and $\forall_{n \geqslant n_0}(F_j(n) \leqslant F_i(n))$, it follows that $L_s - 1 \in \mathcal{T}$ and $L_s - 1 \geqslant F_j$. On the basis of Theorem 1, taking the relation $j \in \theta W_j$ into account, we can conclude that there exists a $k \in \theta W_j \subseteq \omega A$ such that $F_k = L_s - 1$. Since $L_s \approx \lambda n\,[\max(n+1, F_i(n)+1)]$, it follows that $F_k \approx F_i$. Consequently, $F_i \in \mathcal{F}_A$ and $i \in \Omega \mathcal{F}_A(f)$. The theorem is proved.

We get the following corollaries from Theorem 4 and the Remark.

COROLLARY 1. Each infinite recursive set has infimum.

COROLLARY 2. There exists a recursively enumerable nonrecursive set that has infimum.

It has been shown in [7] that each infinite recursively enumerable set can be decomposed into two disjoint infinite nonspeedable sets. Therefore, the following corollary holds.

COROLLARY 3. Each infinite recursively enumerable set can be decomposed into two disjoint infinite sets, each of which has infimum.

The following problem is still unsolved: Does there exist an infinite recursively enumerable set that does not have infimum?

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# THEORY OF ZERMELO WITHOUT POWER SET AXIOM AND THE THEORY OF ZERMELO—FRENKEL WITHOUT POWER SET AXIOM ARE RELATIVELY CONSISTENT


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Introduction. The theory $ZF$ of Zermelo—Frenkel without the axiom of choice is consistent with the theory $Z$ of Zermelo without the axiom of choice because the totality of all sets of rank $< \omega + \omega$ is a model of $Z$ in $ZF$ [1]. In view of Gödel's second theorem, the theories $ZF$ and $Z$ are therefore not relatively consistent (assuming, of course, the consistency of the theory $ZF$). The content of the present paper is the proof of the following theorem, which shows that the situation is different if we remove the power set axiom from the theories under consideration:

THEOREM. The theories $Z^-$ and $ZF^-$, obtained from $Z$ and $ZF$, respectively, by omitting the power set axiom, are relatively consistent. Moreover, $ZF^-$ has an interpretation in $Z^-$.

The consistency of $Z^-$ and $ZF^-$ relative to each other (and for a series of other theories) was announced without proof in [2] with reference to [3], where an equivalent result is stated without proof. The author does not know if proofs of these theorems have been published.


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The proof of our theorem proceeds as follows. In Sec. 1 we discuss a special intermediary theory T obtained from $Z^-$ by adding special cases of the axiom of replacement and an axiom assuring the countability of all sets. These additional axioms enable us to prove the axiom of replacement when the range is transitive. There is also a sketch of how one obtains an interpretation of T in $Z^-$ based on the idea of representing countable sets by means of special binary relations on $\omega$ (see [2]).

Even though the full axiom of replacement is not available in T, this theory is strong enough to allow the definition of a constructive hierarchy $(L_\alpha \colon \alpha \in \mathrm{On})$ and to prove its main properties. This is done in Sec. 2.

Finally, in Sec. 3, we introduce a particular collection $\Omega$ of ordinals (either all of On, or a proper initial segment of On); we then prove that the class $L^* = \bigcup_{\alpha \in \Omega} L_\alpha$ is a model of $ZF^-$. The fixed point Theorem 3.3 plays a crucial role in the verification of the axiom of replacement in L*.

Our set-theoretic notation follows [4]. Lower-case letters are reserved for sets, upper-case letters may denote both classes and sets. Classes can be eliminated from our discussion (at the expense of simplicity), as indicated in [4]. Classes F which satisfy the definition of a function (i.e., each element of F is a pair, and $(u, v_1) \in F \wedge (u, v_2) \in F \rightarrow v_1 = v_2$) will be referred to as functionals, while the word function is reserved for those functionals which are sets.

1. Theory T. If $\varepsilon$ is a binary relation (i.e., a set of pairs) we use the following notation: $\mathrm{dom}\ \varepsilon = \{u \colon \exists v\ (u \varepsilon v)\}$ ($u \varepsilon v$ stands for $(u, v) \in \varepsilon$), $\mathrm{rng}\ \varepsilon = \{v \colon \exists u\ (u \varepsilon v)\}$, $|\varepsilon| = \mathrm{dom}\ \varepsilon \cup \mathrm{rng}\ \varepsilon$. We call a binary relation $\varepsilon$ an EF-relation if the following two conditions are satisfied:

(1) if $x \subseteq |\varepsilon|$ is nonempty, then there exists $u \in x$ with $\neg v \varepsilon u$ for all $v \in x$;

(2) if $u, v \in |\varepsilon|$, and for all $w \in |\varepsilon|$ the equivalence $w \varepsilon u \leftrightarrow w \varepsilon v$ holds, then u = v.

Condition (2) is the condition of extensionality and condition (1) is the condition of well-foundedness of the relation $\varepsilon$. If the set x is transitive, then the relation $\in$ on x is an EF-relation [4].

If $\varepsilon$ and $\delta$ are EF-relations and f is a bijection of $|\varepsilon|$ onto $|\delta|$ with $u \varepsilon v \leftrightarrow f(u) \delta f(v)$ for every pair $u, v \in |\varepsilon|$, we will say that f maps $\varepsilon$ into $\delta$. If there exists a bijection which maps $\varepsilon$ into $\delta$ we shall write $\varepsilon = {}^*\delta$.

If $\varepsilon$ is an EF-relation and $u \in |\varepsilon|$, we put

$$p_\varepsilon(u) = \{v \in |\varepsilon| \colon \text{there exist } w_0, w_1, \ldots, w_n \text{ such that } v = w_0 \varepsilon w_1 \varepsilon \ldots \varepsilon w_n = u\}$$

(n = 0 is possible, in which case u = v), and $\lambda_\varepsilon(u) = \varepsilon \cap (p_\varepsilon(u) \times p_\varepsilon(u))$ (the restriction of $\varepsilon$ to $p_\varepsilon(u)$). It is evident that $\lambda_\varepsilon(u)$ is an EF-relation with $|\lambda_\varepsilon(u)| = p_\varepsilon(u)$.

If $\varepsilon$ is an EF-relation with $\varepsilon = \lambda_\varepsilon(u)$ (or, what amounts to the same, $|\varepsilon| = p_\varepsilon(u)$) for some $u \in |\varepsilon|$, then it is easily seen that u is unique; we write $u = t(\varepsilon)$. In this case $\varepsilon$ is called a bounded EF-relation (BEF-relation). If $\varepsilon$ and $\delta$ are BEF-relations, $w = t(\delta)$, and there exists $v \in |\delta|$ with $\varepsilon = {}^*\lambda_\delta(v)$ and $v \delta w$, then we write $\varepsilon \in {}^*\delta$.

After these preliminary definitions we now introduce the theory T, which consists of all axioms of $Z^-$ together with the following three axioms:

Sq: for every x there exists (the set) x × x;

Cnt: for every nonempty x there exists a function from the natural numbers $\omega$ to x;

Repr: for every EF-relation $\varepsilon$ there exists a transitive set x such that $\varepsilon = {}^*\varepsilon_x$, where $\varepsilon_x = \{(u, v) \colon u, v \in x \text{ and } u \in v\}$.

We note that Sq and Repr are theorems in $ZF$ (for Repr, see [4]).

We show that the theory T can be interpreted in $Z^-$. First of all we note that T can be interpreted in $Z^- + (\omega \times \omega \text{ exists})$. In fact, let D be the class of all BEF-relations $\varepsilon$ with $|\varepsilon| \subseteq \omega$. For every $\in$-formula we denote by $\varphi^*$ the formula obtained from $\varphi$ by replacing $=$ and $\in$ throughout by $=^*$ and $\in^*$, respectively, and by relativizing all quantifiers and free variables to D.

It is easy, although rather laborious, to check that for every axiom $\varphi$ in the theory T one can find a formula $\varphi^*$ in $Z^- + (\omega \times \omega$ exists); we omit these verifications — the reader can find similar material in [2]. But this means that T can be interpreted in this theory. We remark that the axiom "$\omega \times \omega$ exists" is used only in the construction of the formula (axiom of infinity)*.

To eliminate from this reasoning the assumption that $\omega \times \omega$ exists we define $\langle m, n\rangle = 2^m(2n + 1) - 1$ for $m, n \in \omega$. Then we replace in the definitions dealing with EF-relations and in the definition of the class D every pair (m, n) by the "quasipair" $\langle m, n\rangle$. All the previous reasoning remains valid after this replacement, and the condition that $\omega \times \omega$ exists is therefore superfluous.

Thus we have proved the following

THEOREM 1.1. The theory $T$ can be interpreted in $Z^-$. In particular, $T$ and $Z^-$ are relatively consistent.

From now to the end of the paper all our considerations take place in the theory T. We note that the axioms of $T$ enable us to use the following operations: $x \times y$, dom f, rng f, $f''x$ (the image of x under f), $f|_x$ (the restriction of f to x), the result being a set in each case. The role of the additional axioms of $T$ is visible from the proof of the following lemma:

LEMMA 1.2. Let $\Phi$ be a functional and assume that the set $z \subseteq \operatorname{dom} \Phi$ is such that the class $X = \Phi''z$ is transitive. Then X is a set.

Proof. In view of the axiom Cnt (together with the axioms of $Z^-$) we can assume without loss of generality that $z \subseteq \omega$. We may further assume that $\Phi$ is a bijection (otherwise the functional

$$\Phi' = \{(m, p) \in \Phi: \forall n \in z\, (n < m \rightarrow (n, p) \notin \Phi)\}$$

is a bijection from some $z' \subseteq z$ onto X).

It is then easy to verify that the relation

$$\varepsilon = \{(m, n): m, n \in \omega \text{ and } \Phi(m) \in \Phi(n)\}$$

is an EF-relation with $|\varepsilon| = z$ (we note that $\varepsilon$ is a set by virtue of the axiom of separation and Sq). In view of the axiom Repr we can therefore find a transitive x and a bijection f which maps $\varepsilon_x$ onto $\varepsilon$. If we put

$$\Psi = \{(u, \Phi(f(u))): u \in x\},$$

we obtain: $\Psi$ is a bijection from x onto X such that for all u, $v \in x$ the equivalence $u \in v \leftrightarrow \Psi(u) \in \Psi(v)$ holds.

In this situation standard $\in$-induction applies (on account of the axiom of regularity) and one proves without difficulty that $\Psi(u) = u$ for all $u \in x$. Consequently X = x is a set; QED.

2. Constructible Sets. We consider the following Gödel operations:

$$F_0(x, y) = \{x, y\};\quad F_1(x, y) = x \times y;$$
$$F_2(x, y) = x - y;\quad F_3(x, y) = \operatorname{dom} x;$$
$$F_4(x, y) = \{(u, v) \in x: u \in v\};$$
$$F_5(x, y) = \{(a, b, c): (b, c, a) \in x\};$$
$$F_6(x, y) = \{(a, b, c): (c, b, a) \in x\};$$
$$F_7(x, y) = \{(a, b, c): (a, c, b) \in x\};$$
$$F_8(x, y) = x \cup y;\quad F_9(x, y) = x''y.$$

Operations $F_1$ to $F_7$ are taken from [4], and $F_8$ and $F_9$ are added for technical reasons. We remark that if x and y are sets and $i \leqslant 9$, and $F_i(x, y)$ is also a set (apply Sq and the axioms $Z^-$).

A class or a set X is called G-closed if for all $x, y \in X$ and $i \leqslant 9$ we have $F_i(x, y) \in X$. For all X we denote by S(X) the closure of X with respect to the Gödel operations, i.e., the smallest (in the sense of inclusion) G-closed class which contains X as a subclass. Moreover, $X \subseteq S(X)$ (since $x = F_2(x, F_2(x, x))$), and if the class X is transitive, then so is S(X) (because of the operation $F_0$).

LEMMA 2.1 (Sharpening (1.2)). Let F be a functional, $x \subseteq \text{dom } F$, and assume the class $Y = S(F''x)$ is transitive. Then Y and F"x are sets.

Proof. The axiom Cnt provides a function f from $\omega$ onto x. We define for $m, n \in \omega$ and $i \leqslant 9$

$$h(m, n, i) = 10(2^m(2n+1)-1)+i.$$

It is easy to see that h is a bijection from $\omega \times \omega \times \{0, \dots 9\}$ onto $\omega$, that $h(m, n, i) \geqslant m$, and $h(m, n, i) \geqslant n$.

On the set $\omega$ we now define the functional $\Phi$ as follows: $\Phi(2l) = F(f(l))$ for all $l \in \omega$, and if $k = h(m, n, i)$ then $\Phi(2k+1) = F_i(\Phi(m), \Phi(n))$. Due to the choice of f and h we find that $\Phi$ maps $\omega$ onto Y. But the class Y is transitive by hypothesis. Thus Y is a set in view of 1.2. $F''x \subseteq Y$ is therefore also a set by virtue of the axiom of separation. This proves the lemma.

COROLLARY 2.2. If the set x is transitive, then S(x) is also a set.

Proof. Take $F = \{(u, u): u \in x\}$ and apply the lemma.

Now we can deal with the hierarchy of constructible sets. We recall that this hierarchy consists of sets $L_\alpha$ which are defined by induction on $\alpha$, for all ordinals $\alpha$, as follows: $L_0 = \varnothing$, and if $\alpha > 0$ and $L_\gamma$ is already defined for all $\gamma < \alpha$, then $L_\alpha = S(\{L_\gamma : \gamma < \alpha\})$.

The possibility of definition by transfinite induction is usually based on the axiom of replacement, which is not available in the theory T. However, we are able to establish within the framework of T that for all ordinals $\alpha$, $L_\alpha$ is defined and is a set. This is the content of the following lemma.

LEMMA 2.3. If the sequence $(L_\gamma : \gamma < \alpha)$ is already constructed in accordance with the above definition (but without necessarily being a set), then $L_\alpha = S(\{L_\gamma : \gamma < \alpha\})$ is a set.

Before we prove this lemma we note the following properties of the sets $L_\alpha$ (they hold, strictly speaking, only for those ordinals $\alpha$ for which $L_\alpha$ exists; but after Lemma 2.3 is proved they will be valid for all ordinals $\alpha$):

2.3a. If $\alpha$ is a limit ordinal, then $L_\alpha = \bigcup_{\gamma<\alpha} L_\gamma$.

2.3b. $L_{\alpha+1} = S(L_\alpha \cup \{L_\alpha\})$.

2.3c. $L_\alpha$ is a transitive and G-closed set.

2.3d. If $\gamma < \alpha$, then $L_\gamma \in L_\alpha$ and $L_\gamma \subseteq L_\alpha$.

The first two statements are evident from the definition and the following two are proved by induction on $\alpha$ and $\gamma$ using the first two.

Proof of Lemma 2.3. In view of 2.2 and 2.3b it suffices to consider the case of a limit ordinal $\alpha$. Then $L_\alpha = \bigcup_{\gamma<\alpha} L_\gamma$ is a transitive class by 2.3b and 2.3d. Applying Lemma 2.1 with $x = \alpha$ and $F = (L_\gamma : \gamma < \alpha)$, we find that $L_\alpha$ is a set as required.

Thus $L_\alpha$ is well-defined and is a set for all $\alpha \in \text{On}$. We introduce the class $L = S(\{L_\gamma : \gamma \in \text{On}\})$ of all constructible sets and state two properties of this class which follow readily from 2.3a-2.3c.

2.3e. $$L = \bigcup_{\gamma \in \text{On}} L_\gamma.$$

2.3f. The class L is transitive and G-closed.

We now consider the question of constructive definability in the class L and in the sets $L_\alpha$. If X is a class and $\varphi$ a formula, we denote by $\varphi^X$ the relativization of $\varphi$ to X (i.e., the quantifiers $\exists u$, $\forall u$ are replaced by $\exists u \in X$ and $\forall u \in X$, and the conjunction $y \in X$ is introduced wherever a free variable y occurs). If $Y \subseteq X$, and there is a formula $\varphi(x)$ with parameters from X (or, perhaps, without parameters) such that $Y = \{x \in X : \varphi^X(x)\}$, then we shall say that Y is definable in X, or that Y is a class in X.

Let $\alpha \in \text{On}$. A function f which is defined on $\alpha$ and such that $f(\gamma) = S(f''\gamma)$ for all $\gamma < \alpha$ is called a constructing $\alpha$-function (c. $\alpha$-f.). (A particular c. $\alpha$-f. is the sequence $f_\alpha = (L_\gamma : \gamma < \alpha)$.) We denote the following formula by $C(\gamma, z)$:

$$\gamma \in \mathrm{On} \wedge \exists f\,(f \text{ is c. } (\gamma+1)\text{- f. and } z = f(\gamma)).$$

THEOREM 2.4. $(L_\gamma\colon \gamma \in \mathrm{On}) = \{(\gamma, z) \in L\colon C^L(\gamma, z)\}$. If $\alpha$ is a limit ordinal, then

$$(L_\gamma\colon \gamma < \alpha) = \{(\gamma, z) \in L_\alpha : C^{L_\alpha}(\gamma, z)\}.$$

Therefore $(L_\gamma : \gamma \in \mathrm{On})$ is a class in L and $f_\alpha = (L_\gamma\colon \gamma < \alpha)$ is a class in $L_\alpha$ if $\alpha$ is a limit ordinal.

Proof. Let A be the first claim and $A_\alpha$ ($\alpha \in \mathrm{On}$ a limit ordinal) the second claim of the theorem. Let $B_\alpha$ be the auxiliary statement $f_\alpha \in L_{\alpha+1}$. We will show the relationship between A, $A_\alpha$, and $B_\alpha$.

Proposition (1). If $(\forall \beta \in \mathrm{On})\, B_\beta$, then A. If $\alpha$ is a limit ordinal and $(\forall \beta < \alpha)\, B_\beta$, then $A_\alpha$.

We prove only the first statement; the proof of the second one is similar. It follows from 2.3f and the definition of S that if $x, y \in L$ and $S(x) \in L$, then $(y = S(x))^L \leftrightarrow y = S(x)$. The definition of operations $F_3$ and $F_9$ and the definition of c. $\alpha$-f. yield: if $\gamma, f \in L$, then

$$(f \text{ is c. } \gamma\text{-f.})^L \leftrightarrow f \text{ is c. } \gamma\text{-f.}$$

Now A follows immediately from the definition of formula C.

Proposition (2). If $\alpha$ is a limit ordinal and $A_\alpha$ holds, then $B_\alpha$ holds as well.

Indeed, $A_\alpha$ says that $f_\alpha$ is a class in $L_\alpha$. Therefore, Proposition (2) is a consequence of the following lemma:

LEMMA 2.5. If the class X is transitive and G-closed (e.g., X = $L_{\alpha+1}$), $x \in X$ (e.g., x = $L_\alpha$) and the set $y \subseteq x$ is defined in x, then $y \in X$.

The proof of this lemma runs essentially along the lines of the proof of [4], or of the theorem [1].

Proposition (3). If $B_\alpha$, then $B_{\alpha+1}$.

Indeed, we have, on the one hand, that $f_{\alpha+1} = f_\alpha \cup \{(\alpha, L_\alpha)\}$, and therefore $f_{\alpha+1}$ is obtained from $f_\alpha$ and $L_\alpha$ by composing the operations $F_0$, $F_3$, and $F_8$. On the other hand, $f_\alpha \in L_{\alpha+2}$ by 2.3d and $B_\alpha$, and $L_\alpha \in L_{\alpha+2}$ because of 2.3d. The proof is completed by applying 2.3c.

It is now evident how to use Propositions (1)-(3) and induction on $\alpha$ to prove $B_\alpha$ for all $\alpha$, $A_\alpha$ for all limit ordinals $\alpha$, and A. The theorem is now proved.

COROLLARY 2.6. $\mathrm{On} \subseteq L$. If $\alpha$ is a limit ordinal, then $\alpha \subseteq L_\alpha$.

Proof. By virtue of the theorem the class $F = (L_\gamma\colon \gamma \in \mathrm{On})$ is defined in L. In particular, $F \subseteq L$. Therefore, we have also $\mathrm{On} = \mathrm{dom}\, F \subseteq L$ (the class L is transitive, 2.3f). The verification of the second statement is similar.

Now we can prove the following theorem about the well-ordering of the class L and of the sets $L_\alpha$:

THEOREM 2.7. There exists a well-ordering of the class L which is defined in L. If $\alpha$ is a limit ordinal then there exists a well-ordering of the set $L_\alpha$ which is defined in $L_\alpha$.

The proof is given only for L; the case of $L_\alpha$ is similar. We use induction on $n \in \omega$ to define a class $K_n \subseteq L$ and a well-ordering $<_n$ on this class as follows. Put $K_0 = \{L_\gamma\colon \gamma \in \mathrm{On}\}$ and define $L_\gamma <_0 L_\delta$, if $L_\gamma \in L_\delta$ (i.e., if $\delta < \gamma$, 2.3d).

Assume that $K_n$ and $<_n$ are already defined. Put

$$K_{n+1} = \{(u, v, i)\colon u, v \in K_n \text{ and } i \leqslant 9\},$$

and define $<_{n+1}$ as the lexicographic ordering (using $<_n$ in the first two components and the usual ordering of natural numbers in the third).

Moreover, we define for all $p \in K = \bigcup_{n\in\omega} K_n$ a set F(p) as follows. If $p = L_\gamma \in K_0$, then $F(p) = L_\gamma$. If p = (u, v, i) and the sets F(u), F(v) are already defined, then F(p) = $F_i$(F(u), F(v)). From the definition of L we see that F is a functional from $K$ ($\subseteq L$) onto L.

Thus, for every $x \in L$ we can define n(x) as the smallest of those $n \in \omega$ for which x = F(p) for some $p \in K_n$, and we can define p(x) as $<_{n(x)}$ — the smallest of those $p \in K_{n(x)}$ with x = F(p). Now we can introduce the required ordering on L by putting x < y if

$$n(x) < n(y) \vee (n(x) = n(y) \wedge p(x) <_{n(x)} p(y)).$$

One verifies without difficulty that < is indeed a well-ordering of the class L. That < is defined in L is seen by a tedious check of the definability in L of all intermediate constructions (taking into account that L is G-closed). The initial class $\{L_\gamma: \gamma \in On\} = K_0$, which occurs in the construction of L, is defined in L by Theorem 2.4. We leave the details to the reader.

3. Interpretation of $ZF^-$ in T. We recall that our considerations take place in the theory T. Let Ω denote the totality of all ordinals α such that for every $\beta \leqslant \alpha$, $\beta > 0$, there exists a function $g \in L_{\beta+1}$ from ω to $L_\beta$. It is clear that either Ω = On or Ω is an ordinal. The following lemma shows that in the second case Ω is a limit ordinal.

LEMMA 3.1. $0 \in \Omega$. If $\alpha \in \Omega$, then $\alpha + 1 \in \Omega$.

Proof. It is clear that $0 \in \Omega$. Now assume that $\alpha \in \Omega$. To prove that $\alpha + 1 \in \Omega$ we have to construct a function $g \in L_{\alpha+2}$ from ω onto $L_{\alpha+1}$.

If α > 0, we proceed as follows. By hypothesis, there exists a function $f \in L_{\alpha+1}$ from ω onto $L_\alpha$. Put $g(0) = L_\alpha$, $g(2l+2) = f(l)$ for all $l$, and if k = h(m, n, i) (for the definition of h see the proof of 2.1), then $g(2k+1) = F_i(g(m), g(n))$. Thus g is a function from ω onto the set $L_{\alpha+1} = S(\{L_\alpha\} \cup \operatorname{rng} f)$ (by 2.3b and the choice of f and h). On the other hand, g is definable in $L_{\alpha+1}$, and therefore $f \in L_{\alpha+1}$ and $L_\alpha \in L_{\alpha+1}$. Hence $g \in L_{\alpha+2}$ by 2.5; QED.

If α = 0 we define g somewhat differently: $g(0) = \varnothing$, and if k = h(m, n, i), then $g(k+1) = F_i(g(m), g(n))$. Again g is a function from ω onto $L_1 = S(\{\varnothing\})$, definable in $L_1$ (as $L_1$ is the totality of all sets of finite rank). This concludes the proof of the lemma.

The class (or, maybe, the set) $L^* = \bigcup_{\alpha\in\Omega} L_\alpha$ will be our model of $ZF^-$. We note some properties of this class.

3.2a. Either Ω = On and L* = L, or else Ω is a limit ordinal and $L^* = L_\Omega$.

This follows from Lemma 3.1, and 2.3a and 2.3e.

3.2b. In both cases L* is a transitive G-closed class and $\Omega \subseteq L^*$.

This is a consequence of 3.2a, 2.3c, 2.3f, and 2.6.

Now it is evident that in L* the axioms of pairing, of regularity, and of extensionality are satisfied. We shall consider the remaining axioms of $ZF^-$.

3.2c. The axiom of infinity holds in L*. In particular, $\omega \in L^*$.

Indeed, $L_1 = S(\{\varnothing\})$ is the totality of all sets of finite rank. Hence $\omega \subseteq L_1$ and ω is defined in $L_1$. Consequently, $\omega \in L_2$ by 2.5. But $L_2 \subseteq L^*$ because of 3.1.

3.2d. The axiom of unions holds in L*.

Let $x \in L^*$, i.e., $x \in L_\alpha$ for some $\alpha \in \Omega$. Since $L_\alpha$ is transitive we get $\bigcup x \subseteq L_\alpha$. Hence $\bigcup x$ is a class in $L_\alpha$, $\bigcup x \in L_{\alpha+1}$ by Lemma 2.5, and $\bigcup x \in L^*$ by 3.1.

3.2e. If $x \in L^*$, there exists a function $g \in L^*$, defined on ω and such that $x \subseteq \operatorname{rng} g$.

Indeed, $x \in L^*$, and therefore $x \in L_\alpha$ for some $\alpha \in \Omega$. Thus $x \subseteq L_\alpha$ by 2.3c. Now use Lemma 3.1 together with the definition of Ω.

3.2f. The class $(L_\gamma: \gamma \in \Omega)$ is defined in L*.

3.2g. There exists a well-ordering of the class L* which is definable in L*.

These two statements follow from 3.2a and Theorems 2.4, 2.7.

In what follows $<^*$ denotes the well-ordering of the class L* which is defined in L* (its existence being guaranteed by 3.2g).

It remains to verify the axiom of replacement in L*. We make the following definition. Assume that $\beta \in \Omega$, and the functional F is such that whenever $z \in L_\beta \cap \operatorname{dom} F$, then $F(z) \in L_\beta$; then β is called a fixed point for F. The following theorem plays a key role in the verification of the axiom of replacement in L*.

THEOREM 3.3 (Fixed Point Theorem). For every collection of functionals $G_0, G_1, \ldots, G_k$ defined in L* and every $\alpha \in \Omega$ there exists an ordinal $\beta \geqslant \alpha$, $\beta \in \Omega$, which is a fixed point for every $G_i$.

The proof is based on two lemmas.

LEMMA 3.3a. If the functional $F\colon \omega \to \Omega$ is definable in L*, then there exists an ordinal $\alpha \in \Omega$ such that $\operatorname{rng} F \subseteq \alpha$.

Proof. Assume the contrary. Then $L^* = \bigcup_{n\in\omega} L_{F(n)}$. If $n \in \omega$ we denote by $g_n$ the smallest (in the sense of $<^*$) of the functions $g \in L^*$ for which $\operatorname{dom} g = \omega$ and $L_{F(n)} \subseteq \operatorname{rng} g$ (such functions exist by 3.2e). If $l = 2^n(2m+1) - 1$, then we put $G(l) = g_n(m)$.

The choice of $g_n$ implies that G is a functional from $\omega$ onto L*. Moreover, G is definable in L*, because of 3.2f and the definability of F and $<^*$.

If $\Omega$ = On, then the class L* is a set because of the transitivity of L* together with Lemma 1.2. Hence the class On ($\subseteq L^*$ by 3.2b) is also a set by the axiom of separation; but this is impossible.

If, on the other hand, $\Omega \in$ On, then L* = $L_\Omega$; and it follows from 2.5 and the definability of G that $G \in L_{\Omega+1}$. Thus, $\Omega \in \Omega$ by definition of $\Omega$. But this, too, is impossible, and the lemma is established.

LEMMA 3.3b. Assume that the functional F is defined in L* and $x \in L^*$; then there exists an ordinal $\alpha \in \Omega$, such that the set $F''x = \{F(u)\colon u \in x \cap \operatorname{dom} F\}$ is contained in $L_\alpha$.

Proof. We may assume, without loss of generality, that dom F = L*. According to 3.2e there exists a function $g \in L^*$ such that $\operatorname{dom} g = \omega$ and $x \subseteq \operatorname{rng} g$. For every n let G(n) denote the smallest ordinal $\gamma \in \Omega$ such that $F(g(n)) \in L_\gamma$. Then G is a functional from $\omega$ to $\Omega$ which is definable in L* (by definition of F and 3.2f), and such that $g \in L^*$ (we note that $\Omega \subseteq L^*$, 3.2b). Application of the preceding lemma and 2.3d completes the proof.

We return to the proof of Theorem 3.3. If $\gamma \in \Omega$ is an ordinal, Lemma 3.3b allows us to select $\delta \in \Omega, \delta \geqslant \gamma$ such that $G_i''L_\gamma \subseteq L_\delta$ for all $i \leqslant k$. Let G($\gamma$) denote the smallest of these ordinals $\delta$. Now we define F(0) = $\alpha$ and F(n + 1) = G(F(n)) by induction on n. The functional $F\colon \omega \to \Omega$ is definable in L* by virtue of 3.2f and the choice of $G_i$. In view of Lemma 3.3a the ordinal $\beta = \bigcup \operatorname{rng} F$ is therefore contained in $\Omega$. This ordinal $\beta$ is evidently also a fixed point for all $G_i$, and satisfies the inequality $\beta \geqslant \alpha$. This completes the proof of the theorem.

Let $x_1, \ldots, x_m$ be all free variables in the formula $\varphi(x_1, \ldots, x_m)$, which is free of parameters. The ordinal $\beta \in \Omega$ is said to reflect $\varphi$ if for all $p_1, \ldots, p_m \in L_\beta$ the equivalence

$$\varphi^{L_\beta}(p_1, \ldots, p_m) \leftrightarrow \varphi^{L^*}(p_1, \ldots, p_m)$$

holds.

COROLLARY 3.4 (Reflection Principle). If the formula $\varphi$ has no parameters and $\alpha \in \Omega$, then there exists an ordinal $\beta \in \Omega$, $\beta \geqslant \alpha$, which reflects $\varphi$.

Proof. Without loss of generality, we may assume that $\forall$ does not contain universal quantifiers $\forall$ (otherwise $\varphi$ may be replaced by $\neg\exists\neg$). Let $\psi_0, \ldots, \psi_k$ be all the subformulas of $\varphi$ which begin with $\exists$ ($\varphi$ is considered as a subformula of itself). With every $i \leqslant k$, we associate a functional $G_i$ as follows.

Let $\psi_i$ be the formula $\exists y \chi(y, x_1, \ldots, x_m)$ ($\chi$ and m will of course depend on i). Assume that $p = (p_1, \ldots, p_m) \in L^*$ and that there exists a set $q \in L^*$, satisfying $\chi^{L^*}(q, p_1, \ldots, p_m)$; then we denote by $G_i(p)$ the smallest (with respect to $<^*$) of these q. Each functional $G_i$ is definable in L* since the ordering $<^*$ is definable in L*.

By Theorem 3.3 there exists an ordinal $\beta \in \Omega$, $\beta \geqslant \alpha$, which is a fixed point for all $G_i$. Now one can easily verify by induction on the number of logical symbols in the subformula that $\beta$ reflects every subformula of the formula $\varphi$, and in particular that $\beta$ reflects $\varphi$; QED.

COROLLARY 3.5. If $\alpha \in \Omega$, and $X \subseteq L_\alpha$ is a class in L*, then $X \in L^*$ (axiom of separation in L*).

Proof. Assume that the formula $\varphi(y, x_1, \ldots, x_m)$ is free of parameters and that the sets $p_1, \ldots, p_m \in L^*$ satisfy

$$X = \{y \in L^*\colon \varphi^{L^*}(y, p_1, \ldots, p_m)\}.$$

By Corollary 3.4 there exists an ordinal $\beta \in \Omega, \beta' \geqslant \alpha$, which reflects $\varphi$ and such that all $p_i$ belong to $L_\beta$. Then

$$X = \{y \in L_\beta: \varphi^{L_\beta}(y, p_1, \ldots, p_m)\},$$

i.e., X is definable in $L_\beta$. Now apply Lemmas 2.5 and 3.1 to conclude the proof.

Now we have gathered all that is necessary for the proof of the axiom of replacement in L*. Assume that the functional F is definable in L*, $x \in L^*$, $x \subseteq \mathrm{dom}\, F$. We must show that $F''x \in L^*$. But according to Lemma 3.3b there exists an ordinal $\alpha \in \Omega$ such that $F''x \subseteq L_\alpha$. The result now follows from 3.5.

Thus, in (the class or the set) L* all axioms of the theory $ZF^-$ are satisfied (one can show that the axiom of choice and the axiom of constructibility hold also in L*). We can rephrase this result as follows: if λ*(x) is a formula and L* is the canonically defined class $L^* = \{x: \lambda^*(x)\}$, then for every axiom $\varphi$ of $ZF^-$ the relativization of $\varphi$ with respect to λ*(x) is provable in T. This means also that the theory $ZF^-$ has an interpretation in T. This result together with Theorem 1.1 yields the theorem stated in the introduction.

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# EXTENDING EMBEDDINGS AND HOMEOMORPHISMS OF SUBSETS OF THE TIKHONOV CUBE

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This work concerns one of the fundamental questions of shape theory (see [1, 2]): the question of characterizing spaces of the same shape which is a generalization of the notion of homotopy type. The proof of the main theorem (Theorem 3) requires applying techniques of the theory of Q-manifolds, i.e., separable metric spaces in which each point has a neighborhood homeomorphic to an open subset of the Hilbert cube Q. This technique has been developed for the last decade, mainly, by American authors (see [3]).

It has been shown in [4] that Z-sets X and Y lying in some Q-manifolds M and N, respectively, and having comparable proper shapes have arbitrarily close homeomorphic closed neighborhoods in M and N, respectively, and these neighborhoods are themselves Q-manifolds. Our goal is to prove a similar theorem for bicompacts. To prove this theorem, we will have to go along the way indicated in the case of F- and Q-manifolds. Just as in the case of F- and Q-manifolds, we will need theorems on extending embedding and on replacing homotopies by isotopies (see [5-7]).

The Tikhonov cube $I^\Omega$, where $\Omega$ is the set of indices and $\tau = |\Omega| \geqslant \aleph_0$ is the cardinality of this set, will be represented in the following form: $I^\Omega = \prod_{\omega\in\Omega} I_\omega$, $I_\omega = [-1, 1]$. If $\Omega' \subset \Omega$, then $I^{\Omega'} = \prod_{\omega\in\Omega'} I_\omega, I_\omega = [-1, 1]$. $\mathfrak{A}$ will denote the set of finite subsets of the index set $\Omega$, ordered by inclusion. Let I denote the segment [0, 1], $\mathrm{id}_X$ the identity map of the set X onto itself, and Cl, Bd, Int the operators of closure, boundary, and interior of a set, respectively.

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