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Borel and Countably Determined Reducibility in Nonstandard Domain

By

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Abstract. We consider, in a nonstandard domain, reducibility of equivalence relations in terms of the Borel reducibility $\leqslant_{\mathrm{B}}$ and the countably determined (CD, for brevity) reducibility $\leqslant_{\mathrm{CD}}$. This reveals phenomena partially analogous to those discovered in modern "standard" descriptive set theory. The $\leqslant_{\mathrm{CD}}$-structure of CD sets (partially) and the $\leqslant_{\mathrm{B}}$-structure of Borel sets (completely) in ${}^*\mathbb{N}$ are described. We prove that all "countable" (i.e., those with countable equivalence classes) CD equivalence relations (ERs) are CD-smooth, but not all are B-smooth: the relation $x \mathsf{M}_{\mathbb{N}} y$ iff $|x-y| \in \mathbb{N}$ is a counterexample. Similarly to the Silver dichotomy theorem in Polish spaces, any CD equivalence relation on ${}^*\mathbb{N}$ either has at most continuum-many classes (and this can be witnessed, in some manner, by a countably determined function) or there is an infinite internal set of pairwise inequivalent elements. Our study of *monadic* equivalence relations, i.e., those of the form $x \mathsf{M}_U y$ iff $|x-y| \in U$, where U is an additive countably determined cut (initial segment) demonstrates that these ERs split in two linearly $\leqslant_{\mathrm{B}}$-(pre)ordered families, associated with countably cofinal and countably coinitial cuts. The equivalence $u \,\mathsf{FD}\, v$ iff $u \,\Delta\, v$ is finite, on the set of all hyperfinite subsets of ${}^*\mathbb{N}$, $\leqslant_{\mathrm{B}}$-reduces all "countably cofinal" ERs but does not $\leqslant_{\mathrm{CD}}$-reduce any of "countably coinitial" ERs.

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Classical descriptive set theory (DST, for brevity) is mainly concentrated on sets in Polish (complete separable) spaces, see Kechris [12]. It was discovered in the 1980s that ideas of classical DST can be meaningfully developed in a very different setting of nonstandard analysis, where Polish spaces are replaced by internal hyperfinite sets. This alternative version of descriptive set theory is called "hyperfinite", or "nonstandard" DST. It allows to define Borel and projective hierarchies of subsets of a fixed infinite internal (for instance, hyperfinite) domain in quite the same manner as "Polish", i.e., classical DST does, but beginning with internal sets at the initial level rather than open sets. Generally, the structures studied by the "nonstandard" DST appear to be similar, in some aspects, to those considered in the "Polish" descriptive set theory, but different in some other

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aspects. As for the proofs, they are mainly based on very different and rather combinatorial ideas, and (countable) Saturation, of course, see Keisler et al. [14]. "Nonstandard" DST also involves objects which hardly have any direct analogy in the "Polish" setting, like countably determined sets, leading to a remarkably interesting mixture of "Polish" and nonstandard concepts and methods.

This paper is written in an attempt to find nonstandard analogs of concepts which attract a lot of attention in "Polish" DST nowadays: the structure of definable (usually, Borel or analytic) equivalence relations in terms of Borel (sometimes more complicated) reducibility of associated quotient structures. Our results will be related to countably determined (or CD), in particular, Borel sets and equivalence relations on ${}^*\mathbb{N}$ and hyperfinite domains, and the reducibility by countably determined, in particular, by Borel maps.

It is an important difference comparing to "Polish" DST that while classically all uncountable Polish spaces are Borel isomorphic, hence indistinguishable w. r. t. topics in Borel reducibility, in "nonstandard" setting any two infinite hyperfinite sets X, Y admit a Borel bijection iff $\frac{\#X}{\#Y} \simeq 1$ and admit a CD bijection iff $\frac{\#X}{\#Y}$ is neither infinitesimal nor infinitely large (Proposition 2.2). This makes the structure of CD equivalence relations dependent not only on their intrinsic nature, i.e., the method of definition, but also on the size of the domain, which can be any internal infinite hyperfinite subset of ${}^*\mathbb{N}$. (Yet there is an idea of domain-independent approach, see the last remark in Section 16.)

This effect shows up already at the level of *B-smooth* ERs (those which admit a Borel enumeration of equivalence classes), which leads us to the study of Borel sets in terms of the relation $X \leqslant_{\mathrm{B}} Y$ meaning the existence of a Borel injection $\vartheta : X \to Y$. We prove (Theorem 3.1) that any Borel subset of ${}^*\mathbb{N}$ admits a Borel bijection onto a Borel cut (initial segment) in ${}^*\mathbb{N}$, therefore, any two Borel sets are comparable via the existence of a Borel injection, and generally there is a comprehensive classification of Borel subsets of ${}^*\mathbb{N}$ modulo $\equiv_{\mathrm{B}}$ (in other words, *Borel cardinalities*).

A complete classification of countably determined sets modulo $\equiv_{\mathrm{CD}}$ is not known, yet we show (Theorem 4.1) that, for any CD set $X \subseteq {}^*\mathbb{N}$, *either* there is a unique *additive* CD cut $C \subseteq {}^*\mathbb{N}$ (which can be equal to $\mathbb{N}$ or ${}^*\mathbb{N}$ itself) with $X \equiv_{\mathrm{CD}} C$, *or* there is a hyperinteger $c \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $c/\mathbb{N} <_{\mathrm{CD}} X <_{\mathrm{CD}} c\mathbb{N}$. (There are open questions related to *or* case, see Section 5.) Further, we prove (Theorem 5.2) that any CD set $X \subseteq c\mathbb{N}$ with $c/\mathbb{N} \leqslant_{\mathrm{CD}} X$ satisfies $X \equiv_{\mathrm{CD}} M$, where M is a union of *monads* – sets of the form $x + (c/\mathbb{Z})$, $x \in c\mathbb{N}$, but whether such a set can satisfy $c/\mathbb{N} <_{\mathrm{CD}} X <_{\mathrm{CD}} c\mathbb{N}$ is not known.

In "Polish" theory, some most elementary examples of non-smooth (in the sense of Borel enumerations, of course) ERs belong to the type of *countable* ones, i.e., with all equivalence classes at most countable. We prove (Theorem 6.1) that, on the contrary, in the "nonstandard" DST any countable countably determined ER E admits a CD *transversal*, i.e., a set which has exactly one common element with each E-class, hence, is CD-smooth (but not necessarily has a Borel transversal and is B-smooth, i.e., with a Borel enumeration of the equivalence classes). This generalizes a recent theorem of Jin [7] that the (countable) equivalence relation $\mathsf{M}_{\mathbb{N}}$ defined on ${}^*\mathbb{N}$ by $x\,\mathsf{M}_{\mathbb{N}}\,y$ iff $|x-y|\in\mathbb{N}$ admits a countably determined transversal and is CD-smooth. On the other hand, by a typical measure-theoretic argument, $\mathsf{M}_{\mathbb{N}}$ is not Borel-smooth and does not admit a Borel transversal; this is a transparent demonstration of differences between Borel and countably determined structures.

It is known from studies in ‘‘Polish’’ descriptive set theory that any coanalitic (Silver [18]), resp., analytic (Burgess [1]) equivalence relation on a Polish space has $\leqslant\aleph_0$, resp., $\leqslant\aleph_1$ equivalence classes, or admits an uncountable closed set of pairwise inequivalent elements. Theorems of this sort are called *dichotomy theorems*. Our Theorem 7.1 has obvious similarities to these results: it asserts that a CD equivalence relation either admits a reduction to another equivalence relation having a rather small number of equivalence classes or admits a rather large internal set of pairwise inequivalent elements. Here the largeness and smallness are formulated in terms of a given countably cofinal additive cut $U\subseteq{}^*\mathbb{N}$. As a matter of fact, a true dichotomy, that is, the two cases are incompatible, is obtained only for *exponential* cuts U, i.e., those satisfying $x\in U\Longrightarrow 2^x\in U$. In particular, in the case $U=\mathbb{N}$ (Corollary 9.1), any CD equivalence relation E either admits a reduction to an equivalence relation on the (standard) reals or else admits an infinite internal pairwise inequivalent set, and the two cases are incompatible. (Note that, in the first case, E has at most $\mathfrak{c}$-many equivalence classes, which, generally speaking, cannot be improved to any smaller cardinality. However, if E is Borel then the number of E-classes is, in this case, either exactly $\mathfrak{c}$ or $\leqslant\aleph_0$, Theorem 9.2, furthermore, if E is $\mathbf{\Sigma}^0_1$ then the number of E-classes is $\leqslant\aleph_0$ in this case, Lemma 7.2.)

An important class of countably determined ERs which contains mostly non-CD-smooth relations, is the class of *monadic* equivalence relations. Given an additive cut (initial segment) $U\subseteq{}^*\mathbb{N}$, we define $x\,\mathsf{M}_U\,y$ iff $|x-y|\in U$, for all $x,y\in{}^*\mathbb{N}$. Thus, the M_U-class of any x is the *U-monad* of x in the sense of [15]. Since any additive CD cut (with the trivial exceptions of $\emptyset$ and ${}^*\mathbb{N}$) is either countably cofinal or countably coinitial (i.e., of the form, resp., $\bigcup_n[0,a_n)$ or $\bigcap_n[0,a_n)$, where $\{a_n\}_{n\in\mathbb{N}}$ is strictly increasing, resp., decreasing sequence of hyperintegers), countably determined monadic ERs split into two distinct families of *countably cofinal* and *countably coinitial* monadic ERs.

Our study of the reducibility phenomena among monadic equivalence relations in Sections 10–14 (summarized in Theorem 10.3) shows that ERs are mutually comparable within each of these two families, in such a way that $\mathsf{M}_U\leqslant_{\mathrm{B}}\mathsf{M}_V$ iff $\mathsf{M}_U\leqslant_{\mathrm{CD}}\mathsf{M}_V$ iff $\operatorname{rate}U\subseteq\operatorname{rate}V$, where, for any additive cut $\emptyset\neq U\subsetneqq{}^*\mathbb{N}$, rate U, the *rate* of U, is another, not necessarily additive cut, equal to the *thickness* of the cut $\log U$. In each of the two families, there is a subclass of $\leqslant_{\mathrm{B}}$-minimal (and $\leqslant_{\mathrm{CD}}$-minimal) ERs, namely, those generated by cuts of the form $c\mathbb{N}$ or $c/\mathbb{N}$, $c\in{}^*\mathbb{N}$ (in, resp., countably cofinal, coinitial case). Further, among all monadic ERs only those of the form $\mathsf{M}_{c\mathbb{N}}$ are CD-smooth (and all of them even admit a CD transversal, essentially by Jin [7]), but none of them is Borel-smooth. In addition, there is no relationship, in terms of $\leqslant_{\mathrm{B}}$ or $\leqslant_{\mathrm{CD}}$, between countably cofinal and countably coinitial ERs except that we have $\mathsf{M}_{c\mathbb{N}}\leqslant_{\mathrm{CD}}\mathsf{M}_V$ for any countably coinitial equivalence relation M_V.

Finally, we show in Section 15 that monadic ERs induced by countably cofinal cuts admit a natural upper $\leqslant_{\mathrm{B}}$-bound, namely, the equivalence relation of equality of hyperfinite subsets of ${}^*\mathbb{N}$ modulo a finite set. We denote this ER by FD; it has some analogy with the equivalence relation of equality of infinite subsets of $\mathbb{N}$ modulo a finite set, extensively studied in "Polish" descriptive set theory. We prove that $\mathsf{M}_U <_{\mathrm{B}} \mathsf{FD}$ holds for any countably cofinal additive cut U but fails for any countably coinitial additive U. It is not clear whether FD is a *minimal* upper bound for countably cofinal monadic ERs: this and some other open problems are considered in the final Section 16.

Note added in proof. After this paper was, essentially, accomplished in Spring 2002, the authors were informed by Pavol Zlatoš[1] that some of our results were earlier obtained by the followers of AST, the *alternative set theory* of Vopenka. More exactly, Theorem 3.1 belongs to Kalina and Zlatoš [10], Corollary 9.1 follows from an unpublished result of Vencovská, and some results overlapping with Theorem 4.1 were obtained by Kalina and Zlatoš [10]. Yet in order to preserve the integrity of the paper, we decided to keep the results, adding suitable comments and references (prepared also with the help of Zlatoš).

It is worthwhile to note that AST rather adequately describes the structure of hyperfinite, internal, Borel, and countably determined subsets of ${}^*\mathbb{N}$ in the assumption of the continuum-hypothesis CH (called the *two-cardinal hypothesis* in AST), so that any reasonable descriptive set theoretic argument in AST, not using the two-cardinal hypothesis, can be converted, by a certain change of terminology, into a proof by means of hyperfinite descriptive set theory; accordingly, if the two-cardinal hypothesis is involved then the result is a consequence of CH (and can also be treated in terms of consistency).

1. Notation

The set of all functions $f : Y \to X$ is denoted by ${}^{Y}X$, and x^y will denote the arithmetical power operation in standard and nonstandard domains. The set of all finite binary sequences is ${}^{<\omega}2 = \bigcup_{n \in \mathbb{N}} {}^{n}2$. $s^\wedge a$ is the extension of a finite sequence s by a new rightmost term a. The length of a finite sequence s is denoted by CH s. $f''X = \{f(x) : x \in X \cap \operatorname{dom} f\}$ is the *f-image* of a set X. $f^{-1}(Y) = \{x \in \operatorname{dom} f : f(x) \in Y\}$ is the *f-preimage* of a set Y. If P is a set of pairs then $x\ P\ y$ and $P(x, y)$ mean that $\langle x, y\rangle \in P$.

Some acquaintance of the reader with "hyperfinite" descriptive set theory is assumed; we give [14] as the basic reference. All "nonstandard" notions below, for instance ${}^*\mathbb{N}$, are related to a fixed countably saturated "nonstandard universe" (e.g., a nonstandard superstructure, as in [16]), whose elements will be referred to as *nonstandard* (internal or external) sets.

In the remainder, we typically use letters like i, j, k, m, n (with indices) for elements of $\mathbb{N}$, and letters like a, b, c, h, x, y, z for elements of ${}^*\mathbb{N}$. $\mathrm{P}_{\mathrm{int}}(X)$ is the

[1] Whom we also thank for many important remarks and corrections to the text.

set of all internal subsets of a nonstandard set X. If X, Y are internal sets then $({}^{Y}X)_{\text{int}}$ is the set of all *internal* $f : Y \to X$.

Numbers $c \in {}^*\mathbb{N}$ (standard or nonstandard) will be systematically identified with the sets $[0, c) = \{x : x < c\}$ of all smaller numbers. We shall often use $({}^{c}2)_{\text{int}}$, instead of the more pedantical $({}^{[0,c)}2)_{\text{int}}$ to denote the (internal) set of all internal functions $\xi : c = [0, c) \to 2$.

The number of elements of a hyperfinite set X is $\#X \in {}^*\mathbb{N}$. Let $r \simeq q$ mean that the difference $r - q$ is infinitesimal. For any bounded hyperrational α (i.e., $|\alpha| < c$ for some $c \in \mathbb{N}$) there is a unique standard real number r, denoted by $\operatorname{st}\alpha$, the *standard part*[2] of α, such that $\alpha \simeq r$. If α is unbounded then put $\operatorname{st}\alpha = +\infty$.

Borel classes $\mathbf{\Sigma}^0_1$, $\mathbf{\Pi}^0_1$ consist of countable unions, resp., intersections of internal sets. *Borel sets* form the least σ-algebra containing all internal sets; thus, all sets in $\mathbf{\Sigma}^0_1 \cup \mathbf{\Pi}^0_1$ are Borel. Following Henson [4], sets of the form

$$X = \bigcup_{b \in B} \left(\bigcap_{m \in b} X_m \cap \bigcap_{m \notin b} \bar{X}_m \right), \quad \text{where all sets } X_m \text{ are internal,}$$
$$B \subseteq P(\mathbb{N}), \quad \text{and} \quad \bar{X}_m = \bigcup_n X_n \backslash X_m, \tag{†}$$

are called *countably determined*, in brief CD. (Any reasonable version of this concept for Polish spaces yields the collection of all sets of the space.) There are several slightly different ways to define this class of sets, for instance (see, e.g. [7]),

$$X = \bigcup_{f \in F} \bigcap_{m \in \mathbb{N}} X_{f \restriction m}, \quad \text{where all } X_s,\ s \in {}^{<\omega}2, \text{ are internal,}$$
$$F \subseteq {}^{\mathbb{N}}2, \quad \text{and} \quad X_t \subseteq X_s \text{ whenever } s \subset t. \tag{‡}$$

To convert (‡) to (†), let B consist of all sets $b \subseteq {}^{<\omega}2$ containing a subset of the form $\{f \restriction m : m \in \mathbb{N}\}, f \in F$, and apply any bijection ${}^{<\omega}2$ onto $\mathbb{N}$. To convert (†) to (‡), put $X_s = \bigcap_{k<m} X'_k$ for any $s = \langle i_0, \ldots, i_{m-1} \rangle \in {}^{<\omega}2$, where $X'_k = X_k$ whenever $i_k = 1$ and $X'_k = {}^*\mathbb{N} \backslash X_k$ otherwise, then let $F \subseteq {}^{\mathbb{N}}2$ be the set of all characteristics functions of sets in B.

All Borel sets are countably determined, but not conversely. A *map* is Borel, resp., CD, if it has a Borel, resp., CD graph.

Remark 1.1. As usual, all elements of internal sets are internal sets themselves. It follows that all elements of Borel and CD sets are internal sets, and domains and ranges of Borel and CD maps consist of internal elements. □

In the AST vocabulary [2, 10, 11], hyperfinite sets (subsets of ${}^*\mathbb{N}$) are called just *sets*, arbitrary sets *classes*, internal sets *Sd-classes*, countably determined sets *real classes* (Čuda and Vopenka [2]); there are other differences. It is worth to note that the use of the word "class" in AST is in drastic contradiction with the meaning of this word in descriptive set theory (both classical and "hyperfinite"), where

[2] Standard parts can be defined for all bounded hyperreals, of course, but hyperreals are not considered in this article.

it normally denotes various collections of pointsets (like the class of all Borel, projective, countably determined sets).

Initial segments of ${}^*\mathbb{N}$ (including $\emptyset$, $\mathbb{N}$, ${}^*\mathbb{N}$) are called *cuts*. A cut U is *additive* if $x+y \in U$ whenever $x, y \in U$. Given a CD cut U, the sets

$$U\mathbb{N} = \bigcup_{n \in \mathbb{N}, x \in U} [0, xn] \quad \text{and} \quad U/\mathbb{N} = \bigcap_{n \in \mathbb{N}, x \in U} \left[0, \frac{x}{n}\right]$$

are additive CD cuts, $U/\mathbb{N} \subseteq U \subseteq U\mathbb{N}$, $U/\mathbb{N}$ is the largest additive cut included in U while $U\mathbb{N}$ is the smallest additive cut including U. In particular, let $c/\mathbb{N} = [0,c)/\mathbb{N}$ and $c\mathbb{N} = [0,c)\mathbb{N}$ for any $c \in {}^*\mathbb{N}$.

If U is a cut then $2^U = \bigcup_{a \in U} [0, 2^a)$ is an additive cut (and 2^U is multiplicative, that is, closed under products, provided U is additive). On the other hand, if U is an additive cut then $\log U = \{h : 2^h \in U\}$[3] is also a cut (not necessarily additive) and $U = 2^{\log U}$, $\log 2^U = U$.

Internal cuts are $\emptyset$, ${}^*\mathbb{N}$, and those of the form $c = [0,c)$, $c \in {}^*\mathbb{N}$. Non-internal cuts can be obtained with the following general procedure. Put

$$\sup X = \bigcup_{x \in X} [0, x] \quad \text{and} \quad \inf X = \bigcap_{x \in X} [0, x),$$

for any set $\emptyset \neq X \subseteq {}^*\mathbb{N}$; these are, resp., the least cut containing all elements of X and the largest cut disjoint from X. (Note that if X contains a least element a then $\inf X = [0,a)$, similarly, if X contains a largest element b then $\sup X = [0,b]$. Recall that intervals $[0,a)$ and $[0,b]$ of ${}^*\mathbb{N}$ are identified with numbers, resp., a and $b+1$.)

If $X = \{a_n : n \in \mathbb{N}\}$ is countable then we use $\sup_n a_n$ and $\inf_n a_n$ instead of $\sup X$ and $\inf X$; cuts of these forms are, resp., *countably cofinal* (or internal if $X = \{a_n : n \in \mathbb{N}\}$ contains a maximal element) and *countably coinitial* (or internal if X contains a minimal element). Both types consist of Borel sets of classes resp. $\mathbf{\Sigma}^0_1$ and $\mathbf{\Pi}^0_1$.

Cuts of the form $c + \mathbb{N} = \{c+n : n \in \mathbb{N}\}$ and $c - \mathbb{N} = \{c-n : n \in \mathbb{N}\}$ ($c \notin \mathbb{N}$) are countably cofinal, resp., coinitial, but not additive (unless $c \in \mathbb{N}$ in $c + \mathbb{N}$). The following simple result is known at least since [2].

Lemma 1.2. *Any CD cut $\emptyset \neq U \subsetneqq {}^*\mathbb{N}$ is either countably cofinal or countably coinitial or contains a maximal element* (*and then is internal*).

Proof. Let $U = \bigcup_{f \in F} \bigcap_{m \in \mathbb{N}} X_{f \restriction m}$, where F and the sets X_s are as in (‡). By Saturation, $U = \sup U = \bigcup_{f \in F} \bigcap_m U_{f \restriction m}$, where $U_s = \sup X_s$, hence, $U_s = [0, \mu_s]$, where $\mu_s = \max X_s \in {}^*\mathbb{N}$ for all $s \in {}^{>\omega}2$. If there is any $f \in F$ with $U = \bigcap_m U_{f \restriction m}$ then the sequence $\{\mu_{f \restriction m}\}_{m \in \mathbb{N}}$ witnesses that U is countably coinitial, or contains a maximal element if the sequence is eventually constant. Otherwise, by Saturation, for any $f \in F$ there is a number $m_f \in \mathbb{N}$ such that $\mu_{f \restriction m_f} \in U$. Let $S = \{f \restriction m_f : f \in F\}$; this is a countable set and easily $U = \bigcap_{s \in S} [0, \mu_s]$, so that U is either countably cofinal or contains a maximal element. □

[3] log means only $\log_2$ in this paper.

Lemma 1.3 (cf. 2.1, 2.2, 2.3 in [9]). *Let $X \subseteq {}^*\mathbb{N}$ be a countably determined set and $U \subseteq {}^*\mathbb{N}$ an additive cut of countable cofinality. Then*:

(i) *either $X \leqslant_{\mathrm{CD}} U$ or X contains an internal subset Y with $\#Y \notin U$*;

(ii) *either X is bounded (i.e., $X \subseteq h$ for some $h \in {}^*\mathbb{N}$) or X contains an unbounded internal subset.*

Proof. [4](i) Suppose that $X = \bigcup_{f \in F} \bigcap_n X_{f \restriction n}$, where $F \subseteq {}^{\mathbb{N}}2$ and X_s are as in (‡) of Section 1. Let S consist of all $s \in {}^{<\omega}2$ with $\#X_s \in U$. If there is $f \in F$ such that $f \restriction n \notin S$ for all n then by Saturation $\bigcap_n X_{f \restriction n}$ contains an internal subset Y with $\#Y \notin U$. Otherwise we have the "either" case.

(ii) A similar argument, with S being the set of all $s \in {}^{<\omega}2$ such that X_s is unbounded in ${}^*\mathbb{N}$. □

Taking $U = \mathbb{N}$, we obtain the following corollary, originally due to Henson [4], Čuda and Vopenka [2] (see a short proof in [14], Proposition 2.5).

Corollary 1.4. *Any countably determined set $X \subseteq {}^*\mathbb{N}$ is either at most countable or contains an infinite internal subset.* □

2. Equivalence Relations and Reducibility: Preliminaries

Suppose that E, F are countably determined equivalence relations[5] (ERs, for brevity) on (also countably determined) sets X, Y. We write $\mathsf{E} \leqslant_{\mathrm{CD}} \mathsf{F}$, in words: E *is CD-reducible to* F, iff there is a CD map (called: *reduction*) $\vartheta : X \to Y$[6] such that we have $x \mathsf{E} x' \Longleftrightarrow \vartheta(x) \mathsf{F} \vartheta(y)$ for all $x, x' \in X$.[7] We write $\mathsf{E} \equiv_{\mathrm{CD}} \mathsf{F}$ if both $\mathsf{E} \leqslant_{\mathrm{CD}} \mathsf{F}$ and $\mathsf{F} \leqslant_{\mathrm{CD}} \mathsf{E}$, and $\mathsf{E} <_{\mathrm{CD}} \mathsf{F}$ iff $\mathsf{E} \leqslant_{\mathrm{CD}} \mathsf{F}$ but not $\mathsf{F} \leqslant_{\mathrm{CD}} \mathsf{E}$. Changing "countably determined" and "CD" to "Borel" in these definitions, we obtain the relations $\leqslant_{\mathrm{B}}$, $\equiv_{\mathrm{B}}$, $<_{\mathrm{B}}$ of *Borel* reducibility.

Informal meaning of $\mathsf{E} \leqslant_{\mathrm{CD}} \mathsf{F}$ and $\mathsf{E} \leqslant_{\mathrm{B}} \mathsf{F}$ is that *F has at least as many equivalence classes as* E, *and this is witnessed by a CD*, *resp.*, *Borel map*.

For any set A, the *equality relation* $\mathsf{D}(A)$ (D from "diagonal") is defined on A by $x \mathsf{D}(A) y$ iff $x = y$. These are the simplest of ERs; in many aspects $\mathsf{D}(A)$ can be identified with A.

Similarly to the "Polish" descriptive set theory, we say that an ER E on a set X is *CD-smooth*, resp., *B-smooth*, if $\mathsf{E} \leqslant_{\mathrm{CD}} \mathsf{D}({}^*\mathbb{N})$, resp., $\mathsf{E} \leqslant_{\mathrm{B}} \mathsf{D}({}^*\mathbb{N})$, i.e., there is a countably determined, resp., Borel map ϑ, with $X \subseteq \operatorname{dom} \vartheta$ and $\operatorname{ran} \vartheta \subseteq {}^*\mathbb{N}$ such that $x \mathsf{E} x'$ iff $\vartheta(x) = \vartheta(x')$: this means that E-classes admit a CD enumeration by hyperintegers.

A *transversal* of an equivalence relation E is a set which has exactly one common element with each E-equivalence class. Easily any Borel ER E on a set $X \subseteq {}^*\mathbb{N}$, having a Borel transversal $W \subseteq X$, is B-smooth: let $\vartheta(x)$ be the only

[4] The result follows from Theorem 4.1, but we prefer to present a short direct proof.

[5] That is, equivalence relations which, as sets of pairs, are countably determined sets. The notion of *Borel* equivalence relation is understood similarly.

[6] To apply $\leqslant_{\mathrm{CD}}$ to non-CD relations, we should have used the existence of a CD map ϑ with $X \subseteq \operatorname{dom} \vartheta$ and $\vartheta'' X \subseteq Y$, but we shall not consider anything more complicated than CD below, in fact, mainly Borel ERs will be considered.

[7] It would be not less reasonable, but obviously longer, to write $X/\mathsf{E} \leqslant_{\mathrm{CD}} Y/\mathsf{F}$.

element of W equivalent to x. Similarly any CD equivalence relation E having a CD transversal is CD-smooth.

For any Borel sets X, Y, let $X \leqslant_{\mathrm{B}} Y$ mean that there is a Borel injection $\vartheta : X \to Y$. Accordingly, let $X \equiv_{\mathrm{B}} Y$ mean that both $X \leqslant_{\mathrm{B}} Y$ and $Y \leqslant_{\mathrm{B}} X$, and $X <_{\mathrm{B}} Y$ will mean that $X \leqslant_{\mathrm{B}} Y$ but not $Y \leqslant_{\mathrm{B}} X$. Changing "Borel" to "CD", we obtain $\leqslant_{\mathrm{CD}}$, $\equiv_{\mathrm{CD}}$, $<_{\mathrm{CD}}$, stronger relations between countably determined sets. Note that the superposition of two Borel, or two CD maps is in the same class (basically, because the classes of Borel and CD sets are closed under finite intersections), therefore, $\leqslant_{\mathrm{B}}$, $<_{\mathrm{B}}$, $\leqslant_{\mathrm{CD}}$, $<_{\mathrm{CD}}$ are order relations while $\equiv_{\mathrm{B}}$, $\equiv_{\mathrm{CD}}$ are equivalence relations.

Obviously $X \leqslant_{\mathrm{B}} Y$ iff $\mathsf{D}(X) \leqslant_{\mathrm{B}} \mathsf{D}(Y)$, thus, the $\leqslant_{\mathrm{B}}$-structure of Borel sets is in a sense equal to the $\leqslant_{\mathrm{B}}$-structure of B-smooth equivalence relations, and the same for the CD case.

Lemma 2.1. *Let X, Y be Borel sets. Then $X \equiv_{\mathrm{B}} Y$ iff there is a Borel bijection of X onto Y. Similarly, if X, Y are CD sets then $X \equiv_{\mathrm{CD}} Y$ iff there is a CD bijection of X onto Y.*

Proof (see [2] for the CD case and [10] for the Borel case). Apply the Cantor–Bernstein argument. To see that it yields a bijection of necessary type, recall that the image $\operatorname{ran}\vartheta$ of a CD, resp., Borel injection ϑ is equal to $\operatorname{dom}(\vartheta^{-1})$, hence, is still a CD, resp., Borel set [14, 2.10]. $\square$

Thus, $X \equiv_{\mathrm{CD}} Y$ can be interpreted as saying that the sets X, Y have the same *CD-cardinality*; the latter then can be defined as the $\equiv_{\mathrm{CD}}$-class of X. Similarly, $X \equiv_{\mathrm{B}} Y$ means that X, Y have the same *Borel cardinality.*

The following result presents an alternative description of the relations $\equiv_{\mathrm{CD}}$, $\equiv_{\mathrm{B}}$ restricted to ${}^*\mathbb{N}$ (i.e., acting only on hyperfinite sets; recall that any $x \in {}^*\mathbb{N}$ is identified with the set $[0, x) = \{y \in {}^*\mathbb{N} : 0 \leqslant y < x\}$).

Proposition 2.2 (See [2] and [14] for CD case; [5] and [10] for Borel case). *Suppose that $x, y \in {}^*\mathbb{N} \setminus \mathbb{N}$. Then $x \equiv_{\mathrm{B}} y$ iff* $\operatorname{st}\frac{x}{y} = 1$, *and $x \equiv_{\mathrm{CD}} y$ iff* $0 < \operatorname{st}\frac{x}{y} < +\infty$. $\square$

(Note that if at least one of $x, y \in {}^*\mathbb{N}$ belongs to $\mathbb{N}$ then $x \equiv_{\mathrm{B}} y$ iff $x \equiv_{\mathrm{CD}} y$ iff $x = y$ by obvious reasons.) It follows that the relations $x \equiv_{\mathrm{B}} y$ and $x \equiv_{\mathrm{CD}} y$ on ${}^*\mathbb{N}$ are Borel. We show below that the first of them is not CD-smooth while the other one is CD-smooth but not B-smooth.

3. Borel Cardinalities

Our first goal is to study the $\leqslant_{\mathrm{B}}$-structure of Borel sets in ${}^*\mathbb{N}$. The following theorem shows that any infinite Borel subset of ${}^*\mathbb{N}$ is $\equiv_{\mathrm{B}}$-equivalent to a unique Borel cut of some kind.

Theorem 3.1. *For any Borel set $X \subseteq {}^*\mathbb{N}$ there is a Borel cut $U \subseteq {}^*\mathbb{N}$ with $X \equiv_{\mathrm{B}} U$, actually, there is a minimal Borel cut U satisfying $X \equiv_{\mathrm{B}} U$.*

As we mentioned in the introduction, this theorem, together with Corollary 3.4 below, was proved, in the frameworks of AST, by Kalina and Zlatoš [10] (Theorems 4.10 and 4.12). Kalina and Zlatoš obtained other related results in

[8, 9, 11], in particular, in the assumptions of the theorem, there also exists a *maximal* Borel cut U satisfying $X \equiv_{\mathrm{B}} U$ [9, Theorem 1.10]. Those arguments in AST do not involve the two-cardinal hypothesis (essentially, CH), hence, can be considered as valid proofs in hyperfinite descriptive set theory.

Our proof of the theorem has many similarities with the proof in [10]. We precede the proof by two auxiliary lemmas. The first of them says that $\leqslant_{\mathrm{B}}$ is sometimes preserved under unions and intersections.

Lemma 3.2 (Essentially from Zivaljevic [19]). *Suppose that A_n, B_n are hyperfinite sets, and $b_n = \#B_n \leqslant a_n = \#A_n$ for each n. Then*

(i) *if $A_{n+1} \subseteq A_n$ and $B_{n+1} \subseteq B_n$ for each n then $\bigcap_n B_n \leqslant_{\mathrm{B}} \bigcap_n A_n$;*
(ii) *if $A_n \subseteq A_{n+1}$ and $B_n \subseteq B_{n+1}$ for each n then $\bigcup_n B_n \leqslant_{\mathrm{B}} \bigcup_n A_n$.*

Proof. (i) For any n there is an internal bijection $f : A_0$ onto $[0, a_0)$ such that $f''A_k = [0, a_k)$ for all $k \leqslant n$. By Saturation, there is an internal bijection $f : A_0$ onto $[0, a_0)$ with $f''A_n = [0, a_n)$ for all $n \in \mathbb{N}$. We conclude that $\bigcap_n A_n \equiv_{\mathrm{B}} U = \bigcap_n [0, a_n)$. Also, $\bigcap_n B_n \equiv_{\mathrm{B}} D = \bigcap_n [0, b_n)$. However $D \subseteq U$.

(ii) Arguing the same way, we prove that $\bigcup_n A_n \equiv_{\mathrm{B}} U = \bigcup_n [0, a_n)$ and $\bigcup_n B_n \equiv_{\mathrm{B}} D = \bigcup_n [0, b_n)$, but again $D \subseteq U$. $\square$

If $U \subseteq V \subseteq {}^*\mathbb{N}$ are cuts then we write $U \approx V$ iff $\frac{x}{y} \simeq 1$ for all $x, y \in V \setminus U$. (Thus, if $U = [0, a)$ and $V = [0, b)$ then $U \approx V$ iff $\frac{a}{b} \simeq 1$.) The next lemma (cf. 4.7 in [10]) says that this is a necessary and sufficient condition for $U \equiv_{\mathrm{B}} V$.

Lemma 3.3. (i) *If U, V are Borel cuts then $U \equiv_{\mathrm{B}} V$ iff $U \approx V$.*

(ii) *Any $\approx$-class of Borel cuts contains a $\subseteq$-minimal cut, in particular, any* additive *Borel cut is $\approx$-isolated, i.e., $U \not\approx V$ for any cut $V \neq U$.*

Proof. (i) Let, say, $U \subseteq V$. Suppose that $U \equiv_{\mathrm{B}} V$. Take any $x < y$ in $V \setminus U$. Then $x \equiv_{\mathrm{B}} y$, hence, $\frac{x}{y} \simeq 1$ by Proposition 2.2. Suppose, conversely, that $U \approx V$. Take any $x \in V \setminus U$. Let c be the entire part of $x/2$; then easily $c \in U$. Let $A = \{a \in {}^*\mathbb{N} : \frac{a}{c} \simeq 0\}$. We observe that $A \subsetneqq U$ and the difference $D = V \setminus U$ satisfies $D \subseteq X^+ \cup X^-$, where $X^+ = \{x + a : a \in A\}$ and $X^- = \{x - a : a \in A\}$. Define $f(z)$ for any $z \in V$ as follows. If $z \in U \setminus A$ then $f(z) = z$. If $z \in D \cap X^+$ then $z = x + a$, $a \in A$, and we define $f(z) = 3a$ (a number in A). If $z \in D \cap X^-$, but $z \neq x$, then $z = x - a$, $a \in A \setminus \{0\}$, and we define $f(z) = 3a + 1$ (still a number in A). Finally, if $x \in A$ then let $f(x) = 3x + 2$. Easily f is a Borel injection $V \to U$.

(ii) Let $\tilde{U}$ be the set of all $x \in U$ such that there is $y \in U$, $y > x$ with $\frac{x}{y} \not\simeq 1$. This is a cut, moreover, a projective set, hence, countably determined, which implies that $\tilde{U}$ is actually Borel by Lemma 1.2. Easily $\tilde{U} \approx U$. Finally, note that for any $x \in \tilde{U}$ there exists $x' \in \tilde{U}$, $x' > x$, with $\frac{x'}{x} \not\simeq 1$: indeed, let $x' = \frac{x+y}{2}$, where $y \in U$, $y > x$, $\frac{y}{x} \not\simeq 1$. This suffices to infer that $V \not\approx \tilde{U}$ for any cut $V \subsetneqq \tilde{U}$. In other words, $\tilde{U}$ is the $\subseteq$-least cut $\equiv_{\mathrm{B}}$-equivalent to U, as required. That $\tilde{U} = U$ for any additive cut U is a simple exercise. $\square$

Proof of Theorem 3.1. Lemma 3.3 allows us to concentrate on the first assertion of the theorem. Since all Borel sets are countably determined, we can present a given Borel set $X \subseteq {}^*\mathbb{N}$ in the form $X = \bigcup_{f \in F} \bigcap_n X_{f \restriction n}$, where F and the sets

$X_s \subseteq {}^*\mathbb{N}$ are as in (‡) of Section 1. If there is an $f \in F$ such that all sets $X_{f \restriction n}$ are unbounded in ${}^*\mathbb{N}$ then, by Saturation, there is an *internal* unbounded set $Y \subseteq X_f = \bigcap_n X_{f \restriction n}$. Then obviously $Y \equiv_{\mathrm{B}} {}^*\mathbb{N}$, hence, $X \equiv_{\mathrm{B}} {}^*\mathbb{N}$.

We assume henceforth that X *is bounded in* ${}^*\mathbb{N}$ – then it can be assumed that all sets X_s are also bounded, hence, hyperfinite. Let $\nu_s = \#X_s$.

Let C be the set of all $c \in {}^*\mathbb{N}$ such that there is $f \in F$ and an internal injection $\varphi : [0, c) \to X_f = \bigcap_n X_{f \restriction n}$. Easily C is a cut, and a countably determined set. (By Saturation, for any internal Y to be internally embeddable in X_f it suffices that $\#Y \leqslant \nu_{f \restriction m}$ for any m.)

We claim that $C \leqslant_{\mathrm{B}} X$. Indeed if there is $f \in F$ such that $C \subseteq [0, \nu_{f \restriction n})$ for all n then immediately $C \leqslant_{\mathrm{B}} X_f$ by Lemma 3.2(i). Otherwise for any $f \in F$ there is $n_f \in \mathbb{N}$ such that $\nu_{f \restriction n_f} \in C$. As $X_{f \restriction n_f}$ is an internal set with $\#X_{f \restriction n_f} = \nu_{f \restriction n_f}$, no internal set Y with $\#Y > \nu_{f \restriction n_f}$ admits an internal injection in X_f, hence, the countable set $\{\nu_{f \restriction n_f} : f \in F\}$ is cofinal in C, so that $C = \bigcap_k [0, z_k)$, where all z_k belong to C. However for any k there is an internal $R_k \subseteq X$ with $\#R_k = z_k$. Lemma 3.2(ii) implies $C \leqslant_{\mathrm{B}} \bigcup_k R_k$.

In continuation of the proof of the theorem, we have the following cases.

Case 1. C is not additive. Then there is $c \in C$ such that $c\mathbb{N} = U$ and $2c \notin C$. Prove that $X \leqslant_{\mathrm{B}} c\mathbb{N}$. By Lemma 3.2(ii), it suffices to cover X by a countable union $\bigcup_j Y_j$ of internal sets Y_j with $\#Y_j \leqslant 2c$ for all j. For this it suffices to prove that for any $f \in F$ there is m such that $\nu_{f \restriction m} = \#X_{f \restriction m} \leqslant 2c$. To prove this, assume, on the contrary, that $f \in F$ and $\nu_{f \restriction m} \geqslant 2c$ for all m; we obtain, by Saturation, an internal subset $Y \subset X_f$ with $\#Y = 2c \notin C$, contradiction. We return to this case below.

In the remainder, we assume that C is additive.

Case 2. C is countably cofinal. Arguing as in Case 1, we find that for any $f \in F$ there is m such that $\nu_{f \restriction m} = \#X_{f \restriction m} \in C$. (Otherwise, using Saturation and the assumption of countable cofinality, we obtain an internal subset $Y \subseteq X_f$ with $\#Y \notin C$, contradiction.) Thus, X can be covered by a countable union $\bigcup_j Y_j$ of internal sets Y_j with $\#Y_j \in C$ for all j. It follows, by Lemma 3.2(ii), that $X \leqslant_{\mathrm{B}} C$. Since $C \leqslant_{\mathrm{B}} X$ has been established, we have $X \equiv_{\mathrm{B}} C$, so that $U = C$ proves the theorem.

Case 3. C is (additive and) countably coinitial, and there exists a decreasing sequence $\{h_k\}_{k \in \mathbb{N}}$, coinitial in ${}^*\mathbb{N} \setminus C$, such that $\frac{h_k}{h_{k-1}}$ is infinitesimal for all $k \in \mathbb{N}$. For any $k \in \mathbb{N}$, if $f \in F$ then there is m with $\nu_{f \restriction m} \leqslant h_{k+1}$ (otherwise, by Saturation, X_f contains an internal subset Y with $\#Y > h_{k+1}$, contradiction), so that X is covered by a countable union of internal sets Y_j with $\#Y_j \leqslant h_{k+1}$ for all j. It follows, by Saturation and because $\frac{h_k}{h_{k-1}}$ is infinitesimal, that, for any k, X can be covered by an internal set R_k with $\#R_k \leqslant h_k$. Now $X \leqslant_{\mathrm{B}} C$ by Lemma 3.2(i), hence, $U = C$ proves the theorem.

Case 4. Finally, $C = c/\mathbb{N}$ for some $c \notin C$. We have $c/\mathbb{N} \leqslant_{\mathrm{B}} X \leqslant_{\mathrm{B}} c\mathbb{N}$ (similarly to Case 2).

To conclude, Cases 2 and 3 led us directly to the result required, while Cases 1 and 4 can be summarized as follows: there is a number $c \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $c/\mathbb{N} \leqslant_{\mathrm{B}} X \leqslant_{\mathrm{B}} c\mathbb{N}$. We can assume that $X \subseteq c\mathbb{N}$.

Let $\mu(Y) = \frac{\#Y}{c}$ be the counting measure on $c\mathbb{N}$. The set X is Borel, hence, Loeb-measurable. If its Loeb measure is ∞ then there is a sequence $\{X_n\}$ of internal subsets of X with $\#X_n = nc$, $\forall n$. It follows that $c\mathbb{N} \leqslant_{\mathrm{B}} X$ by Lemma 3.2, hence, $X \equiv_{\mathrm{B}} U = c\mathbb{N}$, as required.

Suppose that the Loeb measure of X is a (standard) real $r \geqslant 0$. There is an increasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of internal subsets of X and a decreasing sequence $\{B_n\}_{n\in\mathbb{N}}$ of supersets of X such that $\mu(B_n) - \mu(A_n) \to 0$ as $n \to \infty$ (i.e., the difference is eventually less than any fixed standard $\varepsilon > 0$). If $r = 0$ then $\frac{\#B_n}{c} \to 0$, therefore, $\bigcap_n B_n \equiv_{\mathrm{B}} c/\mathbb{N}$ by Lemma 3.2, which implies $X \equiv_{\mathrm{B}} c/\mathbb{N}$ since $c/\mathbb{N} \leqslant_{\mathrm{B}} X$, therefore, $U = c/\mathbb{N}$ proves the theorem.

Finally, assume that $r > 0$. Prove that then $X \equiv_{\mathrm{B}} [0, \mathrm{E}(cr))$, where $\mathrm{E}(\,)$ denotes the entire part in the internal universe. We have $\frac{\#A_n}{c} \to r$ from below and $\frac{\#B_n}{n} \to r$ from above. Let $U = \bigcup_{n\in\mathbb{N}} [0, \#A_n)$ and $V = \bigcap_{n\in\mathbb{N}} [0, \#B_n)$; then $\bigcup_n A_n \equiv_{\mathrm{B}} U$ and $\bigcap_n B_n \equiv_{\mathrm{B}} V$ by Lemma 3.2, while $\mathrm{E}(cr) \in V \setminus U$, hence, it remains to prove that $U \equiv_{\mathrm{B}} V$. It suffices, by Lemma 3.3, to show that $U \approx V$. Let $x < y$ belong to $V \setminus U$. If $\frac{y}{x} \not\simeq 1$ then $\frac{y}{c} - \frac{x}{c}$ is not infinitesimal, which contradicts the fact that $\mu(B_n) - \mu(A_n) \to 0$ because $\mu(A_n) \leqslant \frac{x}{c}$ and $\frac{y}{c} \leqslant \mu(B_n)$ for all n. □

Corollary 3.4. *Any two Borel sets $X, Y \subseteq {}^*\mathbb{N}$ are $\leqslant_{\mathrm{B}}$-comparable.* □

Corollary 3.5 (originally, perhaps, Zivaljevic [19]). *If $c \in {}^*\mathbb{N} \setminus \mathbb{N}$, μ is a finite counting measure on $[0, c)$, and sets $X, Y \subseteq c\mathbb{N}$ are Borel and of non-0 Loeb measure $L(\mu)$ then $X \equiv_{\mathrm{B}} Y$ iff $L(\mu)(X) = L(\mu)(Y)$.*

Proof. See the last paragraph of the proof of the theorem. □

Complete classification of Borel cardinalities. Call a Borel cut $U \subseteq {}^*\mathbb{N}$ *minimal* if $V \not\equiv_{\mathrm{B}} U$ for any cut $V \subsetneqq U$. It follows from Theorem 3.1 that any $\equiv_{\mathrm{B}}$-class of Borel subsets of ${}^*\mathbb{N}$ contains a unique minimal Borel cut, so that minimal Borel cuts can be viewed as *Borel cardinals* (of Borel subsets of ${}^*\mathbb{N}$).

For instance, any additive Borel cut is minimal by Lemma 3.3, hence, a Borel cardinal. But if U is a non-additive minimal Borel cut, then there is a number $c \in U$ with $2c \notin U$, so that $c/\mathbb{N} \subsetneqq U \subsetneqq c\mathbb{N}$, and, accordingly, $c/\mathbb{N} <_{\mathrm{B}} U <_{\mathrm{B}} c\mathbb{N}$, because $c/\mathbb{N}$ and $c\mathbb{N}$ are minimal cuts themselves. (Easily $c\mathbb{N}$ is the least additive cut bigger than $c/\mathbb{N}$.)

To study the structure of minimal Borel cuts between $c/\mathbb{N}$ and $c\mathbb{N}$ for a fixed nonstandard $c \in {}^*\mathbb{N}$, put $y_{cr} = \mathrm{E}(cr)$ for any real $r \in \mathbb{R}$, $r > 0$. Let $U_{cr} = [0, y_{cr}]$. Easily any minimal Borel cut U satisfying $c/\mathbb{N} <_{\mathrm{B}} U <_{\mathrm{B}} c\mathbb{N}$ is equal to U_{cr} for some positive real r, and $\tilde{U}_{cr} \neq \tilde{U}_{cr'}$ for different r, r'. Thus, Borel cardinals of Borel subsets of ${}^*\mathbb{N}$ are either additive Borel cuts or those of the form $\tilde{U}_{cr}$, or, finally, (finite) natural numbers.

4. CD Cardinalities

It can be expected that different Borel cardinalities are "glued" by countably determined maps. Furthermore, the notion of CD cardinality is addressed to a much bigger class of sets, the countably determined sets, which are not necessarily

Loeb measurable and, generally, have more vague nature. In particular, the $\leqslant_{\text{CD}}$-structure of countably determined sets is known only partially.

Theorem 4.1. *If $X \subseteq {}^*\mathbb{N}$ is an infinite countably determined set then either there is a unique additive Borel cut $U \equiv_{\text{CD}} X$ or there is an infinitely large number $c \in {}^*\mathbb{N}$ such that $c/\mathbb{N} <_{\text{CD}} X <_{\text{CD}} c/n$ for all $n \in \mathbb{N}$.*[8]

Thus, any infinite countably determined subset of ${}^*\mathbb{N}$ either is $\equiv_{\text{CD}}$-equivalent to a unique additive CD cut, or at least can be placed between two adjacent additive cuts $c/\mathbb{N}$ and $c\mathbb{N}$ for some $c \in {}^*\mathbb{N}\setminus\mathbb{N}$. While the "either" case is realized on simple examples (for instance, additive Borel cuts themselves), the existence of sets satisfying the "or" case remains, generally speaking, an open problem, see Section 5.

The next lemma (some parts of which can be found in [2]) comprises several facts involved in the proof.

Lemma 4.2. *Let $U \subseteq V \subseteq {}^*\mathbb{N}$ be infinite Borel cuts. Then*

(i) $U \equiv_{\text{CD}} \mathbb{N} \times U$ (the Cartesian product) *and* $U \equiv_{\text{CD}} U\mathbb{N}$ (a cut), *in particular, we have $c \equiv_{\text{CD}} c\mathbb{N}$ for all $c \in {}^*\mathbb{N}\setminus\mathbb{N}$;*

(ii) *if $U \subsetneqq V$ and U is additive, then $V \not\leqslant_{\text{CD}} U$, moreover, there is no CD map $\varphi : U$ onto V.*

Proof. (i) Theorem 6.1 below in Section 6 implies[9] the existence of a CD *transversal* $H \subseteq U$ for the equivalence relation $x \,\mathsf{M}_{\mathbb{N}}\, y$ iff $|x - y| \in \mathbb{N}$; in other words, for any $x \in U$ there is a unique $h_x \in H$ with $|x - h_x| \in \mathbb{N}$. Let $a \mapsto \langle z_a, n_a \rangle$ be a recursive bijection of $\mathbb{Z}$ (the integers) onto $\mathbb{Z} \times \mathbb{N}$. Now, if $x \in U\setminus\mathbb{N}$ then put $a = x - h_x$ and $\vartheta(x) = \langle h_x + z_a, n_a \rangle$. If $x = m \in \mathbb{N}$ then let $\Phi(x) = \langle i_m, j_m \rangle$, where $m \mapsto \langle i_m, j_m \rangle$ is a fixed bijection of $\mathbb{N}$ onto $\mathbb{N} \times \mathbb{N}$. Also, if U has a maximal element μ and $x = \mu - m$, $m \in \mathbb{N}$, then let $\vartheta(x) = \langle \mu - j_m, i_m \rangle$. Easily ϑ is a CD bijection of U onto $U \times \mathbb{N}$.

To prove $U \equiv_{\text{CD}} U\mathbb{N}$, note that if U is additive then even $U = U\mathbb{N}$. Otherwise there is $c \in U$ such that $U\mathbb{N} = \bigcup_n [0, cn)$. Note that $U\mathbb{N} = \bigcup_{n \in \mathbb{N}} U_n$, where $U_n = cn + U$, hence, there is a Borel bijection of $U \times \mathbb{N}$ onto $U\mathbb{N}$.

(ii) Let, on the contrary, $\varphi = \bigcup_{f \in F} \bigcap_m P_{f \restriction m}$ be such a map ($P_s \subseteq U \times V$ are internal sets and $P_s \subseteq P_t$ whenever $t \subset s$.) Then any $P_f = \bigcap_m P_{f \restriction m}$ is still a function, hence, by Saturation, there is a number m_f such that $P_{f \restriction m_f}$ is a function. Thus, there is a countable family of *internal* functions Φ_i, $i \in \mathbb{N}$ defined on U so that $V = \bigcup_i \Phi_i{}''U$. Assuming that $V = [0, c)$, where $c \in {}^*\mathbb{N}\setminus U$, put $c_0 = c$ and, by induction, let c_{n+1} be the entire part of $c_n/2$. Then still $c_n \notin U$ for any n as U is an additive cut, therefore, $V \subseteq \bigcup_i \Phi_i{}''[0, c_{i+2}]$ (we suppose that Φ_i is trivially extended onto $[0, c_{i+2}]$). Yet every $V_i = \Phi_i{}''[0, c_{i+2})$ is an internal set with $\#V_i \leqslant c/2^{i+2}$, hence, by Saturation, $\bigcup_i V_i$ can be covered by an internal set with $c/2$ elements, and cannot cover V. $\square$

[8] Some related results, which do not fully cover this theorem, were obtained in AST, for instance, in [9] (Theorems 2.5, 2.6).

[9] We claim, of course, that Theorem 6.1 is not, in any way, based on the entire content of Sections 4 and 5.

It follows from the lemma that for any infinite $c \in {}^*\mathbb{N}$, all Borel cardinals (as defined in the end of Section 3) between $c/\mathbb{N}$ and $c\mathbb{N}$ are $\equiv_{\mathrm{CD}}$-equivalent to each other and to $c\mathbb{N}$, hence for any Borel set $X \subseteq {}^*\mathbb{N}$ there is a unique *additive* Borel cut U with $X \equiv_{\mathrm{CD}} U$, so that we can define *CD-cardinals* of Borel sets to be just additive Borel cuts in ${}^*\mathbb{N}$. The structure of CD-cardinalities of countably determined sets remains, generally, an open problem.

Proof of Theorem 4.1. We leave it as an easy exercise for the reader to verify that the arguments in the proof of Theorem 3.1 are partially applicable to any countably determined, not necessarily Borel, set $X \subseteq {}^*\mathbb{N}$. More exactly. If X is unbounded in ${}^*\mathbb{N}$ then $X \equiv_{\mathrm{CD}} {}^*\mathbb{N}$. If X is bounded in ${}^*\mathbb{N}$ then either X is $\equiv_{\mathrm{CD}}$-equivalent to an additive Borel cut (Cases 2 and 3) or there is an infinitely large number c with $c/\mathbb{N} <_{\mathrm{CD}} X <_{\mathrm{CD}} c\mathbb{N}$ (Cases 1 and 4, minus the case of $\equiv_{\mathrm{CD}}$ to any of cuts $c/\mathbb{N}$, $c\mathbb{N}$). The further study of the "or" case in the proof of Theorem 3.1, based on the Loeb measurability of Borel sets, does not seem to work for CD sets in general.

Finally, the uniqueness of U in the theorem follows from Lemma 4.2(ii), while $c/n \equiv_{\mathrm{CD}} c\mathbb{N}$ for any standard n follows from Lemma 4.2(i). □

5. On "Singular" CD Sets

Recall that the *CD-cardinality* of a countably determined set X is the $\equiv_{\mathrm{CD}}$-class of X. For the moment, let us consider only the case of *bounded* CD sets $X \subseteq {}^*\mathbb{N}$. Natural (finite) numbers and CD-cardinalities of additive countably determined cuts $U \subseteq {}^*\mathbb{N}$ can be called *regular*, other *singular*. For instance, any $c = [0, c) \in {}^*\mathbb{N}$ has regular CD-cardinality because, by the above, if $c \notin \mathbb{N}$ then $[0, c) \equiv_{\mathrm{CD}} c\mathbb{N}$.

Problem 5.1. Do there exist singular CD-cardinalities? In other words (we refer to Theorem 4.1), given $c \in {}^*\mathbb{N} \setminus \mathbb{N}$, does there exist a countably determined set X satisfying $c/\mathbb{N} <_{\mathrm{CD}} X <_{\mathrm{CD}} c\mathbb{N}$? If yes then are there $\leqslant_{\mathrm{CD}}$-incomparable sets of this sort?

Further (a version communicated to the authors by Zlatoš), is it consistent that such a "singular" CD set X exists for some $c \in {}^*\mathbb{N} \setminus \mathbb{N}$ but does not exist for other $c \in {}^*\mathbb{N} \setminus \mathbb{N}$? □

It follows from a theorem of Čuda and Vopenka [2, p. 651] (see also Theorems 1.5 and 4.14 in [10]) that both questions answer in the positive (i.e., such sets do exist) in the assumption of CH. In other words, the positive answer is consistent (with ZFC as the underlying "standard" set theory). We don't know whether the negative answer is also consistent. Note that if the first question in 5.1 answers in the negative then the structure of CD cardinalities of (countably determined) subsets of ${}^*\mathbb{N}$ turns out to be rather well organized: any infinite CD set $X \subseteq {}^*\mathbb{N}$ is $\equiv_{\mathrm{CD}}$-equivalent to an additive CD cut in ${}^*\mathbb{N}$.

The goal of this Section is to prove that CD subsets of $X \subseteq c\mathbb{N}$ satisfying $c/\mathbb{N} \leqslant_{\mathrm{CD}} X$ (including possible examples for the problem) are $\equiv_{\mathrm{CD}}$-equivalent to sets of rather simple form, which may lead to more fruitful further studies.

Since any $c \in {}^*\mathbb{N} \setminus \mathbb{N}$ belongs to an interval of the form $[2^d, 2^{d+1})$, $d \in {}^*\mathbb{N} \setminus \mathbb{N}$, and then $c\mathbb{N} = 2^d\mathbb{N}$ and $c/\mathbb{N} = 2^d/\mathbb{N}$, we can assume that already $c = 2^d$. In this case, the domain $c = [0, c)$ can be identified with the set $\Xi = ({}^d 2)_{\rm int}$ of all internal maps $\xi : d \to 2$: the map $\xi \mapsto x(\xi) = \sum_{k=0}^{d-1} 2^{d-k-1}\xi(k)$ is an internal bijection of Ξ onto $[0, c)$. For any $s \in {}^{<\omega}2$ put $M_s^d = \{\xi \in \Xi : s \subset \xi\}$. For any $g \in {}^{\mathbb{N}}2$, put $M_g^d = \bigcup_m M_{g \restriction m}^d = \{\xi \in \Xi : \xi \restriction \mathbb{N} = g\}$. Call sets M_g^d *d-monads*. In different terms, the monads M_g^d are equivalence classes of the equivalence relation $\xi \mathsf{E} \eta$ iff $\xi \restriction \mathbb{N} = \eta \restriction \mathbb{N}$ on $({}^d 2)_{\rm int}$[10] (compare Remark 7.4).

For instance, $M_{\mathbf{0}}^d$, where $\mathbf{0} \in {}^{\mathbb{N}}2$ is the constant 0, is a d-monad. Easily $\{x(\xi) : \xi \in M_{\mathbf{0}}^d\} = c/\mathbb{N}$, hence, $c/\mathbb{N} \equiv_{\rm CD} M_{\mathbf{0}}^d \equiv_{\rm CD} M_g^d$ for each $g \in {}^{\mathbb{N}}2$.

Any union $M_G^d = \bigcup_{g \in G} M_g^d$ of d-monads ($G \subseteq {}^{\mathbb{N}}2$) is clearly a CD set.

Theorem 5.2. *Suppose that $c = 2^d \in {}^*\mathbb{N} \setminus \mathbb{N}$. If $X \subseteq c\mathbb{N}$ is a countably determined set then either $X <_{\rm CD} c/\mathbb{N}$ or $X \equiv_{\rm CD} M_G^d$ for some $G \subseteq {}^{\mathbb{N}}2$.*

Proof. As $c = [0, c) \equiv_{\rm CD} c\mathbb{N}$ by Lemma 4.2, we can assume that $X \subseteq c$, moreover, $X = \bigcup_{f \in F} \bigcap_m X_{f \restriction m}$, where $F \subseteq {}^{\mathbb{N}}2$ while $X_s \subseteq c$ are internal sets. We claim that the following can be assumed w.l.o.g.:

(1) $X_t \subseteq X_s$ whenever $s \subset t$ (otherwise put $X_s' = \bigcap_{k < \mathsf{lh}\, u} X_{s \restriction k}$);
(2) $X_{s^\wedge 0} \cap X_{s^\wedge 1} = \emptyset$ for any $s \in {}^{<\omega}2$;
(3) for any $s \in {}^{<\omega}2$, either $\# X_s = c2^{-k}$ for some $k = k_s$ or $\# X_s \in c/\mathbb{N}$;
(4) for any $s^\wedge i \in {}^{<\omega}2$, $\# X_{s^\wedge i} \leqslant \frac{1}{2} \# X_s$ ($s^\wedge i$ is the extension of s by i).

Justification of (2). Sets X_s admit partitions $X_s = \bigcup \mathscr{X}_s$, where $\mathscr{X}_s$ is a finite collection of pairwise disjoint internal subsets of X_s such that

(a) if $s \subset t$ then for any $A \in \mathscr{X}_t$ there is (unique) $B \in \mathscr{X}_s$ with $A \subseteq B$;
(b) if s, $t \in {}^{<\omega}2$ have the same length then any $A \in \mathscr{X}_s$ and $B \in \mathscr{X}_t$ are either equal or disjoint.

Consider the tree A which consists of all finite sequences $\alpha = \langle A_0, \ldots, A_{n-1} \rangle$, $n \in \mathbb{N}$, such that, for some $t \in {}^{<\omega}2$ of length n we have $A_i \in \mathscr{X}_{t \restriction i}$ for all $i \leqslant n$, and in addition $A_{i+1} \subseteq A_i$ for all i. Put $Y_\alpha = A_{n-1}$ for any such an α (note that $n = \mathsf{lh}\,\alpha$ depends on α). Accordingly, let Φ be the set of all functions φ, $\mathrm{dom}\,\varphi = \mathbb{N}$, such that there is $f \in F$ satisfying $\varphi(m) \in \mathscr{X}_{f \restriction m}$ and $\varphi(m+1) \subseteq \varphi(m)$ for any m. It follows from the construction that

$$X = \bigcup_{\varphi \in \Phi} \bigcap_m \varphi(m) = \bigcup_{\varphi \in \Phi} \bigcap_m Y_{\varphi \restriction m}.$$

It remains to observe that A can be $\subseteq$-preservingly embedded in ${}^{<\omega}2$ as a countably branching tree of height ω (in fact, A is finite-branching, of course). Such an embedding transforms the presentation of X in the last displayed formula into a presentation of the form (‡) (Section 1) satisfying (2).

[10] In the notation of [14], $\xi \restriction \mathbb{N}$ is denoted by $\mathrm{st}\,\xi$, the standard part, hence, we have $M_g^d = \mathrm{st}^{-1}(\{g\})$ and $M_G^d = \mathrm{st}^{-1}(G)$.

Justification of (3). Partitions $X_s = \bigcup X'_s$ can be defined, such that X'_s is an at most countable collection of subsets of X_s, of which at most one, say P_s, is a $\mathbf{\Pi}^0_1$ set with $P_s \leqslant_{\mathrm{CD}} c/\mathbb{N}$ while all others are (pairwise disjoint) internal sets of hyperfinite cardinalities of the form $c2^{-k}$, $k \in \mathbb{N}$, and still both (a) and (b) hold (for the collections X'_s). We can drop all sets P_s because this amounts to a total set of $\leqslant_{\mathrm{CD}} c/\mathbb{N}$ elements by Lemma 4.2, which is not essential in the context of the theorem. Then proceed as above.

Justification of (4). A similar argument.

Coming back to the proof of the theorem, let $S = \{f \restriction m : f \in F \wedge m \in \mathbb{N}\}$ (a subset of ${}^{<\omega}2$). In the assumptions (1)–(4), one can define $\sigma_s \in {}^{<\omega}2$ for any $s \in S$ so that (A) if $s^\wedge i \in S$ $(i = 0$ or $1)$ then $\sigma_s{}^\wedge i \subseteq \sigma_{s^\wedge i}$, and (B) if $\#X_s = c2^{-m}$ then $\mathsf{lh}\,\sigma_s = m$. For any $f \in F$, let $g(f) = \bigcup_m \sigma_{f \restriction m} \in {}^{\mathbb{N}}2$. Let $G = \{g(f) : f \in F\}$. Note that (in our assumptions) the sets $X_f = \bigcap_m X_{f \restriction m}$ and $M^d_{g(f)} = \bigcap_n M^d_{g(f) \restriction n} = \bigcap_m M^d_{\sigma f \restriction m}$ are $\equiv_{\mathrm{CD}}$ by Lemma 3.2(i), moreover, by a suitable modification of the proof of Lemma 3.2(i), we find an internal map ϑ such that $\vartheta'' X_f = M^d_{g(f)}$ for any $f \in F$, hence, $\vartheta \restriction X$ is a bijection of X onto M^d_G. □

Thus, if Problem 5.1 answers in the positive then, by the theorem, there exist corresponding examples of the form M^d_G, $G \subseteq {}^{\mathbb{N}}2$. The following rather elementary consideration focuses on $\leqslant_{\mathrm{CD}}$-properties of sets of this form.

Let a number $c = 2^d \in {}^*\mathbb{N} \setminus \mathbb{N}$ be fixed. First of all note that $M^d_g \equiv_{\mathrm{CD}} c/\mathbb{N}$ for any $g \in {}^{\mathbb{N}}2$, see above, therefore, we have $c/\mathbb{N} \leqslant_{\mathrm{CD}} M^d_G$ whenever $\emptyset \neq G \subseteq {}^{\mathbb{N}}2$.

Consider the Lebesgue measure on ${}^{\mathbb{N}}2$ which associates measure 2^{-n} with any Baire interval $B_s = \{f \in {}^{\mathbb{N}}2 : s \subset f\}$, where $s \in {}^{<\omega}2$, $\mathsf{lh}\, s = n$. Let $\overline{\mathrm{mes}}$ and $\underline{\mathrm{mes}}$ be the corresponding outer and inner measures.

For $c/\mathbb{N} <_{\mathrm{CD}} M^d_G$ (strictly), it is necessary and sufficient that $\overline{\mathrm{mes}}\ G > 0$. Indeed, if $\overline{\mathrm{mes}}\ G = 0$ then, by **Saturation**, for any m the set M^d_G can be covered by an internal set X with $\#X \leqslant c/m$, therefore, we have $c/\mathbb{N} <_{\mathrm{CD}} M^d_G$ by Lemma 3.2(i). Conversely, if M^d_G is covered by an internal set X with $\#X \leqslant c/m$, then, for any $g \in G$, there is a number $m_g \in \mathbb{N}$ such that $M^d_{g'} \subseteq X$ whenever $g' \in D_{g \restriction m_g}$, where $D_s = \{g' \in {}^{\mathbb{N}}2 : s \subset g'\}$ for any $s \in {}^{<\omega}2$. But the union $D = \bigcup_{g \in G} D_{g \restriction m_g}$ easily has measure $\leqslant m^{-1}$ in ${}^{\mathbb{N}}2$.

If $\underline{\mathrm{mes}}\ G > 0$ then $M^d_G \equiv_{\mathrm{CD}} [0, c) \equiv_{\mathrm{CD}} c\mathbb{N}$. Indeed, we can assume, by Cantor–Bernstein, that G is a closed subset of ${}^{\mathbb{N}}2$ of positive measure, say, of measure 2^{-m}, $m \in \mathbb{N}$. Then M^d_G is equal to a decreasing intersection $\bigcap_n X_n$, where each X_n is an internal set with $\#X_n \geqslant c2^{-m}$. It follows, by Lemma 3.2, that $[0, c2^{-m}) \leqslant_{\mathrm{CD}} M^d_G$, and so on.

But $\underline{\mathrm{mes}}\, G > 0$ is not a necessary condition for $M^d_G \equiv_{\mathrm{CD}} c\mathbb{N}$. Indeed, let G be a transversal (obtained using the axiom of choice in the underlying "standard" set universe of ZFC for the equivalence relation $f \mathsf{E}_0 g$ iff $f(n) = g(n)$ for all but finite $n \in \mathbb{N}$ $(f, g \in {}^{\mathbb{N}}2)$, an example of a set with $\underline{\mathrm{mes}}\, G = 0$ and $\overline{\mathrm{mes}}\, G = 1$. There is a sequence of internal functions ϑ_n such that $[0, c) = \bigcup_n \vartheta''_n M^d_G$, so that, by an argument similar to Lemma 4.2, we have $M^d_G \equiv_{\mathrm{CD}} [0, c) \equiv_{\mathrm{CD}} c\mathbb{N}$.

Thus, to obtain an anticipated example for Problem 5.1 in the form M^d_G, we have to employ nonmeasurable sets $G \subseteq {}^{\mathbb{N}}2$ with $\underline{\mathrm{mes}}\, G = 0 < \overline{\mathrm{mes}}\, G$ but less "dense" than transversals of E_0. It remains to be seen whether such an approach may lead to a solution of the problem.

Problem 5.3. Which "standard" property of $G, G' \subseteq {}^{\mathbb{N}}2$ is necessary and sufficient for $M^d_G \equiv_{\mathrm{CD}} M^d_{G'}$? □

6. Countable ERs Have Transversals

An equivalence relation E is "*countable*" if any of its equivalence classes, i.e., a set of the form $[x]_{\mathsf{E}} = \{y : x \,\mathsf{E}\, y\}$, $x \in \operatorname{dom} \mathsf{E}$, is at most countable. In "Polish" descriptive set theory, "countable" Borel ERs form a rather rich class whose full structure in terms of Borel reducibility is a topic of deep investigations (see Kechris [13]). In nonstandard setting, the picture is different.

Theorem 6.1. *Any "countable" countably determined equivalence relation* E *on* ${}^*\mathbb{N}$ *admits a countably determined transversal, hence, is CD-smooth.*

Jin [7] proved the result for the ER $x \,\mathsf{M}_{\mathbb{N}}\, y$ iff $|x - y| \in \mathbb{N}$. Our proof of the general result employs a somewhat different idea, although some affinities with Jin's arguments can be traced. Note also that $\mathsf{M}_{\mathbb{N}}$, a typical countable equivalence relation, is not B-smooth (see Lemma 14.1 below), this is the most transparent case when the Borel reducibility is really stronger.

Proof. The CD-smoothness easily follows from the existence of a transversal: just let $\vartheta(x)$ be the only element of a transversal equivalent to x.

To define a transversal, suppose, as usual, that $\mathsf{E} = \bigcup_{f \in F} \bigcap_{m \in \mathbb{N}} P_{f \restriction m}$, where all sets P_s, $s \in {}^{<\omega}2$, are internal subsets of ${}^*\mathbb{N} \times {}^*\mathbb{N}$ with $P_t \subseteq P_s$ whenever $s \subset t$, and $F \subseteq {}^{\mathbb{N}}2$. An ordinary $\mathsf{Saturation}$ argument shows that, because all E-classes are countable and a countable set cannot contain an infinite internal subset, for any $f \in F$ there is a number $m_f \in \mathbb{N}$ such that all cross-sections $P_{f \restriction m_f}(x) = \{y : x \; P_{f \restriction m_f}\, y\}$ are finite. Let $S = \{f \restriction m_f : f \in F\}$; this is a subset of ${}^{<\omega}2$. Then, for any $s \in S$, $k \in \mathbb{N}$, and $x \in {}^*\mathbb{N}$, we can define $f_{sk}(x)$ to be the k-th element (the counting begins with 0) of $P_s(x)$, in the natural order of ${}^*\mathbb{N}$, whenever $\# P_s(x) \geqslant k$, so that f_{sk} is an internal partial function ${}^*\mathbb{N} \to {}^*\mathbb{N}$.

Let $s \in S$ and $k \in \mathbb{N}$, $k \geqslant 1$. For any $x \in {}^*\mathbb{N}$ define an internal decreasing sequence $\{x_{(a)}\}_{a \leqslant a(x)}$ of length $a(x) + 1 \in {}^*\mathbb{N}$ as follows. Put $x_{(0)} = x$. Suppose that $x_{(a)}$ is defined. If $z = f_{sk}(x_{(a)})$ is defined and $z < x_{(a)}$ then put $x_{(a+1)} = z$, otherwise put $a(x) = a$ and end the construction. (Note that eventually the construction stops simply because $x_{(a+1)} < x_{(a)}$.) Put $\nu_{sk}(x) = 0$ if $a(x)$ is even and $\nu_{sk}(x) = 1$ otherwise.

Put $\operatorname{prfl} x = \{\langle s, k\rangle \in S \times \mathbb{N} : \nu_{sk}(x) = 0\}$ the "profile" of any $x \in {}^*\mathbb{N}$.

Lemma 6.2. *If* $x \neq y \in {}^*\mathbb{N}$ *and* $x \,\mathsf{E}\, y$ *then* $\operatorname{prfl} x \neq \operatorname{prfl} y$.

Thus, while it is, generally speaking, possible that different nonstandard numbers have equal "profiles", this cannot happen if they are E-equivalent.

Proof. We can assume that $y < x$. There is an $f \in F$ such that $\langle x, y\rangle \in P_f = \bigcap_m P_{f \restriction m}$. Let $s = f \restriction m_f$, an element of S. Then y belongs to $P_s(x)$, a finite set, say, y is k-th element of $P_s(x)$, in the natural order of ${}^*\mathbb{N}$. In other words, $y = x_{(1)}$, in the sense of the construction above, therefore, $y_{(1)} = x_{(2)}$, etc.; we conclude that $\nu_{sk}(x) \neq \nu_{sk}(y)$. □

Coming back to the theorem, choose an element $r_A \in A$ in any set $\emptyset \neq A \subseteq P(S \times \mathbb{N})$. For any $x \in {}^*\mathbb{N}$, the set $A(x) = \{\operatorname{prfl} y : y \in [x]_{\mathsf{E}}\}$ is a non-empty countable subset of $P({}^{<\omega}2 \times \mathbb{N})$. Then $X = \{x \in {}^*\mathbb{N} : \operatorname{prfl} x = r_{A(x)}\}$ is a transversal for E by Lemma 6.2. To prove that X is countably determined consider the family H which consists of all sets

$$D_{sk} = \operatorname{dom} f_{sk}, \quad X_{sk} = \{x \in {}^*\mathbb{N} : \nu_{sk}(x) = 0\},$$

and $X_{sks'k'} = \{x \in D_{sk} \colon \nu_{s'k'}(f_{sk}(x)) = 0\}$, along with their complements. Let A be the set of all at most countable sets $A \subseteq P(S \times \mathbb{N})$. Obviously $X = \bigcup_{A \in \mathrm{A}} X(A)$, where

$$X(A) = \{x \in X : A(x) = A\} = \{x \in {}^*\mathbb{N} : A(x) = A \wedge \operatorname{prfl} x = r_A\},$$

Lemma 6.3. *Any set* $X(A)$, $A \in \mathrm{A}$, *is countably determined in* H, *in the sense that it can be obtained by* (†) *of Section* 1 *applied to sets in* H.

Proof. Direct straightforward reduction to sets in H shows that $X(A)$ is even Borel in H in a similar sense. The most essential part of the reduction is to express the inclusion $A(x) \subseteq A$ by the formula

$$\forall s \in S \, \forall k \in \mathbb{N} \, (x \in D_{sk} \Longrightarrow \exists r \in A \, (r = \operatorname{prfl} f_{sk}(x))),$$

to avoid a universal quantifier over the equivalence class $[x]_{\mathsf{E}}$. □

On the other hand, the class of all sets countably determined in a fixed *countable* collection H of internal sets is closed under any unions (as well as under complements and intersections): just take the set theoretic union of the "bases" B in the assumption that the assignment of sets in H to indices is fixed once and for all. (Note that the class of all CD sets is closed only under countable unions and intersections!) □

Corollary 6.4. *The equivalence relation* $x \equiv_{\mathrm{CD}} y$ *on* ${}^*\mathbb{N}$ *admits a countably determined transversal.*

Proof. Recall that (for $x, y \in {}^*\mathbb{N} \setminus \mathbb{N}$) $x \equiv_{\mathrm{CD}} y$ iff $0 < \operatorname{st} \frac{x}{y} < +\infty$, Propositon 2.2. It follows that the set $\{2^x : x \in X\}$, where X is any CD transversal for the countable relation $x \, \mathsf{E}_{\mathbb{N}} \, y$ iff $|x - y| \in \mathbb{N}$, is as required. □

On the contrary, the relation $x \equiv_{\mathrm{B}} y$ iff $\operatorname{st} \frac{x}{y} = 1$ does not have a CD transversal. Indeed, suppose that X is a CD transversal for $\equiv_{\mathrm{B}}$ restricted to the set $D = [c, 2c]$, where c is a fixed infinitely large hyperinteger. Note that, for $x, y \in D$, $x \equiv_{\mathrm{B}} y$ is equivalent to $\operatorname{st} \frac{x}{c} = \operatorname{st} \frac{y}{c}$, so that X yields a CD transversal for the equivalence relation of "having the same standard part $\operatorname{st} r$" on the set of hyperrationals $A = \{r = \frac{x}{z} : x \in D\}$, known to be impossible [14, 2.6]. In fact "the same standard part" ER is not CD-smooth and even not $\leqslant_{\mathrm{CD}}$-reducible to any $\boldsymbol{\Sigma}^0_1$ ER; this can be derived from our result in Part 2 of Section 13.

7. Silver–Burgess Dichotomy

It is known (Corollary 1.4) that any countably determined set $X \subseteq {}^*\mathbb{N}$ is countable or else contains an infinite internal subset. Quotient structures ${}^*\mathbb{N}/\mathsf{E}$, where E is a CD equivalence relation, normally consist of non-internal elements, hence, do not contain internal subsets, but we can consider internal pairwise E-inequivalent sets (i.e., sets of pairwise E-inequivalent elements) instead. This leads us to the following theorem,[11] saying that, given a countably determined ER E, either the number of equivalence classes is somehow restricted or there is a rather big pairwise inequivalent set.

Theorem 7.1. *Let E be a CD equivalence relation on ${}^*\mathbb{N}$, and U a countably cofinal additive cut. Then at least one of the following two statements holds*:

(I) *there is a number $h \in {}^*\mathbb{N} \setminus U$ and an internal map $\vartheta : {}^*\mathbb{N} \to ({}^h 2)_{\text{int}}$ such that $\vartheta(x) \upharpoonright U = \vartheta(y) \upharpoonright U \Longrightarrow x \mathrel{\mathsf{E}} y$;*

(II) *there is an internal pairwise E-inequivalent set $Y \subseteq {}^*\mathbb{N}$ with $\#Y \notin U$.*

Moreover, if (II) *holds and U satisfies $x \in U \Longrightarrow 2^x \in U$ then* (I) *fails even for CD maps ϑ.*

The proof follows in Section 8; here, we proceed with corollaries, remarks and related results. The case $U = \mathbb{N}$, especially interesting, will be considered in more detail in Section 9.

The theorem yields a true dichotomy only for "exponential" cuts U, i.e., those satisfying $x \in U \Longrightarrow 2^x \in U$. If this condition fails then (I) and (II) are compatible, for take E to be the equality on $[0, 2^x)$ but $y \mathrel{\mathsf{E}} z$ for all $y, z \geqslant 2^x$. It is an open problem to obtain a true dichotomy in the general case.

Equivalence relations of class $\mathbf{\Sigma}^0_1$ admit the following special result, part (i) of which was known in AST, according to the report of an anonymous referee.

Lemma 7.2. *Assume that E is a $\mathbf{\Sigma}^0_1$ equivalence relation on a subset of ${}^*\mathbb{N}$, and $x \subseteq \operatorname{dom} \mathsf{E}$. Then*:

(i) *if X is $\mathbf{\Pi}^0_1$ then either the quotient X/E is finite or there is an infinite internal pairwise E-inequivalent set $C \subseteq X$;*

(ii) *if X is countably determined then either X/E is at most countable or there is an infinite pairwise E-inequivalent internal set $C \subseteq X$.*

Proof. (i) Let $X = \bigcap_n X_n$ and $\mathsf{E} = \bigcup_n E_n$, all X_n and E_n being internal and $X_{n+1} \subseteq X_n$, $E_n \subseteq E_{n+1}$ for all n. If X/E is infinite then, for any n, there is an internal set $C \subseteq X_n$ with $\#C \geqslant n$, such that $\langle x, y \rangle \notin E_n$ for any two elements $x \neq y$ of C. It remains to apply Saturation.

(ii) Let $X = \bigcup_{f \in F} \bigcap_m X_{f \upharpoonright m}$, where F and X_s are as in (‡) of Section 1. If for any $f \in F$ there is a number m_f such that $X_{f \upharpoonright m}/\mathsf{E}$ is finite then X/E is at most countable. Otherwise there is $f \in F$ such that $X_{f \upharpoonright m_f}/\mathsf{E}$ is infinite for all m, and, arguing

[11] We call it "Silver–Burgess dichotomy" due to obvious analogies with classical theorems of Silver [18] and Burgess [1] for Borel and analytic equivalence relations in Polish spaces.

as in the proof of (i), we obtain an infinite pairwise E-inequivalent internal subset of $X_f = \bigcap_m X_{f \restriction m}$. $\square$

Case (I) of Theorem 7.1 can be converted to a form which may lead to new insights. Suppose the E, U, h, ϑ are as in (I) of Theorem 7.1. Define, for ξ, $\eta \in ({}^h 2)_{\text{int}}$, $\xi \;\phi\; \eta$ iff either there exist x, $y \in {}^*\mathbb{N}$ with $x \;\mathsf{E}\; y$ and $\vartheta(x) \restriction U = \xi \restriction U$, $\vartheta(y) \restriction U = \eta \restriction U$, *or* just $\xi \restriction U = \eta \restriction U$. Then ϕ is an equivalence relation on $({}^h 2)_{\text{int}}$. Further, ϕ is countably determined (because it admits a rather simple definition in terms of E and ϑ, countably determined objects, while the class of all CD sets is closed under logic functions including quantification, [14]), and is *concentrated on* U in the sense that $\xi \restriction U = \eta \restriction U \Longrightarrow \xi \,\phi\, \eta$. Finally, ϑ is a reduction of E to ϕ, in other words, $x \,\mathsf{E}\, y \Longleftrightarrow \vartheta(x) \phi \vartheta(y)$. We conclude that (I) of Theorem 7.1 can be reformulated as follows:

(I′) *there is a number* $h \in {}^*\mathbb{N} \setminus U$ *and a countably determined equivalence relation* ϕ *on* $({}^h 2)_{\text{int}}$, *concentrated on* U, *such that* $\mathsf{E} \leqslant_{\mathrm{B}} \phi$.

Note that (I′) does not depend on the choice of h, i.e., if it holds for some $h \notin U$ then it also holds for any other $h' \notin U$.

There is a somewhat different approach to ERs concentrated on smaller domains. Suppose that X, H are internal sets and $Y \subseteq H$. A function $f : Y \to X$ is *internally extendable* if there exists $\xi \in ({}^H X)_{\text{int}}$ (i.e., $\xi : H \to X$ is an internal function) such that $f = \xi \restriction Y$. If Y itself is internal then this is the same as an internal function. Let $({}^Y X)_{\text{iex}}$ be the set of all internally extendable $f : Y \to X$. This definition obviously does not depend on the choice of an internal superset H of Y. For any n-ary ($n \in \mathbb{N}$) relation W on $({}^Y X)_{\text{iex}}$ we define $W \uparrow H$, an n-ary relation on $({}^H X)_{\text{int}}$, as follows:

Definition 7.3. $W \uparrow H = \{\langle \xi_1, \dots, \xi_n \rangle \in ({}^H X)_{\text{int}}{}^n : W(\xi_1 \restriction Y, \dots, \xi_n \restriction Y)\}$. $\square$

This applies, for instance, for W being a subset of or a binary relation on $({}^Y X)_{\text{iex}}$. Obviously an equivalence relation ϕ on $({}^H X)_{\text{int}}$ is concentrated on Y (in the sense that $\xi \restriction Y = \eta \restriction Y \Longrightarrow \xi \,\phi\, n$) iff $\phi = \mathsf{F} \uparrow H$ for a equivalence relation F on $({}^Y X)_{\text{iex}}$; in fact, F is unique, of course.

Remark 7.4. Transformation 7.3 deserves special comments. Suppose that $Y \subseteq H$ is a non-internal set. Then the original relation W is a subset of $({}^Y X)_{\text{iex}}{}^n$, a set whose elements are non-internal (because so is Y), therefore, W is not immediately a subject of study of "hyperfinite" descriptive set theory in the frameworks of Section 1. On the contrary, $W \uparrow H$ is an n-ary relation on $({}^H X)_{\text{int}}$, an internal set, hence, it can be studied by methods of "hyperfinite" descriptive set theory. However obviously any reasonable property of W can be expressed as a property of $W \uparrow H$ and Y.

Consider, for instance, the equality $\mathsf{D}(({}^Y 2)_{\text{iex}})$ on $({}^Y 2)_{\text{iex}}$ as a subset of $({}^Y 2)_{\text{iex}}{}^2$. (Thus, we take $X = 2 = \{0, 1\}$.) For any internal $H \supseteq Y$ we can define $\mathsf{D}(({}^Y 2)_{\text{iex}}) \uparrow H$, an equivalence relation on $({}^H X)_{\text{int}}$. Obviously, $\langle \xi, \eta \rangle \in \mathsf{D}(({}^Y 2)_{\text{iex}}) \uparrow H$ iff $\xi \restriction Y = \eta \restriction Y$, therefore, $\mathsf{D}(({}^Y 2)_{\text{iex}}) \uparrow H$ is Borel or CD if so is Y. The quotient set $({}^H X)_{\text{int}} / \mathsf{D}(({}^Y 2)_{\text{iex}}) \uparrow H$ can be considered as an adequate "descriptive" answer to the question: how many there exist internally extendable maps $Y \to 2$.

Note finally that all relations of the form $\mathsf{D}(({}^Y 2)_{\text{iex}}) \uparrow H$, H being an internal superset of Y, are clearly pairwise $\equiv_{\mathrm{B}}$-equivalent, where $\equiv_{\mathrm{B}}$ is the Borel bi-reducibility (Section 2). This allows us to use $\mathsf{D}_{\text{ext}}({}^{Y}2)$ (an H-free form) to denote *any* of the relations of the form $\mathsf{D}(({}^{Y}2)_{\text{iex}})\upharpoonright$ ($H \supseteq Y$ internal). We shall refer to $\mathsf{D}_{\text{ext}}({}^{Y}2)$ as *the equality of internally extendable maps* $Y \to 2$. $\square$

8. Silver–Burgess Dichotomy: The Proof

In this Section, we prove Theorem 7.1. Suppose that $\mathsf{E} = \bigcup_{f\in F}\bigcap_{m\in\mathbb{N}} P_{f\upharpoonright m}$, where P_s are internal subsets of ${}^*\mathbb{N}\times{}^*\mathbb{N}$ with $P_t \subseteq P_s$ whenever $s\subset t$, as in (‡) of Section 1, while $F\subseteq {}^{\mathbb{N}}2$. We can w.l.o.g. assume that the sets P_s are symmetric, i.e., $P_s = {P_s}^{-1}$: indeed, if this is not the case, then, as E itself is symmetric,

$$\mathsf{E} = \mathsf{E}\cup\mathsf{E}^{-1} = \bigcup_{f\in F}\bigcap_{m\in\mathbb{N}}(P_{f\upharpoonright m}\cup P^{-1}_{f\upharpoonright m}),$$

where the sets $P_{f\upharpoonright m}\cup {P_{f\upharpoonright m}}^{-1}$ are symmetric.

Since E is an equivalence relation, we have

$$\exists f\in F\,\exists z\forall m(x\,P_{f\upharpoonright m}\,z\wedge y\,P_{f\upharpoonright m}\,z)\Longrightarrow x\,\mathsf{E}\,y.$$

By $\mathsf{Saturation}$, this can be rewritten as

$$\forall T\in A(F)\,\exists s\in T\,\exists z(x\,P_s\,z\wedge y\,P_s\,z)\Longrightarrow x\,\mathsf{E}\,y, \tag{1}$$

where $A(F)$ is the collection of all sets $T\subseteq\{f\upharpoonright m : f\in F\wedge m\in\mathbb{N}\}$ such that $T\cap\{f\upharpoonright m : m\in\mathbb{N}\}\neq\emptyset$ for each $f\in F$.

Now suppose that (II) fails, i.e., there is no internal pairwise E-inequivalent set Y with $\#Y\notin U$. To show that then (I) holds, fix an increasing sequence $\{a_n\}_{n\in\mathbb{N}}$ cofinal in U. In our assumptions, we have

$$\forall Y\in\mathrm{P}_{\text{int}}({}^*\mathbb{N})(\forall k(\#Y>a_k)\Longrightarrow\exists x\neq y\in Y\,\exists f\in F\,\forall m(x\ P_{f\upharpoonright m}y))$$

where $\mathrm{P}_{\text{int}}({}^*\mathbb{N}) = \{Y\subseteq{}^*\mathbb{N} : Y \text{ is internal}\}$. The expression to the right of $\Longrightarrow$ can be consecutively transformed (using $\mathsf{Saturation}$ and the fact that $P_t\subseteq P_s$ provided $s\subset t$) to $\exists f\in F\,\forall m\,\exists x\neq y\in Y\ (x\ P_{f\upharpoonright m}\ y)$, and then to

$$\forall T\in A(F)\,\exists s\in T\,\exists x\neq y\in Y(x\ P_s\ y),$$

which leads us to the following, for every $T\in A(F)$:

$$\forall Y\in\mathrm{P}_{\text{int}}({}^*\mathbb{N})(\forall k(\#Y>a_k)\Longrightarrow\exists s\in T\,\exists x\neq y\in Y(x\ P_s\ y)).$$

Applying $\mathsf{Saturation}$ once again, we obtain, for any set $T\in A(F)$, a number $k(T)\in\mathbb{N}$ and a finite set $S_T\subseteq T$ such that

$$\forall Y\in\mathrm{P}_{\text{int}}({}^*\mathbb{N})(\#Y>a_{k(T)}\Longrightarrow\exists s\in S_T\,\exists x\neq y\in Y(x\ P_s\ y)).$$

Since the sets P_s are assumed to be symmetric, we conclude that for any $T\in A(F)$ there exists an internal set $Z_T\subseteq{}^*\mathbb{N}$ satisfying $\#Z_T\leqslant a_{k(T)}$ and

$$\forall x\in{}^*\mathbb{N}\,\exists z\in Z_T\,\exists s\in S_T(x\ P_s\ z). \tag{2}$$

Yet (2), as a property of Z_T, depends only on S_T, a finite subset of $T\subseteq{}^{<\omega}2$, not on T itself, hence, we can choose sets Z_T so that there are only countably many different among them. Then the sets $Z = \bigcup_{T\in A(F)} Z_T\subseteq{}^*\mathbb{N}$ and $Z\times{}^{<\omega}2$ are countable unions of hyperfinite sets whose numbers of elements belong to U. Therefore, as U is an additive cut, $Z \times {}^{<\omega}2$ can be covered by a hyperfinite set $H \subseteq {}^*\mathbb{N} \times {}^*({}^{<\omega}2)$ with $h = \#H \notin U$. (Note that in this case there is an internal bijection between h and H, therefore, there is an internal injection of $Z \times {}^{<\omega}2$ into U.)

Recall that P_s is an internal subset of ${}^*\mathbb{N} \times {}^*\mathbb{N}$ for any $s \in {}^{<\omega}2$. By Saturation, the map $s \mapsto P_s$ admits an internal extension to ${}^*({}^{<\omega}2)$, so that we can assume that $P_s \subseteq {}^*\mathbb{N} \times {}^*\mathbb{N}$ is internally defined for all $s \in {}^*({}^{<\omega}2)$. Let, for any $x \in {}^*\mathbb{N}$, $\vartheta(x)$ be an element of $({}^H2)_{\text{int}}$ defined as follows: $\vartheta(x)(z,s) = 1$ iff $x\ P_s\ z$. Then ϑ is an internal map ${}^*\mathbb{N} \to ({}^H2)_{\text{int}}$, hence, it remains to show that, for $x, y \in {}^*\mathbb{N}$, $\vartheta(x) = \vartheta(y)$ implies $x \mathsf{E} y$.

Assuming that $\vartheta(x) = \vartheta(y)$, consider any $T \in A(F)$. Choose, by (2), $z \in Z$ and $s \in T$ with $x P_s z$ – then $\vartheta(x)(z,s) = \vartheta(y)(z,s) = 1$, hence, we also have $y\ P_s\ z$. Since $T \in A(F)$ was arbitrary, we have $x \mathsf{E} y$ by (1).

Thus, at least one of (I), (II) of Theorem 7.1 holds.

Now let us show that (I), even for CD maps ϑ, contradicts (II) provided the cut U satisfies $x \in U \Longrightarrow 2^x \in U$. Indeed, otherwise there is a hyperfinite set $X \subseteq {}^*\mathbb{N}$ with $\#X = k \notin U$, a number $h \notin U$, and a CD map $\vartheta : X \to ({}^h2)_{\text{int}}$ such that $\vartheta(x) \restriction U = \vartheta(y) \restriction U \Longrightarrow x = y$ holds for all $x, y \in X$. In our assumptions, there is a number $\ell \notin U$, $\ell < \min\{h,k\}$, such that $2^{2\ell} < k$. The CD map $\tau(x) = \vartheta(x) \restriction [0,\ell)$ then satisfies $\tau(x) = \tau(y) \Longrightarrow x = y$, in other words, τ is a CD injection of X into $R = ({}^\ell2)_{\text{int}}$, a hyperfinite set satisfying $\#R = 2^\ell$. In other words, we have $k \leqslant_{\text{CD}} 2^\ell$. However, by the choice of ℓ, $(2^\ell)^2 = 2^{2\ell} < k$, which contradicts Proposition 2.2. □

9. Silver–Burgess Dichotomy: The Case $U = \mathbb{N}$

In the case when $U = \mathbb{N}$, Theorem 7.1 implies:[12]

Corollary 9.1. *If* E *is a CD equivalence relation on* ${}^*\mathbb{N}$ *then exactly one of the following two statements holds*:

$(\mathrm{I}_\mathbb{N})$ *there is* $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ *and an internal map* $\vartheta : {}^*\mathbb{N} \to ({}^h2)_{\text{int}}$ *such that* $\vartheta(x) \restriction \mathbb{N} = \vartheta(y) \restriction \mathbb{N} \Longrightarrow x \mathsf{E} y$ (then E has $\leqslant\mathfrak{c}$-many equivalence classes);

$(\mathrm{II}_\mathbb{N})$ *there is an infinite internal pairwise* E*-inequivalent set* $Y \subseteq {}^*\mathbb{N}$.

Moreover, if $(\mathrm{II}_\mathbb{N})$ *holds then* $(\mathrm{I}_\mathbb{N})$ *fails even for CD maps* ϑ.

Our observations will use Definition 7.3 and other notation of Section 7.

Note that, by Saturation, $({}^\mathbb{N}2)_{\text{iex}} = {}^\mathbb{N}2$, moreover, if F is any equivalence relation on ${}^\mathbb{N}2$ then, for any number $h \in {}^*\mathbb{N} \setminus \mathbb{N}$, $\mathsf{F} \uparrow [0,h)$ is a countably determined ER on $({}^h2)_{\text{int}}$, concentrated on $\mathbb{N}$, therefore, (I_N) of Corollary 9.1 can be rewritten as follows:

$(\mathrm{I}_\mathbb{N}{}')$ *there is a number* $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ *and an equivalence relation* F *on* ${}^\mathbb{N}2$ *such that* $\mathsf{E} \leqslant_\mathrm{B} F \uparrow [0,h)$.

As in Section 7, $(\mathrm{I}_\mathbb{N}{}')$ does not depend on the choice of $h \notin \mathbb{N}$.

[12] Alternatively, the corollary follows from a result which appeared in some papers related to AST in the 90s, for instance, [17], with a reference to an unpublished work of Vencovská.

Obviously any F in $(\mathrm{I}_{\mathbb{N}}{}')$ has at most $\mathfrak{c}$-many equivalence classes, this is why any E in $(\mathrm{I}_{\mathbb{N}})$ has at most $\mathfrak{c}$-many equivalence classes. We cannot reduce this cardinality, because the equivalence relation E on $({}^{h}2)_{\text{int}}$ (where $h \in {}^*\mathbb{N} \setminus \mathbb{N}$) defined by $\xi \,\mathsf{E}\, \eta$ iff $\xi \restriction \mathbb{N} = \eta \restriction \mathbb{N}$ has exactly $\mathfrak{c}$-many equivalence classes and does not admit an infinite pairwise inequivalent set (by 2.6 in [14]).

Note that if CH fails, and $\mathfrak{k} < \mathfrak{c}$, then there is a CD equivalence relation having exactly $\mathfrak{k}$ classes. Indeed, let F be an ER on ${}^{\mathbb{N}}2$ having exactly $\mathfrak{k}$ classes. Take any $h \in {}^*\mathbb{N} \setminus \mathbb{N}$. Then $\mathsf{E} = \mathsf{F} \uparrow h$ is a CD equivalence relation on $({}^{h}2)_{\text{int}}$ with exactly $\mathfrak{k}$ equivalence classes. What about Borel (nonstandard) ERs? The following question was addressed to the authors by J. Steprans and I. Farah in the course of our meeting during LC'02 (Münster, August 2002). The following theorem gives a partial answer.

Theorem 9.2. *Let* E *be a Borel equivalence relation on* ${}^*\mathbb{N}$. *Then, in the case* $(\mathrm{I}_{\mathbb{N}})$ *of Corollary* 9.1, E *has either* $\leqslant \aleph_0$ *or exactly* $\mathfrak{c}$ *equivalence classes.*

Proof. Our plan is to show that E has as many classes as a certain Borel ER F on a Polish space does; this implies the result by Silver [18].

Let us return to the arguments in Section 8. As now $U = \mathbb{N}$, the set Z turns out to be countable, and so is $Z \times {}^{<\omega}2 = \{\langle z_n, s_n \rangle : n \in \mathbb{N}\}$. Choose $h \in {}^*\mathbb{N} \setminus \mathbb{N}$. The sequence of pairs $\langle z_n, s_n \rangle$ can be extended to a hyperfinite (internal) sequence $\{\langle z_n, s_n \rangle\}_{n \leqslant h}$ of pairs $\langle z_n, s_n \rangle \in {}^*\mathbb{N} \times {}^*({}^{<\omega}2)$. Assuming that $P_s \subseteq {}^*\mathbb{N} \times {}^*\mathbb{N}$ is internally defined for all $s \in {}^*({}^{<\omega}2)$, we let, for any $x \in {}^*\mathbb{N}$, $\vartheta(x)$ be an element of $({}^{h}2)_{\text{int}}$ defined so that $\vartheta(x)(n) = 1$ iff $x \, P_{s_n} \, z_n$. Then $\vartheta : {}^*\mathbb{N} \to ({}^{h}2)_{\text{int}}$ is an internal map and, as in Section 8, $\vartheta(x) \restriction \mathbb{N} = \vartheta(y) \restriction \mathbb{N}$ implies $x \,\mathsf{E}\, y$, for all x, $y \in {}^*\mathbb{N}$. We put

$$D = \{f \in {}^{\mathbb{N}}2 : \exists x \in {}^*\mathbb{N}(\vartheta(x) \restriction \mathbb{N} = f)\}$$

and, for f, $g \in D$, define $f \,\mathsf{F}\, g$ iff there exist x, $y \in {}^*\mathbb{N}$ with $x \,\mathsf{E}\, y$ such that $\vartheta(x) \restriction \mathbb{N} = f$ and $\vartheta(y) \restriction \mathbb{N} = g$.

Note that D is a closed subset of ${}^{\mathbb{N}}2$ by Saturation and because ϑ is an internal map. Thus, it remains to shown that F is a Borel equivalence relation on D (in the sense of the standard Polish topology on ${}^{\mathbb{N}}2$). Fortunately, by a theorem in [14], it suffices to prove that $\phi = \mathsf{F} \uparrow [0, h)$, an equivalence relation on $({}^{h}2)_{\text{int}}$ concentrated on $\mathbb{N}$, is Borel as a subset of $[0, h)^2$. (This is because the map $\xi \mapsto \xi \restriction \mathbb{N}$ is the standard part map on $({}^{h}2)_{\text{int}}$.) By definition,

$$\begin{aligned} \xi \,\phi\, n \quad &\text{iff} \quad \exists x, y \in {}^*\mathbb{N}(x \,\mathsf{E}\, y \wedge \vartheta(x) \restriction \mathbb{N} = \xi \restriction \mathbb{N} \wedge \vartheta(y) \restriction \mathbb{N} = \eta \restriction \mathbb{N}) \\ &\text{iff} \quad \xi \restriction \mathbb{N} \in D \wedge \eta \restriction \mathbb{N} \in D \wedge \\ &\qquad\quad \forall x, y \in {}^*\mathbb{N}(\vartheta(x) \restriction \mathbb{N} = \xi \restriction \mathbb{N} \wedge \vartheta(y) \restriction \mathbb{N} = \eta \restriction \mathbb{N} \Longrightarrow x \,\mathsf{E}\, y). \end{aligned}$$

As E is Borel, the first equivalence proves that ϕ is analytic (that is, $\mathbf{\Sigma}_1^1$) while the second one shows (because D is closed) that ϕ is also coanalytic (that is, $\mathbf{\Pi}_1^1$), which together implies that ϕ is Borel by the "hyperfinite" Souslin theorem [14]. $\square$

10. Monadic Equivalence Relations

Any additive cut $U \subseteq {}^*\mathbb{N}$ defines a *monadic* equivalence relation $x \mathsf{M}_U y$ iff $|x-y| \in U$ on ${}^*\mathbb{N}$. (If U is not additive then M_U may not be an ER.) Classes of M_U-equivalence, that is, sets of the form $[x]_U = \{y : |x-y| \in U\}$, $x \in {}^*\mathbb{N}$, are called *U-monads* [15], all of them are convex subsets of ${}^*\mathbb{N}$.

It follows from Lemma 1.2 that there are two types of countably determined monadic ERs M_U: *countably confinal* and *countably coinitial*, according to the type of the cut U. (The only exceptions are $\mathsf{M}_\emptyset$, the equality on ${}^*\mathbb{N}$, and $\mathsf{M}_{{}^*\mathbb{N}}$, the relation which makes all elements of ${}^*\mathbb{N}$ equivalent.) It turns out that the relations between monadic ERs in terms of $\leqslant_{\mathrm{CD}}$ are determined by the relative rate of growth or decrease of cuts.

Definition 10.1. For any cut $U \subseteq {}^*\mathbb{N}$ put

$$\operatorname{rate} U = \inf_{a \in \log U,\, a' \notin \log U} a' - a,^{13}$$

the rate of U, where, we recall, $\log U = \{a : 2^a \in U\}$, and inf (as well as sup in Proposition 10.2) were defined in Section 1. □

Note that if U is a countably cofinal or countably coinitial *additive* cut then $\log U$ is still a countably cofinal, resp., countably coinitial cut, but not necessarily additive, and $U = \bigcup_{a \in \log U} [0, 2^a)$.

Proposition 10.2. *We have*

$$\operatorname{rate} U = \bigcap_{a \in \log U} \sup_{\substack{a' \in \log U \\ a' > a}} a' - a, \quad \operatorname{rate} U = \bigcap_{a \notin \log U} \sup_{\substack{a \notin \log U \\ a < a'}} a' - a,$$

provided $U \subseteq {}^*\mathbb{N}$ is a countably cofinal, resp., countably *coinitial additive cut.*

Additive cuts of lowest possible rate are obviously those of the form $U = c\mathbb{N}$, $c \in {}^*\mathbb{N}$ and $U = c/\mathbb{N}$, $c \in {}^*\mathbb{N} \setminus \mathbb{N}$, which we call *slow*; they satisfy $\operatorname{rate} U = \mathbb{N}$. Other additive cuts will be called *fast*.

Theorem 10.3. *Suppose that U, V are additive countably determined cuts in ${}^*\mathbb{N}$ other than $\emptyset$ and ${}^*\mathbb{N}$. Then $\mathsf{D}({}^*\mathbb{N}) \leqslant_{\mathrm{B}} \mathsf{M}_U$. In addition,*

(i) *if both U, V are countably cofinal or both countably coinitial then M_U and M_V are $\leqslant_{\mathrm{B}}$-comparable, and $\mathsf{M}_U \leqslant_{\mathrm{B}} \mathsf{M}_V$ iff $\mathsf{M}_U \leqslant_{\mathrm{CD}} \mathsf{M}_V$ iff* $\operatorname{rate} U \subseteq \operatorname{rate} V$, *in particular, if U is slow then $\mathsf{M}_U \leqslant_{\mathrm{CD}} \mathsf{M}_V$*;

(ii) *if U is countably cofinal and V countably coinitial then $\mathsf{M}_V \not\leqslant_{\mathrm{CD}} \mathsf{M}_U$ and $\mathsf{M}_U \not\leqslant_{\mathrm{B}} \mathsf{M}_V$, while $\mathsf{M}_U \leqslant_{\mathrm{CD}} \mathsf{M}_V$ holds iff U is slow*;

(iii) *M_U is not B-smooth, and M_U is CD-smooth if and only if U is countably cofinal and slow*;

(iv) *for any countably sequence of countably cofinal fast cuts U_n there are countably cofinal fast cuts U, V with $\mathsf{M}_U <_{\mathrm{B}} \mathsf{M}_{U_n} <_{\mathrm{B}} \mathsf{M}_V$, $\forall n$, and the same for countably coinitial cuts.*

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[13] The right-hand side of the displayed formula, as a function of $\log U$, is known, for instance, from papers on AST; it can be called *the thickness* of $\log U$.

This theorem, which explains the $\leqslant_{\rm CD}$-structure of monadic equivalence relations, will be the focal point in the remainder. According to the theorem, countably determined monadic ERs form two distinct linearly $\leqslant_{\rm CD}$-(pre) ordered domains, one of which contains countably cofinal and the other one countably coinitial ERs, each has slow ERs as the $\leqslant_{\rm CD}$-least element, and there is no $\leqslant_{\rm CD}$-connection between them except that any slow countably cofinal ER (it is necessarily CD-smooth) is $\leqslant_{\rm CD}$-reducible to any countably coinitial ER. In addition, each of the domains is neither countably $\leqslant_{\rm CD}$-cofinal nor countably $\leqslant_{\rm CD}$-coinitial in its fast part. (It can be shown that each of the domains is also dense and countably saturated, i.e., contains no gaps of countable character.)

To check that $\mathsf{D}({}^*\mathbb{N}) \leqslant_{\rm B} \mathsf{M}_U$ for any additive countably determined cut U choose a number $c \notin U$; then $x \mapsto xc$ is an internal, hence, Borel reduction of $\mathsf{D}({}^*\mathbb{N})$ to M_U, in other words, $x = x'$ iff $xc \mathsf{M}_U x'c$. This argument works for both countably cofinal and countably coinitial cuts U.

The proof of more complicated parts of the theorem begins with a couple of auxiliary results.

11. Two Preliminary Facts

The first result will be a connection between monadic ERs and certain natural equivalence relations on dyadic sequences. Let ${}^*\mathbb{S}$ be the (internal) set of all internal sequences $\varphi \in {}^{{}^*\mathbb{N}}2$ such that the set $\{a : \varphi(a) = 1\}$ is hyperfinite.

Consider an additive cut $\emptyset \neq U \neq {}^*\mathbb{N}$. Then $\log U = \{a \in {}^*\mathbb{N} : 2^a \in U\}$ is still a cut (not necessarily additive). Define the equivalence relation $\mathsf{R}_{\log U}$ on ${}^*\mathbb{S}$ as follows: $\varphi \,\mathsf{R}_{\log U} \psi$ iff $\varphi \restriction ({}^*\mathbb{N} \setminus \log U) = \psi \restriction ({}^*\mathbb{N} \setminus \log U)$. The realtion $\mathsf{R}_{\log U}$ is a version of $\mathsf{D}_{\rm ext}({}^{{}^*\mathbb{N} \setminus \log U}2)$ (see Remark 7.4) defined on ${}^*\mathbb{S}$.

Proposition 11.1. *In this case*, $\mathsf{M}_U \equiv_{\rm B} \mathsf{R}_{\log U}$.

Proof. For any $x \in {}^*\mathbb{N}$ there is a unique $\sigma = \sigma_x \in {}^*\mathbb{S}$ with $x = \sum_{z \in {}^*\mathbb{N}} 2^z \sigma(z)$ in ${}^*\mathbb{N}$. (The essential domain of summability here is a hyperfinite set because $\sigma \in {}^*\mathbb{S}$.) The map $x \mapsto \sigma_x$ is not yet a reduction of M_U to $\mathsf{R}_{\log U}$ because of a little discrepancy. Let ${}^*\mathbb{S}_{\log U}$ be the set of all $\sigma \in {}^*\mathbb{S}$ which are not eventually 1 in $\log U$, i.e., the set $\{a \in \log U : \sigma(a) = 0\}$ is cofinal in $\log U$. Let $\Omega_{\log U}$ be the set of all $x \in {}^*\mathbb{N}$ such that $\sigma_x \in {}^*\mathbb{S}_{\log U}$.

We assert that $x \,\mathsf{M}_U\, x' \iff \sigma_x \mathsf{R}_{\log U} \sigma_{x'}$ for all $x, x' \in \Omega_{\log U}$. (Consider any $x < x'$ in $\Omega_{\log U}$. If $d = x' - x \in U$ then $d < 2^a$ for some $a \in \log U$. As $\sigma_x \in {}^*\mathbb{S}_{\log U}$, there is $b \in \log U$, $b > a$, with $\sigma_x(b) = 0$. But easily $\sigma_x(z) = \sigma_{x'}(z)$ for any $z > b$, hence, $\sigma_x \mathsf{R}_{\log U} \sigma_{x'}$. The converse is obvious.)

Yet for any $x \notin \Omega_{\log U}$ there is $\tilde{x} \in \Omega_{\log U}$ with $|x - \tilde{x}| \in U$: put $\tilde{x} = x + 2^{a+1}$, where a is the largest number in $\log U$ with $\sigma_x(a) = 0$. For $x \in \Omega_{\log U}$ put $\tilde{x} = x$. The map $\vartheta(x) = \sigma_{\tilde{x}}$ is a Borel reduction of M_U to $\mathsf{R}_{\log U}$.

Finally, the map $f(\sigma) = \sum_{z \in {}^*\mathbb{N}} 2^{2z} \sigma(z)$ is a reduction of $\mathsf{R}_{\log U}$ to M_U. (The factor 2 in $2z$ helps to avoid the trouble with values only $\notin \Omega_{\log U}$.) $\square$

Remark 11.2. Choose $d \notin \log U$. A slight modification of the same argument proves that $\mathsf{M}_U \equiv_{\rm B} \mathsf{D}_{\rm ext}({}^{d \setminus U}2) \times \mathsf{D}({}^*\mathbb{N})$. (As usual, $d = [0, d)$.) $\square$

Proposition 11.3. *If U, V are additive countably cofinal cuts and* $\operatorname{rate} U \subseteq \operatorname{rate} V$ *then there are increasing sequences $\{a_n\}$, $\{b_n\}$, cofinal, resp., in* $\log U$, $\log V$, *with $b_{n+1} - b_n \geqslant a_{n+1} - a_n$ for all n.*

Similarly, if U, V are additive countably coinitial cuts and $\operatorname{rate} U \subseteq \operatorname{rate} V$ *then there are decreasing sequences $\{a_n\}$, $\{b_n\}$, coinitial, resp., in ${}^*\mathbb{N} \setminus \log U$ and ${}^*\mathbb{N} \setminus \log V$, with $b_n - b_{n+1} \geqslant a_n - a_{n+1}$ for all n.*

Proof. We concentrate on the case of increasing sequences, the case of decreasing sequences is similar. Choose any increasing sequences $\{\alpha_n\}$ and $\{\beta_k\}$, cofinal, resp., in $\log U$, $\log V$. Then, by Proposition 10.2,

$$\operatorname{rate} U = \bigcap_n \sup_{n' \geqslant n} \alpha_{n'} - \alpha_n \subseteq \operatorname{rate} V = \bigcap_k \sup_{k' \geqslant k} \beta_{k'} - \beta_k,$$

therefore, for $k_0 = 0$ there exists n_0 such that

$$\forall n' > n_0 \; \exists k' > k_0 \; (\alpha_{n'} - \alpha_{n_0} \leqslant \beta_{k'} - \beta_{k_0}). \tag{3.0}$$

If (*Case 1*) we also have $\forall k' > k_0 \; \exists n' > n_0 \; (\alpha_{n'} - \alpha_{n_0} \geqslant \beta_{k'} - \beta_{k_0})$, then the sequences $\{a_i\}$ and $\{b_i\}$ defined by $a_i = \alpha_{n_0} + \beta_{k_0+i} - \beta_{k_0}$ and $b_i = \beta_{k_0+i}$ prove the lemma. Otherwise (*Case 2*) there is $k_1 > k_0$ such that $\alpha_{n'} - \alpha_{n_0} < \beta_{k_1} - \beta_{k_0}$ for all $n' > n_0$. Choose, by Proposition 10.2, $n_1 > n_0$ so that

$$\forall n' > n_1 \; \exists k' > k_1 \; (\alpha_{n'} - \alpha_{n_1} \leqslant \beta_{k'} - \beta_{k_1}). \tag{3.1}$$

If we have now Case 1, i.e., $\forall k' > k_1 \; \exists n' > n_1 \; (\alpha_{n'} - \alpha_{n_1} \geqslant \beta_{k'} - \beta_{k_1})$, then, as above, the lemma holds immediately. Thus, we can assume that there is $k_2 > k_1$ with $\alpha_{n'} - \alpha_{n_1} < \beta_{k_2} - \beta_{k_1}$ for all $n' > n_1$. Choose $n_2 > n_1$ for k_2 as above. *And so on.*

In the course of this construction, either the required result comes up just at some step, or we obtain increasing sequences $\{n_i\}$ and $\{k_i\}$ such that $\alpha_{n'} - \alpha_{n_i} \leqslant \beta_{k_{i+1}} - \beta_{k_i}$ for all $n' > n_i$ and $i \in \mathbb{N}$. Let $a_i = \alpha_{n_i}$, $b_i = \beta_{k_i}$. □

12. Countably Cofinal Monadic Relations

The goal of this section is to prove the part of (i) of Theorem 10.3 related to countably cofinal cuts and associated monadic equivalence relations.

Choose increasing sequences $\{a_n\}$, $\{b_k\}$ in ${}^*\mathbb{N}$ with $U = \sup_n 2^{a_n}$ and $V = \sup_k 2^{b_k}$. (Note that $\log U = \sup_n a_n$ and $\log V = \sup_k b_k$.)

Part 1. Suppose that $\mathsf{M}_U \leqslant_{\text{CD}} \mathsf{M}_V$. Then $\mathsf{R}_{\log U} \leqslant_{\text{CD}} \mathsf{R}_{\log V}$ by Proposition 11.1. Let $\vartheta : {}^*\mathbb{S} \to {}^*\mathbb{S}$ be a CD reduction of $\mathsf{R}_{\log U}$ to $\mathsf{R}_{\log V}$, thus, $\varphi \mathsf{R}_{\log U} \varphi'$ iff $\vartheta(\varphi) \mathsf{R}_{\log V} \vartheta(\varphi')$ for all $\varphi, \varphi' \in {}^*\mathbb{S}$. The graph of ϑ has the form $\bigcup_{f \in F} C_f$, where $F \subseteq {}^{\mathbb{N}}2$ and $C_f = \bigcap_m C_{f \restriction m}$ for any $f \in {}^{\mathbb{N}}2$, sets C_s, $s \in {}^{<\omega}2$, are internal, and $C_t \subseteq C_s \subseteq {}^*\mathbb{S} \times {}^*\mathbb{S}$ for $s \subset t$, as in (‡) of Section 1.

Consider any $f \in F$. Then C_f is a subset of the graph of ϑ, hence, by the choice of ϑ, for any $k \in \mathbb{N}$ we have, for all $\varphi, \varphi', \psi, \psi' \in {}^*\mathbb{S}$,

$$\forall m (\varphi \, C_{f \restriction m} \psi \wedge \varphi' \, C_{f \restriction m} \psi') \wedge \psi \restriction_{\geqslant b_k} = \psi' \restriction_{\geqslant b_k} \Longrightarrow \exists n (\varphi \restriction_{\geqslant a_n} = \varphi' \restriction_{\geqslant a_n}),$$

where $\sigma \restriction_{\geqslant c} = \sigma \restriction ({}^*\mathbb{N} \setminus [0, c))$ for $\sigma \in {}^*\mathbb{S}$ and $c \in {}^*\mathbb{N}$. Then, by Saturation,

$$\forall k \; \exists n \; \exists m \; \forall \varphi, \varphi', \psi, \psi' \in {}^*\mathbb{S} :$$

$$\varphi \, C_{f \restriction m} \psi \wedge \varphi' C_{f \restriction m} \psi' \wedge \psi \restriction_{\geqslant b_k} = \psi' \restriction_{\geqslant b_k} \Longrightarrow \varphi \restriction_{\geqslant a_n} = \varphi' \restriction_{\geqslant a_n}. \tag{4}$$

A similar (symmetric) argument also yields the following:

$$\forall n\ \exists k\ \exists m\, \varphi, \varphi', \psi, \psi' \in {}^*\mathbb{S} : \\ \varphi\ C_{f\restriction m}\psi \wedge \varphi' C_{f\restriction m}\psi' \wedge \varphi \restriction {\geqslant a_n} = \varphi' \restriction {\geqslant a_n} \Longrightarrow \psi \restriction {\geqslant b_k} = \psi' \restriction {\geqslant b_k}. \quad (5)$$

Suppose, towards the contrary, that rate $U \not\subseteq$ rate V, hence, rate $V \subsetneqq$ rate U. Then $\mathbb{N} \subsetneqq U$, thus U is a fast cut, and we can suppose that $a_{n+1} - a_n$ is infinitely large for all n. As rate $V \subsetneqq$ rate U, there is an index k such that $\sup_{k'>k} b_{k'} - b_k \subsetneqq$ rate U. Let n, m be numbers defined for this k by (4). By the choice of k, there exists a number $n' > n$ such that $a_{n'} - a_n > b_{k'} - b_k$ for any $k' > k$, hence, in fact, $a_{n'} - a_n > \ell + b_{k'} - b_k$ for any $m' > m$ and any $\ell \in \mathbb{N}$. Finally, choose $k' > k$ and $m' > m$ according to (5) but w. r. t. n'. Put $C(f) = C_{f\restriction m'}$. Then we have, for all $\langle\varphi, \psi\rangle$, $\langle\varphi', \psi'\rangle$ in $C(f)$:

$$\left.\begin{array}{ll} \psi \restriction {\geqslant b_k} = \psi' \restriction {\geqslant b_k} \Longrightarrow \varphi \restriction {\geqslant a_n} = \varphi' \restriction {\geqslant a_n}, & \text{and} \\ \psi \restriction {\geqslant b_{k'}} \neq \psi' \restriction {\geqslant b_{k'}} \Longrightarrow \varphi \restriction {\geqslant a_{n'}} \neq \varphi' \restriction {\geqslant a_{n'}} & \end{array}\right\}. \quad (6)$$

We have ${}^*\mathbb{S} = \operatorname{dom}\vartheta = \bigcup_{f\in F} X(f)$, where $X(f) = \operatorname{dom} C(f)$, hence, by Saturation, there is a finite set $F' \subseteq F$ such that ${}^*\mathbb{S} = \bigcup_{f\in F'} X(f)$. Let us show that all sets $X(f)$ are too small for a finite union of them to cover ${}^*\mathbb{S}$.

Call an internal set $X \subseteq {}^*\mathbb{S}$ *small* iff

$(*)$ there is a number $h \in {}^*\mathbb{N}\setminus\mathbb{N}$ such that, for any internal map $\sigma \in {}^{{}^*\mathbb{N}\setminus[0,h)}2$ the set $X_\sigma = \{\varphi \in X : \varphi \restriction {\geqslant h} = \sigma\}$ satisfies $2^{-h}\#X_\sigma \simeq 0$.

Proposition 12.1. ${}^*\mathbb{S}$ *is not a union of finitely many small internal sets.* □

It remains to show that any set $X(f)$ is small, with $h = a_{n'}$ in the notation above. (Note that $a_{n'}$ depends on f, of course.) Take any $\langle\varphi, \psi\rangle \in C(f)$ and let $\sigma = \varphi \restriction {\geqslant a_{n'}}$, $\tau = \psi \restriction {\geqslant b_{k'}}$. By (6), each $\langle\varphi', \psi'\rangle \in C(f)$ with $\varphi' \restriction {\geqslant a_{n'}} = \sigma$ satisfies $\psi' \restriction {\geqslant b_{k'}} = \tau$. Let us divide the domain $\Psi = \{\psi' \in {}^*\mathbb{S} : \psi' \restriction {\geqslant b_{k'}} = \tau\}$ into subsets $\Psi_w = \{\psi' \in \Psi : \psi' \restriction [b_k, b_{k'}) = w\}$, where $w \in {}^{[b_k, b_{k'})}2$ is any internal map, totally $2^{b_{k'} - b_k}$ of the sets Ψ_w. For any such Ψ_w, the set $\Phi_w = \{\varphi' : \exists\, \psi' \in \Psi_w \langle\varphi', \psi'\rangle \in C(f)\}$ contains at most 2^{a_n} elements by the first implication in (6), therefore, the whole set $X(f)_\sigma = \{\varphi' \in X(f) : \varphi' \restriction {\geqslant a_{n'}} = \sigma\}$ contains at most $2^{a_n + b_{k'} - b_k}$ elements of the set $X(f)$, which is less than $2^{a_{n'} - \ell}$ for any $\ell \in \mathbb{N}$, hence, $X(f)$ is small, as required.

Part 2. Suppose that rate $U \subseteq$ rate V, and derive $\mathsf{R}_{\log U} \leqslant_{\mathrm{B}} \mathsf{R}_{\log V}$. We can assume, by Proposition 11.3, that $a_{n+1} - a_n \leqslant b_{n+1} - b_n$ for all $n \in \mathbb{N}$. By Robinson's lemma, there is a number $N \in {}^*\mathbb{N}\setminus\mathbb{N}$ and internal extensions $\{a_\nu\}_{\nu \leqslant N}$ and $\{b_\nu\}_{\nu \leqslant N}$ of sequences $\{a_n\}_{n\in\mathbb{N}}$ and $\{b_n\}_{n\in\mathbb{N}}$, both being increasing hyperfinite sequences satisfying $a_{\nu+1} - a_\nu \leqslant b_{\nu+1} - b_\nu$ for all $\nu < N$. Now we are ready to define a Bord reduction ϑ of $\mathsf{R}_{\log U}$ to $\mathsf{R}_{\log V}$.

If $\varphi \in {}^*\mathbb{S}$ then define $\vartheta(\varphi) = \psi \in {}^*\mathbb{S}$ as follows:

1) $\psi \restriction [0, b_0)$ is constant 0 (not imporant);
2) $\psi(b_\nu + h) = \varphi(a_\nu + h)$ whenever $\nu < N$ and $h < a_{\nu+1} - a_\nu$;
3) $\psi \restriction [b_\nu + a_{\nu+1} - a_\nu, b_{\nu+1})$ is constant 0 for any $\nu < N$;
4) $\psi(b_N + z) = \varphi(a_N + z)$ for all $z \in {}^*\mathbb{N}$.

Thus, to define ψ, we move each piece $\varphi \upharpoonright [a_\nu, a_{\nu+1})$ of φ so that it begins with the b_ν-th position in ψ, and fill the rest of $[b_\nu, b_{\nu+1})$ by 0s; in addition, $\psi \upharpoonright [b_N, \infty)$ is a shift of $\varphi \upharpoonright [p_N, \infty)$. That ϑ is a Borel reduction of $\mathsf{R}_{\log U}$ to $\mathsf{R}_{\log V}$ is a matter of routine verification.

13. Countably Coinitial Monadic Relations

That the equivalence $\mathsf{M}_U \leqslant_{\mathrm{B}} \mathsf{M}_V \Longleftrightarrow \mathsf{M}_U \leqslant_{\mathrm{CD}} \mathsf{M}_V \Longleftrightarrow \operatorname{rate} U \subseteq \operatorname{rate} V$ of (i) of Theorem 10.3 holds for any pair of countably coinitial cuts U, V, can be verified the same way as for countably cofinal cuts in Section 12 (with rather obvious amendments which account for the fact that now decreasing rather than increasing sequences $\{a_n\}$, $\{b_k\}$ are considered). We leave this to the reader, and concentrate, in this section, on (ii) (the incomparability between countably cofinal and countably coinitial ERs), except for its Borel part.

Suppose that $U = \sup_n 2^{a_n}$ and $V = \inf_k 2^{b_k}$, where $\{a_n\}$ and $\{b_k\}$ are resp. (strictly) increasing and decreasing sequences of hyperintegers. Note that then $\log U = \sup a_n$ and $\log V = \inf b_k$.

Part 1. Assuming that $\{a_n\}$ is fast, prove that $\mathsf{M}_U \not\leqslant_{\mathrm{CD}} \mathsf{M}_V$. We prove a more general result: $\mathsf{M}_U \not\leqslant_{\mathrm{CD}} \mathsf{E}$ for any $\mathbf{\Pi}^0_1$ equivalence relation E on ${}^*\mathbb{N}$. It suffices (Proposition 11.1) to show that $\mathsf{R}_{\log U} \not\leqslant_{\mathrm{CD}} \mathsf{E}$. Suppose, towards the contrary, that $\vartheta : {}^*\mathbb{S} \to {}^*\mathbb{N}$ is a CD reduction of $\mathsf{R}_{\log U}$ to E, so that $\varphi \mathsf{R}_{\log U} \varphi' \Longleftrightarrow \vartheta(\varphi) \mathsf{E} \vartheta(\varphi')$ for all φ, $\varphi' \in {}^*\mathbb{S}$. The graph of ϑ has the form $\bigcup_{f \in F} C_f$, where $F \subseteq {}^{\mathbb{N}}2$ and $C_f = \bigcap_m C_{f \upharpoonright m}$ for any f, all sets C_s, $s \in {}^{<\omega}2$, are internal, and $C_t \subseteq C_s \subseteq {}^*\mathbb{S} \times {}^*\mathbb{N}$ whenever $s \subset t$. Let $\mathsf{E} = \bigcap_k E_k$, where E_k are internal sets and $E_{k+1} \subseteq E_k$ for all k. As $\{a_n\}$ is fast, we can assume that $a_{n+1} - a_n$ is infinitely large for any $n \in \mathbb{N}$.

By the choice of ϑ, for any $f \in F$ we have:

$$\begin{aligned}\forall \varphi, \varphi' \in {}^*\mathbb{S}\ \forall x, x' \in {}^*\mathbb{N} : \quad & \forall m (\varphi\ C_{f \upharpoonright m} x \wedge \varphi'\ C_{f \upharpoonright m} x') \Longrightarrow \\ & (\exists n (\varphi \upharpoonright {\geqslant a_n} = \varphi' \upharpoonright {\geqslant a_n}) \Longleftrightarrow \forall k (x\ E_k\ x')). \quad (7)\end{aligned}$$

Applying Saturation here, with the implication $\Longleftarrow$ in the equivalence in the second line, we obtain numbers m, n, k (which depend on f) such that

$$\varphi\ C_{f \upharpoonright m}\ x \wedge \varphi'\ C_{f \upharpoonright m}\ x' \wedge x\ E_k\ x' \Longrightarrow \varphi \upharpoonright {\geqslant a_n} = \varphi' \upharpoonright {\geqslant a_n}$$

for all φ, $\varphi' \in {}^*\mathbb{S}$ and x, $x' \in {}^*\mathbb{N}$. Further, applying Saturation to (7) with the implication $\Longrightarrow$ in the second line, with fixed numbers k and $n+1$, we find $m'(f) \geqslant m$ such that, for all φ, $\varphi' \in {}^*\mathbb{S}$ and x, $x' \in {}^*\mathbb{N}$,

$$\varphi\ C_{f \upharpoonright m'(f)} x \wedge \varphi' C_{f \upharpoonright m'(f)} x' \wedge \varphi \upharpoonright {\geqslant a_{n+1}} = \varphi' \upharpoonright {\geqslant a_{n+1}} \Longrightarrow x\ E_k\ x'.$$

Let $X(f) = \operatorname{dom} C_{f \upharpoonright m'(f)}$. It follows from the choice of $m'(f)$ that

$$\forall\, \varphi, \varphi' \in X(f) : \quad \varphi \upharpoonright {\geqslant a_{n+1}} \neq \varphi' \upharpoonright {\geqslant a_{n+1}} \vee \varphi \upharpoonright {\geqslant a_n} = \varphi' \upharpoonright {\geqslant a_n},$$

therefore, $X(f)$ is small (see the definition before Proposition 12.1) because $a_{n(f)+1} - a_{n(f)}$ is infinitely large. This leads to a contradiction as in Section 12.

Part 2. Assuming that $\{a_n\}$ is slow, prove $\mathsf{M}_U \leqslant_{\rm CD} \mathsf{M}_V$. In this case, $U = c\mathbb{N}$ for some c. Recall that $\mathsf{M}_{\mathbb{N}}$ has a countably determined transversal A by Theorem 6.1. Then $B = \{ac : a \in A\}$ is obviously a CD transversal for M_U, hence M_U is CD-smooth (use the map sending any x to the only element of B equivalent to x), in other words, $\mathsf{M}_U \leqslant_{\rm CD} \mathsf{D}({}^*\mathbb{N})$. However $\mathsf{D}({}^*\mathbb{N}) \leqslant_{\rm CD} \mathsf{M}_V$ for any V, see the very end of Section 10.

Part 3. Prove that $\mathsf{M}_V \not\leqslant_{\rm CD} \mathsf{M}_U$ in any case. First of all, we can assume that V is a slow countably coinitial cut, because if V is such while V' is any countably coinitial cut then $\mathsf{M}_V \leqslant_{\rm CD} \mathsf{M}_{V'}$ by (i) of Theorem 10.3. Thus, let $V = \inf_k 2^{d-k} = \bigcap_k [0, 2^{d-k})$, where $d \in {}^*\mathbb{N} \setminus \mathbb{N}$; then $\log V = \inf_k d - k = \bigcap_k [0, d-k)$. It suffices to prove that $\mathsf{R}_{\log V} \not\leqslant_{\rm CD} \mathsf{M}_U$ (Proposition 11.1). We show that, even more, $\mathsf{R}_{\log V} \not\leqslant_{\rm CD} \mathsf{E}$ for any $\mathbf{\Sigma}^0_1$ equivalence relation E on ${}^*\mathbb{N}$.

Suppose, on the contrary, that $\mathsf{R}_{\log V} \leqslant_{\rm CD} \mathsf{E}$.

Consider an auxiliary equivalence relation R, defined on $\Xi = ({}^d 2)_{\rm int}$ (all internal maps $[0, d) \to 2$) as follows: $\sigma \,\mathsf{R}\, \tau$ iff $\sigma \restriction (d \setminus \log V) = \tau \restriction (d \setminus \log V)$.[14] For any $\sigma \in \Xi$ let $\tilde\sigma \in {}^*\mathbb{S}$ be its extension by 0s. The map $\sigma \to \tilde\sigma$ is a reduction of R to $\mathsf{R}_{\log V}$, hence, in our assumptions, $\mathsf{R} \leqslant_{\rm CD} \mathsf{E}$. Let $\vartheta : \Xi \to {}^*\mathbb{N}$ be a CD reduction of E to $\mathsf{R}_{\log U}$. Then $\vartheta = \bigcup_{f \in F} \bigcap_m C_{f \restriction m}$, where $F \subseteq {}^{\mathbb{N}}2$ while C_s, $s \in {}^{<\omega}2$, are internal subsets of $\Xi \times {}^*\mathbb{N}$ with $C_s \subseteq C_t$ whenever $t \subset s$. Finally, let $\mathsf{E} = \bigcup_n E_n$, where $E_n \subseteq {}^*\mathbb{N} \times {}^*\mathbb{N}$ are internal sets and $E_n \subseteq E_{n+1}$, $\forall n$.

For any $f \in F$, we have, by the choice of ϑ,

$$\forall \sigma, \sigma' \in \Xi \, \forall x, x' \in {}^*\mathbb{N} : \\ \forall m (\sigma \ C_{f \restriction m} x \wedge \sigma' \ C_{f \restriction m} x') \wedge \forall k (\sigma \restriction {}_{\geqslant d-k} = \sigma' \restriction {}_{\geqslant d-k}) \Longrightarrow \exists n (x \ E_n \ x'),$$

where $\sigma \restriction {}_{\geqslant d-k} = \sigma \restriction [d-k, d)$. Using **Saturation**, we obtain numbers $k = k(f)$, $n = n(f)$, $m = m(f)$ such that

$$\sigma \ C_{f \restriction m} x \wedge \sigma' \ C_{f \restriction m} x' \wedge \sigma \restriction {}_{\geqslant d-k(f)} = \sigma' \restriction {}_{\geqslant d-k(f)} \Longrightarrow x \ E_n \ x'. \tag{8}$$

We put $C(f) = C_{f \restriction m(f)}$ and $R(f) = \operatorname{ran} C(f)$. It follows from (8) that the set $R(f)$ can contain at most $2^{k(f)}$, a finite number, of pairwise E-inequivalent elements (because so is the number of all restrictions $\sigma \restriction {}_{\geqslant d-k(f)}$, $\sigma \in \Xi$). On the other hand, since the graph of ϑ is covered by countably many sets of the form $C(f)$, the full image $\operatorname{ran} \vartheta = \{\vartheta(\sigma) : \sigma \in \Xi\}$ is covered by countably many sets of the form $R(f)$ (even if F itself is uncountable), so that $\operatorname{ran} \vartheta$ contains only countably many pairwise E-inequivalent elements. Yet R admits continuum-many pairwise R-inequivalent elements in Ξ, contradiction.

14. Remaining Parts of the Theorem on Monadic ERs

We continue with the following result, which proves the $\leqslant_{\rm B}$-statement in (ii) of Theorem 10.3 and ends the proof of (ii) of Theorem 10.3 in general.

[14] Thus, R is $\mathsf{D}_{\rm ext}({}^{\{d-k : k \in \mathbb{N}\}}2)$ (see Remark 7.4), which is isomorphic to just $\mathsf{D}_{\rm ext}({}^{\mathbb{N}}2)$ on Ξ via the bijection $\{i_z\}_{z<d} \mapsto \{i_{d-1-z}\}_{z<d}$ of Ξ. In terms of this bijection, the partition of Ξ into R-classes is equal to the partition into d-monads M^d_g as in Section 5.

Lemma 14.1. *If U is an additive countably cofinal cut and* E *a* $\mathbf{\Pi}^0_1$ *equivalence relation then* $\mathsf{M}_U \not\leqslant_{\mathrm{B}} \mathsf{E}$.

It follows that $\mathsf{M}_U \not\leqslant_{\mathrm{B}} \mathsf{M}_V$ provided V is any countably coinitial cut.

Proof. Since obviously $\mathbb{N} = \operatorname{rate} \mathbb{N} \subseteq \operatorname{rate} U$, it can be assumed that $U = \mathbb{N}$. Let $\mathsf{E} = \bigcap_n E_n$, each $E_n \subseteq {}^*\mathbb{N} \times {}^*\mathbb{N}$ internal and $E_{n+1} \subseteq E_n$, $\forall n$. Fix $c \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let $\vartheta : [0,c) \to {}^*\mathbb{N}$ be a Borel reduction of $\mathsf{M}_{\mathbb{N}} \restriction [0,c)$ to E. As any Borel (generally, any analytic) set, the graph of ϑ has the form $\bigcup_{f \in {}^{\mathbb{N}}\mathbb{N}} \bigcap_m C_{f \restriction m}$, where ${}^{\mathbb{N}}\mathbb{N}$ is the set of all ω-sequences of natural numbers, all sets $C_u \subseteq [0,c) \times {}^*\mathbb{N}$, $u \in {}^{<\omega}\mathbb{N}$, are internal, ${}^{<\omega}\mathbb{N} =$ all finite sequences of natural numbers, and $C_v \subseteq C_u$ whenever $u \subset v$ (see [14]).

Applying a simple measure-theoretic argument, we can find a sequence of numbers $\{j_m\}_{m \in \mathbb{N}}$ in $\mathbb{N}$ such that the set $X = \operatorname{dom} \vartheta'$ has Loeb measure $\geqslant \frac{1}{2}$, where $\vartheta' = \bigcup_{f \in F} \bigcap_m C_{f \restriction m}$ and $F = \{f \in {}^{\mathbb{N}}\mathbb{N} : \forall m (f(m) \leqslant j_m\}$. By Koenig's lemma, $\vartheta' = \bigcap_m C_m$, where $C_m = \bigcup_u C_u$, where the union is taken over all sequences u of length m such that $u(k) \leqslant j_k$ for all $k < m$, so that each C_m is internal and (the graph of) ϑ' is a $\mathbf{\Pi}^0_1$ set. Also, $\vartheta' = \vartheta \restriction X$, where $X \subseteq [0,c)$ is a Borel set of Loeb measure $\geqslant \frac{1}{2}$.

Since ϑ is a reduction (and $\vartheta' \subseteq \vartheta$ a partial one), we have

$$\begin{aligned} &\forall x, x' \in X \, \forall y, y' \in {}^*\mathbb{N} : \\ &\qquad \forall m (x \, C_m y \wedge x' \, C_m \, y') \Longrightarrow (\exists k (|x - x'| < k) \Longleftrightarrow \forall n (y \; E_n \; y')). \end{aligned}$$

Applying Saturation with $\Longleftarrow$ instead of $\Longleftrightarrow$ in the second line, we find numbers m, n, k such that

$$\forall x, x' \in X \, \forall y, y' \in {}^*\mathbb{N} : \quad x \; C_m \; y \wedge x' \; C_m y' \wedge y \; E_n \; y' \Longrightarrow |x - x'| < k.$$

Applying Saturation with $\Longrightarrow$ instead of $\Longleftrightarrow$, and fixed numbers n and $4k$, we find a number $m' \geqslant m$ such that

$$\forall x, x' \in X \, \forall y, y' \in {}^*\mathbb{N} : \quad x \, C_{m'} \, y \wedge x' \, C_{m'} \, y' \wedge |x - x'| < 4k \Longrightarrow y \; E_n \; y'.$$

It follows that $|x - x'| < k \vee x - x'| \geqslant 4k$ holds for all x, $x' \in X$, which contradicts the assumption that X has measure $\geqslant \frac{1}{2}$. □

(iii) of Theorem 10.3. Note that every CD-smooth ER is $\leqslant_{\mathrm{CD}}$-reducible to $\mathsf{M}_{\mathbb{N}}$ because $\mathsf{D}({}^*\mathbb{N}) \equiv_{\mathrm{B}} \mathsf{M}_{\mathbb{N}}$, see above. It follows, by (ii) of Theorem 10.3 already proved, that M_V is not CD-smooth (hence, not B-smooth), provided V is an additive countably coinitial cut.

If $U = c\mathbb{N}$ is a slow additive countably cofinal cut then M_U is CD-smooth, see Part 2 in Section 13. If U is a fast additive countably cofinal cut then $\operatorname{rate} U \not\subseteq \operatorname{rate} \mathbb{N} = \mathbb{N}$, and the non-CD-smoothness of M_U follows as above for countably coinitial cuts. Finally, that M_U is not B-smooth for any additive countably cofinal cut U follows from Lemma 14.1.

(iv) of Theorem 10.3. It suffices, by (i), to prove the following:

Lemma 14.2. *Suppose that, for any n, U_n is a fast countably cofinal cut. Then there are fast countably cofinal cuts U and V such that* $\operatorname{rate} U \subsetneqq \operatorname{rate} U_n \subsetneqq \operatorname{rate} V$ *for any n. The same for fast countably coinitial cuts.*

Proof. In the case of countably cofinal cuts, let $\{a^n_k\}_{k\in\mathbb{N}}$ be an increasing sequence cofinal in $\log U_n$. We can assume that $d^n_k = a^n_{k+1} - a^n_k$ is infinitely large for all n, k. By countable Saturation, there are numbers a, $b \in {}^*\mathbb{N}\setminus\mathbb{N}$ such that $a < d^n_k < b$ for all n, k. Put $U = \sup_k 2^{k\sqrt{a}}$ and $V = \sup_k 2^{kb}$. $\square$

15. An Upper Bound for Countably Cofinal Relations

In classical descriptive set theory, the equivalence relation E_0, defined on ${}^{\mathbb{N}}2$ so that $x \mathrel{\mathsf{E}_0} y$ iff $x(n) = y(n)$ for all but finite n, plays a distinguished role in the structure of Borel ERs, in particular, because it is the least, in the sense of Borel reducibility, non-smooth Borel equivalence relation. It would be a rather bold prediction to expect any analogous result in the "nonstandard" setting, yet a reasonable nonstandard version of E_0 attracts some interest, giving a natural upper bound for countably cofinal monadic ERs.

For ξ, $n \in {}^*\mathbb{S}$ define: $\xi \mathrel{\mathsf{FD}} \eta$ iff $\xi(x) = \eta(x)$ for all but finite $x \in {}^*\mathbb{N}$ (FD from "finite difference").

Lemma 15.1. *If $U \subseteq {}^*\mathbb{N}$ is an additive countably cofinal cut then $\mathsf{M}_U \leqslant_{\mathrm{B}} \mathsf{FD}$. If $V \subseteq {}^*\mathbb{N}$ is an additive countably coinitial cut then $\mathsf{M}_V \not\leqslant_{\mathrm{CD}} \mathsf{FD}$.*

Proof. That $\mathsf{M}_V \not\leqslant_{\mathrm{CD}} \mathsf{FD}$ follows from the argument in Part 3 of Section 13 because FD is obviously a $\boldsymbol{\Sigma}^0_1$ relation. As for the first statement, suppose that $U = \sup_n 2^{a_n}$, where $\{a_n\}$ is an increasing sequence in ${}^*\mathbb{N}$; accordingly, $\log U = \sup a_n = \bigcup_n [0, a_n]$. It suffices to prove that $\mathsf{R}_{\log U} \leqslant_{\mathrm{B}} \mathsf{FD}$.

The sequence $\{a_n\}$ admits an internal $*$-extension $\{a_\nu\}_{\nu\leqslant N}$, where $N \in {}^*\mathbb{N}\setminus\mathbb{N}$, still an increasing hypersequence of elements of ${}^*\mathbb{N}$. Let, for any $\varphi \in {}^*\mathbb{S}$, $\vartheta(\varphi)$ be the (internal, hyperfinite) set of all restricted maps $\varphi \restriction [a_\nu, \infty)$, $\nu \leqslant N$, where $[a, \infty) = {}^*\mathbb{N}\setminus[0, a)$. By definition, $\varphi \mathsf{R}_{\log U} \psi$ iff the symmetric difference $\vartheta(\varphi)\Delta\vartheta(\psi)$ is finite. Yet ϑ takes values in the set of all hyperfinite subsets of a certain internal hyper-countable set (because ${}^*\mathbb{S}$ itself is hyper-countable) which can be identified with ${}^*\mathbb{N}$. $\square$

Corollary 15.2. *If U is as in the lemma then $\mathsf{M}_U <_{\mathrm{B}} \mathsf{FD}$.*

Proof. Use the lemma and (iv) of Theorem 10.3. $\square$

We don't know whether FD is an *exact* upper bound for countably cofinal monadic ERs, but still the lower $\leqslant_{\mathrm{B}}$-cone of FD contains many ERs not reducible to countably cofinal monadic ones, at least, all hyperfinite restrictions of FD are such. For any hyperfinite set $D \subseteq {}^*\mathbb{N}$ let $\mathsf{FD} \restriction D$ be the restriction of FD to the domain $({}^{D}2)_{\mathrm{int}}$, so that $\xi \mathrel{\mathsf{FD}\restriction D} \eta$ iff $\{d \in D : \xi(d) \neq \eta(d)\}$ is finite. Easily $\mathsf{FD} \restriction D \leqslant_{\mathrm{B}} \mathsf{FD}$, moreover, $FD \restriction D <_{\mathrm{B}} \mathsf{FD}$ because any possible CD reduction of FD to $\mathsf{FD} \restriction D$ must be a bijection on any set $X \subseteq {}^*\mathbb{S}$ of pairwise FD-inequivalent elements, but we can take X to be internal and hyper-infinite, which leads to a contradiction because there is no CD injection from a hyper-infinite (internal) set in a hyperfinite set (say, by Lemma 4.2).

Theorem 15.3. *If D is an infinite hyperfinite set and U an additive countably cofinal cut then $\mathsf{FD} \restriction D \not\leqslant_{\mathrm{CD}} \mathsf{M}_U$.*

Proof. In the course of the proof, it is more convenient to view $\mathsf{FD}\restriction D$ as an equivalence on $\mathrm{P}_{\mathrm{int}}(D)$ defined so that $u\;\mathsf{FD}\restriction D\;v$ iff $u\,\Delta\,v$ is finite. Let, on the contrary, $\vartheta:\mathrm{P}_{\mathrm{int}}(D)\to{}^*\mathbb{N}$ be a countably determined reduction of $\mathsf{FD}\restriction D$ to M_U. Assume that $D=[0,K)$ for some $K\in{}^*\mathbb{N}\setminus\mathbb{N}$. The graph of ϑ has the form $\bigcup_{f\in F}\bigcap_m P_{f\restriction m}$, where $F\subseteq{}^{\mathbb{N}}2$ and $P_s\subseteq\mathrm{P}_{\mathrm{int}}(D)\times{}^*\mathbb{N}$ are as in (‡) of Section 1. Let $X_s=\operatorname{dom}P_s$ and $X_f=\operatorname{dom}P_f$, where $P_f=\bigcap_m P_{f\restriction m}$.

Applying countable $\mathsf{Saturation}$, we find a number $\nu\in{}^*\mathbb{N}\setminus\mathbb{N}$ which is less than K and moreover, ${}^*\rho(\nu)<K$ for any standard recursive function ρ. Say that a set $Z\subseteq\mathrm{P}_{\mathrm{int}}(D)$ is *large* if there is an internal set $I\subseteq D$ such that $\#I=2\nu$ and $[I]^{\nu}\subseteq Z$, where $[I]^{\nu}$ is the set of all internal subsets $Y\subseteq I$ with $\#Y=\nu$. Then it is a consequence of the Ramsey theorem (in the nonstandard domain) that, for any $k\in\mathbb{N}$ and any internal partition $\mathrm{P}_{\mathrm{int}}(D)=Z_1\cup\cdots\cup Z_k$ at least one of the sets Z_i is large.

We observe that there is $f\in F$ such that all sets $X_{f\restriction m}$ are large. (Otherwise let $X_{f\restriction m_f}$ be non-large for any $f\in F$. Since $\operatorname{dom}\vartheta=\mathrm{P}_{\mathrm{int}}(D)$, it follows from $\mathsf{Saturation}$ that $\mathrm{P}_{\mathrm{int}}(D)$ is a finite union of non-large sets of the form $X_{f\restriction m_f}$, contradiction with the above.) Then, by $\mathsf{Saturation}$, X_f itself is large, so that there is an internal set $I\subseteq X_f$ such that $\#I=2\nu$ and $[I]^{\nu}\subseteq X_f$.

Note that $P_f\subseteq\vartheta$, hence, P_f is a function, actually, $P_f=\vartheta\restriction X_f$. In addition, by $\mathsf{Saturation}$, there is n such that $\varphi=P_{f\restriction n}$ is already a function (internal). Then clearly $P_f=\varphi\restriction X_f$, therefore, $\vartheta\restriction[I]^{\nu}=\varphi\restriction[I]^{\nu}$, which implies that $\vartheta\restriction[I]^{\nu}$ is an internal map. Use this fact to derive a contradiction.

Let $I=\{a_1,\dots,a_{2\nu}\}$ in the increasing order. For any $z=1,\dots,\nu$, let $u_z=\{a_z,\dots,a_{z+\nu-1}\}$ and $u_{\nu+z}=\{a_1,\dots,a_{z-1},a_{\nu+z},\dots,a_{2\nu}\}$ (in particular, $u_{\nu+1}=\{a_{\nu+1},\dots,a_{2\nu}\}$). Put $h_z=\vartheta(u_z)$. Easily the sets u_z are internal and $\#u_z=\nu$ for all z, moreover, $\#(u_z\,\Delta\,u_{z+1})=2$, hence, $u_z\;\mathsf{FD}\restriction D\;u_{z+1}$ for each $z<2\nu$, so that $|h_z-h_{z+1}|\in U$ because ϑ is a reduction, and, by the same reasons, $|h_{2\nu}-h_1|\in U$. On the other hand, $\#(u_1\,\Delta\,u_{\nu+1})=2\nu\notin\mathbb{N}$, hence, $|h_1-h_{\nu+1}|\notin U$.

To conclude, we have two hyperintegers h_1 and $h_{\nu+1}$, with $|h_1-h_{\nu+1}|\notin U$, connected by two internal chains, $h_1,h_2,\dots,h_\nu,\tau_{\nu+1}$ and $h_{\nu+1},\dots,h_{2\nu},h_1$, in which each link has length in U. Obviously there is an index z, $1<z\leqslant\nu$, such that $|h_z-h_{\nu+z}|\in U$. However by definition $\#(u_z\,\Delta\,u_{\nu+z})=2\nu\notin\mathbb{N}$, hence, $|h_z-h_{\nu+z}|\notin U$ for any z, contradiction. □

Thus, we have the following two classes of countably determined equivalence relations strictly $\leqslant_{\mathrm{B}}$-below FD: 1) ERs of the form M_U, where $U\subseteq{}^*\mathbb{N}$ is an additive countably cofinal cut, 2) ERs of the form $\mathsf{FD}\restriction[0,c)$, where $c\in{}^*\mathbb{N}\setminus\mathbb{N}$. It follows from our analysis that there is no ER in the first class $\leqslant_{\mathrm{CD}}$-compatible with a ER in the second class. Is there anything below FD essentially different from these two classes?

16. Final Remarks and Problems

This final Section contains few scattered remarks and questions, mainly implied by analogies with "Polish" descriptive set theory.

Back to CD-cardinalities. Problem 5.1 (Section 5) is, perhaps, the most interesting. Our analysis in the end of Section 5 shows that, for M^d_G to satisfy

$M_G^d \equiv_{\rm CD} c/\mathbb{N}$ it is necessary and sufficient that $G \subseteq {}^{\mathbb{N}}2$ is a set of Lebesgue measure 0. It is an interesting problem *to find a reasonable necessary and sufficient condition for M_G^d to sastisfy $M_G^d \equiv_{\rm CD} c\mathbb{N} \equiv_{\rm CD} [0,c)$*. Can $M_G^d \equiv_{\rm CD} [0,c)$ hold in the case when $\aleph_0 < \operatorname{card} G < 2^{\aleph_0}$? Do these problems depend on the basic properties of the (standard) continuum in essential way?

How many $\equiv_{\rm B}$-classes of Borel subsets of ${}^*\mathbb{N}$ do exist?[15] To answer such a question in the spirit of modern descriptive set theory, one has to define an equivalence relation, say, E, on ${}^*\mathbb{N}$ (in the "Polish" DST, on a Polish space), whose equivalence classes naturally represent $\equiv_{\rm B}$-classes of Borel subsets of ${}^*\mathbb{N}$, and classify E in terms of best known, "canonical" ERs (see [6, 13]).

It follows from Theorem 3.1 that Borel subsets of ${}^*\mathbb{N}$ are represented, modulo $\equiv_{\rm B}$, by sets of the following three classes: 1) ${}^*\mathbb{N}$ and cuts of the form $c = [0,c)$, $c \in {}^*\mathbb{N}$; 2) additive countably cofinal cuts; 3) additive countably coinitial cuts.

The first class naturally leads to $\equiv_{\rm B} \restriction {}^*\mathbb{N}$, i.e., the relation on ${}^*\mathbb{N}$ defined so that $x \equiv_{\rm B} y$ iff there is a Borel bijection of $[0,x)$ onto $[0,y)$ iff $\frac{x}{y} \simeq 1$. Can it be characterized in terms of relations $\mathsf{D}_{\rm ext}({}^D2)$, $D \subseteq {}^*\mathbb{N}$? We conjecture that the relation $\equiv_{\rm B} \restriction {}^*\mathbb{N}$ is $\equiv_{\rm B}$-equivalent to $\mathsf{D}({}^*\mathbb{N}) \times \mathsf{D}_{\rm ext}({}^{\mathbb{N}}2)$.

To approach the second class, fix $d \in {}^*\mathbb{N} \setminus \mathbb{N}$ and let D be the set of all increasing internal maps $\xi : d \to {}^*\mathbb{N}$ satisfying $\xi(x+1) \geqslant x\xi(x)$ for all $x < d-1$, so that any additive countably cofinal cut U has the form $U = U(\xi) = \bigcap_{n \in \mathbb{N}} \xi(n)$ for some (not unique) $\xi \in D$. Define $\xi \mathrel{\mathsf{E}} \eta$ iff $U(\xi) = U(\eta)$. This is a $\mathbf{\Pi}^0_2$ equivalence relation; can it be described in terms of relations of the form $\mathsf{D}(X)$ and $\mathsf{D}_{\rm ext}({}^X2)$? Third class can be studied similarly, but with decreasing sequences and $\xi(x+1) \leqslant \xi(x)/x$ for all x, but does this lead to an equivalence relation $\equiv_{\rm B}$-equivalent to E?

Equalities of internally extendable maps. Recall that $\mathsf{D}_{\rm ext}({}^X2)$ is the equivalence relation of equality of internally extendable maps $X \to 2$, Remark 7.4. This class of ERs contains, for instance, all monadic ERs (Proposition 11.1, it suffices to take complements of CD cuts as sets X), hence, study of its properties in terms of $\leqslant_{\rm CD}$ appears interesting and important. When $\mathsf{D}_{\rm ext}({}^X2) \leqslant_{\rm CD} \mathsf{D}_{\rm ext}({}^Y2)$? The results for monadic ERs show that the answer has little to do with, for instance, the inclusion $X \subseteq Y$. Our study of monadic equivalence relations can be rather routinely generalized on ERs $\mathsf{D}_{\rm ext}({}^X2)$ for sets $X \subseteq {}^*\mathbb{N}$ of classes $\mathbf{\Sigma}^0_1$ and $\mathbf{\Pi}^0_1$ (generalization of resp. countably coinitial and countably cofinal monadic ERs). For instance, it turns out that $\mathsf{D}_{\rm ext}({}^X2)$ is not CD-smooth for any non-internal $\mathbf{\Sigma}^0_1$ set $X \subseteq {}^*\mathbb{N}$, as well as for any non-internal $\mathbf{\Pi}^0_1$ set $X \subseteq {}^*\mathbb{N}$ not of the form $H \setminus C$, where H is internal and C is countable. Can $\mathsf{D}_{\rm ext}({}^X2)$ be CD-smooth for a set $X \subseteq {}^*\mathbb{N}$ not in $\mathbf{\Pi}^0_1$?

A hyperfinite continuum-hypothesis. Theorem 4.1 implies that, given $c \in {}^*\mathbb{N} \setminus \mathbb{N}$, there is no *regular* (see Section 5) CD-cardinality strictly between those of $c/\mathbb{N}$ and $c\mathbb{N}$ (it is a question whether there are *singular* ones there). Are there any other similar pairs in the $\leqslant_{\rm CD}$-structure? A natural analogy with the

[15] This question can be addressed to $\equiv_{\rm CD}$-classes of CD sets as well, but perhaps it is premature to search for an answer until Problem 5.1 is solved.

continuum-hypothesis leads to the following question. Let U be an additive CD cut in ${}^*\mathbb{N}$. (Or, generally, any Cd subset of ${}^*\mathbb{N}$, but then the problem is most likely more difficult.) Does there exist any countably determined ER E with $\mathsf{D}(U) <_{\text{CD}} \mathsf{E} <_{\text{CD}} \mathsf{D}_{\text{ext}}({}^U2)$? Since $\mathsf{D}(U)$ is the equality on U while $\mathsf{D}_{\text{ext}}({}^U2)$ is the equality of internally extendable maps $U \to 2$, the double inequality can be seen to represent the fact that the CD-cardinality of the quotient space of E is strictly between the CD-cardinality of U and its natural "power cardinality". This equation deserves a brief consideration.

Let $d \in {}^*\mathbb{N} \setminus U$, so that $\mathsf{D}_{\text{ext}}({}^U2)$ can be seen as the relation on $({}^d2)_{\text{int}}$ defined so that $\xi \; \mathsf{D}_{\text{ext}}({}^U2) \eta$ iff $\xi \restriction U = \eta \restriction U$. Then $\mathsf{D}(U) \leqslant_{\text{CD}} \mathsf{D}_{\text{ext}}({}^U2)$ can be witnessed by the map $x \mapsto \xi_x$, where $\xi_x \in ({}^d2)_{\text{int}}$ is the characteristic function of the singleton $\{x\}$. If $U = H \setminus C$, where H is internal while C countable, then we can prove, using Lemma 4.2, that, paradoxically, $\mathsf{D}(U) \equiv_{\text{CD}} \mathsf{D}_{\text{ext}}({}^U2)$. Otherwise (see a remark above) $\mathsf{D}_{\text{ext}}({}^U2)$ is not CD-smooth, hence, $\mathsf{D}(U) <_{\text{CE}} \mathsf{D}_{\text{ext}}({}^U2)$ strictly. Further, if there is a number $c \in U$ with $2^c \notin U$ then easily there are plenty of numbers $a < 2^c$, $a \notin U$ with $\mathsf{D}(U) <_{\text{CD}} \mathsf{D}(a) <_{\text{CD}} \mathsf{D}_{\text{ext}}({}^U2)$, thus, the "continuum-hypothesis" fails.

Now suppose that U is exponentially closed, so that $c \in U \Longrightarrow 2^c \in U$. Then (Lemma 4.2 applied) there is no CD set $X \subseteq {}^*\mathbb{N}$ with $\mathsf{D}(U) <_{\text{CD}} \mathsf{D}(X) <_{\text{CD}} \mathsf{D}_{\text{ext}}({}^U2)$, but is there any other countably determined ER E strictly $\leqslant_{\text{CD}}$-between $\mathsf{D}(U)$ and $\mathsf{D}_{\text{ext}}({}^U2)$?

Another family of equivalence relations. For any cut $\emptyset \neq U \subsetneqq {}^*\mathbb{N}$, take $c \notin U$ and define, for (internal) $\xi, \eta \in ({}^c2)_{\text{int}}$, $\xi \; \mathsf{F}_U \; \eta$ iff there are numbers $a \in U$ and $b \notin U$, $b \leqslant c$ such that $\xi \restriction [a,b) = \eta \restriction [a,b)$. If U is countably determined then it belongs to $\mathbf{\Sigma}^0_1$ or $\mathbf{\Pi}^0_1$, subsequently, F_U can be transformed to $\mathbf{\Sigma}^0_2$ using Saturation. Anything about the $\leqslant_{\text{CD}}$-structure of this family?

Smoothness and transversals. Our general method to establish smoothness was to find a suitable transversal. Recall, in this context, that $\mathsf{M}_{\mathbb{N}}$ admits a countably determined transversal by Theorem 6.1, hence, is CD-smooth, but is not B-smooth (Lemma 14.1), hence, does not admit a Borel transversal. However the existence of a transversal is not a necessary condition for the smoothness. Indeed, there exist Borel and B-smooth equivalence relations which do not admit even a countably determined transversal! An example can be easily extracted from the observation made in [14, 4.8] that there is a $\mathbf{\Pi}^0_2$ set in ${}^*\mathbb{N} \times {}^*\mathbb{N}$ which does not admit a CD uniformization.

"Fine structure" of equivalence relations. Is the ER FD defined in Section 15 in any sense $\leqslant_{\text{CD}}$-minimal over countably cofinal monadic ERs?

Is there any result analogous to the Glimm – Effros dichotomy (see [3] or [13]) of "Polish" descriptive set theory, in the same way as our Theorem 7.1 is analogous to the Silver – Burgess dichotomy? we conjecture that any Borel equivalence relation E on ${}^*\mathbb{N}$ or satisfies $\mathrm{D}_{\text{ext}}({}^{\mathbb{N}}2) \leqslant_{\text{B}} \mathsf{E}$.

Theorem 6.1 says that any countable CD equivalence relation is CD-smooth. What is the $\leqslant_{\text{B}}$-structure of countable *Borel* ERs?

Ergodic theory. Let $c \in {}^*\mathbb{N} \setminus \mathbb{N}$. The relation $\mathsf{M}_{\mathbb{N}} \restriction [0,c)$ on $[0, c)$ has certain similarities with the Vitali equivalence $x \, \mathsf{VIT} \, y$ iff $x - y$ is rational on $\mathbb{R}$, for instance, Borel non-smoothness, the nonexistence of Borel transversals, perhaps, the $\leqslant_{\mathrm{B}}$-minimality amongst all non-smooth ERs. However $\mathsf{M}_{\mathbb{N}} \restriction [0,c)$ lacks the following relevant property of VIT: while every VIT-invariant Borel subset of $\mathbb{R}$ has Lebesgue measure 0 or its complement has measure 0, there exist plenty of $\mathsf{M}_{\mathbb{N}}$-invariant Borel subsets of $[0,c)$ having Loeb measure, for instance, $1/2$: just consider the cut $[0, \frac{c}{2} + \mathbb{N})$ as a subset of $[0,c)$. (We consider the Loeb measure associated with the counting measure $\mu(X) = \frac{\#X}{c}$ for internal subsets of $[0,c)$.) Are there naturally defined "nonstandard" ERs which, unlike $\mathsf{M}_{\mathbb{N}}$, satisfy this property? Henson and Ross [5, 2.3] ask whether there exists a bijection $f : [0,c) \xrightarrow{\text{onto}} [0,c)$ ergodic in the sense that for any Loeb measurable set $X \subseteq [0,c)$ such that $X \,\Delta\, f''X$ has Loeb measure 0, the set X itself has Loeb measure either 0 or 1; they prove that Borel bijections (i.e., with a Borel graph, as usual) are not ergodic.

Domain-independent version. Define $\mathsf{E} \leqslant'_{\mathrm{B}} \mathsf{F}$ if $\mathsf{E} \times \mathsf{D}({}^*\mathbb{N}) \leqslant_{\mathrm{B}} \mathsf{E} \times \mathsf{D}({}^*\mathbb{N})$. With this definition, we have, for instance, $\mathsf{D}(X) \equiv'_{\mathrm{B}} \mathsf{D}(Y)$ for any infinite hyperfinite X, Y, and $\mathsf{M}_{\mathbb{N}} \restriction a \equiv'_{\mathrm{B}} \mathsf{M}_{\mathbb{N}} \restriction b \equiv'_{\mathrm{B}} \mathsf{M}_{\mathbb{N}}$ for any $a, b \in {}^*\mathbb{N} \setminus \mathbb{N}$, leading to structures less contaminated by the dependence on the size of the domain.

There is another possible way to the same goal. Unlike the case of Polish spaces, it is not true in the nonstandard domain that any Borel-measurable function (i.e., here, it means that all preimages of internal sets are Borel) is Borel in the sense that its graph is Borel. It is known that, for rather good nonstandard "universes", for instance, those satisfying *the Isomorphism Property*, for any two infinite hyperfinite sets X, Y there is a bijection $f : X \xrightarrow{\text{onto}} Y$ such that the images and preimages of internal sets are Borel. (Such a bijection cannot be even countably determined unless the fraction $\frac{\#X}{\#Y}$ is neither infinitesimal nor infinitely large.) As mentioned in [5], such a bijection induces an isomorphism of the entire structure of Borel and countably determined sets.

This naturally leads to the reducibility via Borel-measurable maps. Is $\mathsf{M}_{\mathbb{N}}$ Borel-measurable reducible to $\mathsf{D}({}^*\mathbb{N})$ in a nonstandard "universe" satisfying the Isomorphism Property?

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