5. A. V. Chakmazyan, "A Norden normalized submanifold with parallel field of normal directions in P_n," Dokl. Akad. Nauk SSSR, 236, No. 4, 816-819 (1977).
6. A. P. Norden, Spaces with Affine Connection [in Russian], Nauka, Moscow (1976).
7. N. M. Ostianu, "Distributions of m-dimensional linear elements in a space with projective connection. II," in: Memoirs of the Geometry Seminar [in Russian], No. 3, VINITI Akad. Nauk SSSR, Moscow (1971), pp. 95-114.
8. N. M. Ostianu, V. V. Ryzhkov, and P. I. Shveikin, "Sketch of the scientific investigations of German Fedorovich Laptev," in: Memoirs of the Geometry Seminar [in Russian], No. 4, VINITI Akad. Nauk SSSR (1973), pp. 7-70.
9. K. Nomizu, Lie Groups and Differential Geometry [in Russian], Moscow (1960).
10. V. V. Gol'dberg, "A normalization of p-conjugate systems of n-dimensional projective space," in: Memoirs of the Geometry Seminar [in Russian], No. 1, VINITI Akad. Nauk SSSR (1966), pp. 89-109.
11. V. T. Bazylev, "Multidimensional nets and their transformations," in: Geometry, 1963 [in Russian], Itogi Nauki i Tekhniki, VINITI Akad. Nauk SSSR, Moscow (1965), pp. 138-164.

GENERALIZATION OF P.S. NOVIKOV'S THEOREM ON CROSS SECTIONS OF BOREL SETS

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In [1] Novikov has proved that for each plane Borel set the collection of all points of this set that belong to its closed vertical cross sections is a CA-set. We prove that this result remains valid if in place of closed cross sections we consider the cross sections that belong to an arbitrary fixed class of Borel hierarchy.

We start with definitions. The Baire space I consists of all functions defined on the set of natural numbers $\omega = \{0, 1, 2, \ldots\}$ with values in ω, i.e., of all infinite sequences of natural numbers [2]. We will consider subsets of the Baire plane I^2, and not of the Euclidean plane OXY. However, the obtained results are easily carried over to subsets of the Euclidean plane by using the fact that the Baire space is homeomorphic to the set of all irrational points of the Euclidean line (see [2]).

The sets that can be obtained from open subsets of the given space I^m ($m \geqslant 1$) with the help of the operations of countable union, countable intersection, and complementation are called Borel sets. These sets are organized into a Borel hierarchy that consists of the classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, and $\mathbf{\Delta}^0_\alpha$ defined for each ordinal α, $1 \leqslant \alpha < \omega_1$, by induction over α in the following manner:

$\mathbf{\Sigma}^0_1$ is the family of all open sets;

$\mathbf{\Pi}^0_\alpha$ is the family of the complements of all sets in $\mathbf{\Sigma}^0_\alpha$;

$\mathbf{\Sigma}^0_\alpha$ (for $\alpha \geqslant 2$) is the family of all countable unions of sets of the classes $\mathbf{\Pi}^0_\beta$ with $\beta < \alpha$;

$$\mathbf{\Delta}^0_\alpha = \mathbf{\Sigma}^0_\alpha \cap \mathbf{\Pi}^0_\alpha.$$

The projections of closed subsets P of I^{m+1} on the space I^m form the class of $\mathbf{\Sigma}^1_1$-sets (i.e., of A-sets in the classical system of notation, adopted in [1]). The complements of sets of the class $\mathbf{\Sigma}^1_1$ are called $\mathbf{\Pi}^1_1$-sets (or CA-sets).

Let $P \subseteq I^2$. Let us put in correspondence with each point $x \in I$ the cross section

$$P_x = \{y \in I: \langle x, y\rangle \in P\}$$

of the set P by the vertical line with abscissa x. For each class K of the Borel hierarchy we set

$$P_{(K)} = \{\langle x, y\rangle \in P: P_x \in K\}$$

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which is the set of all points of P that lie on the vertical cross sections of this set that (the cross sections) belong to this class K.

THEOREM 1. Let K be one of the classes $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$, and $\mathbf{\Delta}^0_\alpha$. Then for each Borel set $P \subseteq I^2$, the set $P_{(K)}$ belongs to the class $\mathbf{\Pi}^1_1$.

The proof of this theorem uses the coding of Borel sets by means of the points of the space I. This coding puts in correspondence with each point f of a certain special $\mathbf{\Pi}^1_1$-set $W \subseteq I$ (set of Borel codes) a set $B_f \subseteq I$ in the following manner (see [3]):

(1) If f(0) = 0, then $B_f = U_{f(1)}$, where $\{U_i : i \in \omega\}$ is a certain (once for every) fixed numbering of all basic clopen subsets of I.

(2) If f(0) = 1 and the set $B_{(f)_m}$ is already defined for all $m \in \omega$, then $B_f = \bigcup_{m \in \omega} B_{(f)_m}$. Here $(f)_m \in I$ is the function (a point of I) defined on ω by the condition

$$(f)_m (k) = f(2^m (2k+1)) \quad \text{for all} \quad k.$$

(3) If f(0) = 2, g(k + 1) = f(k) for all k, and the set B_g is defined, then $B_f = I - B_g$.

The set of all points $f \in I$ such that the set $B_f \subseteq I$ can be defined in accordance with the conditions (1)-(3) is denoted by W. If $f \in W$, then B_f is a Borel set. Conversely, each Borel subset B of I has the form B_f for a suitable code $f \in W$.

The following lemma is the crucial stage in the proof of Theorem 1.

LEMMA 1 [4]. The set $W(\mathbf{\Sigma}^0_\alpha) = \{f \in W : B_f \in \mathbf{\Sigma}^0_\alpha\}$ belongs to the class $\mathbf{\Pi}^1_1$ for each ordinal $\alpha < \omega_1$.

This lemma implies the following corollary.

COROLLARY 1. The sets $W(\mathbf{\Pi}^0_\alpha)$ and $W(\mathbf{\Delta}^0_\alpha)$, defined in the same manner, also belong to the class $\mathbf{\Pi}^1_1$.

Proof of Corollary 1. We put in correspondence with each point $g \in I$ the point $2^\frown g \in I$ by the equalities $(2^\frown g)(0) = 2$ and $(2^\frown g)(k+1) = g(k)$. Then, obviously,

$$W(\mathbf{\Pi}^0_\alpha) = \{g \in I : 2^\frown g \in W(\mathbf{\Sigma}^0_\alpha),$$

whence it follows from Lemma 1 that $W(\Pi^0_\alpha) \in \Pi^1_1$. (Here we have used the fact that the class $\mathbf{\Pi}^1_1$, like any projective or Borel class, is closed with respect to the operation of taking continuous preimage (see, e.g., [5]); the mapping $g \mapsto 2^\wedge g$ is obviously continuous.)

Finally, it is clear that $W(\mathbf{\Delta}^0_\alpha) = W(\mathbf{\Sigma}^0_\alpha) \cap W(\mathbf{\Pi}^0_\alpha)$.

We pass immediately to the proof of Theorem 1. We prove that each Borel subset P of I^2 has the following property:

(*) There exists a continuous function F: $I \to W$ such that $P_x = B_{F(x)}$ for all $x \in I$.

The Borel subsets of I^2 are obtained by repeated application of the operations of countable union and complementation from basic sets of the form $U_i \times U_j$. Therefore, it is required to prove that

(a) each set P of the form $U_i \times U_j$ satisfies (*);

(b) if $P \subseteq I^2$ satisfies (*), then the complement $I^2 - P$ also satisfies (*); and, finally,

(c) if the sets $P_0, P_1, P_2, \ldots \subseteq I^2$ satisfy (*), then their union P also satisfies (*).

If $P = U_i \times U_j$, then F(x) denotes the point $\omega \times \{0\}$ for $x \in I - U_i$ (it is assumed that U_0 is the empty set) and the point $f \in I$, such that f(1) = j and f(k) = 0 for $k \neq 1$, for $x \in U_i$. The constructed function F proves (a).

To prove (b), let us observe that if the function F ensures (*) for the set P, then, setting $G(x) = 2^\frown F(x)$ for all $x \in I$, we get the function G, desired in the sense of (*) for the set $I^2 - P$.

Finally, if the functions $F_0, F_1, F_2, \ldots$ ensure (*) for the sets $P_0, P_1, P_2, \ldots$, then for each $x \in I$ we should denote by F(x) the point $f \in I$ such that f(0) = 1 and $(f)_m = F_m(x)$ $\forall m$ (these relations define f uniquely). This function F is the desired one for the union of the sets P_m.

Thus, in fact, each Borel subset P of I^2 has the property (*). In particular, a certain continuous function F: I → W ensures (*) for the set P from the condition of Theorem 1. Now, we have

$$P_{(K)} = \{\langle x, y\rangle \in P: F(x) \in W(K)\}$$

for each Borel class K, which gives $P_{(K)} \in \mathbf{\Pi}_1^1$ by virtue of Lemma 1 and Corollary 1.

Remark. Under the conditions of Theorem 1, the projection Pr P_K of the set P_K on the first axis is the intersection of the $\mathbf{\Pi}_1^1$-set $\{x: F(x) \in W(K)\}$ with the $\mathbf{\Sigma}_1^1$-set Pr P. Thus, if, in addition, we know that Pr P is a Borel set, then we can assert that Pr $P_{(K)} \in \mathbf{\Pi}_1^1$.

LITERATURE CITED

1. P. S. Novikov, "On projections of certain B-sets," Dokl. Akad. Nauk SSSR, 23, No. 9, 863-864 (1939).
2. P. S. Aleksandrov, Introduction to Set Theory and General Topology [in Russian], Nauka, Moscow (1977).
3. R. M. Solovay, "A model of set theory in which every set of reals is Lebesgue measurable," Ann. Math., 92, 1-56 (1970).
4. A. Louveau, "La hierarchie Borelienne des ensembles Δ_1^1," C. R. Acad. Sci. Paris, 285, 601-604 (1977).
5. K. Kuratowski and A. Mostowski, Set Theory, North-Holland, Amsterdam (1967).

GRAPHS GENERATED BY INCONSISTENT SYSTEMS OF LINEAR INEQUALITIES

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In this article we examine properties of a class of graphs generated by inconsistent systems of linear inequalities in order to determine all the maximal consistent subsystems which they contain (maximal under inclusion).

In [1, 2] Chernikov discusses the theory of convolution for systems of linear inequalities, which he uses to construct an algorithm for finding all minimal inconsistent subsystems of an inconsistent system. All the maximal consistent subsystems were found there by considering inclusion-maximal sets of inequalities, none of which is completely contained in any minimal inconsistent subsystem. A modified method for solving this combinatorial problem is proposed in [3].

The convolution method greatly increases the number of inequalities in the convolutions, so that its application to interesting practical problems is limited.

Another approach is suggested in [4] which is based on the following fact: The complement of a maximal consistent subsystem with respect to the entire system is a consistent subsystem. This property can be exploited to devise an iterative procedure for identifying maximal consistent subsystems. We develop this method further in this work by introducing the concept of the graph of a maximal consistent subsystem of an inconsistent system of linear inequalities.

We will consider a homogeneous inconsistent system of strict linear inequalities of rank n over the space $\mathbf{R}^n$:

$$\langle a_i, x\rangle > 0, \quad i \in J = \{1, 2, \ldots, m\}, \tag{1}$$

where

$$\| a_i \| = 1, a_i \neq -a_j, i \neq j, i, j \in J.$$

The set M of indices i, $M \subseteq J$, appearing in (1), is called the index of w.

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