Math. Log. Quart. **63,** No. 1–2, 19–31 (2017) / **DOI** 10.1002/malq.201500020

A Groszek-Laver pair of undistinguishable E_0-classes

Mohammad Golshani[1]*, **Vladimir Kanovei**[2,3]**, and **Vassily Lyubetsky**[2,4]***

[1] School of Mathematics, Institute for Research in Fundamental Sciences, P.O. Box 19395-5746, Tehran, Iran
[2] Institute for Information Transmission Problems, Russian Academy of Sciences, Bolshoy Karetny per. 19, Moscow 127051, Russia
[3] Institute of Economics and Finance, Moscow State University of Railway Engineering, 9b9 ul. Obrazcova, Moscow 127994, Russia,
[4] Moscow State University, Department of Mechanics and Mathematics, GSP-1, 1 Leninskiye Gory, Moscow 119991, Russia

Received 7 March 2015, revised 14 October 2015, accepted 21 October 2015
Published online 30 March 2017

A generic extension $\mathbf{L}[x, y]$ of the constructible universe $\mathbf{L}$ by reals x, y is defined, in which the union of E_0-classes of x and y is a lightface Π^1_2 set, but neither of these two E_0-classes is separately ordinal-definable.

1 Introduction

Let a *Groszek-Laver pair* be any unordered OD (ordinal-definable) pair $\{X, Y\}$ of sets $X, Y \subseteq \omega^\omega$ such that neither of X, Y is separately OD. As demonstrated in [4], if $\langle x, y\rangle$ is a Sacks×Sacks generic pair of reals over $\mathbf{L}$, the constructible universe, then their degrees of constructibility $X = [x]_{\mathbf{L}} \cap \omega^\omega$ and $Y = [y]_{\mathbf{L}} \cap \omega^\omega$ form such a pair in $\mathbf{L}[x, y]$; the set $\{X, Y\}$ is definable as the set of all $\mathbf{L}$-degrees of reals, $\mathbf{L}$-minimal over $\mathbf{L}$.

As the sets X, Y in this example are obviously uncountable, one may ask whether there can consistently exist a Groszek-Laver pair of *countable* sets. The next theorem answers this question in the positive in a rather strong way: both sets are E_0-classes in the example! (Recall that the equivalence relation E_0 is defined on 2^ω as follows: $x \mathrel{\mathsf{E}_0} y$ iff $x(n) = y(n)$ for all but finite n.)

Theorem 1.1 *It is true in a suitable generic extension* $\mathbf{L}[x, y]$ *of* $\mathbf{L}$*, by a pair of reals* $x, y \in 2^\omega$ *that the union of* E_0*-equivalence classes* $[x]_{\mathsf{E}_0} \cup [y]_{\mathsf{E}_0}$ *is* Π^1_2*, but neither of the sets* $[x]_{\mathsf{E}_0}, [y]_{\mathsf{E}_0}$ *is separately* OD.

The forcing we employ is a conditional product $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ of an "E_0-large tree"[1] version $\mathbb{P}$ of a forcing notion, introduced in [14] to define a model with a Π^1_2 E_0-class containing no OD elements. The forcing in [14] was a clone of Jensen's minimal Π^1_2 real singleton forcing [9] (cf. [8, § 28A]), but defined on the base of the Silver forcing instead of the Sacks forcing. The crucial advantage of Silver's forcing here is that it leads to a Jensen-type forcing naturally closed under the 0-1 flip at any digit, so that the corresponding extension contains a Π^1_2 E_0-class of generic reals instead of a Π^1_2 generic singleton as in [9].

In another relevant note [13] it is demonstrated that a countable OD set of reals (not an E_0-class), containing no OD elements, exists in a generic extension of $\mathbf{L}$ via the countable finite-support product of Jensen's [9] forcing itself. The existence of such a set was discussed as an open question on mathoverflow [5] and on the Foundations of Mathematics (FOM) mailing list [3], and the result in [13] was conjectured by Enayat [3] on the base of his study of finite-support products of Jensen's forcing in [2].

The remainder of the paper is organized as follows:

We introduce E_0-large perfect trees in $2^{<\omega}$ in § 2, study their splitting properties in § 3, and consider E_0-large-tree forcing notions in § 4, i.e., collections of E_0-large trees closed under both restriction and action of a group of transformations naturally associated with E_0.

* E-mail: golshani.m@gmail.com
** Corresponding author; e-mail: kanovei@googlemail.com.
*** E-mail: lyubetsk@iitp.ru

[1] An E_0-large tree is a perfect tree $T \subseteq 2^{<\omega}$ such that $\mathsf{E}_0 \restriction [T]$ is not smooth, cf. [11, 10.9].

If $\mathbb{P}$ is an E_0-large-tree forcing notion then the *conditional product forcing* $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is a part of the full forcing product $\mathbb{P} \times \mathbb{P}$ which contains all conditions $\langle T, T' \rangle$ of trees $T, T' \in \mathbb{P}$, E_0-connected in some way. This key notion, defined in § 5, goes back to early research on the Gandy-Harrington forcing [6, 7].

The basic E_0-large-tree forcing $\mathbb{P}$ employed in the proof of Theorem 1.1 is defined, in $\mathbf{L}$, in the form $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{U}_\xi$ in § 10. The model $\mathbf{L}[x, y]$ which proves the theorem is then a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic extension of $\mathbf{L}$; it is studied in § 11. The elements $\mathbb{U}_\xi$ of this inductive construction are countable E_0-large-tree forcing notions in $\mathbf{L}$.

The key issue is as follows: given a subsequence $\{\mathbb{U}_\eta\}_{\eta<\xi}$ and accordingly the union $\mathbb{P}_{<\xi} = \bigcup_{\eta<\xi} \mathbb{U}_\eta$, to define the next level $\mathbb{U}_\xi$. We maintain this task in § 7 with the help of a well-known splitting/fusion construction, modified so that it yields E_0-large perfect trees. Generic aspects of this construction lead to the c.c.c. of forcing notions $\mathbb{P}$ and $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ and to rather simple reading of real names, but most of all to the crucial property that if $\langle x, y \rangle$ is a pair of reals $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic over $\mathbf{L}$ then any real $z \in \mathbf{L}[x, y]$ $\mathbb{P}$-generic over $\mathbf{L}$ belongs to $[x]_{\mathsf{E}_0} \cup [y]_{\mathsf{E}_0}$. This is Lemma 11.4, proved on the base of preliminary results of § 9.

The final § 12 briefly discusses some related topics.

2 E_0-large trees

Let $2^{<\omega}$ be the set of all strings (finite sequences) of numbers 0,1, including the empty string Λ. If $t \in 2^{<\omega}$ and $i = 0, 1$ then $t^\frown i$ is the extension of t by i as the rightmost term. If $s, t \in 2^{<\omega}$ then $s \subseteq t$ means that t extends s, $s \subset t$ means proper extension, and $s^\frown t$ is the concatenation. If $s \in 2^{<\omega}$ then $\mathrm{lh}(s)$ is the length of s, and we let $2^n = \{s \in 2^{<\omega} : \mathrm{lh}(s) = n\}$ (strings of length n).

Let any $s \in 2^{<\omega}$ *act* on 2^ω so that $(s \cdot x)(k) = x(k) + s(k) \pmod 2$ whenever $k < \mathrm{lh}(s)$ and simply $(s \cdot x)(k) = x(k)$ otherwise. If $X \subseteq 2^\omega$ and $s \in 2^{<\omega}$ then, as usual, let $s \cdot X = \{s \cdot x : x \in X\}$. Similarly, if $s, t \in 2^{<\omega}$ and $\mathrm{lh}(s) = m \le n = \mathrm{lh}(t)$, then define $s \cdot t \in 2^n$ so that $(s \cdot t)(k) = t(k) + s(k) \pmod 2$ whenever $k < m$ and $(s \cdot t)(k) = t(k)$ whenever $m \le k < n$. If $m > n$ then let simply $s \cdot t = (s \restriction n) \cdot t$. Note that $\mathrm{lh}(s \cdot t) = \mathrm{lh}(t)$ in both cases. Let $s \cdot T = \{s \cdot t : t \in T\}$ for $T \subseteq 2^{<\omega}$. If $T \subseteq 2^{<\omega}$ is a tree and $s \in T$ then put $T \restriction_s = \{t \in T : s \subseteq t \vee t \subseteq s\}$.

Let PT be the set of all *perfect* trees $\varnothing \ne T \subseteq 2^{<\omega}$ (those with no endpoints and no isolated branches). If $T \in \mathrm{PT}$ then there is a largest string $s \in T$ such that $T = T \restriction_s$; it is denoted by $s = \mathrm{stem}(T)$ (the *stem* of T); we have $s^\frown 1 \in T$ and $s^\frown 0 \in T$ in this case. If $T \in \mathrm{PT}$ then

$$[T] = \{a \in 2^\omega : \forall n \, (a \restriction n \in T)\} \subseteq 2^\omega$$

is the perfect set of all *paths through* T; clearly $[S] \subseteq [T]$ iff $S \subseteq T$.

Let LT (large trees) be the set of all *special* E_0*-large trees*: those $T \in \mathrm{PT}$ such that there is a double sequence of non-empty strings $q^i_n = q^i_n(T) \in 2^{<\omega}$, $n < \omega$ and $i = 0, 1$, such that

1. we have $\mathrm{lh}(q^0_n) = \mathrm{lh}(q^1_n) \ge 1$ and $q^i_n(0) = i$ for all n;
2. the tree T consists of all substrings of strings of the form $r^\frown q_0^{i(0)\frown} q_1^{i(1)\frown} \ldots {}^\frown q_n^{i(n)}$ in $2^{<\omega}$, where $r = \mathrm{stem}(T)$, $n < \omega$, and $i(0), i(1), \ldots, i(n) \in \{0, 1\}$.

We let $\mathrm{spl}_0(T) = \mathrm{lh}(r)$ and then by induction $\mathrm{spl}_{n+1}(T) = \mathrm{spl}_n(T) + \mathrm{lh}(q^i_n)$, so that $\mathrm{spl}(T) = \{\mathrm{spl}_n(T) : n < \omega\} \subseteq \omega$ is the set of *splitting levels* of T. Then

$$[T] = \{a \in 2^\omega : a \restriction \mathrm{lh}(r) = r \wedge \forall n \left(a \restriction [\mathrm{spl}_n(T), \mathrm{spl}_{n+1}(T)) = q^0_n \text{ or } q^1_n\right)\}.$$

Lemma 2.1 *Assume that* $T \in \mathrm{LT}$ *and* $h \in \mathrm{spl}(T)$. *Then*

(i) if $u, v \in 2^h \cap T$ *then* $T \restriction_v = (u \cdot v) \cdot T \restriction_u$ *and* $(u \cdot v) \cdot T = T$;
(ii) if $\sigma \in 2^{<\omega}$ *then* $T = \sigma \cdot T$ *or* $T \cap (\sigma \cdot T)$ *is finite.*

P r o o f . (ii) Suppose that $T \cap (\sigma \cdot T)$ is infinite. Then there is an infinite branch $x \in [T]$ such that $y = \sigma \cdot x \in [T]$, too. We can assume that $\mathrm{lh}(\sigma)$ is equal to some $h = \mathrm{spl}_n(T)$. (If $\mathrm{spl}_{n-1}(T) < h < \mathrm{spl}_n(T)$ then extend σ by $\mathrm{spl}_n(T) - h$ zeros.) Then $\sigma = (x \restriction h) \cdot (y \restriction h)$. It remains to apply (i). □

Example 2.2 If $s \in 2^{<\omega}$ then $T[s] = \{t \in 2^{<\omega} : s \subseteq t \vee t \subset s\}$ is a tree in LT, $\mathrm{stem}(T[s]) = s$, and $q_n^i(T[s]) = \langle i \rangle$ for all n, i. Note that $T[\Lambda] = 2^{<\omega}$ (the full binary tree), and $T[\Lambda]\restriction_s = (2^{<\omega})\restriction_s = T[s]$ for all $s \in 2^{<\omega}$.

3 Splitting of large trees

The *simple splitting* of a tree $T \in \mathrm{LT}$ consists of smaller trees $T(\to 0) = T\restriction_{\mathrm{stem}(T)^\frown 0}$ and $T(\to 1) = T\restriction_{\mathrm{stem}(T)^\frown 1}$, so that $[T(\to i)] = \{x \in [T] : x(h) = i\}$, where $h = \mathrm{spl}_0(T) = \mathrm{lh}(\mathrm{stem}(T))$. Clearly $T(\to i) \in \mathrm{LT}$ and $\mathrm{spl}(T(\to i)) = \mathrm{spl}(T)\backslash\{\mathrm{spl}_0(T)\}$.

Lemma 3.1 *If $R, S, T \in \mathrm{LT}$, $S \subseteq R(\to 0)$, $T \subseteq R(\to 1)$, $\sigma \in 2^{<\omega}$, $T = \sigma \cdot S$, and $\mathrm{lh}(\sigma) \leq \mathrm{lh}(\mathrm{stem}(S)) = \mathrm{lh}(\mathrm{stem}(T))$ then $U = S \cup T \in \mathrm{LT}$, $\mathrm{stem}(U) = \mathrm{stem}(R)$, and $S = U(\to 0)$, $T = U(\to 1)$.*

The splitting can be iterated, so that if $s \in 2^n$ then we define $T(\to s) = T(\to s(0))(\to s(1))(\to s(2))\ldots(\to s(n-1))$. We separately define $T(\to \Lambda) = T$, where Λ is the empty string as usual.

Lemma 3.2 *In terms of Example 2.2, for all s, we have $T[s] = (2^{<\omega})(\to s) = (2^{<\omega})\restriction_s$. Generally if $T \in \mathrm{LT}$ and $2^n \subseteq T$ then $T(\to s) = T\restriction_s$ for all $s \in 2^n$.*

If $T, S \in \mathrm{LT}$ and $n \in \omega$ then let $S \subseteq_n T$ (S *n-refines* T) mean that $S \subseteq T$ and $\mathrm{spl}_k(T) = \mathrm{spl}_k(S)$ for all $k < n$. In particular, $S \subseteq_0 T$ iff simply $S \subseteq T$. By definition if $S \subseteq_{n+1} T$ then $S \subseteq_n T$ (and $S \subseteq T$), too.

Lemma 3.3 *Suppose that $T \in \mathrm{LT}$, $n < \omega$, and $h = \mathrm{spl}_n(T)$. Then*

(i) we have $T = \bigcup_{s\in 2^n} T(\to s)$ and $[T(\to s)] \cap [T(\to t)] = \varnothing$ for all $s \neq t$ in 2^n;
(ii) if $S \in \mathrm{LT}$ then $S \subseteq_n T$ iff $S(\to s) \subseteq T(\to s)$ for all strings $s \in 2^{\leqslant n}$ iff $S \subseteq T$ and $S \cap 2^h = T \cap 2^h$;
(iii) if $s \in 2^n$ then $\mathrm{lh}(\mathrm{stem}(T(\to s))) = h$ and there is a string $u[s] \in 2^h \cap T$ such that $T(\to s) = T\restriction_{u[s]}$;
(iv) if $u \in 2^h \cap T$ then there is a string $s[u] \in 2^n$ s.t. $T\restriction_u = T(\to s[u])$;
(v) if $s_0 \in 2^n$ and $S \in \mathrm{LT}$, $S \subseteq T(\to s_0)$, then there is a unique tree $T' \in \mathrm{LT}$ such that $T' \subseteq_n T$ and $T'(\to s_0) = S$.

P r o o f. (iii) Define $u[s] = \mathrm{stem}(T)^\frown q_0^{s(0)}(T)^\frown q_1^{s(1)}(T)^\frown \ldots {}^\frown q_{n-1}^{s(n-1)}(T)$.
(iv) Define $s = s[u] \in 2^n$ by $s(k) = u(\mathrm{spl}_k(T))$ for all $k < n$.
(v) Let $u_0 = u[s_0] \in 2^h$. Following Lemma 2.1, define T' so that $T' \cap 2^h = T \cap 2^h$, and if $u \in T \cap 2^h$ then $T'\restriction_u = (u \cdot u_0) \cdot S$; in particular $T'\restriction_{u_0} = S$. □

Lemma 3.4 (Fusion) *Suppose that $\cdots \subseteq_5 T_4 \subseteq_4 T_3 \subseteq_3 T_2 \subseteq_2 T_1 \subseteq_1 T_0$ is an infinite decreasing sequence of trees in* LT. *Then*

(i) we have $T = \bigcap_n T_n \in \mathrm{LT}$;
(ii) if $n < \omega$ and $s \in 2^{n+1}$ then $T(\to s) = T \cap T_n(\to s) = \bigcap_{m\geq n} T_m(\to s)$.

P r o o f. Both parts are clear, just note that $\mathrm{spl}(T) = \{\mathrm{spl}_n(T_n) : n < \omega\}$. □

4 Large-tree forcing notions

Let a *large-tree forcing notion* be any set $\mathbb{P} \subseteq \mathrm{LT}$ such that

(4.1) if $u \in T \in \mathbb{P}$ then $T\restriction_u \in \mathbb{P}$;
(4.2) if $T \in \mathbb{P}$ and $s \in 2^{<\omega}$ then $s \cdot T \in \mathbb{P}$.

We shall typically consider large-tree forcing notions $\mathbb{P}$ containing the full tree $2^{<\omega}$. In this case, $\mathbb{P}$ contains all trees $T[s]$ of Example 2.2 by Lemma 3.2. Any large-tree forcing notion $\mathbb{P}$ can be viewed as a forcing notion (if $T \subseteq T'$ then T is a stronger condition), and then it adds a real in 2^ω. If $\mathbb{P} \subseteq \mathrm{LT}$, $T \in \mathrm{LT}$, $n < \omega$, and all split trees $T(\to s)$, $s \in 2^n$, belong to $\mathbb{P}$, then we say that T is an *n-collage over* $\mathbb{P}$. Let $\mathrm{LC}_n(\mathbb{P})$ be the set of all trees $T \in \mathrm{LT}$ which are n-collages over $\mathbb{P}$, and $\mathrm{LC}(\mathbb{P}) = \bigcup_n \mathrm{LC}_n(\mathbb{P})$. Note that $\mathrm{LC}_n(\mathbb{P}) \subseteq \mathrm{LC}_{n+1}(\mathbb{P})$ by (4.1).

Lemma 4.1 *Assume that $\mathbb{P} \subseteq \mathrm{LT}$ is a large-tree forcing notion and $n < \omega$. Then*

(i) if $T \in \mathrm{LT}$ and $s_0 \in 2^n$ then $T(\to s_0) \in \mathbb{P}$ iff $T \in \mathrm{LC}_n(\mathbb{P})$;
(ii) if $P \in \mathrm{LC}_n(\mathbb{P})$, $s_0 \in 2^n$, $S \in \mathbb{P}$, and $S \subseteq P(\to s_0)$, then there is a tree $Q \in \mathrm{LC}_n(\mathbb{P})$ such that $Q \subseteq_n P$ and $Q(\to s_0) = S$;
(iii) if $P \in \mathrm{LC}_n(\mathbb{P})$ and a set $D \subseteq \mathbb{P}$ is open dense in $\mathbb{P}$, then there is a tree $Q \in \mathrm{LC}_n(\mathbb{P})$ such that $Q \subseteq_n P$ and $Q(\to s) \in D$ for all $s \in 2^n$;
(iv) if $P \in \mathrm{LC}_n(\mathbb{P})$, $S, T \in \mathbb{P}$, $s, t \in 2^n$, $S \subseteq P(\to s^\frown 0)$, $T \subseteq P(\to t^\frown 1)$, $\sigma \in 2^{<\omega}$, and $T = \sigma \cdot S$, then there is a tree $Q \in \mathrm{LC}_{n+1}(\mathbb{P})$, $Q \subseteq_{n+1} P$, such that $Q(\to s^\frown 0) \subseteq S$ and $Q(\to t^\frown 1) \subseteq T$.

Recall that a set $D \subseteq \mathbb{P}$ is *open dense* in $\mathbb{P}$ iff, first, if $S \in \mathbb{P}$ then there is a tree $T \in D$, $T \subseteq S$, and, secondly, if $S \in \mathbb{P}$, $T \in D$, and $S \subseteq T$, then $S \in D$, too.

P r o o f. (i) If $T \in \mathrm{LC}_n(\mathbb{P})$ then by definition $T(\to s_0) \in \mathbb{P}$. To prove the converse, let $h = \mathrm{spl}_n(T)$, and let $h[s] \in 2^h \cap T$ satisfy $T(\to s) = T\restriction_{u[s]}$ for all $s \in 2^n$ by Lemma 3.3(iii). If $T(\to s_0) \in \mathbb{P}$ then $T(\to s) = T\restriction_{u[s]} = (u[s] \cdot u[s_0]) \cdot T\restriction_{u[s]}$ by Lemma 2.1, so $T(\to s) \in \mathbb{P}$ by (4.2). Thus $T \in \mathrm{LC}_n(\mathbb{P})$.

(ii) By Lemma 3.3(v) there is a tree $Q \in \mathrm{LT}$ such that $Q \subseteq_n P$ and $Q(\to s_0) = S$. We observe that Q belongs to $\mathrm{LC}_n(\mathbb{P})$ by (i).

(iii) Apply (ii) consecutively 2^n times (all $s \in 2^n$).

(iv) We first consider the case when $t = s$. If $\mathrm{lh}(\sigma) \le L = \mathrm{lh}(\mathrm{stem}(S)) = \mathrm{lh}(\mathrm{stem}(T))$ then by Lemma 3.1 $U = S \cup T \in \mathrm{LT}$, $\mathrm{stem}(U) = \mathrm{stem}(P(\to s))$, and $U(\to 0) = S$, $U(\to 1) = T$. Lemma 3.3(v) yields a tree $Q \in \mathrm{LT}$ such that $Q \subseteq_n P$ and $Q(\to s) = U$, hence $\mathrm{stem}(Q(\to s)) = \mathrm{stem}(P(\to s))$ by the above. This implies $\mathrm{spl}_n(Q) = \mathrm{spl}_n(P)$ by Lemma 3.3(iii), and hence $Q \subseteq_{n+1} P$. And finally $Q \in \mathrm{LC}_{n+1}(\mathbb{P})$ by (i) since $Q(\to s^\frown 0) = S \in \mathbb{P}$.

Now suppose that $\mathrm{lh}(\sigma) > L$. Take any string $u \in S$ with $\mathrm{lh}(u) \ge \mathrm{lh}(s)$. The set $S' = S\restriction_u \subseteq S$ belongs to $\mathbb{P}$ and obviously $\mathrm{lh}(\mathrm{stem}(S')) \ge \mathrm{lh}(\sigma)$. It remains to follow the case already considered for the trees S' and $T' = \sigma \cdot S'$.

Finally consider the general case $s \ne t$. Let $h = \mathrm{spl}_n(P)$, $H = \mathrm{spl}_{n+1}(P)$. Let $u = u[s]$ and $v = u[t]$ be the strings in $P \cap 2^h$ defined by Lemma 3.3(iii) for P, so that $P\restriction_u = P(\to s)$ and $P\restriction_v = P(\to t)$, and let $U, V \in 2^H \cap P$ be defined accordingly so that $P\restriction_U = P(\to s^\frown 1)$ and $P\restriction_V = P(\to t^\frown 1)$. Let $\varrho = u \cdot v$. Then $P(\to s) = \varrho \cdot P(\to t)$ by Lemma 2.1. However we have $U = u^\frown\tau$ and $V = v^\frown\tau$ for one and the same string τ, see the proof of Lemma 3.3(iii). Therefore $U \cdot V = u \cdot v = \varrho$ and $P(\to s^\frown 1) = \varrho \cdot P(\to t^\frown 1)$ still by Lemma 2.1.

It follows that the tree $T_1 = \varrho \cdot T$ satisfies $T_1 \subseteq P(\to s^\frown 1)$. Applying the result for $s = t$, we get a tree $Q \in \mathrm{LC}_{n+1}(\mathbb{P})$, $Q \subseteq_{n+1} P$, such that $Q(\to s^\frown 0) \subseteq S$ and $Q(\to s^\frown 1) \subseteq T_1$. Then by definition $\mathrm{spl}_k(P) = \mathrm{spl}_k(Q)$ for all $k \le n$, and $Q(\to s) \subseteq P(\to s)$ for all $s \in 2^{n+1}$ by Lemma 3.3(ii). Therefore the same strings u, v satisfy $Q\restriction_u = Q(\to s)$ and $Q\restriction_v = Q(\to t)$. The same argument as above implies $Q(\to t^\frown 1) = \varrho \cdot Q(\to s^\frown 1)$. We conclude that $Q(\to t^\frown 1) \subseteq \varrho \cdot T_1 = T$, as required. □

5 Conditional product forcing

Along with any large-tree forcing notion $\mathbb{P}$, we shall consider the *conditional product* $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, which by definition consists of all pairs $\langle T, T' \rangle$ of trees $T, T' \in \mathbb{P}$ such that there is a string $s \in 2^{<\omega}$ satisfying $s \cdot T = T'$. We order $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ componentwise so that $\langle S, S' \rangle \le \langle T, T' \rangle$ ($\langle S, S' \rangle$ is stronger) iff $S \subseteq T$ and $S' \subseteq T'$.[2]

Remark 5.1 The conditional product $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ forces a pair of $\mathbb{P}$-generic reals. Indeed if $\langle T, T' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ with $s \cdot T = T'$ and $S \in \mathbb{P}$, $S \subseteq T$, then there is a tree $S' = s \cdot S \in \mathbb{P}$ (we make use of (4.2)) such that $\langle S, S' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ and $\langle S, S' \rangle \le \langle T, T' \rangle$.

But $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic pairs are not necessarily generic in the sense of the true forcing product $\mathbb{P} \times \mathbb{P}$. Indeed, if say $\mathbb{P} =$ Sacks (all perfect trees) then any $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$-generic pair $\langle x, y \rangle$ has the property that x, y belong to same E_0-invariant Borel sets coded in the ground universe, while for any uncountable and co-uncountable Borel set U coded in the ground universe there is a $\mathbb{P} \times \mathbb{P}$-generic pair $\langle x, y \rangle$ with $x \in U$ and $y \notin U$.

[2] Conditional product forcing notions of this kind were considered in [6,7,10] and some other papers with respect to the Gandy-Harrington and similar forcings, and recently in [15] with respect to many forcing notions.

Lemma 5.2 *Assume that $\mathbb{P}$ is a large-tree forcing notion, $n \geq 1$, $P \in \mathrm{LC}_n(\mathbb{P})$, and a set $D \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is open dense in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$. Then there is a tree $Q \in \mathrm{LC}_n(\mathbb{P})$ such that $Q \subseteq_n P$ and $\langle Q(\to s), Q(\to t)\rangle \in D$ whenever $s, t \in 2^n$ and $s(n-1) \neq t(n-1)$.*

Proof. (Compare to Lemma 4.1(iii).) Let $s, t \in 2^n$ be any pair with $s(n-1) \neq t(n-1)$. By the density there is a condition $\langle S, T\rangle \in D$ such that $S \subseteq P(\to s)$ and $T \subseteq P(\to t)$. Note that $T = \sigma \cdot S$ for some $s \in 2^{<\omega}$ since $\langle S, T\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$. Applying Lemma 4.1(iv) ($n+1$ there corresponds to n here) we obtain a tree $P' \in \mathrm{LC}_n(\mathbb{P})$ such that $P' \subseteq_n P$ and $P'(\to s) \subseteq S$, $P'(\to t) \subseteq T$. Then $\langle P'(\to s), P'(\to t)\rangle \in D$, as D is open. Consider all pairs $s, t \in 2^n$ with $s(n-1) \neq t(n-1)$ one by one. □

Lemma 5.3 *Assume that $\mathbb{P}$ is a large-tree forcing notion, $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $n < \omega$, $s, t \in 2^n$. Then $\langle T(\to s), T'(\to t)\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$.*

Proof. Let $\sigma \in 2^{<\omega}$ satisfy $\sigma \cdot T = T'$. Note that $\mathrm{spl}(T) = \mathrm{spl}(T')$, hence we define $h = \mathrm{spl}_n(T) = \mathrm{spl}_n(T')$. By Lemma 3.3(iii), there are strings $u \in 2^h \cap T$ and $v \in 2^h \cap T'$ such that $T(\to s) = T\restriction_u$ and $T'(\to t) = T'\restriction_v$. Then obviously $\sigma \cdot T\restriction_u = T'\restriction_{v'}$, where $v' = \sigma \cdot u$. On the other hand $T'\restriction_v = (v \cdot v') \cdot T'\restriction_{v'}$ by Lemma 2.1. It follows that $T'\restriction_v = (v \cdot v' \cdot \sigma) \cdot T\restriction_u$, as required. □

Corollary 5.4 *Assume that $\mathbb{P}$ is a large-tree forcing notion. Then $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ forces $\dot{\boldsymbol{x}}_{\mathrm{left}} \not\mathrel{\mathsf{E}_0} \dot{\boldsymbol{x}}_{\mathrm{right}}$, where $\langle \dot{\boldsymbol{x}}_{\mathrm{left}}, \dot{\boldsymbol{x}}_{\mathrm{right}}\rangle$ is a name of the $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic pair.*

Proof. Otherwise a condition $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ forces $\dot{\boldsymbol{x}}_{\mathrm{right}} = \sigma \cdot \dot{\boldsymbol{x}}_{\mathrm{left}}$, where $\sigma \in 2^{<\omega}$. Find n and $s, t \in 2^n$ such that $T'(\to t) \cap (\sigma \cdot T(\to s)) = \varnothing$ and apply the lemma. □

6 Multitrees

Let a *multitree* be any sequence $\varphi = \{\langle \tau_k^\varphi, h_k^\varphi\rangle\}_{k<\omega}$ such that

(6.1) if $k < \omega$ then $h_k^\varphi \in \omega \cup \{-1\}$, and the set $|\varphi| = \{k : h_k^\varphi \neq -1\}$ (the *support* of φ) is finite;
(6.2) if $k \in |\varphi|$ then $\tau_k^\varphi = \langle T_k^\varphi(0), T_k^\varphi(1), \dots, T_k^\varphi(h_k^\varphi)\rangle$, where each $T_k^\varphi(n)$ is a tree in LT and $T_k^\varphi(n) \subseteq_n T_k^\varphi(n-1)$ whenever $1 \leq n \leq h_k^\varphi$, while if $k \notin |\varphi|$ then simply $\tau_k^\varphi = \Lambda$ (the empty sequence).

In this context, if $n \leq h_k^\varphi$ and $s \in 2^n$ then let $T_k^\varphi(s) = T_k^\varphi(n)(\to s)$.

Let φ, ψ be multitrees. Say that φ *extends* ψ, symbolically $\psi \preccurlyeq \varphi$, if $|\psi| \subseteq |\varphi|$, and, for every $k \in |\psi|$, we have $h_k^\varphi \geq h_k^\psi$ and τ_k^φ extends τ_k^ψ, so that $T_k^\varphi(n) = T_k^\psi(n)$ for all $n \leq h_k^\psi$;

If $\mathbb{P}$ is a large-tree forcing notion, then let $\mathrm{MT}(\mathbb{P})$ (*multitrees over* $\mathbb{P}$) be the set of all multitrees φ such that $T_k^\varphi(n) \in \mathrm{LC}_n(\mathbb{P})$ whenever $k \in |\varphi|$ and $n \leq h_k^\varphi$.

7 Jensen's extension of a large-tree forcing notion

Let ZFC' be the subtheory of ZFC including all axioms except for the power set axiom, plus the axiom saying that $\wp(\omega)$ exists. (Then ω_1, 2^ω, and sets like PT exist as well.)

Definition 7.1 Let $\mathfrak{M}$ be a countable transitive model of ZFC'. Suppose that $\mathbb{P} \in \mathfrak{M}$, $\mathbb{P} \subseteq \mathrm{LT}$ is a large-tree forcing notion. Then $\mathrm{MT}(\mathbb{P}) \in \mathfrak{M}$. A set $D \subseteq \mathrm{MT}(\mathbb{P})$ is *dense in* $\mathrm{MT}(\mathbb{P})$ iff for any $\psi \in \mathrm{MT}(\mathbb{P})$ there is a multitree $\varphi \in D$ such that $\psi \preccurlyeq \varphi$.

Consider any $\preccurlyeq$-increasing sequence $\Phi = \{\varphi(j)\}_{j<\omega}$ of multitrees

$$\varphi(j) = \{\langle \tau_k^{\varphi(j)}, h_k^{\varphi(j)}\rangle\}_{k<\omega} \in \mathrm{MT}(\mathbb{P}),$$

generic over $\mathfrak{M}$ in the sense that it intersects every set D, $D \subseteq \mathrm{MT}(\mathbb{P})$, dense in $\mathrm{MT}(\mathbb{P})$, which belongs to $\mathfrak{M}$. Then in particular Φ intersects every set

$$D_{kp} = \{\varphi \in \mathrm{MT}(\mathbb{P}) : k \in |\varphi| \wedge h_k^\varphi \geq p\}$$

for $k, p < \omega$. Therefore if $k < \omega$ then by definition there is an infinite sequence

$$\cdots \subseteq_5 T_k^\Phi(4) \subseteq_4 T_k^\Phi(3) \subseteq_3 T_k^\Phi(2) \subseteq_2 T_k^\Phi(1) \subseteq_1 T_k^\Phi(0)$$

of trees $T_k^\Phi(n) \in \mathrm{LC}_n(\mathbb{P})$, such that, for any j, if $k \in |\varphi(j)|$ and $n \le h_k^{\varphi(j)}$ then $T_k^{\varphi(j)}(n) = T_k^\Phi(n)$. If $n < \omega$ and $s \in 2^n$ then we let $T_k^\Phi(s) = T_k^\Phi(n)(\to s)$; then $T_k^\Phi(s) \in \mathbb{P}$ since $T_k^\Phi(n) \in \mathrm{LC}_n(\mathbb{P})$. Then it follows from Lemma 3.4 that

$$U_k^\Phi = \bigcap_n T_k^\Phi(n) = \bigcap_n \bigcup_{s\in 2^n} T_k^\Phi(s) \tag{1}$$

is a tree in LT (not necessarily in $\mathbb{P}$), as well as the trees $U_k^\Phi(\to s)$, and still by Lemma 3.4,

$$U_k^\Phi(\to s) = U_k^\Phi \cap T_k^\Phi(s) = \bigcap_{n\ge \mathrm{lh}(s)} T_k^\Phi(n)(\to s) = \bigcap_{n\ge \mathrm{lh}(s)} \bigcup_{t\in 2^n,\, s\subseteq t} T_k^\Phi(t)\,, \tag{2}$$

and obviously $U_k^\Phi = U_k^\Phi(\to \Lambda)$.

Define a set of trees $\mathbb{U} = \{\sigma \cdot U_k^\Phi(\to s) : k < \omega \wedge s \in 2^{<\omega} \wedge \sigma \in 2^{<\omega}\} \subseteq \mathrm{LT}$.

The next few simple lemmas show useful effects of the genericity of Φ; their common motto is that the extension from $\mathbb{P}$ to $\mathbb{P}\cup\mathbb{U}$ is rather innocuous.

Lemma 7.2 *Both $\mathbb{U}$ and the union $\mathbb{P}\cup\mathbb{U}$ are large-tree forcing notions; $\mathbb{P}\cap\mathbb{U} = \varnothing$.*

P r o o f. To prove the last claim, let $T \in \mathbb{P}$ and $U = U_k^\Phi(\to s) \in \mathbb{U}$. (If $U = \sigma\cdot U_k^\Phi(\to s)$, $\sigma \in 2^{<\omega}$, then replace T by $\sigma \cdot T$.) The set $D(T,k)$ of all multitrees $\varphi \in \mathrm{MT}(\mathbb{P})$, such that $k \in |\varphi|$ and $T \setminus T_k^\varphi(n)(\to s) \ne \varnothing$, where $n = h_k^\varphi$, belongs to $\mathfrak{M}$ and obviously is dense in $\mathrm{MT}(\mathbb{P})$. Now any multitree $\varphi(j) \in D(T,k)$ witnesses that $T\setminus U_k^\Phi(\to s) \ne \varnothing$. □

Lemma 7.3 *The set $\mathbb{U}$ is dense in $\mathbb{U}\cup\mathbb{P}$. The set $\mathbb{U}\times_{\mathsf{E}_0}\mathbb{U}$ is dense in $(\mathbb{P}\cup\mathbb{U})\times_{\mathsf{E}_0}(\mathbb{P}\cup\mathbb{U})$.*

P r o o f. Suppose that $T \in \mathbb{P}$. The set $D(T)$ of all multitrees $\varphi \in \mathrm{MT}(\mathbb{P})$, such that $T_k^\varphi(0) = T$ for some k, belongs to $\mathfrak{M}$ and obviously is dense in $\mathrm{MT}(\mathbb{P})$. It follows that $\varphi(j) \in D(T)$ for some j, by the choice of Φ. Then $T_k^\Phi(\Lambda) = T$ for some k. However by construction $U_k^\Phi(\to\Lambda) = U_k^\Phi \subseteq T_k^\Phi(\Lambda)$.

Now suppose that $\langle T, T'\rangle \in \mathbb{P}\times_{\mathsf{E}_0}\mathbb{P}$, so that $T' = \sigma\cdot T$, $\sigma \in 2^{<\omega}$. By Lemma 7.2 ($\mathbb{P}\cap\mathbb{U} = \varnothing$) it is impossible that one of the trees T, T' belongs to $\mathbb{P}$ and the other one to $\mathbb{U}$. Therefore we can assume that $T, T' \in \mathbb{P}$. By the first claim of the lemma, there is a tree $U \in \mathbb{U}$, $U \subseteq T$. Then $U' = \sigma\cdot U \in \mathbb{U}$ and still $U' = \sigma \cdot U$, hence $\langle U, U'\rangle \in \mathbb{U}\times_{\mathsf{E}_0}\mathbb{U}$, and it extends $\langle T, T'\rangle$. □

Lemma 7.4 *If $k, \ell < \omega$, $k \ne \ell$, and $\sigma \in 2^{<\omega}$ then $U_k^\Phi \cap (\sigma\cdot U_\ell^\Phi) = \varnothing$.*

P r o o f. The set $D'(k,\ell)$ of all multitrees $\varphi \in \mathrm{MT}(\mathbb{P})$, such that $k, \ell \in |\varphi|$ and $T_k^\varphi(n) \cap (\sigma\cdot T_\ell^\varphi(m)) = \varnothing$ for some $n \le h_k^\varphi$, $m \le h_\ell^\varphi$, belongs to $\mathfrak{M}$ and is dense in $\mathrm{MT}(\mathbb{P})$. So $\varphi(j) \in D'(k,\ell)$ for some $j < \omega$. But then for some n, m we have $U_k^\Phi \cap (\sigma\cdot U_\ell^\Phi) \subseteq T_k^{\varphi(j)}(n) \cap (\sigma \cdot T_\ell^{\varphi(j)}(m)) = \varnothing$. □

Corollary 7.5 *If $\langle U, U'\rangle \in \mathbb{U}\times_{\mathsf{E}_0}\mathbb{U}$ then there exist: $k < \omega$, strings $s, s' \in 2^{<\omega}$ with* $\mathrm{lh}(s) = \mathrm{lh}(s')$*, and strings $\sigma, \sigma' \in 2^{<\omega}$, such that $U = \sigma\cdot U_k^\Phi(\to s)$ and $U' = \sigma'\cdot U_k^\Phi(\to s')$.*

P r o o f. By definition, we have $U = \sigma\cdot U_k^\Phi(\to s)$ and $U' = \sigma'\cdot U_{k'}^\Phi(\to s')$, for suitable $k, k' < \omega$ and $s, s', \sigma, \sigma' \in 2^{<\omega}$. As $\langle U, U'\rangle \in \mathbb{U}\times_{\mathsf{E}_0}\mathbb{U}$, it follows from Lemma 7.4 that $k' = k$, hence $U' = \sigma\cdot U_k^\Phi(\to s')$. Therefore $\sigma\cdot U_k^\Phi(\to s) = \tau\cdot\sigma'\cdot U_k^\Phi(\to s')$ for some $\tau \in 2^{<\omega}$. In other words, $U_k^\Phi(\to s) = \tau'\cdot U_k^\Phi(\to s')$, where $\tau' = \sigma\cdot\sigma'\cdot\tau \in 2^{<\omega}$. It easily follows that $\mathrm{lh}(s) = \mathrm{lh}(s')$. □

The two following lemmas show that, due to the generic character of extension, those pre-dense sets which belong to $\mathfrak{M}$, remain pre-dense in the extended forcing.

Let $X \subseteq^{\mathrm{fin}} \bigcup D$ mean that there is a finite set $D' \subseteq D$ with $X \subseteq \bigcup D'$.

Lemma 7.6 *If a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P}$ is pre-dense in $\mathbb{P}$, and $U \in \mathbb{U}$, then $U \subseteq^{\mathrm{fin}} \bigcup D$. Moreover D is pre-dense in $\mathbb{U}\cup\mathbb{P}$.*

P r o o f. We can assume that D is in fact open dense in $\mathbb{P}$. (Otherwise replace it with the set $D' = \{T \in \mathbb{P} : \exists\, S \in D\,(T \subseteq S)\}$ which also belongs to $\mathfrak{M}$.)

We can also assume that $U = U_k^\Phi(\to s) \in \mathbb{U}$, where $k < \omega$ and $s \in 2^{<\omega}$. (The general case, when $U = \sigma\cdot U_k^\Phi(\to s)$ for some $\sigma \in 2^{<\omega}$, is reducible to the case $U = U_k^\Phi(\to s)$ by substituting the set $\sigma\cdot D$ for D.)

The set $\Delta \in \mathfrak{M}$ of all multitrees $\varphi \in \mathrm{MT}(\mathbb{P})$ such that $k \in |\varphi|$, $\mathrm{lh}(s) < h = h_k^\varphi$, and $T_k^\varphi(h)(\to t) \in D$ for all $t \in 2^h$, is dense in $\mathrm{MT}(\mathbb{P})$ by Lemma 4.1(iii) and the open density of D. Therefore there is an index j such that $\varphi(j) \in \Delta$. Let $h(j) = h_k^{\varphi(j)}$. Then the tree $S_t = T_k^{\varphi(j)}(h(j))(\to t) = T_k^\Phi(h(j))(\to t) = T_k^\Phi(t)$ belongs to D for all $t \in 2^{h(j)}$. We conclude that

$$U = U_k^\Phi(\to s) \subseteq U_k^\Phi \subseteq \bigcup_{t\in 2^{h(j)}} T_k^\Phi(t) \subseteq \bigcup_{t\in 2^{h(j)}} S_t = \bigcup D',$$

where $D' = \{S_t : t \in 2^{h(j)}\} \subseteq D$ is finite.

To prove the pre-density claim, pick a string $t \in 2^{h(j)}$ with $s \subset t$. Then $V = U_k^\Phi(\to t) \in \mathbb{U}$ and $V \subseteq U$. However $V \subseteq T_k^\Phi(t) = S_t \in D$. Thus V witnesses that U is compatible with $S_t \in D$ in $\mathbb{U} \cup \mathbb{P}$, as required. □

Lemma 7.7 *If a set $D \in \mathfrak{M}$, $D \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is pre-dense in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ then D is pre-dense in $(\mathbb{P} \cup \mathbb{U}) \times_{\mathsf{E}_0} (\mathbb{P} \cup \mathbb{U})$.*

P r o o f. Let $\langle U, U'\rangle \in \mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$; the goal is to prove that $\langle U, U'\rangle$ is compatible in $(\mathbb{P} \cup \mathbb{U}) \times_{\mathsf{E}_0} (\mathbb{P} \cup \mathbb{U})$ with a condition $\langle T, T'\rangle \in D$. By Corollary 7.5, there exist: $k < \omega$ and strings $s, s', \sigma, \sigma' \in 2^{<\omega}$ such that $\mathrm{lh}(s) = \mathrm{lh}(s')$ and $U = \sigma \cdot U_k^\Phi(\to s)$, $U' = \sigma' \cdot U_k^\Phi(\to s')$. As in the proof of the previous lemma, we can assume that $\sigma = \sigma' = \Lambda$, so that $U = U_k^\Phi(\to s)$, $U' = U_k^\Phi(\to s')$. (The general case is reducible to this case by substituting the set $\{\langle \sigma \cdot T, \sigma' \cdot T'\rangle : \langle T, T'\rangle \in D\}$ for D.)

Assume that D is in fact open dense.

Consider the set $\Delta \in \mathfrak{M}$ of all multitrees $\varphi \in \mathrm{MT}(\mathbb{P})$ such that $k \in |\varphi|$, $\mathrm{lh}(s) = \mathrm{lh}(s') = n < h = h_k^\varphi$, and $\langle T_k^\varphi(h)(\to u), T_k^\varphi(h)(\to u')\rangle \in D$ whenever $u, u' \in 2^h$ and $u(h-1) \neq u'(h-1)$. The set Δ is dense in $\mathrm{MT}(\mathbb{P})$ by Lemma 5.2. Therefore $\varphi(j) \in \Delta$ for some j, so that if $u, u' \in 2^{h(j)}$, where $h(j) = h_k^{\varphi(j)} > n$, and $u(h(j) - 1) \neq u'(h(j) - 1)$, then

$$\langle T_k^{\varphi(j)}(h(j))(\to u), T_k^{\varphi(j)}(h(j))(\to u')\rangle = \langle T_k^\Phi(u), T_k^\Phi(u')\rangle \in D.$$

Now, as $h(j) > n$, let us pick $u, u' \in 2^{h(j)}$ such that $u(h(j) - 1) \neq u'(h(j) - 1)$ and $s \subset u$, $s' \subset u'$. Then $\langle T_k^\Phi(u), T_k^\Phi(u')\rangle \in D$. On the other hand, the pair $\langle U_k^\Phi(\to u), U_k^\Phi(\to u')\rangle$ belongs to $\mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$ by Lemma 5.3,

$$\langle U_k^\Phi(\to u), U_k^\Phi(\to u')\rangle \leq \langle U_k^\Phi(\to s), U_k^\Phi(\to s')\rangle,$$

and finally we have $\langle U_k^\Phi(\to u), U_k^\Phi(\to u')\rangle \leq \langle T_k^\Phi(u), T_k^\Phi(u')\rangle$. We conclude that the given condition $\langle U_k^\Phi(\to s), U_k^\Phi(\to s')\rangle$ is compatible with the condition $\langle T_k^\Phi(u), T_k^\Phi(u')\rangle \in D$, as required. □

8 Real names

In this section, we assume that $\mathbb{P}$ is a large-tree forcing notion and $2^{<\omega} \in \mathbb{P}$. It follows by (4.1) that all trees $T[s] = (2^{<\omega})(\to s)$ (see Example 2.2) also belong to $\mathbb{P}$.

Recall that $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ adds a pair of reals $\langle x_{\mathrm{left}}, x_{\mathrm{right}}\rangle \in 2^\omega \times 2^\omega$.

Arguing in the conditions of Definition 7.1, the goal of the following Theorem 9.3 will be to prove that, for any $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-name c of a real in 2^ω, it is forced by the extended forcing $(\mathbb{P} \cup \mathbb{U}) \times_{\mathsf{E}_0} (\mathbb{P} \cup \mathbb{U})$ that c does not belong to sets of the form $[U]$, where U is a tree in $\mathbb{U}$, unless c is a name of one of reals in the E_0-class of one of the generic reals $x_{\mathrm{left}}, x_{\mathrm{right}}$ themselves.

We begin with a suitable notation.

Definition 8.1 A $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-*real name* is a system $\mathbf{c} = \{C_n^i\}_{n<\omega,\, i<2}$ of sets $C_n^i \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ such that each set $C_n = C_n^0 \cup C_n^1$ is pre-dense in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ and any conditions $\langle S, S'\rangle \in C_n^0$ and $\langle T, T'\rangle \in C_n^1$ are incompatible in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$. If a set $G \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic at least over the collection of all sets C_n then we define $\mathbf{c}[G] \in 2^\omega$ so that $\mathbf{c}[G](n) = i$ iff $G \cap C_n^i \neq \varnothing$.

Any $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\mathbf{c} = \{C_n^i\}$ induces (can be understood as) a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-name (in the ordinary forcing notation) for a real in 2^ω.

Definition 8.2 (Actions) Strings in $2^{<\omega}$ can act on names $\mathbf{c} = \{C_n^i\}_{n<\omega, i<2}$ in two ways, related either to conditions or to the output. If $\sigma, \sigma' \in 2^{<\omega}$ then define a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\langle \sigma, \sigma'\rangle \circ \mathbf{c} = \{\langle \sigma, \sigma'\rangle \cdot C_n^i\}$, where $\langle \sigma, \sigma'\rangle \cdot C_n^i = \{\langle \sigma \cdot T, \sigma' \cdot T'\rangle : \langle T, T'\rangle \in C_n^i\}$ for all n, i. If $\varrho \in 2^{<\omega}$ then define a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\varrho \cdot \mathbf{c} = \{C\varrho_n^i\}$, where $C\varrho_n^i = C_n^{1-i}$ whenever $n < \mathrm{lh}(\varrho)$ and $\varrho(n) = 1$, but $C\varrho_n^i = C_n^i$ otherwise.

Both actions are idempotent. The difference between them is as follows. If $G \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic set then $(\langle \sigma, \sigma' \rangle \circ \mathbf{c})[G] = \mathbf{c}[\langle \sigma, \sigma' \rangle \circ G]$, where $\langle \sigma, \sigma' \rangle \circ G = \{\langle \sigma \cdot T, \sigma' \cdot T' \rangle : \langle T, T' \rangle \in G\}$, while $(\varrho \cdot \mathbf{c})[G] = \varrho \cdot (\mathbf{c}[G])$.

Example 8.3 Define a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\dot{\boldsymbol{x}}_{\text{left}} = \{C_n^i\}_{n<\omega, i<2}$ such that each set $C_n^i \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ contains all pairs of the form $\langle T[s], T[t] \rangle$, where $s, t \in 2^{n+1}$ and $s(n) = i$, and a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\dot{\boldsymbol{x}}_{\text{right}} = \{C_n^i\}_{n<\omega, i<2}$ such that accordingly each set $C_n^i \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ contains all pairs $\langle T[s], T[t] \rangle$, where $s, t \in 2^{n+1}$ and now $t(n) = i$.

Then $\dot{\boldsymbol{x}}_{\text{left}}, \dot{\boldsymbol{x}}_{\text{right}}$ are names of the $\mathbb{P}$-generic reals x_{left}, resp., x_{right}, and each name $\sigma \cdot \dot{\boldsymbol{x}}_{\text{left}}$ ($\sigma \in 2^{<\omega}$) induces a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-name of the real $\sigma \cdot (x_{\text{left}}[G])$; the same for $_{\text{right}}$.

9 Direct forcing a real to avoid a tree

Let $\mathbf{c} = \{C_n^i\}$, $\mathbf{d} = \{D_n^i\}$ be $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real names. Say that a condition $\langle T, T' \rangle \in \mathrm{LT} \times_{\mathsf{E}_0} \mathrm{LT}$:

1. *directly forces* $\mathbf{c}(n) = i$, where $n < \omega$, $i = 0, 1$, if $\langle T, T' \rangle \leq \langle S, S' \rangle$ for some $\langle S, S' \rangle \in C_n^i$;
2. *directly forces* $s \subset \mathbf{c}$, where $s \in 2^{<\omega}$, iff for all $n < \mathrm{lh}(s)$, $\langle T, T' \rangle$ directly forces $\mathbf{c}(n) = i$, where $i = s(n)$;
3. *directly forces* $\mathbf{d} \neq \mathbf{c}$, iff there are strings $s, t \in 2^{<\omega}$, incomparable in $2^{<\omega}$ and such that $\langle T, T' \rangle$ directly forces $s \subset \mathbf{c}$ and $t \subset \mathbf{d}$;
4. *directly forces* $\mathbf{c} \notin [U]$, where $U \in \mathrm{PT}$, iff there is a string $s \in 2^{<\omega} \setminus U$ such that $\langle T, T' \rangle$ directly forces $s \subset \mathbf{c}$.

Lemma 9.1 *If $S \in \mathbb{P}$, $\langle R, R' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, and $\mathbf{c}$ is a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name, then there exists a tree $S' \in \mathbb{P}$ and a condition $\langle T, T' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T' \rangle \leq \langle R, R' \rangle$, such that $S' \subseteq S$ and $\langle T, T' \rangle$ directly forces $\mathbf{c} \notin [S']$.*

Proof. Clearly there is a condition $\langle T, T' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T' \rangle \leq \langle R, R' \rangle$, which directly forces $u \subset \mathbf{c}$ for some $u \in 2^{<\omega}$ satisfying $\mathrm{lh}(u) > \mathrm{lh}((\mathrm{stem}(S)))$. There is a string $v \in S$, $\mathrm{lh}(v) = \mathrm{lh}(u)$, incomparable with u. The tree $S' = S\restriction_v$ belongs to $\mathbb{P}$, $S' \subseteq S$ by construction, and obviously $\langle T, T' \rangle$ directly forces $\mathbf{c} \notin [S']$. □

Lemma 9.2 *If $\mathbf{c}$ is a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name, $\sigma \in 2^{<\omega}$, and a condition $\langle R, R' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ directly forces $\sigma \cdot \mathbf{c} \neq \dot{\boldsymbol{x}}_{\text{left}}$, resp., $\sigma \cdot \mathbf{c} \neq \dot{\boldsymbol{x}}_{\text{right}}$, then there is a stronger condition $\langle T, T' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T' \rangle \leq \langle R, R' \rangle$, which directly forces resp. $\mathbf{c} \notin [\sigma \cdot T]$, $\mathbf{c} \notin [\sigma \cdot T']$.*

Proof. We just prove the "left" version, as the "right" version can be proved similarly. So let's assume that $\langle R, R' \rangle$ directly forces $\mathbf{c} \neq \dot{\boldsymbol{x}}_{\text{left}}$. There are incomparable strings $u, v \in 2^{<\omega}$ such that $\langle R, R' \rangle$ directly forces $u \subset \sigma \cdot \mathbf{c}$, hence, $\sigma \cdot u \subset \mathbf{c}$ as well, and also directly forces $v \subset \dot{\boldsymbol{x}}_{\text{left}}$. Then by necessity $v \in R$, hence $T = R\restriction_v \in \mathbb{P}$, but $u \notin T$. Let $T' = \varrho \cdot T$, where $\varrho \in 2^{<\omega}$ satisfies $R' = \varrho \cdot R$. By definition, the condition $\langle T, T' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ directly forces $\mathbf{c} \notin [\sigma \cdot T]$ (witnessed by $s = \sigma \cdot u$), as required. □

Theorem 9.3 *With the assumptions of Definition 7.1, suppose that $\mathbf{c} = \{C_m^i\}_{m<\omega, i<2} \in \mathfrak{M}$ is a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name, and for every $\sigma \in 2^{<\omega}$ the set*

$$D_\sigma = \{\langle T, T' \rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P} : \langle T, T' \rangle \textit{ directly forces } \mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\text{left}} \textit{ and } \mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\text{right}}\}$$

is dense in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$. Let $\langle W, W' \rangle \in (\mathbb{P} \cup \mathbb{U}) \times_{\mathsf{E}_0} (\mathbb{P} \cup \mathbb{U})$ and $U \in \mathbb{U}$. Then there is a stronger condition $\langle V, V' \rangle \in \mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$, $\langle V, V' \rangle \leq \langle W, W' \rangle$, which directly forces $\mathbf{c} \notin [U]$.

Proof. By construction, $U = \varrho \cdot U_K^\Phi(\to s_0)$, where $K < \omega$ and $\varrho, s_0 \in 2^{<\omega}$; we can assume that simply $s_0 = \Lambda$, so that $U = \varrho \cdot U_K^\Phi$. Moreover we can assume that $\varrho = \Lambda$ as well, so that $U = U_K^\Phi$ (for if not then replace $\mathbf{c}$ with $\varrho \cdot \mathbf{c}$).

Further, by Corollary 7.5, we can assume that $W = \sigma \cdot U_L^\Phi(\to t_0) \in \mathbb{U}$ and $W' = \sigma' \cdot U_L^\Phi(\to t_0') \in \mathbb{U}$, where $L < \omega$, $t_0, t_0' \in 2^{<\omega}$, $\mathrm{lh}(t_0) = \mathrm{lh}(t_0')$, and $\sigma, \sigma' \in 2^{<\omega}$. And moreover we can assume that $\sigma = \sigma' = \Lambda$, so that $W = U_L^\Phi(\to t_0)$ and $W' = U_L^\Phi(\to t_0')$ (for if not then replace $\mathbf{c}$ with $\langle \sigma, \sigma' \rangle \circ \mathbf{c}$).

The indices K, L involved can be either equal or different.

There is an index J such that the multitree $\varphi(J)$ satisfies $K, L \in |\varphi(J)|$ and $h_L^{\varphi(J)} \geq h_0 = \mathrm{lh}(t_0) = \mathrm{lh}(t'_0)$, so that the trees $S_0 = T_K^{\varphi(J)}(0) = T_K^{\Phi}(0)$,

$$T_0 = T_L^{\varphi(J)}(h_0)(\to t_0) = T_L^{\Phi}(t_0), \quad T'_0 = T_L^{\varphi(J)}(h_0)(\to t'_0) = T_L^{\Phi}(t'_0)$$

in $\mathbb{P}$ are defined. Note that $U \subseteq S_0$ and $W \subseteq T_0$, $W' \subseteq T'_0$ under the above assumptions.

Let $\mathscr{D}$ be the set of all multitrees $\varphi \in \mathrm{MT}(\mathbb{P})$ such that $\varphi(J) \preccurlyeq \varphi$ and for every pair $t, t' \in 2^n$, where $n = h_L^{\varphi}$, such that $t(n-1) \neq t'(n-1)$, the condition $\langle T_L^{\varphi}(t), T_L^{\varphi}(t')\rangle$ directly forces $\mathbf{c} \notin [T_K^{\varphi}(m)]$, where $m = h_K^{\varphi}$.

Claim 9.4 *The set $\mathscr{D}$ is dense in* $\mathrm{MT}(\mathbb{P})$ *above* $\varphi(J)$.

Proof. Let a multitree $\psi \in \mathrm{MT}(\mathbb{P})$ satisfy $\varphi(J) \preccurlyeq \psi$; the goal is to define a multitree $\varphi \in \mathscr{D}$, $\psi \preccurlyeq \varphi$. Let $m = h_K^{\psi}$, $n = h_L^{\psi}$, $Q = T_K^{\psi}(m)$, $P = T_L^{\psi}(n)$.

Case 1: $K \neq L$. Consider any $s \in 2^{m+1}$ and $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$. By Lemma 9.1, there is a tree $S \in \mathbb{P}$ and a condition $\langle R, R'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ such that $S \subseteq Q(\to s)$, $\langle R, R'\rangle \leq \langle P(\to t), P(\to t')\rangle$, and $\langle R, R'\rangle$ directly forces $\mathbf{c} \notin [S]$. By Lemma 4.1(ii),(iv) there are trees $Q_1 \in \mathrm{LC}_{m+1}(\mathbb{P})$ and $P_1 \in \mathrm{LC}_{n+1}(\mathbb{P})$ such that $Q_1 \subseteq_{m+1} Q$, $P_1 \subseteq_{n+1} P$, $Q_1(\to s) = S$ and $\langle P_1(\to t), P_1(\to t')\rangle \leq \langle R, R'\rangle$.

Repeat this procedure so that all strings $s \in 2^{m+1}$ and all pairs of strings $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$ are considered. We obtain trees $Q' \in \mathrm{LC}_{m+1}(\mathbb{P})$ and $P' \in \mathrm{LC}_{n+1}(\mathbb{P})$ such that $Q' \subseteq_{m+1} Q$, $P' \subseteq_{n+1} P$, and if $s \in 2^{m+1}$ and $t, t' \in 2^{n+1}$, $t(n) \neq t'(n)$, the condition $\langle P'(\to t), P'(\to t')\rangle$ directly forces $\mathbf{c} \notin [Q'(\to s)]$—hence directly forces $\mathbf{c} \notin [Q']$.

Now define a multitree $\varphi \in \mathrm{MT}(\mathbb{P})$ so that $|\varphi| = |\psi|$, $h_k^{\varphi} = h_k^{\psi}$ and $\tau_k^{\varphi} = \tau_k^{\psi}$ for all $k \notin \{K, L\}$, $h_K^{\varphi} = m+1$, $h_L^{\varphi} = n+1$, and $T_K^{\varphi}(m+1) = P'$, $T_L^{\varphi}(n+1) = Q'$ as the new elements of the Kth and Lth components. We have $\varphi \in \mathscr{D}$ and $\psi \preccurlyeq \varphi$ by construction. (Use the fact that $P' \subseteq_{n+1} P$ and $Q' \subseteq_{m+1} Q$.)

Case 2: $L = K$, and hence $m = n$ and $P = Q$. Let $h = \mathrm{spl}_n(P)$. Consider any pair $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$. In our assumptions there is a condition $\langle U, U'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle U, U'\rangle \leq \langle T(\to t), T(\to t')\rangle$, which directly forces both $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\mathrm{left}}$ and $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\mathrm{right}}$ for any $\sigma \in 2^h$. By Lemma 9.2, there is a stronger condition $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T'\rangle \leq \langle U, U'\rangle$, which directly forces both $\mathbf{c} \notin [\sigma \cdot T]$ and $\mathbf{c} \notin [\sigma \cdot T']$ still for all $\sigma \in 2^h$. Then as in Case 1, there is a tree $P_1 \in \mathrm{LC}_{n+1}(\mathbb{P})$, $P_1 \subseteq_{n+1} P$, such that $P_1(\to t) \subseteq T$, $P_1(\to t') \subseteq T'$.

We claim that $\langle T, T'\rangle$ directly forces $\mathbf{c} \notin [P_1]$, or equivalently, directly forces $\mathbf{c} \notin [P_1(\to s^\frown i)]$ for any $s^\frown i \in 2^{n+1}$ (then $s \in 2^n$). Indeed if $s^\frown i \in 2^{n+1}$ then $P_1(\to s^\frown i) = \sigma \cdot P_1(\to t)$ or $= \sigma \cdot P_1(\to t')$ for some $\sigma \in 2^h$ by the choice of h. Therefore $P_1(\to s^\frown i)$ is a subtree of one of the two trees $\sigma \cdot T$ and $\sigma \cdot T'$. The claim now follows from the choice of $\langle T, T'\rangle$. We conclude that the stronger condition $\langle P_1(\to t), P_1(\to t')\rangle \leq \langle T, T'\rangle$ also directly forces $\mathbf{c} \notin [P_1]$.

Repeat this procedure so that all pairs of strings $t, t' \in 2^{n+1}$ with $t(n) \neq t'(n)$ are considered. We obtain a tree $P' \in \mathrm{LC}_{n+1}(\mathbb{P})$ such that $P' \subseteq_{n+1} P$, and if $t, t' \in 2^{n+1}$, $t(n) \neq t'(n)$, then $\langle P'(\to t), P'(\to t')\rangle$ directly forces $\mathbf{c} \notin [P']$.

Similar to Case 1, define a multitree $\varphi \in \mathrm{MT}(\mathbb{P})$ so that $|\varphi| = |\psi|$, $h_k^{\varphi} = h_k^{\psi}$ and $\tau_k^{\varphi} = \tau_k^{\psi}$ for all $k \neq K$, $h_K^{\varphi} = n+1$, and $T_K^{\varphi}(n+1) = P'$ as the new element of the $(K = L)$th component. Then $\varphi \in \mathscr{D}$, $\psi \preccurlyeq \varphi$. □

We come back to the proof of Theorem 9.3. The lemma implies that there is an index $j \geq J$ such that the multitree $\varphi(j)$ belongs to $\mathscr{D}$. Let $n = h_L^{\varphi(j)}$, $m = h_K^{\varphi(j)}$. Pick strings $t, t' \in 2^n$ such that $t_0 \subset t$, $t'_0 \subset t'$, $t(n) \neq t'(n)$. Let

$$T = T_L^{\varphi(j)}(t) = T_L^{\Phi}(t),\, T' = T_L^{\varphi(j)}(t') = T_L^{\Phi}(t'),\, S = T_K^{\varphi(j)}(m) = T_K^{\Phi}(m).$$

Then $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$, $\langle T, T'\rangle \leq \langle T_0, T'_0\rangle$, and $\langle T, T'\rangle$ directly forces $\mathbf{c} \notin [S]$.

Consider the condition $\langle V, V'\rangle \in \mathbb{U} \times_{\mathsf{E}_0} \mathbb{U}$, where $V = U_L^{\Phi}(\to t)$ and $V' = U_L^{\Phi}(\to t')$ belong to $\mathbb{U}$. (Recall that $V = U_L^{\Phi}(\to t)$ and $V' = U_L^{\Phi}(\to t')$, and hence $V' = \sigma \cdot V$ for a suitable $\sigma \in 2^{<\omega}$.) By construction we have both $\langle V, V'\rangle \leq \langle W, W'\rangle$ (as $t_0 \subseteq t, t'$) and $\langle V, V'\rangle \leq \langle T, T'\rangle \leq \langle T_0, T'_0\rangle$. Therefore $\langle V, V'\rangle$ directly forces $\mathbf{c} \notin [S]$. And finally, we have $U \subseteq T_K^{\varphi(j)}(m) = S$, so that $\langle V, V'\rangle$ directly forces $\mathbf{c} \notin [U]$, as required. □

10 Jensen's forcing

In this section, we argue in $\mathbf{L}$, the constructible universe. Let $\leq_{\mathbf{L}}$ be the canonical wellordering of $\mathbf{L}$.

Definition 10.1 In $\mathbf{L}$, following the construction in [9, § 3] *mutatis mutandis*, define, by induction on $\xi < \omega_1$, a countable large-tree forcing notion $\mathbb{U}_\xi \subseteq \mathrm{LT}$ as follows.

Let $\mathbb{U}_0$ consist of all trees of the form $T[s]$, see Example 2.2.

Suppose that $0 < \lambda < \omega_1$, and countable large-tree forcing notions $\mathbb{U}_\xi \subseteq \mathrm{LT}$ are defined for $\xi < \lambda$. Let $\mathfrak{M}_\lambda$ be the least model $\mathfrak{M}$ of ZFC' of the form $\mathbf{L}_\kappa$, $\kappa < \omega_1$, containing $\{\mathbb{U}_\xi\}_{\xi<\lambda}$ and such that $\lambda < \omega_1^{\mathfrak{M}}$ and all sets $\mathbb{U}_\xi$, $\xi < \lambda$, are countable in $\mathfrak{M}$. Then $\mathbb{P}_\lambda = \bigcup_{\xi<\lambda} \mathbb{U}_\xi$ is countable in $\mathfrak{M}$, too. Let $\{\varphi(j)\}_{j<\omega}$ be the $\le_{\mathbf{L}}$-least sequence of multitrees $\varphi(j) \in \mathrm{MT}(\mathbb{P}_\lambda)$, $\preccurlyeq$-increasing and generic over $\mathfrak{M}_\lambda$. Define $\mathbb{U}_\lambda = \mathbb{U}$ as in Definition 7.1. This completes the inductive step.

Let $\mathbb{P} = \bigcup_{\xi<\omega_1} \mathbb{U}_\xi$.

Proposition 10.2 *In $\mathbf{L}$, the sequence $\{\mathbb{U}_\xi\}_{\xi<\omega_1}$ belongs to Δ_1^{HC}.*

Lemma 10.3 *In $\mathbf{L}$, if a set $D \in \mathfrak{M}_\xi$, $D \subseteq \mathbb{P}_\xi$ is pre-dense in $\mathbb{P}_\xi$ then it remains pre-dense in $\mathbb{P}$. Therefore if $\xi < \omega_1$ then $\mathbb{U}_\xi$ is pre-dense in $\mathbb{P}$. If a set $D \in \mathfrak{M}_\xi$, $D \subseteq \mathbb{P}_\xi \times_{\mathsf{E}_0} \mathbb{P}_\xi$ is pre-dense in $\mathbb{P}_\xi \times_{\mathsf{E}_0} \mathbb{P}_\xi$ then it is pre-dense in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$.*

Proof. By induction on $\lambda \ge \xi$, if D is pre-dense in $\mathbb{P}_\lambda$ then it remains pre-dense in $\mathbb{P}_{\lambda+1} = \mathbb{P}_\lambda \cup \mathbb{U}_\lambda$ by Lemma 7.6. Limit steps are obvious. To prove the second claim note that $\mathbb{U}_\xi$ is dense in $\mathbb{P}_{\xi+1}$ by Lemma 7.3, and $\mathbb{U}_\xi \in \mathfrak{M}_{\xi+1}$.

To prove the last claim use Lemma 7.7. □

Lemma 10.4 *In $\mathbf{L}$, if $X \subseteq \mathrm{HC} = \mathbf{L}_{\omega_1}$ then the set W_X of all ordinals $\xi < \omega_1$ such that $\langle \mathbf{L}_\xi\,;\, X \cap \mathbf{L}_\xi\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}\,;\, X\rangle$ and $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ is unbounded in ω_1. More generally, if $X_n \subseteq \mathrm{HC}$ for all n then the set W of all ordinals $\xi < \omega_1$, such that $\langle \mathbf{L}_\xi\,;\, \{X_n \cap \mathbf{L}_\xi\}_{n<\omega}\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}\,;\, \{X_n\}_{n<\omega}\rangle$ and $\{X_n \cap \mathbf{L}_\xi\}_{n<\omega} \in \mathfrak{M}_\xi$, is unbounded in ω_1.*

Proof. Let $\xi_0 < \omega_1$. Let M be a countable elementary submodel of $\mathbf{L}_{\omega_2}$ containing ξ_0, ω_1, X, and such that $M \cap \mathrm{HC}$ is transitive. Let $\varphi : M \xrightarrow{\text{onto}} \mathbf{L}_\lambda$ be the Mostowski collapse, and let $\xi = \varphi(\omega_1)$. Then $\xi_0 < \xi < \lambda < \omega_1$ and $\varphi(X) = X \cap \mathbf{L}_\xi$ by the choice of M. It follows that $\langle \mathbf{L}_\xi\,;\, X \cap \mathbf{L}_\xi\rangle$ is an elementary submodel of $\langle \mathbf{L}_{\omega_1}\,;\, X\rangle$. Moreover, ξ is uncountable in $\mathbf{L}_\lambda$, hence $\mathbf{L}_\lambda \subseteq \mathfrak{M}_\xi$. We conclude that $X \cap \mathbf{L}_\xi \in \mathfrak{M}_\xi$ since $X \cap \mathbf{L}_\xi \in \mathbf{L}_\lambda$ by construction.

The second claim does not differ much: we start with a model M containing both the whole sequence $\{X_n\}_{n<\omega}$ and each particular X_n, and so on. □

Corollary 10.5 *The forcing notions $\mathbb{P}$ and $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ satisfy the c.c.c. in $\mathbf{L}$.*

Proof. (Compare to [9, Lemma 6].) Suppose that $A \subseteq \mathbb{P}$ is a maximal antichain. By Lemma 10.4, there is an ordinal ξ such that $A' = A \cap \mathbb{P}_\xi$ is a maximal antichain in $\mathbb{P}_\xi$ and $A' \in \mathfrak{M}_\xi$. But then A' remains pre-dense, therefore, still a maximal antichain, in the whole set $\mathbb{P}$ by Lemma 10.3. It follows that $A = A'$ is countable. □

11 The model

We view the sets $\mathbb{P}$ and $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ (Definition 10.1) as forcing notions over $\mathbf{L}$.

Lemma 11.1 *A real $x \in 2^\omega$ is $\mathbb{P}$-generic over $\mathbf{L}$ iff $x \in Z = \bigcap_{\xi<\omega_1^{\mathbf{L}}} \bigcup_{U\in\mathbb{U}_\xi} [U]$.*

Proof. (Compare to [9, Lemma 7].) If $\xi < \omega_1^{\mathbf{L}}$ then $\mathbb{U}_\xi$ is pre-dense in $\mathbb{P}$ by Lemma 10.3, therefore any real $x \in 2^\omega$ $\mathbb{P}$-generic over $\mathbf{L}$ belongs to $\bigcup_{U\in\mathbb{U}_\xi} [U]$.

To prove the converse, suppose that $x \in Z$ and prove that x is $\mathbb{P}$-generic over $\mathbf{L}$. Consider a maximal antichain $A \subseteq \mathbb{P}$ in $\mathbf{L}$; we have to prove that $x \in \bigcup_{T\in A} [T]$. Note that $A \subseteq \mathbb{P}_\xi$ for some $\xi < \omega_1^{\mathbf{L}}$ by Corollary 10.5. But then every tree $U \in \mathbb{U}_\xi$ satisfies $U \subseteq^{\mathrm{fin}} \bigcup A$ by Lemma 7.6, so that $\bigcup_{U\in\mathbb{U}_\xi} [U] \subseteq \bigcup_{T\in A} [T]$, and hence $x \in \bigcup_{T\in A} [T]$, as required. □

Corollary 11.2 *In any generic extension of $\mathbf{L}$, the set of all reals in 2^ω $\mathbb{P}$-generic over $\mathbf{L}$ is Π_1^{HC} and Π_2^1.*

Proof. (Compare to [9, Corollary 9].) Use Lemma 11.1 and Proposition 10.2. □

Definition 11.3 From now on, we assume that $G \subseteq \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is a set $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-generic over $\mathbf{L}$, so that the intersection $X = \bigcap_{\langle T,T'\rangle \in G} [T] \times [T']$ is a singleton $X_G = \{\langle x_{\text{left}}[G], x_{\text{right}}[G]\rangle\}$.

Compare the next lemma to [9, Lemma 10]. While Jensen's forcing notion in [9] guarantees that there is a single generic real in the extension, the forcing notion $\mathbb{P}$ we use adds a whole E_0-class (a countable set) of generic reals!

Lemma 11.4 (under the assumptions of Definition 11.3) *If $y \in \mathbf{L}[G] \cap 2^\omega$ then y is a $\mathbb{P}$-generic real over $\mathbf{L}$ iff $y \in [x_{\text{left}}[G]]_{\mathsf{E}_0} \cup [x_{\text{right}}[G]]_{\mathsf{E}_0}$.*

Recall that $[x]_{\mathsf{E}_0} = \{\sigma \cdot x : \sigma \in 2^{<\omega}\}$.

Proof. The reals $x_{\text{left}}[G]$, $x_{\text{right}}[G]$ are separately $\mathbb{P}$-generic (see Remark 5.1). It follows that any real $y = \sigma \cdot x_{\text{left}}[G] \in [x_{\text{left}}[G]]_{\mathsf{E}_0}$ or $y = \sigma \cdot x_{\text{right}}[G] \in [x_{\text{right}}[G]]_{\mathsf{E}_0}$ is $\mathbb{P}$-generic as well since the forcing $\mathbb{P}$ is by definition invariant under the action of any $\sigma \in 2^{<\omega}$.

To prove the converse, suppose towards the contrary that there is a condition $\langle T, T'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ and a $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-real name $\mathbf{c} = \{C^i_n\}_{n<\omega,\, i=0,1} \in \mathbf{L}$ such that $\langle T, T'\rangle$ $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-forces that $\mathbf{c}$ is $\mathbb{P}$-generic while $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ forces both formulas $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\text{left}}$ and $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\text{left}}$ for all $\sigma \in 2^{<\omega}$.

Let $C_n = C^0_n \cup C^1_n$, this is a pre-dense set in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$. It follows from Lemma 10.4 that there exists an ordinal $\lambda < \omega_1$ such that each set $C'_n = C_n \cap (\mathbb{P}_\lambda \times_{\mathsf{E}_0} \mathbb{P}_\lambda)$ is pre-dense in $\mathbb{P}_\lambda \times_{\mathsf{E}_0} \mathbb{P}_\lambda$, and the sequence $\{C'_{ni}\}_{n<\omega,\, i=0,1}$ belongs to $\mathfrak{M}_\lambda$, where $C'_{ni} = C'_n \cap C^i_n$—then C'_n is pre-dense in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ too, by Lemma 10.3. Therefore we can assume that in fact $C_n = C'_n$, that is, $\mathbf{c} \in \mathfrak{M}_\lambda$ and $\mathbf{c}$ is a $(\mathbb{P}_\lambda \times_{\mathsf{E}_0} \mathbb{P}_\lambda)$-real name.

Further, as $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ forces that $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\text{left}}$ and $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\text{right}}$, the set $D(\sigma)$ of all conditions $\langle S, S'\rangle \in \mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ which directly force $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\text{left}}$ and $\mathbf{c} \neq \sigma \cdot \dot{\boldsymbol{x}}_{\text{right}}$, is dense in $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$—for every $\sigma \in 2^{<\omega}$. Therefore, still by Lemma 10.4, we may assume that the same ordinal λ as above satisfies the following: each set $D'(\sigma) = D(\sigma) \cap (\mathbb{P}_\lambda \times_{\mathsf{E}_0} \mathbb{P}_\lambda)$ is dense in $\mathbb{P}_\lambda \times_{\mathsf{E}_0} \mathbb{P}_\lambda$.

Applying Theorem 9.3 with $\mathbb{P} = \mathbb{P}_\lambda$, $\mathbb{U} = \mathbb{U}_\lambda$, and $\mathbb{P} \cup \mathbb{U} = \mathbb{P}_{\lambda+1}$, we conclude that for each tree $U \in \mathbb{U}_\lambda$ the set Q_U of all conditions $\langle V, V'\rangle \in \mathbb{P}_{\lambda+1} \times_{\mathsf{E}_0} \mathbb{P}_{\lambda+1}$ which directly force $\mathbf{c} \notin [U]$, is dense in $\mathbb{P}_{\lambda+1} \times_{\mathsf{E}_0} \mathbb{P}_{\lambda+1}$. As obviously $Q_U \in \mathfrak{M}_{\lambda+1}$, we further conclude that Q_U is pre-dense in the whole forcing $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ by Lemma 10.3. This implies that $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ forces $\mathbf{c} \notin \bigcup_{U \in \mathbb{U}_\lambda} [U]$, hence, forces that $\mathbf{c}$ is not $\mathbb{P}$-generic, by Lemma 11.1. But this contradicts to the choice of $\langle T, T'\rangle$. □

Corollary 11.5 *The set $[x_{\text{left}}[G]]_{\mathsf{E}_0} \cup [x_{\text{right}}[G]]_{\mathsf{E}_0}$ is Π^1_2 set in $\mathbf{L}[G]$. Therefore the two element set $\{[x_{\text{left}}[G]]_{\mathsf{E}_0}, [x_{\text{right}}[G]]_{\mathsf{E}_0}\}$ is* OD *in $\mathbf{L}[G]$.*

Corollary 11.6 *The E_0-classes $[x_{\text{left}}[G]]_{\mathsf{E}_0}$, $[x_{\text{right}}[G]]_{\mathsf{E}_0}$ are disjoint.*

Proof. Corollary 5.4 implies $x_{\text{left}}[G] \not\mathsf{E}_0\, x_{\text{right}}[G]$. □

Lemma 11.7 (still under the assumptions of Definition 11.3) *Neither of the two E_0-classes $[x_{\text{left}}[G]]_{\mathsf{E}_0}$, $[x_{\text{right}}[G]]_{\mathsf{E}_0}$ is* OD *in $\mathbf{L}[G]$.*

Proof. Suppose towards the contrary that there is a condition $\langle T, T'\rangle \in G$ and a formula $\vartheta(x)$ with ordinal parameters such that $\langle T, T'\rangle$ $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-forces that $\vartheta([\dot{\boldsymbol{x}}_{\text{left}}]_{\mathsf{E}_0})$ but $\neg\, \vartheta([\dot{\boldsymbol{x}}_{\text{right}}]_{\mathsf{E}_0})$. However both the formula and the forcing are invariant under actions of strings in $2^{<\omega}$. In particular if $\sigma \in 2^{<\omega}$ then $\langle \sigma \cdot T, \sigma \cdot T'\rangle$ still $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-forces $\vartheta([\dot{\boldsymbol{x}}_{\text{left}}]_{\mathsf{E}_0})$ and $\neg\, \vartheta([\dot{\boldsymbol{x}}_{\text{right}}]_{\mathsf{E}_0})$. We can take σ which satisfies $T' = \sigma \cdot T$; thus $\langle T', T\rangle$ still $(\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P})$-forces $\vartheta([\dot{\boldsymbol{x}}_{\text{left}}]_{\mathsf{E}_0})$ and $\neg\, \vartheta([\dot{\boldsymbol{x}}_{\text{right}}]_{\mathsf{E}_0})$.[3] However $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ is symmetric with respect to the left-right exchange, which implies that conversely $\langle T', T\rangle$ has to force $\vartheta([\dot{\boldsymbol{x}}_{\text{right}}]_{\mathsf{E}_0})$ and $\neg\, \vartheta([\dot{\boldsymbol{x}}_{\text{left}}]_{\mathsf{E}_0})$. The contradiction proves the lemma. □

This concludes the proof of Theorem 1.1.

12 Concluding remarks

First, one may ask whether other Borel equivalence relations E admit results similar to Theorem 1.1. Fortunately this question can be easily solved on the base of the Glimm-Effros dichotomy theorem [6].

[3] This is the argument which does not go through for the full product $\mathbb{P} \times \mathbb{P}$.

Corollary 12.1 *The following is true in the model of Theorem* 1.1. *Let* E *be a Borel equivalence relation on* ω^ω *coded in* $\mathbf{L}$. *Then there exists an* OD *pair of* E*-equivalence classes* $\{[x]_\mathsf{E}, [y]_\mathsf{E}\}$ *such that neither of the classes* $[x]_\mathsf{E}$, $[y]_\mathsf{E}$ *is separately* OD, *iff* E *is not smooth.*

Proof. Suppose first that E is smooth. By the Shoenfield absoluteness theorem, the smoothness can be witnessed by a Borel map $\vartheta : \omega^\omega \to \omega^\omega$ coded in $\mathbf{L}$, hence, ϑ is OD itself. If $p = \{[x]_\mathsf{E}, [y]_\mathsf{E}\}$ is OD in the extension then so is the 2-element set $R = \{\vartheta(z) : z \in [x]_\mathsf{E} \cup [y]_\mathsf{E}\} \subseteq \omega^\omega$, whose both elements (reals), say p_x and p_y, are OD by obvious reasons. Then finally $[x]_\mathsf{E} = \vartheta^{(-1)}(p_x)$ and $[y]_\mathsf{E} = \vartheta^{(-1)}(p_y)$ are OD as required.

Now let E be non-smooth. Then by Shoenfield and the Glimm-Effros dichotomy theorem in [6], there is a continuous, coded by some $r \in \omega^\omega \cap \mathbf{L}$, hence, OD, reduction $\vartheta : 2^\omega \to \omega^\omega$ of E_0 to E, so that we have $a \mathrel{\mathsf{E}_0} b$ iff $\vartheta(a) \mathrel{\mathsf{E}} \vartheta(b)$ for all $a, b \in 2^\omega$. Let, by Theorem 1.1, $\{[a]_{\mathsf{E}_0}, [b]_{\mathsf{E}_0}\}$ be a Π^1_2 pair of non-OD E_0-equivalence classes. By the choice of ϑ, one easily proves that $\{[\vartheta(a)]_\mathsf{E}, [\vartheta(b)]_\mathsf{E}\}$ is a $\Pi^1_2(r)$ pair of non-OD E-equivalence classes. □

Secondly, one may ask what happens with the Groszek-Laver pairs of sets of reals in better known models. For some of them the answer tends to be in the negative. Consider, e.g., the Solovay model of ZFC in which all projective sets of reals are Lebesgue measurable [16]. Arguing in the Solovay model, let $\{X, Y\}$ be an OD set, where $X, Y \subseteq 2^\omega$. Then the set of *four* sets $X \setminus Y$, $Y \setminus X$, $X \cap Y$, $2^\omega \setminus (X \cup Y)$ is still OD, and hence we have an OD equivalence relation E on 2^ω with four (or fewer if say $X \subseteq Y$) equivalence classes. By a theorem of [10][4], either E admits an OD reduction $\vartheta : 2^\omega \to 2^{<\omega_1}$ to equality on $2^{<\omega_1}$ or E_0 admits a continuous reduction to E. The "or" option fails since E has finitely many classes.

The "either" option leads to a finite (not more than 4 elements) OD set $R = \operatorname{ran}(\vartheta) \subseteq 2^{<\omega_1}$. An easy argument shows that then every $r \in R$ is OD, and hence so is the corresponding E-class $\vartheta^{-1}(r)$. It follows that X, Y themselves are OD.

Question 12.2 Is it true in the Solovay model that every *countable* OD set $W \subseteq \wp(\omega^\omega)$ *of sets of reals* contains an OD element $X \in W$ (a set of reals)?

An uncountable counterexample readily exists, for take the set of all non-OD sets of reals. As for sets $W \subseteq \omega^\omega$, any countable OD set of reals in the Solovay model consists of OD elements, e.g., by the result mentioned in Footnote 4.

Thirdly, one may ask whether a forcing similar to $\mathbb{P} \times_{\mathsf{E}_0} \mathbb{P}$ with respect to the results in § 11, exists in ground models other than $\mathbf{L}$ or $\mathbf{L}[x]$, $x \in 2^\omega$. Some coding forcing constructions with perfect trees do exist in such a general frameworks, cf. [1, 12].

Acknowledgements The authors thank Ali Enayat for the interest in the problem and helpful remarks. The authors thank the anonymous referee for many important suggestions that helped to improve the text. The second author (Vladimir Kanovei) thanks IPM for hospitality and support during his visit to Tehran in January 2015 where his part of work was accomplished. The first author was supported in part by IPM Grant #91030417; The second author was supported in part by RFBR Grant #13-01-00006; the third author was supported in part by RNF Grant #14-50-00150.

References

[1] J. Bagaria and V. Kanovei, On coding uncountable sets by reals, Math. Log. Q. **56**(4), 409–424 (2010).
[2] A. Enayat, On the Leibniz-Mycielski axiom in set theory, Fund. Math. **181**(3), 215–231 (2004).
[3] A. Enayat, Ordinal definable numbers, posting on the Foundations of Mathematics mailing list FOM, 23 July 2010.
[4] M. Groszek and R. Laver, Finite groups of OD-conjugates, Period. Math. Hung. **18**, 87–97 (1987).
[5] G. Gulbenkian, A question about ordinal definable real numbers, Mathoverflow posting, 9 March 2010, `http://mathoverflow.net/questions/17608`.
[6] L. A. Harrington, A. S. Kechris, and A. Louveau, A Glimm-Effros dichotomy for Borel equivalence relations, J. Amer. Math. Soc. **3**(4), 903–928 (1990).
[7] L. A. Harrington, D. Marker, and S. Shelah, Borel orderings, Trans. Amer. Math. Soc. **310**(1), 293–302 (1988).
[8] T. Jech, Set Theory, the third millennium revised and expanded edition (Springer, 2003).

[4] To replace the following brief argument, one can also refer to a result by Stern implicit in [17]: in the Solovay model, if an OD equivalence relation E has at least one non-OD equivalence class then there is a pairwise E-inequivalent perfect set.

[9] R. Jensen, Definable sets of minimal degree, in: Mathematical Logic and Foundations of Set Theory; Proceedings of an international colloquium under the auspices of the Israel Academy of Sciences and Humanities, Jerusalem, 11–14 November 1968, edited by Y. Bar-Hillel, Studies in Logic and the Foundations of Mathematics (North-Holland, 1970), pp. 122–128.
[10] V. Kanovei, An Ulm-type classification theorem for equivalence relations in Solovay model, J. Symb. Log. **62**(4), 1333–1351 (1997).
[11] V. Kanovei, Borel Equivalence Relations. Structure and Classification, University Lecture Series Vol. 44 (American Mathematical Society, 2008).
[12] V. Kanovei and V. Lyubetsky, An effective minimal encoding of uncountable sets, Sib. Math. J. **52**(5), 854–863 (2011).
[13] V. Kanovei and V. Lyubetsky, A countable definable set of reals containing no definable elements, preprint, arXiv:1408.3901 (2014).
[14] V. Kanovei and V. Lyubetsky, A definable E_0-class containing no definable elements, Arch. Math. Log. **54**(5), 711–723 (2015).
[15] V. Kanovei, M. Sabok, and J. Zapletal, Canonical Ramsey Theory on Polish Spaces, Cambridge Tracts in Mathematics Vol. 202 (Cambridge University Press, 2013).
[16] R. M. Solovay, A model of set-theory in which every set of reals is Lebesgue measurable, Ann. Math. (2) **92**, 1–56 (1970).
[17] J. Stern, On Lusin's restricted continuum problem, Ann. Math. (2) **120**, 7–37 (1984).

