehat{Y}(b, u).$$

The trees $\boldsymbol{T}(b, u)$, $u \in U(b)$, belong to Perf, and the sets $[\boldsymbol{T}(b, u)] = \widehat{Y}(b, u) \subseteq 2^\omega$ are pairwise disjoint: $[\boldsymbol{T}(b, u)] \cap [\boldsymbol{T}(b, v)] = \emptyset$ for any $u \neq v$ in $U(b)$ (of equal or non-equal length). The following is obvious.

Lemma 3.1 $\boldsymbol{R}^{\mathrm{ev}} = \bigcup_{b \in \boldsymbol{R}^0} \bigcup_{u \in U(b)} [\boldsymbol{T}(b, u)]$.

If $x \in \boldsymbol{R}^{\mathrm{ev}}$, then $b = \boldsymbol{b}^{\mathrm{ev}}(x) \in \boldsymbol{R}^0$, $u = \boldsymbol{u}(x) \in U(b)$, and $x \in [\boldsymbol{T}(b, u)]$.

If $s \in 2^{<\omega}$, then there exists $u \in U(b)$ such that $[\boldsymbol{T}(b, u)] \subseteq [s]$.

Now relativize the construction to any $P \in \mathrm{Perf}$. Let $h_P : 2^\omega \xrightarrow{\text{onto}} [P]$ be a canonical homeomorphism. If $x \in \boldsymbol{R}^{\mathrm{ev}}_P = \{x \in [P] : h_P^{-1}(x) \in \boldsymbol{R}^{\mathrm{ev}}\}$, then put

$$\boldsymbol{b}^{\mathrm{ev}}_P(x) = \boldsymbol{b}^{\mathrm{ev}}(x') \quad \text{and} \quad \boldsymbol{u}_P(x) = \boldsymbol{u}(x'), \quad \text{where } x' = h_P^{-1}(x).$$

Accordingly if $b \in \boldsymbol{R}^0$ and $u \in U(b)$, then let

$$\widehat{Y}_P(b, u) = \{h_P(y) : y \in \widehat{Y}(b, u)\} \quad \text{and} \quad \boldsymbol{T}_P(b, u) = \operatorname{tree}(\widehat{Y}_P(b, u)).$$

Corollary 3.2 (of Lemma 3.1) *Suppose that $P \in \mathrm{Perf}$. Then the following hold:*

(i) $\boldsymbol{R}_P^{\mathrm{ev}} = \bigcup_{b \in \boldsymbol{R}^0} \bigcup_{u \in U(b)} [\boldsymbol{T}_P(b,u)]$, *and if* $x \in \boldsymbol{R}_P^{\mathrm{ev}}$, *then* $b = \boldsymbol{b}_P^{\mathrm{ev}}(x) \in \boldsymbol{R}^0$, $u = \boldsymbol{u}_P(x) \in U(b)$, *and* $x \in [\boldsymbol{T}_P(b,u)]$.

(ii) *If* $b \in \boldsymbol{R}^0$, *then* $\mathrm{Next}_P(b) = \{\boldsymbol{T}_P(b,u) : u \in U(b)\} \subseteq \mathrm{Perf}$ *is an antichain:* $[\boldsymbol{T}_P(b,u)] \cap [\boldsymbol{T}_P(b,v)] = \emptyset$ *for all* $u \neq v$ *in* $U(b)$.

(iii) *If* $s \in P$, *then there is* $u \in U(b)$ *such that* $[\boldsymbol{T}_P(b,u)] \subseteq [P] \cap [s]$.

4 An auxiliary sequence φ

The forcing construction for Theorem 1.1 involves a certain sequence $\varphi \in (2^\omega)^{\omega_1}$.

Fix a recursive enumeration $\mathbb{Q} = \{r_n : n \in \omega\}$ of the rationals. If $\xi < \omega_1$, then let WO_ξ be the set of all $x \in 2^\omega$ such that $\{r_n : x(n) = 1\} \subseteq \mathbb{Q}$ is wellordered by the natural order of $\mathbb{Q}$ similarly to ξ. Put $\mathrm{WO} = \bigcup_{\xi<\omega_1} \mathrm{WO}_\xi$ (codes of ordinals) and $|x| = \xi$ for $x \in \mathrm{WO}_\xi$. If $x \in \mathrm{WO}$ and $|x| = \xi \geq \omega$, then one can concretely define a bijection $\boldsymbol{\beta}_x : \omega \xrightarrow{\text{onto}} \xi$. If $|x| < \omega$, put $\boldsymbol{\beta}_x(n) = n$ for all n.

Definition 4.1 Suppose that $\varphi \in (2^\omega)^\lambda$, $\lambda \leq \omega_1$.

(1) Put $\Omega_1[\varphi] = \{0\} \cup \{\xi+1 : \xi+1 < \lambda\}$ (all successor ordinals $\leq \lambda = \mathrm{dom}\,\varphi$).

(2) Let $\Omega_2[\varphi]$ be the set of all ordinals $\xi \leq \lambda$, $\xi \notin \Omega_1[\varphi]$, such that $\xi < \omega_1^{\mathbb{L}[\varphi\restriction\xi]}$.

(3) Assume that $\xi \in \Omega_2[\varphi]$. Let $\mu_\xi[\varphi] = \mu_\xi[\varphi \restriction \xi]$ be the least ordinal $\mu > \xi$ such that $\mathbb{L}_\mu[\varphi \restriction \xi]$ is a model of ZFC^- (ZFC minus the Power Set Axiom) and ξ is countable already in $\mathbb{L}_\mu[\varphi \restriction \xi]$.

(4) Put $M_\xi[\varphi] = \mathbb{L}_{\mu_\xi[\varphi]}[\varphi \restriction \xi]$. Then $\xi < \mu_\xi[\varphi] < \omega_1^{\mathbb{L}[\varphi\restriction\xi]}$, and $M_\xi[\varphi]$ is a countable transitive model of ZFC^-.

(5) Finally define $\Omega_3[\varphi] = \{\xi : \xi \leq \lambda\} \setminus (\Omega_1[\varphi] \cup \Omega_2[\varphi])$.

Lemma 4.2 *Suppose that* $\xi \leq \lambda \leq \omega_1$, $\varphi \in (2^\omega)^\lambda$, $\xi \in \Omega_2[\varphi]$, *and* M *is a transitive model of* ZFC^- *containing* $\varphi \restriction \xi$ *and the ordinal* $\mu_\xi[\varphi]$. *Then the set* $M_\xi[\varphi]$ *belongs to* M *and is countable in* M.

P r o o f. Let $\kappa = M \cap \mathrm{Ord}$. We may assume without loss of generality that $M = \mathbb{L}_\kappa[\varphi\restriction\xi]$. If all ordinals $\vartheta < \kappa$ are countable in M, then the result is obvious. Otherwise let $\vartheta = \omega_1^M$ be the least ordinal uncountable in M. Then $\xi < \vartheta$ since $\mu_\xi[\varphi] \in M$. Using the same condensation argument as in the proof that $\mathbb{V} = \mathbb{L}$ implies CH, one can prove that ξ is countable in $\mathbb{L}_\vartheta[\varphi \restriction \xi]$, therefore $\mu_\xi[\varphi] < \vartheta$. □

Let $\boldsymbol{R}_0^+$ consist of all $b \in \boldsymbol{R}^0$ such that $((b)_0)_0$ and $((b)_0)_1$ belong to WO. With every $b \in \boldsymbol{R}_0^+$ we associate a sequence $\boldsymbol{\chi}_b \in (2^\omega)^{\lambda+1}$, where $\lambda = |((b)_0)_0|$, by $\boldsymbol{\chi}_b(\lambda) = b$, and if $\eta < \lambda$, then $\boldsymbol{\chi}_b(\eta) = ((b)_0)_{k+2}$, where $k \in \omega$ satisfies $\boldsymbol{\beta}_{((b)_0)_0}(k) = \eta$. The role of $((b)_0)_1$ will be clear a few lines below.

If $\lambda \leq \omega_1$, then let Φ_λ be the set of all functions $\varphi \in (\boldsymbol{R}_0^+)^\lambda$ such that $\varphi(0)$ is the real $x(n) = 0$ for all n, and in addition for any ξ, $0 < \xi < \lambda$, we have:

1) $|((\varphi(\xi))_0)_0| = \xi$ and $\varphi \restriction (\xi+1) = \boldsymbol{\chi}_{\varphi(\xi)}$, so that $\varphi(\xi)$ codes $\varphi \restriction (\xi+1)$,

2) if $\xi \in \Omega_2(\varphi)$, then $((\varphi(\xi))_0)_1 \in \mathrm{WO}_{\mu_\xi[\varphi]}$, so that the real $\varphi(\xi)$ also codes the ordinal $\mu_\xi[\varphi] = \mu_\xi[\varphi \restriction \xi]$ in this case.

Put $\Phi = \bigcup_{\lambda \leq \omega_1} \Phi_\lambda$.

Lemma 4.3 *If* $\lambda \leq \omega_1$, $\varphi \in \Phi_\lambda$, *and* $\xi \in \Omega_3[\varphi]$, *then* $\xi = \omega_1^{\mathbb{L}[\varphi\restriction\xi]}$.
On the other hand, if $\xi \in \Omega_2[\varphi] \cup \Omega_1[\varphi]$, *then* $\xi < \omega_1^{\mathbb{L}[\varphi\restriction\xi]}$.

P r o o f. If $\varphi \in \Phi_\lambda$, then each value $\varphi(\xi)$, $\xi < \lambda$, codes the countability of ξ, therefore $\xi > \omega_1^{\mathbb{L}[\varphi\restriction\xi]}$ is impossible. This proves the first claim. The second claim holds by definition. □

Definition 4.4 Coming back to Theorem 1.1, fix a set $A \subseteq \omega_1$ such that $\mathbb{V} = \mathbb{L}[A]$. The definition of Φ_λ leaves a lot of freedom as to how to define $\varphi(\xi)$, $\xi \in \Omega_1[\varphi]$. This makes it possible to define $\varphi \in \Phi_{\omega_1}$ such that $A \in \mathbb{L}[\varphi]$, so that $\mathbb{V} = \mathbb{L}[\varphi]$ in the ground universe $\mathbb{V}$.

Moreover it can be guaranteed, by a suitable choice of φ, that in any generic extension of $\mathbb{V}$, if $x \in 2^\omega$ and φ is $\Delta_1^{\mathrm{HC}}(x)$, then A is $\Delta_1^{\mathrm{HC}}(x)$ as well.

Fix $\widehat{\varphi} \in \Phi_{\omega_1}$ with all these properties.

5 The forcing

Suppose that $\varphi \in \Phi_{\omega_1}$. Define a forcing notion $\mathbb{P}[\varphi] = \bigcup_{\xi<\omega_1} \mathbb{P}_\xi[\varphi]$ satisfying the following conditions:

(A) If $\xi < \omega_1$, then $\mathbb{P}_\xi[\varphi] \subseteq \mathrm{Perf}$ and $\mathbb{P}_\xi[\varphi]$ is countable.

(B) If $\eta < \xi$ and $S \in \mathbb{P}_\eta[\varphi]$, then there is a tree $T \in \mathbb{P}_\xi[\varphi]$, $T \subseteq S$.

(C) If $\xi \leq \omega_1$, then the subsequence $\{\mathbb{P}_\eta[\varphi]\}_{\eta<\xi}$, hence $\mathbb{P}_{<\xi}[\varphi] = \bigcup_{\eta<\xi} \mathbb{P}_\eta[\varphi]$ as well, belong to $\mathbb{L}[\varphi \restriction \xi]$.

(D) If $\xi \in \Omega_2[\varphi]$, then $\mathbb{P}_\xi[\varphi]$ is an antichain and $T \subseteq^{\mathrm{fin}} \bigcup D$ holds for any $T \in \mathbb{P}_\xi[\varphi]$ and any set $D \subseteq \mathbb{P}_{<\xi}[\varphi]$, $D \in M_\xi[\varphi]$, dense in $\mathbb{P}_{<\xi}[\varphi]$.

Definition 5.1 The construction goes on by induction on ξ.

1^* $\mathbb{P}_0[\varphi]$ consists of a single tree $2^{<\omega}$.

2^* If $\xi = \eta + 1$, then $\mathbb{P}_\xi[\varphi] = \bigcup_{P \in \mathbb{P}_\eta[\varphi]} \mathrm{Next}_P(b)$, where $b = \varphi(\xi)$,

3^* If $\xi \in \Omega_3[\varphi]$, then $\mathbb{P}_\xi[\varphi] = \bigcup_{P \in \mathbb{P}_{<\xi}[\varphi]} \mathrm{Next}_P(b)$, where $b = \varphi(\xi)$.

4^* Suppose that $\xi \in \Omega_2[\varphi]$, that is, $\xi < \omega_1$ is a limit ordinal and $\xi < \omega_1^{\mathbb{L}[\varphi \restriction \xi]}$. In this case the definition of $\mathbb{P}_\xi[\varphi]$ takes more time. We require, following (C), that $\mathbb{P}_{<\xi}[\varphi] \in \mathbb{L}[\varphi \restriction \xi]$. If this fails, then let $\mathbb{P}_\xi[\varphi] = \mathbb{P}_{<\xi}[\varphi]$, yet in fact (see below) this condition will always be satisfied. Consider $M_\xi[\varphi]$, a countable transitive model of ZFC^- (see Definition 4.1). The set $\mathscr{D}$ of all sets $D \subseteq \mathbb{P}_{<\xi}[\varphi]$, $D \in M_\xi[\varphi]$, dense in $\mathbb{P}_{<\xi}[\varphi]$, is countable too, and, under the assumption $\mathbb{P}_{<\xi}[\varphi] \in \mathbb{L}[\varphi \restriction \xi]$ as above, $\mathscr{D}$ belongs to the class $\mathbb{L}[\varphi \restriction \xi]$ and is countable there by Lemma 4.2. Using Lemma 2.2 in $\mathbb{L}[\varphi \restriction \xi]$, we get an antichain $E \in \mathbb{L}[\varphi \restriction \xi]$, $E \subseteq \mathrm{Perf} \setminus \mathbb{P}_{<\xi}[\varphi]$, countable in $\mathbb{L}[\varphi \restriction \xi]$ and satisfying:

1) $T \subseteq^{\mathrm{fin}} \bigcup D$ for all $D \in \mathscr{D}$ and $T \in E$, and

2) the set E is dense in $\mathbb{P}_{<\xi}[\varphi] \cup E$, that is, for each $S \in \mathbb{P}_{<\xi}[\varphi]$ there is $T \in E$ with $T \subseteq S$.

Let $\mathbb{P}_\xi[\varphi]$ be the least of such sets E in the sense of the Gödel wellordering $\leq^{\mathrm{G}}_{\varphi \restriction \xi}$ of $\mathbb{L}[\varphi \restriction \xi]$. This completes Step 4^* and the inductive definition of $\mathbb{P}_\xi[\varphi]$.

Remark 5.2 Note that (B), the density of $\mathbb{P}_\xi[\varphi]$ in $\mathbb{P}_{\leq\xi}[\varphi]$, is guaranteed by the construction in 4^* in the nontrivial case $\xi \in \Omega_2[\varphi]$. (C) follows from a rather obvious absoluteness: if $\xi < \omega_1$, then the initial segment $\{\mathbb{P}_\eta[\varphi]\}_{\eta<\xi}$ can be defined in $\mathbb{L}[\varphi \restriction \xi]$. (D) is also guaranteed by the construction in 4^*. □

Definition 5.3 Choose $\widehat{\varphi} \in \Phi_{\omega_1}$ as in Definition 4.4. Put $\widehat{\mathbb{P}}_\xi = \mathbb{P}_\xi[\widehat{\varphi}]$ for all ξ, and $\widehat{\mathbb{P}} = \mathbb{P}[\widehat{\varphi}] = \bigcup_{\xi<\omega_1} \widehat{\mathbb{P}}_\xi$. This will be our forcing notion, and we order it so that a condition (a perfect tree) $S \in \widehat{\mathbb{P}}$ is stronger than $T \in \widehat{\mathbb{P}}$ if and only if $S \subseteq T$, or equivalently, if $[S] \subseteq [T]$.

Proposition 5.4

(a) *The set* $\widehat{\mathbb{P}} = \mathbb{P}[\widehat{\varphi}]$ *is CO-dense in the sense of Section* 2.

(b) *If* $\vartheta < \omega_1$, *then the set* $\widehat{\mathbb{P}}^{\geq\vartheta} = \bigcup_{\vartheta\leq\xi<\omega_1} \widehat{\mathbb{P}}_\xi$ *is dense in* $\widehat{\mathbb{P}}$.

(c) *If* $\vartheta < \lambda < \omega_1$, *then the set* $\widehat{\mathbb{P}}_{\vartheta,\lambda} = \bigcup_{\vartheta\leq\xi<\lambda} \widehat{\mathbb{P}}_\xi$ *is dense in* $\widehat{\mathbb{P}}_{<\lambda}$.

P r o o f. Use Corollary 3.2(ii). □

It follows from the first claim of Proposition 5.4 that any $\widehat{\mathbb{P}}$-generic set $G \subseteq \widehat{\mathbb{P}}$ defines a real $x_G \in 2^\omega$, the only element of the intersection $\bigcap_{T\in G}[T]$, and then $G = \{T \in \widehat{\mathbb{P}} : x_G \in [T]\}$, so that $\mathbb{L}[G] = \mathbb{L}[x_G]$. Reals of the form x_G ($G \subseteq \widehat{\mathbb{P}}$ being a $\widehat{\mathbb{P}}$-generic set) are called $\widehat{\mathbb{P}}$-*generic reals* themselves.

Lemma 5.5 *Suppose that* M *is a countable transitive model of* ZFC^-, $\lambda \in M$ *is an ordinal, and* $\varphi \in \Phi_\lambda \cap M$. *Then, for any* $\xi < \lambda$, $\mathbb{P}_\xi[\varphi] = (\mathbb{P}_\xi[\varphi])^M$. *If moreover* $\mu_\lambda[\varphi] \in M$, *then* $\mathbb{P}_\lambda[\varphi] = (\mathbb{P}_\lambda[\varphi])^M$ *as well.*

P r o o f. Note that if $\xi < \lambda$ and $\xi \in \Omega_2[\varphi]$, then the ordinal $\mu_\xi[\varphi]$ is coded by $\varphi(\xi)$ by the definition of Φ_ξ. It follows that $\mu_\xi[\varphi] \in M$, and hence the model $M_\xi[\varphi]$ belongs to M and is countable in M by Lemma 4.2. This obviously implies the absoluteness required. □

6 Cardinal preservation

We claim that the forcing $\widehat{\mathbb{P}} = \mathbb{P}[\widehat{\varphi}]$ just defined *preserves cardinals*.

Under the assumptions of Theorem 1.1, it suffices to prove that $\omega_1^{\mathbb{L}[\widehat{\varphi}]}$ remains uncountable in any $\widehat{\mathbb{P}}$-generic extension of $\mathbb{L}[\widehat{\varphi}]$. (Indeed, as $\mathbb{V} = \mathbb{L}[A] = \mathbb{L}[\widehat{\varphi}]$, the GCH holds in $\mathbb{V}$, the ground universe, and hence the forcing notion $\widehat{\mathbb{P}}$ is a set of cardinality just $\aleph_1$ in $\mathbb{V}$.) The next lemma contains a well known sufficient condition for ω_1 to remain uncountable.

Lemma 6.1 *Suppose that $\{D_n : n \in \omega\}$ is a family of dense subsets of $\widehat{\mathbb{P}}$, and $S \in \widehat{\mathbb{P}}$. Then there is a tree $T \in \widehat{\mathbb{P}}$, $T \subseteq S$, such that $T \subseteq^{\text{fin}} \bigcup D_n$ for all n.*

Proof. We prove a more general result: If $\lambda \in \Omega_3[\widehat{\varphi}] \cup \{\omega_1\}$, then the set $\widehat{\mathbb{P}}_{<\lambda}$ (it belongs to $\mathbb{L}[\widehat{\varphi} \restriction \lambda]$ by (C) of Section 5) satisfies the requirement of the lemma inside $\mathbb{L}[\widehat{\varphi} \restriction \lambda]$. We prove this by induction, i.e., we prove the result for some $\lambda \in \Omega_3[\widehat{\varphi}] \cup \{\omega_1\}$ assuming that for all $\lambda' \in \Omega_3[\widehat{\varphi}]$, $\lambda' < \lambda$, it has been established. This holds, for instance, if λ is the least ordinal in $\Omega_3[\widehat{\varphi}]$, or if $\lambda = \omega_1$ and $\Omega_3[\widehat{\varphi}] = \emptyset$. Thus let $\lambda \in \Omega_3[\widehat{\varphi}] \cup \{\omega_1\}$.

Then $\lambda = \omega_1^{\mathbb{L}[\widehat{\varphi} \restriction \lambda]}$ by Lemma 4.3. Suppose that $S \in \widehat{\mathbb{P}}_{<\lambda}$, all sets $D_n \subseteq \widehat{\mathbb{P}}_{<\lambda}$ are dense in $\widehat{\mathbb{P}}_{<\lambda}$, and $\{D_n\}_{n\in\omega} \in \mathbb{L}[\widehat{\varphi} \restriction \lambda]$. This statement then will be true in $\mathbb{L}_\delta[\widehat{\varphi} \restriction \lambda]$ as well, where $\delta = \omega_2^{\mathbb{L}[\widehat{\varphi} \restriction \lambda]}$. Consider, arguing in $\mathbb{L}[\widehat{\varphi} \restriction \lambda]$, a countable transitive submodel of $\mathbb{L}_\delta[\widehat{\varphi} \restriction \lambda]$, containing S and the sequence of sets D_n. Applying the Mostowski collapse to such a submodel, we obtain ordinals $\xi < \zeta < \lambda$ such that $S \in \widehat{\mathbb{P}}_{<\xi}$, while all sets $D_n' = D_n \cap \widehat{\mathbb{P}}_{<\xi}$ are dense in $\widehat{\mathbb{P}}_{<\xi}$ and belong to $N = \mathbb{L}_\zeta[\widehat{\varphi} \restriction \xi]$. We have two cases.

Case 1. $\xi \in \Omega_3[\widehat{\varphi}]$. By the inductive hypothesis the result holds for ξ, and hence a required tree $T \subseteq S$, $T \subseteq^{\text{fin}} \bigcup D_n'$ for all n, exists in $\widehat{\mathbb{P}}_{<\xi}$.

Case 2. $\xi \in \Omega_2[\widehat{\varphi}]$. Note that ξ is uncountable in N since $\lambda = \omega_1^{\mathbb{L}[\widehat{\varphi} \restriction \lambda]}$ is uncountable in $\mathbb{L}[\widehat{\varphi} \restriction \lambda]$ and then in $\mathbb{L}_\delta[\widehat{\varphi} \restriction \lambda]$. It follows that $\zeta < \mu_\xi[\widehat{\varphi}]$. Therefore $N \subseteq M_\xi[\widehat{\varphi}] = \mathbb{L}_{\mu_\xi[\widehat{\varphi}]}[\widehat{\varphi} \restriction \xi]$, so that the sets D_n' belong to $M_\xi[\widehat{\varphi}]$. Now, by (B) of Section 5, there exists $T \in \widehat{\mathbb{P}}_\xi$ satisfying $T \subseteq S$. And by (D) of Section 5, we have $T \subseteq^{\text{fin}} D_n'$ for all n, therefore $T \subseteq^{\text{fin}} D_n$. □

7 Decoding

The following procedure of decoding the values $\widehat{\varphi}(\xi)$ is based on the fact that if $P \in \mathrm{Perf}$, $b \in \boldsymbol{R}^0$, $u \in U(b)$ and $x \in [\boldsymbol{T}_P(b,u)]$, (see Section 3), then $b = \boldsymbol{b}_P^{\text{ev}}(x)$ and $\boldsymbol{u}_P(x) = u$. In other words, if we know a tree $P \in \mathrm{Perf}$ and a point $x \in [\boldsymbol{T}_P(b,u)]$, then the values of $b,\ u,\ [\boldsymbol{T}_P(b,u)]$ can be restored by simple absolute operations.

Definition 7.1 Suppose that $x \in 2^\omega$. We define the following objects by induction on $\alpha < \omega_1$:

(1) a tree $Q_\alpha(x) \in \mathrm{Perf}$ with $x \in [Q_\alpha(x)]$ and an ordinal $\boldsymbol{\lambda}_\alpha(x) < \omega_1$;

(2) a sequence $\boldsymbol{\psi}_\alpha(x) \in \Phi_{\boldsymbol{\lambda}_\alpha(x)}$ such that $\alpha < \beta$ implies $\boldsymbol{\psi}_\alpha(x) \subset \boldsymbol{\psi}_\beta(x)$.

The construction may involve all ordinals $\xi < \omega_1$ or stop on some $\xi^* < \omega_1$.

Beginning. Put $Q_0(x) = 2^{<\omega}$, so that $[Q_0(x)] = 2^\omega$, definitely $x \in [Q_0(x)]$, and $\boldsymbol{\psi}_0(x) = \Lambda$ (the empty sequence), so that $\boldsymbol{\lambda}_0(x) = 0$.

Step $\alpha \to \alpha + 1$. Let $P = Q_\alpha(x)$, $\boldsymbol{\lambda}_\alpha(x)$, $\boldsymbol{\psi}_\alpha(x)$ be defined and satisfy (1) and $\boldsymbol{\psi}_\alpha(x) \in \Phi_{\boldsymbol{\lambda}_\alpha(x)}$ in (2). In particular, $x \in [P]$. Suppose that

$$(*)\qquad \begin{array}{l} x \in \boldsymbol{R}_P^{\text{ev}},\ b = \boldsymbol{b}_P^{\text{ev}}(x) \in \boldsymbol{R}_0^+, \text{ and the sequence } \boldsymbol{\chi}_b \text{ belongs to } \Phi_{\lambda+1}, \\ \text{where } \lambda = |((b)_0)_0| \geq \boldsymbol{\lambda}_\alpha(x), \text{ and satisfies } \boldsymbol{\psi}_\alpha(x) \subset \boldsymbol{\chi}_b. \end{array}$$

If this fails, then the construction stops. If $(*)$ holds, then $u = \boldsymbol{u}_P(x) \in U(b)$ and $x \in [\boldsymbol{T}_P(b,u)]$ by Corollary 3.2. Put $Q_{\alpha+1}(x) = \boldsymbol{T}_P(b,u)$, $\boldsymbol{\psi}_{\alpha+1}(x) = \boldsymbol{\chi}_b$, and $\boldsymbol{\lambda}_{\alpha+1}(x) = \lambda + 1 = \mathrm{dom}\,\boldsymbol{\psi}_{\alpha+1}(x)$.

Limit step. Let $\gamma < \omega_1$ be a limit ordinal. Assume that the values $Q_\alpha(x)$, $\boldsymbol{\psi}_\alpha(x)$ and $\boldsymbol{\lambda}_\alpha(x)$ have been defined for all $\alpha < \gamma$ and satisfy (1) and (2). Put $\boldsymbol{\psi}_\gamma(x) = \bigcup_{\alpha<\gamma} \boldsymbol{\psi}_\alpha(x)$. Then clearly $\psi = \boldsymbol{\psi}_\gamma(x) \in \Phi_\lambda$, where $\lambda = \boldsymbol{\lambda}_\gamma(x) := \sup_{\alpha<\gamma} \boldsymbol{\lambda}_\alpha(x)$. Suppose that

$$(\dagger)\qquad \begin{array}{l} \text{the sequence } \psi = \boldsymbol{\psi}_\gamma(x) \text{ satisfies } \omega_1^{\mathbb{L}[\psi]} > \lambda, \text{ where } \lambda = \mathrm{dom}\,\psi = \boldsymbol{\lambda}_\gamma(x), \\ \text{that is, formally, } \lambda \in \Omega_2[\psi]. \end{array}$$

If this fails, then the construction stops. If $(\dagger)$ holds, then go ahead.

Arguing in $\mathbb{L}[\psi]$, carry out the construction of Definition 5.1 up to the step λ. That is, define sets $\mathbb{P}_\alpha[\psi] \subseteq \mathrm{Perf}$ for all ordinals $\alpha < \lambda$, define $\mathbb{P}_{<\lambda}[\psi]$, and proceed to 4^* of Definition 5.1 (as $\lambda \in \Omega_2[\psi]$). Define an antichain $\mathbb{P}_\lambda[\psi]$ according to 4^*. Now, if

(‡) there is a unique tree $T \in \mathbb{P}_\lambda[\psi]$ such that $x \in [T]$,

then take this T as $Q_\gamma(x)$, otherwise the construction stops.

Lemma 7.2 *Suppose that M is a countable transitive model of ZFC^-, $x \in 2^\omega \cap M$, and $\alpha \in M$ is an ordinal countable in M. If $(\boldsymbol{\psi}_\alpha(x))^M$ and $(Q_\alpha(x))^M$ are defined, then $(\boldsymbol{\psi}_\alpha(x))^M = \boldsymbol{\psi}_\alpha(x)$ and $(Q_\alpha(x))^M = Q_\alpha(x)$.*

Proof. Arguing by induction, the step $\alpha \to \alpha + 1$ is entirely trivial, so we can focus on the limit step. Thus suppose that $\gamma \in M$ is a limit ordinal countable in M and $(\boldsymbol{\psi}_\gamma(x))^M$, $(Q_\gamma(x))^M$ exist in M – and then, of course, $(\boldsymbol{\psi}_\alpha(x))^M$ and $(Q_\alpha(x))^M$ exist in M for any ordinal $\alpha < \gamma$ and, by the inductive hypothesis, $(\boldsymbol{\psi}_\alpha(x))^M = \boldsymbol{\psi}_\alpha(x)$ and $(Q_\alpha(x))^M = Q_\alpha(x)$ for $\alpha < \gamma$.

Clearly enough we have $(\boldsymbol{\psi}_\gamma(x))^M = \boldsymbol{\psi}_\gamma(x) = \bigcup_{\alpha<\gamma} \boldsymbol{\psi}_\alpha(x)$. Therefore $\psi = \boldsymbol{\psi}_\gamma(x)$ belongs to M. On the other hand, $\psi \in \Phi_\lambda$, where $\lambda = \boldsymbol{\lambda}_\gamma(x)$. Then the sequence $\{\mathbb{P}_\xi[\psi]\}_{\xi<\lambda}$ belongs to M as well and coincides with $(\{\mathbb{P}_\xi[\psi]\}_{\xi<\lambda})^M$ by Lemma 5.5. As $(Q_\gamma(x))^M$ exists in M, we conclude that (†) holds in M, so that $\lambda \in \Omega_2[\psi]$ in M, and hence $\lambda \in \Omega_2[\psi]$ in the universe.

That $(Q_\gamma(x))^M$ exists in M means that $(\mathbb{P}_\lambda[\psi])^M$ exists in M as well – basically, $(Q_\gamma(x))^M$ is a unique tree $T \in (\mathbb{P}_\lambda[\psi])^M$ satisfying $x \in [T]$. And by definition, in M, $(\mathbb{P}_\lambda[\psi])^M$ is the Gödel-least countable antichain in Perf in a certain collection of antichains. Then it is quite clear that $(\mathbb{P}_\lambda[\psi])^M$ coincides with the true $\mathbb{P}_\lambda[\psi]$, so finally $(Q_\alpha(x))^M = Q_\alpha(x)$, as required. □

8 The decoding is correct

The following key lemma shows that the decoding procedure of Definition 7.1 restores $\widehat{\varphi}$ assuming that the given x belongs to sets of the form $[T]$, where T belongs to suitably high levels $\widehat{\mathbb{P}}_\lambda$ of the set $\widehat{\mathbb{P}} = \mathbb{P}[\widehat{\varphi}]$.

Lemma 8.1 *Suppose that $\lambda < \omega_1$, $T \in \widehat{\mathbb{P}}_\lambda = \mathbb{P}_\lambda[\widehat{\varphi}]$ and $x \in [T]$. Then there is an ordinal $\alpha \le \lambda$ such that $Q_\alpha(x)$ and $\boldsymbol{\psi}_\alpha(x)$ are defined by Definition 7.1, $T = Q_\alpha(x)$, and we have $\operatorname{dom} \boldsymbol{\psi}_\alpha(x) = \boldsymbol{\lambda}_\alpha(x) = \lambda'$ and $\widehat{\varphi} \restriction \lambda' = \boldsymbol{\psi}_\alpha(x)$, where $\lambda' = \lambda + 1$ whenever $\lambda \in \Omega_1[\widehat{\varphi}] \cup \Omega_3[\widehat{\varphi}]$, but $\lambda' = \lambda$ whenever $\lambda \in \Omega_2[\widehat{\varphi}]$.*

Proof. We argue by induction on λ. If $\lambda = 0$, then by definition $\widehat{\varphi} \restriction 0 = \Lambda = \boldsymbol{\psi}_0(x)$. Now we prove the lemma for some $\lambda > 0$ assuming the result holds already for all $\xi < \lambda$.

Case 1. $\lambda \in \Omega_1[\widehat{\varphi}] \cup \Omega_3[\widehat{\varphi}]$. Then by Definition 5.1 there exists an ordinal $\xi < \lambda$ and a tree $P \in \widehat{\mathbb{P}}_\xi$, such that $T = \boldsymbol{T}_P(b, u)$ for some $u \in U(b)$, where $b = \widehat{\varphi}(\lambda)$. (In particular, $\lambda = \xi + 1$ in the case when $\lambda \in \Omega_1[\widehat{\varphi}]$.) We have $T \subseteq P$, and hence $x \in [P]$, so that the inductive assumption can be applied. It gives an ordinal $\alpha \le \xi$ such that $Q_\alpha(x) = P$ and $\boldsymbol{\psi}_\alpha(x) = \widehat{\varphi} \restriction \xi'$ are well-defined by Definition 7.1, and $\xi' \in \{\xi, \xi + 1\}$. As $x \in [T]$ and $T = \boldsymbol{T}_P(b, u)$, we have $x \in \boldsymbol{R}_P^{\mathrm{ev}}$ by Corollary 3.2, and $b = \widehat{\varphi}(\lambda) = \boldsymbol{b}_P^{\mathrm{ev}}(x) \in \boldsymbol{R}_0^+$. Then by the construction $Q_{\alpha+1}(x) = T$. In addition, as $b = \widehat{\varphi}(\lambda)$, we have $\boldsymbol{\chi}_b = \widehat{\varphi} \restriction (\lambda + 1)$ by the choice of $\widehat{\varphi}$ (Definition 4.4). Therefore $\boldsymbol{\psi}_\alpha(x) \subset \boldsymbol{\chi}_b$ holds, and this implies $\boldsymbol{\psi}_{\alpha+1}(x) = \boldsymbol{\chi}_b = \widehat{\varphi} \restriction (\lambda + 1)$.

Case 2. $\lambda \in \Omega_2[\widehat{\varphi}]$. Then $\widehat{\mathbb{P}}_\lambda$ is an antichain in Perf, $\widehat{\mathbb{P}}_\lambda \in \mathbb{L}[\widehat{\varphi} \restriction \lambda]$, $\widehat{\mathbb{P}}_\lambda$ is countable in $\mathbb{L}[\widehat{\varphi} \restriction \lambda]$, and each $T \in \widehat{\mathbb{P}}_\lambda$ satisfies $T \subseteq^{\mathrm{fin}} \bigcup D$, whenever $D \subseteq \widehat{\mathbb{P}}_{<\lambda}$ is dense in $\widehat{\mathbb{P}}_{<\lambda}$ and belongs to $M_\lambda[\widehat{\varphi}] = \mathbb{L}_{\mu_\lambda[\widehat{\varphi}]}[\widehat{\varphi} \restriction \lambda]$. Recall that $\mu_\lambda[\widehat{\varphi}] < \omega_1$ is equal to the least ordinal ζ such that λ is countable in $\mathbb{L}_\zeta[\widehat{\varphi} \restriction \lambda]$ and $\mathbb{L}_\zeta[\widehat{\varphi} \restriction \lambda]$ satisfies ZFC^-. However all sets of the form $\widehat{\mathbb{P}}_{\vartheta,\lambda} = \bigcup_{\vartheta \le \xi < \lambda} \widehat{\mathbb{P}}_\xi, \vartheta < \lambda$, are dense in $\widehat{\mathbb{P}}_{<\lambda}$ by Proposition 5.4. And all these sets belong to $M_\lambda[\widehat{\varphi}]$. (Indeed the sequence $\{\widehat{\mathbb{P}}_\xi\}_{\xi<\lambda}$ itself belongs to $M_\lambda[\widehat{\varphi}]$ by Lemma 5.5.)

We conclude that $T \subseteq^{\mathrm{fin}} \bigcup \widehat{\mathbb{P}}_{\vartheta,\lambda}$ for all $\vartheta < \lambda$. Therefore, as $x \in [T]$, for any $\vartheta < \lambda$ there is an ordinal ξ, $\vartheta \le \xi < \lambda$, and a condition $S \in \widehat{\mathbb{P}}_\xi$ satisfying $x \in [S]$. Then by the inductive hypothesis there is an ordinal $\alpha(\xi) \le \xi$ such that $Q_{\alpha(\xi)}(x) = S$ and $\boldsymbol{\psi}_{\alpha(\xi)}(x) = \widehat{\varphi} \restriction \xi'$, where $\xi' \in \{\xi, \xi + 1\}$. It follows that there is a limit ordinal $\gamma \le \lambda$, such that $\boldsymbol{\psi}_\gamma(x) = \widehat{\varphi} \restriction \lambda \in \Phi_\lambda$.

Now come back to the last paragraph of Definition 7.1 where, naturally, $\psi = \boldsymbol{\psi}_\gamma(x) = \widehat{\varphi} \restriction \lambda$. The set $\mathbb{P}_{<\lambda}[\psi]$ is then equal to $\widehat{\mathbb{P}}_{<\lambda} = \mathbb{P}_{<\lambda}[\widehat{\varphi}]$, while the antichain $\mathbb{P}_\lambda[\psi]$ is equal to $\widehat{\mathbb{P}}_\lambda = \mathbb{P}_\lambda[\widehat{\varphi}]$. Recall that $T \in \widehat{\mathbb{P}}_\lambda$ and $x \in [T]$. Then by definition $Q_\gamma(x) = T$, as required. □

Corollary 8.2 *Suppose that $x = x_G$ is a $\widehat{\mathbb{P}}$-generic real over the universe $\mathbb{L}[\widehat{\varphi}] = \mathbb{L}[A]$. Then $Q_\alpha(x)$, $\boldsymbol{\psi}_\alpha(x)$ and $\boldsymbol{\lambda}_\alpha(x)$ are defined for every ordinal $\alpha < \omega_1^{\mathbb{L}[\varphi]}$ in accordance with Definition* 7.1*, and we have*

$$\widehat{\varphi} = \bigcup_{\alpha<\omega_1^{\mathbb{L}[\varphi]}} \boldsymbol{\psi}_\alpha(x) \in \mathbb{L}[x].$$

It follows that $\omega_1^{\mathbb{L}[\varphi]} = \omega_1^{\mathbb{L}[\varphi,x]} = \omega_1^{\mathbb{L}[x]}$.

Proof. In $\mathbb{L}[\widehat{\varphi}]$, the sets $\widehat{\mathbb{P}}^{\geq\vartheta} = \bigcup_{\vartheta\leq\xi<\omega_1} \widehat{\mathbb{P}}_\xi$, $\vartheta < \omega_1$, are dense in $\widehat{\mathbb{P}}$. It follows that for every $\vartheta < \omega_1$ there is an ordinal ξ, $\vartheta \leq \xi < \omega_1^{\mathbb{L}[\varphi]}$, and a tree $T \in \widehat{\mathbb{P}}_\xi$ such that $x \in [T]$. We conclude by Lemma 8.1 that $\widehat{\varphi} = \bigcup_{\alpha<\omega_1^{\mathbb{L}[\varphi]}} \boldsymbol{\psi}_\alpha(x)$. Yet the decoding construction of $\boldsymbol{\psi}_\alpha(x)$ is absolute for $\mathbb{L}[x]$, therefore $\widehat{\varphi} \in \mathbb{L}[x]$. □

9 The proof of Theorem 1.1 without minimality and reshaping

Our goal is to prove the following lemma:

Lemma 9.1 *If $x = x_G$ is $\widehat{\mathbb{P}}$-generic over $\mathbb{L}[A] = \mathbb{L}[\widehat{\varphi}]$, then x satisfies part* (i) *of Theorem* 1.1.

Proof. The forcing $\widehat{\mathbb{P}}$ preserves all cardinals, see Section 6. Furthermore, $\widehat{\varphi}$, and then A, belong to $\mathbb{L}[x]$ by Corollary 8.2. To prove that $\widehat{\varphi}$, and then A as well (see Definition 4.4) are $\Delta_1^{\mathrm{HC}}(x)$ in $\mathbb{L}[x]$, note that $\widehat{\varphi}(\xi) = r$ is equivalent to either of the two following formulas:

$$\exists\, M\, \exists\, \alpha \left(\alpha < \omega_1^M \wedge \xi \in M \wedge r \in M \wedge M \vDash \boldsymbol{\psi}_\alpha(x)(\xi) = r\right),$$
$$\forall\, M\, \forall\, \alpha \left(\alpha < \omega_1^M \wedge \xi \in M \wedge r \in M \Rightarrow M \vDash (!\, \boldsymbol{\psi}_\alpha(x)(\xi) \Rightarrow \boldsymbol{\psi}_\alpha(x)(\xi) = r)\right),$$

where M runs over all countable transitive models of ZFC^-, α over ordinals in M, and $!\,\boldsymbol{\psi}_\alpha(x)(\xi)$ means that $\boldsymbol{\psi}_\alpha(x)(\xi)$ is defined. The equivalence follows from Lemma 7.2 and Corollary 8.2. The first displayed formula shows that $\widehat{\varphi}$ is $\Sigma_1^{\mathrm{HC}}(x)$, while the second one shows that $\widehat{\varphi}$ is $\Pi_1^{\mathrm{HC}}(x)$. □

10 The minimality

By a certain modification of the construction we can achieve the *minimality* of $\widehat{\mathbb{P}}$-generic reals, as required by Theorem 1.1. First of all, rather straightforward cardinality estimations reduce the problem (that is, getting (ii) of Theorem 1.1) to the particular case when $Y \subseteq \omega_2$ in the extension. (Recall that the forcing $\widehat{\mathbb{P}}$ preserves cardinals, Section 6.)

Now we start a rather long argument in the ground universe $\mathbb{V} = \mathbb{L}[A] = \mathbb{L}[\widehat{\varphi}]$ of Theorem 1.1 related to names for subsets of ω_2. Suppose that $\mathbb{Q} \subseteq \mathrm{Perf}$ is any forcing notion, not necessarily equal to the forcing $\widehat{\mathbb{P}}$ of Definition 5.3.

Definition 10.1 For any ordinal κ let $\mathrm{Name}_{\mathbb{Q}}[\kappa]$ be the set of all sets $\tau \subseteq \mathbb{Q} \times \kappa$. Each $\tau \in \mathrm{Name}_{\mathbb{Q}}[\kappa]$ is a $\mathbb{Q}$-name for a subset of κ.

If $\tau \in \mathrm{Name}_{\mathbb{Q}}[\kappa]$, $\xi < \kappa$ and $T \in \mathbb{Q}$, then say that T "$\mathbb{Q}$-*forces* $\xi \notin \tau$" if and only if T is incompatible in $\mathbb{Q}$ with every condition $S \in \tau$"$\xi = \{S \in \mathbb{Q} : \langle S, \xi\rangle \in \tau\}$, and say that T "$\mathbb{Q}$-*forces* $\xi \in \tau$" if and only if every condition $T' \in \mathbb{Q}$, $T' \subseteq T$, is compatible in $\mathbb{Q}$ with at least one condition $S \in \tau$"ξ.

Put $\tau[G] = \{\xi < \kappa : (\exists\, T \in G)\, (\langle T, \xi\rangle \in \tau)\}$ for each $\mathbb{Q}$-generic set $G \subseteq \mathbb{Q}$ and $\tau \in \mathrm{Name}_{\mathbb{Q}}[\kappa]$. It is known that for any set $Y \in \mathbb{V}[G]$, $Y \subseteq \kappa$, there is $\tau \in \mathrm{Name}_{\mathbb{Q}}[\kappa]$ in $\mathbb{V}$ such that $Y = \tau[G]$.

A *splitting system* in $\mathbb{Q}$ will be any family $\{T_s\}_{s\in 2^{<\omega}}$ of trees $T_s \in \mathbb{Q}$ satisfying the following two conditions:

(i) $T_{s^\wedge 0} \cup T_{s^\wedge 1} \subseteq T_s$, but $T_{s^\wedge 0} \cap T_{s^\wedge 1} = \emptyset$;

(ii) $\operatorname{diam} T_s \leq \dfrac{1}{\operatorname{lh} s}$.

In this case, $S = \bigcap_n \bigcup_{\operatorname{lh} s=n} T_s \in \mathrm{Perf}$, moreover even $S \cap T_s \in \mathrm{Perf}$ for any string $s \in 2^{<\omega}$, but not necessarily $S \in \mathbb{Q}$. If indeed $S \in \mathbb{Q}$, then say that the system $\{T_s\}_{s\in 2^{<\omega}}$ *converges* (to S) in $\mathbb{Q}$. In this case it is still not necessary that $S \cap T_s \in \mathbb{Q}$ for any string $s \in 2^{<\omega}$, but if $\mathbb{Q}$ is CO-dense in the sense of Section 2, then at least there is a tree $P = P_s \in \mathbb{Q}$ such that $[P] \subseteq [S] \cap [T_s]$.

Definition 10.2 Suppose that $\mathbb{Q} \subseteq \mathrm{Perf}$ is CO-dense, $\kappa \in \mathrm{Ord}$, and $n \in \omega$.

Say that a splitting system $\{T_s\}_{s\in 2^{<\omega}}$ of trees $T_s \in \mathbb{Q}$ *breaks* a name $\tau \in \mathrm{Name}_{\mathbb{Q}}[\kappa]$ in $\mathbb{Q}$ at n, if for each $s \in 2^{<\omega}$ with $\mathrm{lh}\, s = n$:

– either τ is $\mathbb{Q}$-*constant on* T_s in $\mathbb{Q}$, that is, for any $\xi < \kappa$: either T_s "$\mathbb{Q}$-forces $\xi \in \tau$", or T_s "$\mathbb{Q}$-forces $\xi \notin \tau$" in the sense of Definition 10.1;

– or the system is τ-*bijective in* $\mathbb{Q}$ *above* s, that is, for any string $u \in 2^{<\omega}$ with $s \subseteq u$ there exists an ordinal $\xi < \kappa$ such that either $T_{u^\wedge 0}$ "$\mathbb{Q}$-forces $\xi \in \tau$" and $T_{u^\wedge 1}$ "$\mathbb{Q}$-forces $\xi \notin \tau$", or the other way around $T_{u^\wedge 0}$ "$\mathbb{Q}$-forces $\xi \notin \tau$" and $T_{u^\wedge 1}$ "$\mathbb{Q}$-forces $\xi \in \tau$".

In this case, if the system $\{T_s\}_{s\in 2^{<\omega}}$ converges to some $S \in \mathbb{Q}$, then S $\mathbb{Q}$-forces that either $\tau \in \mathbb{V}$ (where $\mathbb{V} = \mathbb{L}[A]$ is the ground universe, as above), or $\dot{x} \in \mathbb{V}[\tau]$, where $\dot{x}$ is a canonical name for x_G, the $\mathbb{Q}$-generic real.

Therefore if the forcing $\widehat{\mathbb{P}}$ of Definition 5.3 has the following additional property (E), then $\widehat{\mathbb{P}}$-generic reals are minimal in the sense of (ii) of Theorem 1.1:

(E) If $P \in \widehat{\mathbb{P}}$ and $\tau \in \mathrm{Name}_{\widehat{\mathbb{P}}}[\omega_2]$, then there is a splitting system $\{T_s\}_{s\in 2^{<\omega}}$ in $\widehat{\mathbb{P}}$ with $T_\Lambda \subseteq P$, which converges in $\widehat{\mathbb{P}}$ and breaks τ in $\widehat{\mathbb{P}}$ at some $n \in \omega$.

Thus what we have to do is to modify the construction of Definition 5.1 so that the additional requirement (E) is satisfied. This is based on the following lemma:

Lemma 10.3 *If a set $\mathbb{Q} \subseteq \mathrm{Perf}$ is CO-dense, $P \in \mathbb{Q}$, and, for every $n < \omega$, we have $\kappa_n \in \mathrm{Ord}$ and $\tau_n \in \mathrm{Name}_{\mathbb{Q}}[\kappa_n]$, then there is a splitting system $\{T_s\}_{s\in 2^{<\omega}}$ in $\mathbb{Q}$ with $T_\Lambda \subseteq P$, which breaks each τ_n in $\mathbb{Q}$ at n.*

Note that the convergence of the system in $\mathbb{Q}$ is not required by the lemma.

P r o o f (sketch). Suppose that $n \in \omega$ and all trees $T_s \in \mathbb{Q}$ with $\mathrm{lh}\, s = n$ have been defined. Split each T_s into two trees $T'_{s^\wedge 0} \in \mathbb{Q}$ and $T'_{s^\wedge 1} \in \mathbb{Q}$ arbitrarily.

For any $s^\wedge i$ ($\mathrm{lh}\, s = n$ and $i = 0, 1$), if there is $P \in \mathbb{Q}$ such that $P \subseteq T'_{s^\wedge i}$ and τ_{n+1} is $\mathbb{Q}$-constant on P in $\mathbb{Q}$, then let $T_{s^\wedge i}$ be equal to any such P. Otherwise put $T_{s^\wedge i} = T'_{s^\wedge i}$. In the latter case it is clear that for any pair of trees $P, Q \in \mathbb{Q}$ with $P \cup Q \subseteq T_{s^\wedge i}$ there exist trees $P', Q' \in \mathbb{Q}$ with $P' \subseteq P$ and $Q' \subseteq Q$, such that, for some $\xi < \kappa_{n+1}$, either P' "$\mathbb{Q}$-forces $\xi \in \tau_{n+1}$" and Q' "$\mathbb{Q}$-forces $\xi \notin \tau_{n+1}$", or vice versa P' "$\mathbb{Q}$-forces $\xi \notin \tau_{n+1}$" and Q' "$\mathbb{Q}$-forces $\xi \in \tau_{n+1}$". This allows us to continue the construction of the splitting system at higher levels, so that the system will be τ_{n+1}-bijective in $\mathbb{Q}$ above $s^\wedge i$. □

To make use of Lemma 10.3 in the construction of $\widehat{\mathbb{P}}$, let us come back to step 4^* in Definition 5.1, where $\mathbb{P}_\xi[\varphi]$ is defined, in the case when $\xi \in \Omega_2[\varphi]$, as the $\le^{\mathrm{G}}_{\varphi\restriction\xi}$-least antichain $E \in \mathbb{L}[\varphi \restriction \xi]$, $E \subseteq \mathrm{Perf}$, satisfying certain properties. From this point on in Section 10, we modify the definition of $\mathbb{P}_\xi[\varphi]$ (that is, step 4^* in Definition 5.1) as follows:

Definition 10.4 Let $E \in \mathbb{L}[\varphi\restriction\xi]$, $E \subseteq \mathrm{Perf}\backslash\mathbb{P}_{<\xi}[\varphi]$, still be the $\le^{\mathrm{G}}_{\varphi\restriction\xi}$-least antichain as indicated. Recall that $M_\xi[\varphi] = \mathbb{L}_{\mu_\xi[\varphi]}[\varphi \restriction \xi]$ is a transitive model of ZFC^- countable in $\mathbb{L}[\varphi \restriction \xi]$ by Lemma 4.2. Applying Lemma 10.3 in $\mathbb{L}[\varphi \restriction \xi]$ for $\mathbb{Q} = \mathbb{P}_{<\xi}[\varphi]$ and the collection C of all sets $\tau \in M_\xi[\varphi]$ such that $\tau \in \mathrm{Name}_{\mathbb{P}_{<\xi}[\varphi]}[\kappa]$ for some $\kappa < \mu_\xi[\varphi]$, we find, for every $P \in E$, a splitting system $\sigma_P = \{T^P_s\}_{s\in 2^{<\omega}}$ in $\mathbb{P}_{<\xi}[\varphi]$ with $T^P_\Lambda \subseteq P$ which breaks each $\tau \in C$ in $\mathbb{P}_{<\xi}[\varphi]$ at some $n = n(\tau)$.

Suppose that the collection $\{\sigma_P\}_{P\in E}$ is chosen as the $\le^{\mathrm{G}}_{\varphi\restriction\xi}$-least among all of them of this sort in $\mathbb{L}[\varphi \restriction \xi]$. Put $P' = \bigcap_{n\in\omega} \bigcup_{\mathrm{lh}\, s=m} T^P_s$ for every $P \in E$, and $\mathbb{P}_\xi[\varphi] = E' = \{P' : P \in E\}$. This ends the modified construction of $\mathbb{P}_\xi[\varphi]$.

We summarize the key property of the modified definition as follows:

Lemma 10.5 *If $\varphi \in \Phi_{\omega_1}$, $\xi \in \Omega_2[\varphi]$, $\kappa < \mu_\xi[\varphi]$, $\tau \in M_\xi[\varphi] \cap \mathrm{Name}_{\mathbb{P}_{<\xi}[\varphi]}[\kappa]$, and $P \in \mathbb{P}_{<\xi}[\varphi]$, then, in $\mathbb{L}[\varphi \restriction \xi]$, there is a splitting system $\{T_s\}_{s\in 2^{<\omega}}$ in $\mathbb{P}_{<\xi}[\varphi]$ with $T_\Lambda \subseteq P$ that converges in $T \in \mathbb{P}_\xi[\varphi]$ and breaks τ in $\mathbb{P}_{<\xi}[\varphi]$ at some n.*

Lemma 10.6 *The modified forcing notion $\widehat{\mathbb{P}} = \mathbb{P}[\widehat{\varphi}]$ satisfies* (E)*, therefore all $\widehat{\mathbb{P}}$-generic reals are minimal in the sense of* (ii) *of Theorem* 1.1.

Proof. Let us show, by induction on λ, that if $\lambda \in \Omega_3[\widehat{\varphi}] \cup \{\omega_1\}$, then the forcing $\widehat{\mathbb{P}}_{<\lambda} = \mathbb{P}_{<\lambda}[\widehat{\varphi}]$ satisfies (E) in $\mathbb{L}[\widehat{\varphi} \upharpoonright \lambda]$. Let λ be as indicated; $\lambda = \omega_1^{\mathbb{L}[\widehat{\varphi} \upharpoonright \lambda]}$ by Lemma 4.3. Consider, arguing in $\mathbb{L}[\widehat{\varphi} \upharpoonright \lambda]$, arbitrary $P \in \widehat{\mathbb{P}}_{<\lambda}$ and $\tau \in \mathrm{Name}_{\widehat{\mathbb{P}}_{<\lambda}}[\kappa]$, where $\kappa = \omega_2^{\mathbb{L}[\widehat{\varphi} \upharpoonright \lambda]}$. This is true then in $\mathbb{L}_\vartheta[\widehat{\varphi} \upharpoonright \lambda]$, where $\vartheta = \omega_3^{\mathbb{L}[\widehat{\varphi} \upharpoonright \lambda]}$. Take a countable elementary submodel $N \subseteq \mathbb{L}_\vartheta[\widehat{\varphi} \upharpoonright \lambda]$, containing $\widehat{\varphi} \upharpoonright \lambda$. P, $\widehat{\mathbb{P}}_{<\lambda}$, τ, and let $h : N \xrightarrow{\text{onto}} N'$ be the Mostowski collapse onto a transitive set $N' = \mathbb{L}_{\vartheta'}[\widehat{\varphi} \upharpoonright \lambda']$, where $\lambda' < \vartheta' < \lambda = \omega_1^{\mathbb{L}[\widehat{\varphi} \upharpoonright \lambda]}$. Standard arguments show that $h(\widehat{\varphi} \upharpoonright \lambda) = \widehat{\varphi} \upharpoonright \lambda'$, $h(P) = P \in \widehat{\mathbb{P}}_{<\lambda'}$ and $\tau' = h(\tau) \in N' \cap \mathrm{Name}_{\widehat{\mathbb{P}}_{<\lambda'}}[\kappa']$, where $\kappa' = h(\kappa)$, and also $\lambda' < \kappa' < \vartheta'$. Finally $\lambda' = \omega_1^{N'}$ and $\kappa' = \omega_2^{N'}$.

Case 1. $\lambda' \in \Omega_3[\widehat{\varphi}]$. Then $\lambda' = \omega_1^{\mathbb{L}[\widehat{\varphi} \upharpoonright \lambda']}$ by Lemma 4.3. By the inductive hypothesis, the forcing $\widehat{\mathbb{P}}_{<\lambda'}$ satisfies (E) in $\mathbb{L}[\widehat{\varphi} \upharpoonright \lambda']$, and hence there is a splitting system $\{T_s\}_{s \in 2^{<\omega}} \in \mathbb{L}[\widehat{\varphi} \upharpoonright \lambda']$ in $\widehat{\mathbb{P}}_{<\lambda'}$ with $T_\Lambda \subseteq P$, which converges in $\widehat{\mathbb{P}}_{<\lambda'}$ and breaks τ' in $\widehat{\mathbb{P}}_{<\lambda'}$ at some n. Then, as h^{-1} is an elementary embedding, the same splitting system breaks τ in $\widehat{\mathbb{P}}_{<\lambda}$ at n. That is, $\widehat{\mathbb{P}}_{<\lambda}$ satisfies (E), as required.

Case 2. $\lambda' \in \Omega_2[\widehat{\varphi}]$. In this case $\lambda' < \omega_1^{\mathbb{L}[\widehat{\varphi} \upharpoonright \lambda']}$. Recall that by definition λ' is countable in the model $M_{\lambda'}[\widehat{\varphi}] = \mathbb{L}_{\mu_{\lambda'}[\widehat{\varphi}]}[\widehat{\varphi} \upharpoonright \lambda']$. On the other hand, still $\lambda' = \omega_1^{N'}$, where $N' = \mathbb{L}_{\vartheta'}[\widehat{\varphi} \upharpoonright \lambda']$. It follows that $\vartheta' < \mu_{\lambda'}[\widehat{\varphi}]$, $N' \in M_{\lambda'}[\widehat{\varphi}]$ and $\tau' \in M_{\lambda'}[\widehat{\varphi}]$. By Lemma 10.5, there is, in $\mathbb{L}[\varphi \upharpoonright \lambda']$, a splitting system $\{T_s\}_{s \in 2^{<\omega}}$ in $\mathbb{P}_{<\lambda'}[\widehat{\varphi}]$ with $T_\Lambda \subseteq P$ which converges to some $T \in \mathbb{P}_{\lambda'}[\widehat{\varphi}]$ and breaks τ' in $\mathbb{P}_{<\lambda'}[\widehat{\varphi}]$ at some n. Then, as h^{-1} is an elementary embedding, the same splitting system breaks τ in $\widehat{\mathbb{P}}_{<\lambda}$, and obviously $T \in \widehat{\mathbb{P}}_{<\lambda}$. □

On the other hand, the modification as in Definition 10.4 does not affect the cardinal preservation result in Section 6. Moreover, after a related modification in the decoding procedure of Section 7, Lemmas 8.1 and 9.1 remain correct.

11 Getting a reshaping club

Here we sketch the proof of (iii) of Theorem 1.1. Let $C(x) = \{\boldsymbol{\lambda}_\alpha(x) : \alpha < \omega_1 \text{ and } \boldsymbol{\lambda}_\alpha(x) \text{ is defined}\}$, coming back to Section 7. Recall that the sequence of ordinals $\boldsymbol{\lambda}_\alpha(x)$ is strictly increasing and continuous at limit steps. On the other hand, it follows from Corollary 8.2 that if x is a $\widehat{\mathbb{P}}$-generic real, then $\boldsymbol{\lambda}_\alpha(x)$ is defined for all $\alpha < \omega_1$, and hence $C(x)$ is a club in $\mathbb{L}[x] = \mathbb{L}[A][x]$.

On the other hand, at least all limit-position elements of $C(x)$ (those $\boldsymbol{\lambda}_\alpha(x)$ with limit indices α) by definition belong to $\Omega_2[\widehat{\varphi}]$ by (†) in Section 7. In other words, $C'(x) \subseteq \Omega_2[\widehat{\varphi}]$, where $C'(x)$ is the club of all limit-position elements of $C(x)$ (x continues to be a $\widehat{\mathbb{P}}$-generic real). It remains to note this: the construction of $\widehat{\varphi}$ can be amended so that in addition to everything above, we have $A \cap \gamma \in \mathbb{L}[\widehat{\varphi} \upharpoonright \gamma]$ for all ordinals γ. And then we get (iii) of Theorem 1.1.

12 Which forcing notions don't do it

The forcing we apply to prove Theorem 1.1 was obtained as the result of a complicated inductive construction of length ω_1. Therefore one can ask whether a more naturally defined forcing notion can code a given subset of ω_1 by a generic real as in Theorem 1.1 even in the absence of the minimality claim.

We begin with a simple theorem which shows that the Sacks forcing, generally speaking, does not produce the extensions as in Theorem 1.1.

Theorem 12.1 *Suppose that ϑ is an inaccessible cardinal in $\mathbb{L}$, and $A \subseteq \vartheta$ codes over $\mathbb{L}$ the Levy collapse of all uncountable cardinals $< \vartheta$. Let x be a Sacks-generic real over $\mathbb{L}[A]$. Then $\omega_1^{\mathbb{L}[x]} < \vartheta$, and therefore $A \notin \mathbb{L}[x]$.*

Proof. Suppose that a tree $T \in \mathrm{Perf} \cap \mathbb{L}[A]$ Sacks-forces the opposite. Then T is essentially a real in $\mathbb{L}[A]$, hence $\omega_1^{\mathbb{L}[T]} < \omega_1 = \vartheta$ in $\mathbb{L}[A]$, and moreover $\omega_2^{\mathbb{L}[T]} < \vartheta$ as well. Therefore the set of all sets $D \subseteq \mathrm{Perf} \cap \mathbb{L}[T]$, $D \in \mathbb{L}[T]$, dense in $\mathrm{Perf} \cap \mathbb{L}[T]$, is countable in $\mathbb{L}[A]$. This allows to define, by means of a known splitting construction (see for example the proof of Lemma 2.2 above), a tree $S \in \mathrm{Perf} \cap \mathbb{L}[A]$, $S \subseteq T$, such that any real $x \in [S]$ is $\mathrm{Perf} \cap \mathbb{L}[T]$-generic, that is, Sacks-generic, over $\mathbb{L}[T]$. Therefore $\omega_1^{\mathbb{L}[T,x]} = \omega_1^{\mathbb{L}[T]} < \vartheta = \omega_1^{\mathbb{L}[A]}$ for any $x \in [S] \subseteq [T]$, which contradicts the choice of T. □

The next theorem contains a more general outlook of the negative side. For the definitions of the notions involved in the statement of the theorem, please refer to the quoted papers in the theorem's proof.

Theorem 12.2 *Suppose that ϑ and A are as in Theorem* 12.1. *Let $\mathbb{P}$ be a forcing notion in $\mathbb{L}[A]$. Suppose also that at least one of the following six assumptions holds in $\mathbb{L}[A]$:*

(1) *ϑ is a remarkable cardinal in $\mathbb{L}$, there are no inaccessible cardinals above ϑ, and $\mathbb{P}$ is semiproper;*

(2) *ϑ is a remarkable cardinal in $\mathbb{L}$, and $\mathbb{P}$ is proper;*

(3) *ϑ is a weakly-compact cardinal in $\mathbb{L}$, and $\mathbb{P}$ is ccc;*

(4) *ϑ is a Mahlo cardinal in $\mathbb{L}$, and $\mathbb{P}$ is σ-linked;*

(5) *$\mathbb{P}$ is strongly-$\boldsymbol{\Sigma}^1_3$ and absolutely-ccc;*

(6) *$\mathbb{P}$ is $\boldsymbol{\Sigma}^1_3$ and $\boldsymbol{\Pi}^1_2$-strongly proper.*

Let x be a real that belongs to a $\mathbb{P}$-generic extension of $\mathbb{L}[A]$. Then $\omega_1^{\mathbb{L}[x]} < \vartheta = \omega_1^{\mathbb{L}[A]}$, and therefore $A \notin \mathbb{L}[x]$.

The class of forcing notions satisfying (6) includes a variety of *arboreal* forcing notions such as Sacks forcing, Miller forcing, Mathias forcing, Laver forcing, etc. (see [3]), so, accordingly, this theorem includes the previous one.

P r o o f. The theorem follows from several results which show, under each respective assumption, from (1) to (6), that $\mathbb{L}[A]$ has the property that every real x that belongs to a forcing extension of $\mathbb{L}[A]$ by $\mathbb{P}$ is *small-generic* over $\mathbb{L}$. This means that that there is a forcing notion $Q \in \mathbb{L}$ of cardinality strictly less than ϑ such that x belongs to a Q-generic extension of $\mathbb{L}$. Hence $\omega_1^{\mathbb{L}[x]} < \vartheta$.

For (1) and (2), the theorem follows from results of Schindler [15] and [16], respectively.

For (3) this is a consequence of a result of Kunen (see [10]).

In the case of (4), the theorem follows from Bagaria and Bosch [2]. And for $\mathbb{P}$ satisfying (5) or (6), it follows from the results of Bagaria and Bosch [1], and Bagaria and Di Prisco [3], respectively. □

The results just quoted show, in fact, that the theory of $\mathbb{L}(\mathbb{R})$ is absolute under forcing with $\mathbb{P}$. Thus we have the following, somewhat more general, result:

Theorem 12.3 *Suppose ϑ is an inaccessible cardinal in $\mathbb{L}$, and G is $\mathrm{Coll}(\omega;<\vartheta)$-generic over $\mathbb{L}$. In $\mathbb{L}[G]$, let Γ be the class of forcing notions for which the following holds: For every $\mathbb{P} \in \Gamma$ and every set H $\mathbb{P}$-generic over $\mathbb{L}[G]$, $\mathbb{L}[G] \equiv_{\Sigma^1_4} \mathbb{L}[G][H]$, that is, lightface-$\Sigma^1_4$-sentences are absolute between $\mathbb{L}[G]$ and $\mathbb{L}[G][H]$. Then no $\mathbb{P} \in \Gamma$ can force $\mathbb{L}[G][H] \subseteq \mathbb{L}[x]$, for x a real.*

P r o o f. Suppose $\mathbb{P} \in \Gamma$ forces $\mathbb{L}[G][H] = \mathbb{L}[x]$, where x is a real. Clearly $\mathbb{L}[x]$ satisfies the sentence

$$(\exists\, a \in \omega^\omega)(\forall\, b \in \omega^\omega)(b \in \mathbb{L}[a]),$$

which is easily seen to be Σ^1_4. Hence, $\mathbb{L}[G]$ must also satisfy it. But this is clearly not the case. □

13 The "non-Mahlo" case and ccc coding

Coming back to the definition of $\widehat{\mathbb{P}}$, one may note that while the set of ordinals $\Omega_2[\widehat{\varphi}]$ is unbounded in ω_1 for any $\widehat{\varphi} \in (2^\omega)^{\omega_1}$, the nature of the set $\Omega_3[\widehat{\varphi}]$ is somewhat less clear, and it can be even empty provided the true ω_1 is not a Mahlo cardinal in $\mathbb{L}$! This leads to the following ccc version of our main theorem:

Theorem 13.1 *Suppose that $A \subseteq \omega_1$, $\mathbb{V} = \mathbb{L}[A]$, and we have:*

(†) *there is a club $C \subseteq \omega_1$ such that $\alpha < \omega_1^{\mathbb{L}[A\cap\alpha]}$ for all $\alpha \in C$.*

Then we can strengthen Theorem 1.1 *by the claim: the forcing notion is ccc.*

P r o o f. If $\xi < \omega_1$, then let α_ξ be the ξ-th ordinal in C. And by (†) let x_ξ be the Gödel-least real in $\mathbb{L}[A \cap \alpha_\xi]$ which effectively codes the set $A \cap \alpha_\xi$. The sequence $\sigma = \{x_\xi\}_{\xi<\omega_1}$ obviously satisfies $\xi \le \alpha_\xi < \omega_1^{\mathbb{L}[\sigma\restriction\xi]}$ for all limit $\xi < \omega_1$. In other words, $\Omega_3[\sigma] = \emptyset$. And still $\mathbb{L}[\sigma] = \mathbb{L}[A]$. Now it does not take much effort to transform σ into a sequence $\widehat{\varphi} \in \Phi_{\omega_1}$ as in Definition 4.4, and with the additional property that $\xi < \omega_1^{\mathbb{L}[\widehat{\varphi}\restriction\xi]}$ for all $\xi < \omega_1$, that is, $\Omega_3[\widehat{\varphi}] = \emptyset$. Then all limit ordinals $\xi < \omega_1$ belong to $\Omega_2[\widehat{\varphi}]$.

Let us use this intermediate result to prove that the forcing $\widehat{\mathbb{P}} = \mathbb{P}[\widehat{\varphi}]$ is ccc.

Let $E \subseteq \widehat{\mathbb{P}}$ be a maximal antichain. Clearly $E \in \mathbb{L}_{\omega_2}[A] = \mathbb{L}_{\omega_2}[\widehat{\varphi}]$. By a condensation argument, there are limit ordinals $\xi < \lambda < \omega_1$ such that the set $E' = E \cap \widehat{\mathbb{P}}_{<\xi}$ is a maximal antichain in $\widehat{\mathbb{P}}_{<\xi}$, $E' \in \mathbb{L}_\lambda[\widehat{\varphi} \restriction \xi]$, and ξ is uncountable in $\mathbb{L}_\lambda[\widehat{\varphi} \restriction \xi]$, so that $\lambda < \mu_\xi[\widehat{\varphi}]$ and accordingly $E' \in M_\xi[\widehat{\varphi}] = \mathbb{L}_{\mu_\xi[\widehat{\varphi}]}[\widehat{\varphi} \restriction \xi]$. However $\xi \in \Omega_2[\widehat{\varphi}]$ (see above). It follows by definition that any condition $P \in \widehat{\mathbb{P}}_\xi$ satisfies $P \subseteq^{\text{fin}} \bigcup E'$ by (D) of Section 5, thus $P \subseteq^{\text{fin}} \bigcup E$ as well.

It remains to show that if $\xi < \lambda < \omega_1$ and $T \in \widehat{\mathbb{P}}_\lambda$, then $T \subseteq^{\text{fin}} \bigcup \widehat{\mathbb{P}}_\xi$; then $T \subseteq^{\text{fin}} \bigcup E$ by the above, and $E = E'$ is countable.

We prove this last claim by induction on λ. The step $\lambda \to \lambda + 1$ is trivial since by definition for any $T \in \widehat{\mathbb{P}}_{\lambda+1}$ there exists a tree $S \in \widehat{\mathbb{P}}_\lambda$ with $T \subseteq S$. Thus suppose that $\lambda > \xi$ is a limit ordinal, and the result holds for any λ' satisfying $\xi < \lambda' < \lambda$. Once again, $\lambda \in \Omega_2[\widehat{\varphi}]$. Therefore (see Case 2 in the proof of Lemma 8.1) any $T \in \widehat{\mathbb{P}}_\lambda$ satisfies $T \subseteq^{\text{fin}} \bigcup \widehat{\mathbb{P}}_{\vartheta,\lambda}$ for all ϑ, $\xi \le \vartheta < \lambda$, and hence it satisfies $T \subseteq^{\text{fin}} \bigcup \widehat{\mathbb{P}}_\xi$ by the inductive hypothesis, as required. □

Let us remark that if $A \subseteq \omega_1$, $V = \mathbb{L}[A]$ and (†) holds for A, then A can be coded by a real using the Jensen-Solovay almost-disjoint forcing, which is σ-centered, hence ccc. (See, e.g., [4, Section 2] for details.)

The status of (†) with respect to other large cardinal hypotheses is not fully known. To avoid trivialities, note that if there is a real x such that $\omega_1^{\mathbb{L}[x]} = \omega_1$, then (†) holds for *every* $A \subseteq \omega_1$ with $x \in \mathbb{L}[A]$. Thus it can be assumed that the true ω_1 is inaccessible in $\mathbb{L}[x]$ for any real x, in brief "inaccessible to reals". Then one may consider the following question:

Suppose that κ is an inaccessible cardinal in $\mathbb{L}$. Let $\mathbb{L}[G]$ be a Levy-collapse extension of $\mathbb{L}$ (then $\kappa = \omega_1^{\mathbb{L}[G]}$, and in $\mathbb{L}[G]$ it is true that there is a set $A \subseteq \omega_1$ such that $\mathbb{V} = \mathbb{L}[A]$). Is it true then in $\mathbb{L}[G]$ that there is a set $A \subseteq \omega_1$ such that $\mathbb{V} = \mathbb{L}[A]$ and (†)?

The following simple lemma shows that a sufficient condition for the positive answer is that κ is not too large.

Lemma 13.2 *If the true ω_1 is not a Mahlo cardinal in $\mathbb{L}$, then* (†) *holds for every set $A \subseteq \omega_1$.*

P r o o f. The non-Mahlo assumption means that there is a club $C_0 \in \mathbb{L}$, $C_0 \subseteq \omega_1$, which consists of $\mathbb{L}$-singular limit ordinals, that is, limit ordinals which are not regular cardinals in $\mathbb{L}$. Now, given $A \subseteq \omega_1$, let C_A be the club subset of ω_1 consisting of all α such that $\alpha \le \omega_1^{L[A\cap\alpha]}$. Then, letting $C = C_0 \cap C_A$, we have that every $\alpha \in C$ remains singular, and therefore countable, in $\mathbb{L}[A \cap \alpha]$. Hence (†) holds. □

It is natural to ask if the converse also holds, that is, suppose that $A \subseteq \omega_1$, $\mathbb{V} = \mathbb{L}[A]$, and (†) holds for A. Is then κ necessarily non-Mahlo in $\mathbb{L}$? The answer is no. For suppose $A \subseteq \omega_1$ codes the Levy-collapse of a Mahlo cardinal over $\mathbb{L}$. Then, as the next theorem shows, one can force over $\mathbb{L}[A]$ while preserving ω_1 and add a set $X \subseteq \omega_1$ such that $\xi < \omega_1^{\mathbb{L}[A\cap\xi, X\cap\xi]}$ for all $\xi < \omega_1$. Thus, if A' codes A and X so that for every limit ξ, $A' \cap \xi$ codes $A \cap \xi$ and $X \cap \xi$, then we have that, in $\mathbb{L}[A']$, (†) holds for A', and ω_1 is Mahlo in $\mathbb{L}$.

The same argument shows that κ may have any other large-cardinal property compatible with $V = \mathbb{L}$. Notice however that, as a consequence of Theorems 13.1 and 15.4, if $\mathbb{L}[A]$, with $A \subseteq \omega_1$, is the result of collapsing a weakly-compact cardinal over $\mathbb{L}$, then (†) cannot hold for A in $\mathbb{L}[A]$.

Suppose now that $V = \mathbb{L}[A]$ for some $A \subseteq \omega_1$, and ω_1 is a Mahlo cardinal in $\mathbb{L}$. If (†) holds for A, then by Theorem 13.1 we can code A by a real satisfying (i) – (iii) of Theorem 1.1 by means of a ccc forcing. However, if (†) does not hold for A, then (†) cannot hold either in any forcing extension that preserves stationary subsets of ω_1. Thus we have the following corollary:

Corollary 13.3 *Assume $V = \mathbb{L}[A]$ for some $A \subseteq \omega_1$. Then the following are equivalent:*

1. (†) *holds for A.*
2. *A can be coded by a real satisfying Theorem* 1.1 (i) – (iii) *by means of a ccc* (*stationary preserving*) *forcing.*

14 Reshaping

For the sake of completeness, we present here a proof of the reshaping theorem, which follows the same inductive argument as in several proofs above. The result, originally due to Jensen, is given in two slightly different versions, (i) and (ii) of the following theorem.

Theorem 14.1 *Suppose that $A \subseteq \omega_1$ and $\mathbb{V} = \mathbb{L}[A]$. For each of the following two (incompatible) conditions, there is a cardinal preserving generic extension $\mathbb{L}[A][X]$ of $\mathbb{L}[A]$ by a generic set $X \subseteq \omega_1$ which does not add new reals to $\mathbb{L}[A]$.*

(i) (strong reshaping) *X is a club and $\xi < \omega_1^{\mathbb{L}[A\cap\xi]}$ for any limit $\xi \in X$;*

(ii) (weak reshaping) *$\xi < \omega_1^{\mathbb{L}[A\cap\xi, X\cap\xi]}$ for any $\xi < \omega_1$, and X preserves stationarity of sets $S \in \mathbb{L}[A]$, $S \subseteq \omega_1$.*

Proof.

(i) Let P (the forcing) be the set of all closed, at most countable sets $p \subseteq \omega_1$ such that $\xi < \omega_1^{\mathbb{L}[A\cap\xi]}$ for any limit ordinal $\xi \in p$. The order $p \le q$ (p is stronger than q) if and only if $q \subseteq p$ and $\max q < \min(p \setminus q)$. We have to prove that P does not add new reals. Note by the way that P is not necessarily ω-closed!

Let τ be a P-name for a real in 2^ω. Thus $\tau \subseteq P \times \omega \times \{0,1\}$, and $\langle p,n,i\rangle \in \tau$ if and only if p forces $\tau(n) = i$. If $\gamma \le \omega_1$, then put

$$P \restriction \gamma = P \cap \mathbb{L}_\gamma[A\cap\gamma] \quad \text{and} \quad \tau \restriction \gamma = \{\langle p,n,i\rangle \in \tau : p \in P \restriction \gamma\};$$

thus $P \restriction \omega_1 = P$ and $\tau \restriction \omega_1 = \tau$. Let $\Omega_3(A) = \{\xi < \omega_1 : \omega_1^{\mathbb{L}[A\cap\xi]} = \xi\}$. We claim that *for any ordinal $\gamma \in \Omega_3(A) \cup \{\omega_1\}$, if $\tau \restriction \gamma$ belongs to $\mathbb{L}[A\cap\gamma]$, then $P \restriction \gamma$ decides $\tau \restriction \gamma$ over $\mathbb{L}[A\cap\gamma]$*. Taking $\gamma = \omega_1$, we get the result required.

The proof of the claim uses the same type of induction as in the proofs of Lemmas 6.1 and 10.6. Thus, one proves the claim for $\gamma = \omega_1$, assuming it holds for every ordinal $\gamma \in \Omega_3(A)$. Let $p^* \in P$. The sets A, P, τ, p^* belong to $\mathbb{L}_{\omega_2}[A]$. Let $M \prec \mathbb{L}_{\omega_2}[A]$ be a countable elementary submodel containing these sets. Let $f : M \xrightarrow{\text{onto}} N$ be an $\in$-isomorphism onto a transitive set N. Then $\gamma = f(\omega_1)$ is a countable limit ordinal, and $N = \mathbb{L}_\delta[A\cap\gamma]$, where δ is the least ordinal not in N. Moreover, $f(\xi) = \xi$ for any $\xi \in M \cap \omega_1$, $f(p^*) = p^* \in N$, $f(A) = A \cap \gamma$, $f(P) = P \restriction \gamma$, and $f(\tau) = \tau \restriction \gamma \in \mathbb{L}_\delta[A\cap\gamma] = N$.

Case 1. $\gamma < \omega_1^{\mathbb{L}[A\cap\gamma]}$. Let $\{\xi_0 < \xi_1 < \xi_2 < \cdots\} \in \mathbb{L}[A\cap\gamma]$ be an arbitrary increasing cofinal sequence in γ. Note that for any $p \in P \restriction \gamma$ and n there is a condition $q \in P \restriction \gamma$, $q \le p$, which decides $\tau(n)$ (i.e., one of the triples $\langle q,n,0\rangle$, $\langle q,n,1\rangle$ belongs to τ) and has $\sup q > \xi_n$. Arguing in $\mathbb{L}[A\cap\gamma]$ (where γ is countable by the Case 1 assumption!), this allows us to define, by induction, a decreasing sequence $p^* \ge p_0 \ge p_1 \ge p_2 \ge \cdots$ of conditions $p_n \in P \restriction \gamma$ such that each p_n decides $\tau(n)$ and satisfies $\sup p_n > \xi_n$. Then $q = \bigcup_n p_n \cup \{\gamma\}$ is a condition in P which decides all values $\tau(n)$.

Case 2. $\gamma \in \Omega_3(A)$. Then by the inductive hypothesis there is a condition $q \in P \restriction \gamma$ which decides all values $\tau(n)$ and satisfies $q \le p^*$.

(ii) In this case the forcing notion $P \subseteq 2^{<\omega_1}$ consists of all maps $p : \operatorname{dom} p \to \{0,1\}$ defined on an ordinal $\delta = \operatorname{dom} p < \omega_1$ and satisfying $\gamma < \omega_1^{\mathbb{L}[A\cap\gamma, p\restriction\gamma]}$ for all $\gamma \le \delta$, ordered by inclusion. We prove first that P does not add new reals. Once again, let $\tau \subseteq P \times \omega \times \{0,1\}$ be a P-name for a real. Define $P \restriction \gamma$, $\tau \restriction \gamma$, $\Omega_3(A)$ as above. It suffices to prove that *for any ordinal $\gamma \in \Omega_3(A) \cup \{\omega_1\}$, if $\tau \restriction \gamma$ belongs to $\mathbb{L}[A\cap\gamma]$, then $P \restriction \gamma$ decides $\tau \restriction \gamma$ over $\mathbb{L}[A\cap\gamma]$*.

The proof goes on by induction on γ, as above. Thus, we prove the claim for $\gamma = \omega_1$, assuming it holds for every ordinal $\gamma \in \Omega_3(A)$. Let $p^* \in P$. The sets A, P, τ, p^* belong to $\mathbb{L}_{\omega_2}[A]$. Let $M \prec \mathbb{L}_{\omega_2}[A]$ be a countable elementary submodel containing these sets. Define f, $\gamma = f(\omega_1)$, δ, and $N = \mathbb{L}_\delta[A\cap\gamma]$ as in the proof of (i).

Case 1. $\gamma < \omega_1^{\mathbb{L}[A\cap\gamma]}$. Let $\{\xi_0 < \xi_1 < \xi_2 < \cdots\} \in \mathbb{L}[A\cap\gamma]$ be any increasing cofinal sequence in γ. For any $p \in P \restriction \gamma$ and n there is a condition $q \in P \restriction \gamma$, $q \le p$, which decides $\tau(n)$ and has $\sup q > \xi_n$. Arguing in $\mathbb{L}[A\cap\gamma]$, define a decreasing sequence $p^* \ge p_0 \ge p_1 \ge p_2 \ge \cdots$ of conditions $p_n \in P \restriction \gamma$ such that each p_n decides $\tau(n)$ and satisfies $\sup p_n > \xi_n$. Then $q = \bigcup_n p_n$ is a condition in P which decides all values $\tau(n)$.

Case 2. $\gamma \in \Omega_3(A)$. Then by the inductive hypothesis there is a condition $q \in P \restriction \gamma$, $q \le p^*$, which decides all values $\tau(n)$ and satisfies $q \le p^*$.

In continuation of the proof of (ii), let $S \subseteq \omega_1$ be a stationary set in $\mathbb{L}[A]$. We shall see that S remains stationary in the extension. Suppose that $C \subseteq P \times \omega_1$ is a P-name for a closed unbounded subset of ω_1. Let $p^* \in P$. Consider the set $\mathscr{M}$ of all countable elementary submodels $M \prec \mathbb{L}_{\omega_2}[A]$ containing A, P, C, p^*. For any $M \in \mathscr{M}$, let $\gamma_M = M \cap \omega_1$; then $\gamma_M = f_M(\omega_1)$, where f_M is the collapse function from M onto a transitive set. Once again it suffices to prove that *for any ordinal $\gamma \in (\Omega_3(A) \cap \mathscr{M}) \cup \{\omega_1\}$, $P \restriction \gamma$ forces "$C \restriction \gamma \cap (S\cap\gamma) \ne \emptyset$ over $\mathbb{L}[A\cap\gamma]$"*.

The proof of this claim goes on by the same type of induction on γ. We prove the claim for $\gamma = \omega_1$, assuming it holds for every ordinal $\gamma \in \Omega_3(A) \cap \mathscr{M}$. Quite obviously, $F = \{\gamma_M : M \in \mathscr{M}\}$ is a club in ω_1, and hence there is an ordinal $\gamma = \gamma_M \in F \cap S$ for some $M \in \mathscr{M}$, such that $p^* \in P \restriction \gamma$.

Case 1. $\gamma < \omega_1^{\mathbb{L}[A \cap \gamma]}$. Let $\{\xi_0 < \xi_1 < \xi_2 < \cdots\} \in \mathbb{L}[A \cap \gamma]$ be an arbitrary increasing cofinal sequence in γ. For $p \in P \restriction \gamma$ and $\alpha < \gamma$ there exist a condition $q \in P \restriction \gamma$, $q \le p$, and an ordinal β, $\alpha < \beta < \gamma$, such that q forces $\beta \in C$ and $\sup q > \xi_n$. Arguing in $\mathbb{L}[A \cap \gamma]$, define a decreasing sequence $p^* \ge p_0 \ge p_1 \ge p_2 \ge \cdots$ of conditions $p_n \in P \restriction \gamma$ and an increasing sequence of ordinals β_n such that each p_n forces $\beta_n \in C$, $\sup p_n > \xi_n$, and $\xi_n < \beta_n < \gamma$. Then $q = \bigcup_n p_n$ is a condition in P which forces $\beta_n \in C$ for all n, and hence forces $\gamma = \sup_n \beta_n \in C$. On the other hand, $\gamma \in S$.

Case 2. $\gamma \in \Omega_3(A)$. Then by the inductive hypothesis there exist a condition $q \in P \restriction \gamma$, $q \le p^*$, and an ordinal $\vartheta \in F \cap S \cap \gamma$ such that q $P \restriction \gamma$-forces $\vartheta \in C$. Then obviously q P-forces $\vartheta \in C$, as required. □

15 When the coding forcing is proper

Here we prove the following version of Theorem 1.1. For the definition and basic properties of remarkable cardinals see [15] or [16].

Theorem 15.1 *Suppose that $A \subseteq \omega_1$, $\mathbb{V} = \mathbb{L}[A]$, and the true ω_1 is not a remarkable cardinal in $\mathbb{L}$. Then in Theorem* 1.1 *we may require the forcing to be proper, with the minimality condition* (ii) *only in the weak sense, that is, for sets $Y \subseteq \omega$ (that is, reals) only. We cannot guarantee that* (iii) *of Theorem* 1.1 *holds, for since the forcing preserves stationary subsets of ω_1, a club C as in* (iii) *exists in $\mathbb{L}[x]$ if and only if it exists in $\mathbb{L}[A]$.*

Proof (sketch). The ideas of the proof are from Schindler [15, 16]. By collapsing some ordinal to ω_1 by σ-closed forcing, if necessary, we may assume that the non-remarkability of ω_1 in $\mathbb{L}$ is witnessed by some $\vartheta < \omega_2$.

Further, since $0^\sharp$ does not exist – as otherwise every Silver indiscernible is remarkable in $\mathbb{L}$ – we can collapse a singular strong limit δ of uncountable cofinality to ω_1 by means of a σ-closed forcing, and then again with σ-closed forcing we can produce a subset W of ω_1 such that $\mathbb{L}_{\omega_2}[W] = H(\omega_2)$.

In this forcing extension the set of all $X \in [\mathbb{L}_{\omega_2}[W]]^\omega$ such that

$$(\mathbb{L}_\beta[W \cap \alpha], \in, W \cap \alpha) \cong (X, \in W \cap X) \preccurlyeq (\mathbb{L}_{\omega_2}[W], \in, W),$$

where β is not a cardinal in $\mathbb{L}[W \cap \alpha]$, contains a club C.

Let $\mathbb{P}$ be the weak reshaping forcing, as above. The following lemma is proved in [16].

Lemma 15.2 *$\mathbb{P}$ is proper.*

Proof. Note that since $\mathbb{P} \in \mathbb{L}_{\omega_2}[W] = H(\omega_2)$, it will be enough to show that for every $X \in C$ and every $p \in X \cap \mathbb{P}$, there is a condition $q \le p$ such that for every $\mathbb{P}$-name $\dot\alpha$ for an ordinal, if $\dot\alpha \in X$, then $q \Vdash \dot\alpha \in X$.

So fix X and let $\{\dot\alpha_i : i < \omega\}$ be an enumeration of all $\mathbb{P}$-names for ordinals that belong to X. We have an isomorphism $\pi : (\mathbb{L}_\beta[W \cap \alpha], \in, W \cap \alpha) \cong (X, \in W \cap X)$, given by the transitive collapse of X, where β is not a cardinal in $\mathbb{L}[W \cap \alpha]$. Fix a condition $p \in \mathbb{P}$ in X. Note that $p \in \mathbb{L}_\beta[W \cap \alpha]$. We will define a sequence $\{p_i\}_{i<\omega}$ of conditions, all in $\mathbb{L}_\beta[W \cap \alpha]$, such that $p_0 = p$, $p_{i+1} \le p_i$, and $p_i \Vdash \dot\alpha_i \in X$. We will then see that $q := \bigcup_{i<\omega} p_i$ is a condition, thus completing the proof.

Suppose first that $\alpha = \omega_1^{\mathbb{L}_\beta[W \cap \alpha]}$. Note that in $\mathbb{L}[W \cap \alpha]$, β has cardinality α, because β is not a cardinal in $\mathbb{L}[W \cap \alpha]$. Hence there are α-many club subsets of α in $\mathbb{L}_\beta[W \cap \alpha]$. Let E be a diagonal intersection of all these clubs. Let us fix a sequence $\{\bar\alpha_i\}_{i<\omega}$ cofinal in α. Now suppose p_i is already given, and $p_i \in \mathbb{L}_\beta[W \cap \alpha]$. Thus $\mathrm{dom}(p_i) < \alpha$. Working inside $\mathbb{L}_\beta[W \cap \alpha]$, for each $\mathrm{dom}(p_i) \le \delta < \alpha$ we choose a $p^\delta \le p_i$ such that (i) $p^\delta \Vdash \pi^{-1}(\dot\alpha_i) \in \mathbb{L}_\beta[W \cap \alpha]$, (ii) $\mathrm{dom}(p^\delta) > \max\{\bar\alpha_i, \delta\}$, (iii) $p^\delta(\lambda) = 0$ for all limit ordinals λ with $\mathrm{dom}(p_i) \le \lambda < \delta$, and (iv) $p^\delta(\delta) = 1$. Let $D \in \mathbb{L}_\beta[W \cap \alpha]$ be a club subset of α such that for every $\eta \in D$, if $\delta < \eta$, then $\mathrm{dom}\,(p^\delta) < \eta$.

Now back in $\mathbb{L}[W \cap \alpha]$, since $D \in \mathbb{L}_\beta[W \cap \alpha]$ there is some $\delta \in E$ such that $E \setminus D \subseteq \delta$. Set $p_{i+1} = p^\delta$, and let for future reference $\delta_{i+1} = \delta$. Then we have that $p_{i+1} \Vdash \dot\alpha_i \in X$. Moreover, since every ordinal in E greater than δ belongs to D, we have $\mathrm{dom}(p_{i+1}) < \mathtt{min}\{\gamma \in E : \gamma > \delta\}$. So, for all limit ordinals λ in $E \cap (\mathrm{dom}(p_{i+1}) \setminus \mathrm{dom}(p_i))$ we have that $p_{i+1}(\lambda) = 1$ if and only if $\lambda = \delta_{i+1}$, since all such λ are $\le \delta$.

Now let $q := \bigcup_{i<\omega} p_i$. We will show that q is a condition. The construction of the p_i clearly gives that $\mathrm{dom}(q) = \alpha$. So it only remains to show that $\mathbb{L}[W \cap \alpha, q] \models$ "α is countable". By the construction of the p_i's, we have that the set of all limit $\lambda \in E \cap (\mathrm{dom}(q) \setminus \mathrm{dom}(p))$ such that $q(\lambda) = 1$ is precisely the set $\{\delta_{i+1} : i < \omega\}$, which is a cofinal subset of E. But since E, $\{\delta_{i+1} : i < \omega\} \in \mathbb{L}[W \cap \alpha, q]$, this witnesses the countability of α.

If $\alpha < \omega_1^{\mathbb{L}_\beta[W\cap\alpha]}$, then the definition of q is much simpler. Indeed, we can take any $q := \bigcup_{i<\omega} p_i$ with $p_0 = p$ and $p_{i+1} \leq p_i$ for all $i < \omega$, such that $p_i \Vdash \dot{\alpha}_i \in X$, and with $\mathrm{dom}(q) = \alpha$. Since α is countable in $\mathbb{L}[W \cap \alpha]$, it is also countable in $\mathbb{L}[W \cap \alpha, q]$, and so q is a condition. □ [Lemma 15.2]

Now force with $\mathbb{P}$, and then with the coding forcing, which is ccc. Thus, the whole extension is a proper extension of $\mathbb{V}$. □ [Theorem 15.1]

Corollary 15.3 *The following are equiconsistent, modulo* ZFC*:*

(i) *There exists a remarkable cardinal.*

(ii) *For some $A \subseteq \omega_1$, in $\mathbb{L}[A]$ there is no semiproper* (*proper*) *forcing notion coding A by a real.*

P r o o f. Suppose κ is remarkable. Then it is remarkable in $\mathbb{L}$. Let λ be the least inaccessible cardinal above κ, if any such cardinal exists. Force with the Levy collapse Coll$(\omega, <\kappa)$ over $\mathbb{L}_\lambda$, or over $\mathbb{L}$ if there are no inaccessible cardinals above κ. So the forcing extension is of the form $\mathbb{L}_\lambda[A]$ or $\mathbb{L}[A]$, for some $A \subseteq \omega_1$. Now by (1) of Theorem 12.2 above, (ii) follows.

For the converse, fix A as in (ii) and suppose ω_1 is not remarkable in $\mathbb{L}$. Then the previous theorem shows that one can force with a proper forcing notion over $\mathbb{L}[A]$ to code A by a real, thus contradicting (ii). □

Theorem 15.4 *The following are equiconsistent, modulo* ZFC*:*

(i) *There exists a weakly-compact cardinal.*

(ii) *For some $A \subseteq \omega_1$, in $\mathbb{L}[A]$ there is no ccc* (*satisfying property $\mathcal{K}$*) *forcing notion coding A by a real.*

P r o o f. Suppose κ is weakly-compact. Then it is weakly-compact in $\mathbb{L}$. Let $A \subseteq \kappa$ code a generic for the Levy-collapse Coll$(\omega, <\kappa)$ over $\mathbb{L}$. By a result of Kunen (see [10] or [2]), in $\mathbb{L}[A]$ the theory of the reals is absolute under ccc forcing. Thus, by Theorem 12.3, A cannot be coded by a real using a ccc forcing notion.

For the converse, fix A as in (ii) and suppose ω_1 is not weakly-compact in $\mathbb{L}$. By Theorem 13.1 and Lemma 13.2 we may assume, without loss of generality, that ω_1 is Mahlo in $\mathbb{L}$. But now by a result of Harrington and Shelah [10], we can code A by a real using an Aronszajn tree in $\mathbb{L}$, so that the forcing is ccc (and in fact it has property $\mathcal{K}$), thus contradicting (ii). □

Theorem 15.5 *The following are equiconsistent, modulo* ZFC*:*

(i) *There exists a Mahlo cardinal.*

(ii) *For some $A \subseteq \omega_1$, in $\mathbb{L}[A]$ there is no σ-linked* (*σ-centered*) *forcing notion coding A by a real.*

P r o o f. Suppose κ is Mahlo and $A \subseteq \kappa$ codes a generic for the Levy-collapse Coll$(\omega, <\kappa)$ over $\mathbb{L}$. Then by [2], in $\mathbb{L}[A]$ the theory of the reals is absolute under σ-linked forcing. Hence, by Theorem 12.3, A cannot be coded by a real using a σ-linked forcing notion.

Conversely, suppose A is as in (ii) and ω_1 is not Mahlo in $\mathbb{L}$. By Lemma 13.2, (†) holds for A. So one can use almost-disjoint forcing, which is σ-centered, to code A by a real (see our remark after the proof of Theorem 13.1). □

16 Remarks and questions

One may want to strengthen the minimality requirement in (ii) of Theorem 1.1 as follows: $x \in \mathbb{L}[Y]$ or $Y \in \mathbb{V}$. Such a *strong minimality* would mean that x, a generic real, adds only one (its own) degree of constructibility to the ground universe $\mathbb{V}$. But this is, generally speaking, impossible.

Indeed, suppose that, in Theorem 1.1, $\omega_1^{\mathbb{L}} < \omega_1$ holds in the ground universe $\mathbb{V}$. Then the set C of all reals $y \in 2^\omega$ Cohen-generic over $\mathbb{L}$ is an uncountable $\mathbf{\Pi}^0_2$ set. Therefore there exists a continuous bijective map $f : 2^\omega \longrightarrow C$. Now, suppose that $x \in 2^\omega$, $x \notin \mathbb{V}$, is a cardinal-preserving generic real over $\mathbb{V}$. Identifying f with its extension in $\mathbb{V}[x]$, we conclude that $y = f(x) \in \mathbb{V}[x] \setminus \mathbb{V}$. Thus if x is strongly minimal in the sense just defined, then we have $\mathbb{L}[x] = \mathbb{L}[y]$, not merely $\mathbb{V}[x] = \mathbb{V}[y]$ as with the original minimality of Theorem 1.1(ii). Yet y is Cohen-generic over $\mathbb{L}$ while x collapses $\omega_1^{\mathbb{L}}$ under the assumptions above, a contradiction.

On the other hand, there is a nontrivial case in which such a strengthening of minimality is possible. Indeed let $\mathbb{V}$ be an iterated Sacks extension of $\mathbb{L}$, of length ω_1 and with countable support. This is still a universe of the form $\mathbb{V} = \mathbb{L}[A]$, where $A \subseteq \omega_1 = \omega_1^{\mathbb{L}}$. Now, let $x \in 2^\omega$ be a real Sacks-generic over $\mathbb{V}$, so that $\mathbb{V}[x] = \mathbb{L}[A, x]$ is an iterated Sacks extension of $\mathbb{L}$ of length $\omega_1 + 1$. It is known (see, for instance, [14]), that in this case x adds just one $\mathbb{L}$-degree to $\mathbb{V}$ (its own degree), so that x is strongly minimal over $\mathbb{V}$.

We finish with a couple of open problems.

Question 16.1 (inspired by Theorem 12.1) If a is a real Cohen-generic over a model $\mathfrak{M}$ and b Sacks-generic over $\mathfrak{M}[a]$, is it necessary that $a \in \mathfrak{M}[b]$?

Question 16.2 Can Theorem 1.1 be improved by the requirement x is lightface $\varDelta^1_3$ in $\mathbb{V}[x] = \mathbb{L}[A, x]$? A positive answer may lead to further studies in the direction of [9].

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