CARDINALITY OF THE SET OF VITALI EQUIVALENCE CLASSES

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An equivalence relation defined by letting $x \sim y$ if and only if the difference $y - x$ is rational, which was used by Vitali to construct a set which is not Lebesgue measurable (see [1]), partitions the real axis R into pairwise-nonintersecting countable classes $[x] = \{y: y \sim x\} = \{x + q: q \text{ is rational}\}$. What is the cardinality $\mathfrak{m}$ of the set M of all classes [x]?

If we accept the axiom of choice (AC) then the answer is simple; in this case we have $\mathfrak{m} = \mathfrak{c}$, where $\mathfrak{c}$ is the cardinality of the continuum. Indeed, $\mathfrak{c} = \mathfrak{m}\aleph_0$ by construction, and AC implies that $\varkappa\aleph_0 = \varkappa$ for every infinite cardinality $\varkappa$. However, the character of this argument, which does not in any way suggest an explicit one-to-one correspondence between M and some continual set (say, R), let Luzin to make the following remark in [2, Sec. 64]: "It seems natural..., that the set M would have the cardinality of the continuum... . In reality, the cardinality of the set M (if discussed properly) is completely unknown; we do not know how to construct a mapping of M onto the continuum."

However, an inequality $\mathfrak{c} \leqslant \mathfrak{m}$ holds regardless of AC. Namely, to every number x, 1, with a decimal notation $x = 0.x_1x_2x_3x_4...$ we can associate a number x^* with a decimal notation

$$x^* = 0.x_1 0 x_1 x_2 00 x_1 x_2 x_3 000 x_1 x_2 x_3 x_4 0000... \tag{1}$$

It is easy to see that the class $[x^*]$ contains only one number, namely, x^* itself, of form (1) for some sequence of decimal digits x_i. Therefore, $[x^*] \neq [y^*]$ if $x \neq y$, i.e., we have obtained a one-to-one mapping of a half-interval [0, 1] onto a portion of the set M. This implies that $\mathfrak{c} \leqslant \mathfrak{m}$.

The inverse inequality $\mathfrak{m} \leqslant \mathfrak{c}$ appears even more natural than the one proven above, but the possibility of it being true is limited by the following theorem, which we refer to below [2, Sec. 64].

Sierpinski's Theorem [3, p. 147]. If a set M is linearly ordered than there exists $X \subseteq R$. which is not Lebesgue measurable.

COROLLARY. A strict inequality $\mathfrak{c} < \mathfrak{m}$ is compatible with the axioms of a system ZF + DC.

Recall that ZF is the Zermelo–Frenkel theory without the axiom of choice, and DC is the principle of dependent choice that allows a countable sequence of selections in a rather general situation, which is usually used to prove certain fundamental assumptions of analysis based on AC in cases where an application of the "full" axiom of choice is undesirable (see [1; 4, Sec. 20]).

Indeed, Solovei in [5] established the compatibility of ZF + DC with a hypothesis LM that states that every set $X \subseteq R$ is Lebesgue measurable, and Sierpinski's theorem shows that LM implies that $\mathfrak{c} < \mathfrak{m}$, since otherwise an equality $\mathfrak{c} = \mathfrak{m}$ would define on M the natural order of R.

To measure the "width" of the apparently possible gap between the cardinalities $\mathfrak{c}$ and $\mathfrak{m}$, we use the Hartogs function

$$H(\theta) = \min \{\aleph_\xi : \aleph_\xi \leqslant \theta \text{ does not hold}\}$$

[4, Sec. 4] that projects the class of all cardinalities θ onto a sequence of alephs (i.e., cardinalities of fully ordered sets; such cardinalities are called cardinals). Under the AC all cardinalities are cardinals and $H(\theta) = \theta^+$ is the cardinal next in magnitude after θ.

THEOREM. Let $\mathcal{M}$ be a countable transitive model of the theory ZFC together with the constructivity axiom and $\varkappa \leqslant \lambda$ a pair of cardinals that are noncountably co-final in $\mathcal{M}$, cannot be written as θ^+ for some cardinal θ of countable cofinality in $\mathcal{M}$, and $\aleph_2^{\mathcal{M}} \leqslant \varkappa$. Then

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there exists a generic cardinal-preserving extension $\mathcal{N}$ of the model $\mathcal{M}$, that satisfies all axioms of ZF, principle DC, and equations $H(\mathfrak{c}) = \varkappa$, $H(\mathfrak{m}) = \lambda$.

COROLLARY. An inequality $H(\mathfrak{c}) < H(\mathfrak{m})$ is compatible with ZF + DC.

We note that cardinals $H(\mathfrak{c})$ and $H(\mathfrak{m})$ have noncountable cofinality and so cannot follow immediately after the countably co-final cardinals in ZF + DC, so therefore the corresponding conditions on $\varkappa$ and λ in the statement of the theorem are necessary.

We also note that $\aleph_0 < \mathfrak{c}$, so therefore, in general, $\aleph_1 \leqslant H(\mathfrak{c})$. The case $H(\mathfrak{c}) = \aleph_1$ is not covered by our theorem and in principle cannot be realized on cardinal-preserving extensions of a constructive model, since, as it can be easily shown, the latter equation implies in ZF + DC a strict unattainability of the "real" $\aleph_1$ in the constructive universe.

However, the consistency of a relation $\aleph_1 = H(\mathfrak{c}) < H(\mathfrak{m})$ can be proved using models that combine an identification of cardinals of the type carried out in the construction of Solovei's model mentioned above and a corresponding symmetrization. Incidentally, Solovei's model itself satisfies $H(\mathfrak{c}) = H(\mathfrak{m}) = \aleph_1$.

Proof of the Theorem. As a set of restricting conditions on the construction of the desired model we take a λ-cohen m.v.u.

$$P = \{p\colon p \text{ is a function such that } \operatorname{dom} p \subseteq \lambda \times \omega \text{ is finite and } \operatorname{ran} p \subseteq \{0,1\}\}$$

[1, p. 119]. We fix a set $G \subseteq P$ which is P-generic on $\mathcal{M}$ and note two known properties of generic extensions of $\mathcal{M}[G]$:

cardinals of the model $\mathcal{M}$ are also cardinals in $\mathcal{M}[G]$ and their cofinalities are also preserved; (2)

every set $u \in \mathcal{M}[G]$, $u \subseteq \lambda$ can be covered by a set $u' \in \mathcal{M}$, $u' \subseteq \lambda$ that has the same cardinality in $\mathcal{M}[G]$ as u. (3)

The proof is based on the chain countability condition (c.c.c.) of the set P (see [1, Chap. 4, Sec. 3] and [4, Sec. 18]).

Let $a_\xi(k) = i \leftrightarrow \exists p \in G\,(p(\xi, k) = i)$ for $\xi < \lambda$; then a_ξ is a function from ω to {0, 1}. We identify every such function a with a proper fraction whose binary expansion has digits a(0), a(1), a(2),..., and use this identification to denote the Vitali class $[a_\xi]$ of a_ξ. We also note that

$$a_\xi \neq a_\eta \text{ and } [a_\xi] \neq [a_\eta] \text{ for } \xi \neq \eta. \tag{4}$$

A model $\mathcal{N}$ needed for the proof of the theorem is obtained by a symmetrization in $\mathcal{M}[G]$ using two automorphism groups.

The first group is $B = \{b \in \mathcal{M}\colon b$ is a bijection from λ onto λ}.

The automorphisms of the second group are constructed using sets belonging to $Z = \{z \subseteq \lambda \times \omega\colon z$ is finite}.

Let $b \in B$ and $z \in Z$. If $\xi < \lambda$ then let

$$za_\xi(k) = \begin{cases} a_\xi(k) & \text{for } \langle \xi, k\rangle \notin z, \\ 1 - a_\xi(k) & \text{for } \langle \xi, k\rangle \in z. \end{cases}$$

The finiteness of z implies the finiteness of the set of all indices ξ such that $za_\xi \neq a_\xi$, and if ξ is such an index then the set of k such that $za_\xi(k) \neq a_\xi(k)$ is also finite. In particular, this implies that $[za_\xi] = [a_\xi]$ for all ξ.

Suppose also that $u \subseteq \lambda$. Define in $\mathcal{M}[G]$ sequences $zb \mid u = \langle za_{b(\xi)}\colon \xi \in u\rangle$ and $[zb \mid u] = \langle [za_{b(\xi)}]\colon \xi \in u\rangle$: actually, the above arguments imply that [zb|u] does not depend on z, and this notation method is used for consistency.

Finally, let

$$C_\theta = \{u \subseteq \lambda\colon u \in \mathcal{M} \text{ has cardinality } < \theta \text{ in } \mathcal{M}\};$$
$$W_\varkappa = \{zb \mid u\colon z \in Z,\ b \in B,\ u \in C_\varkappa\};$$
$$[W_\lambda] = \{[zb \mid u]\colon z \in Z,\ b \in B,\ u \in C_\lambda\}.$$

The desired properties are possessed by a model

$$\mathcal{N} = HOD\,(W_\varkappa \cup [W_\lambda] \cup \{W_\varkappa, [W_\lambda]\}) \text{ in } \mathcal{M}[G],$$

containing all sets that are hereditarily defined (i.e., together with every element of their transitive closures) in $\mathcal{M}[G]$ with formulas whose variables are ordinals, sets $W_\varkappa$ and $[W_\lambda]$, and the elements of these sets.

The model $\mathcal{N}$ satisfies all axioms of ZF [6]. Furthermore, $\mathcal{M} \subseteq \mathcal{N} \subseteq \mathcal{M}[G]$ (the first inclusion holds because the constructivity axiom holds in $\mathcal{M}$, which also implies that $\mathcal{N} \subseteq HOD$ in $\mathcal{M}[G]$). Consequently, assertion (2) also holds for the extension $\mathcal{N}$ of the model $\mathcal{M}$.

LEMMA 1. The principle DC holds in the model $\mathcal{N}$.

Proof. Using the known method of calculation of DC in models constructed using ordinal definability [4, Sec. 20; 5, Sec. III.2], we see that to check DC in $\mathcal{N}$ it suffices to prove the following two statements:

$$\text{if } g \in \mathcal{M}[G],\ g\colon \omega \to \mathcal{M},\ \text{then } g \in \mathcal{N}; \tag{5}$$

$$\text{if } g \in \mathcal{M}[G];\ g\colon \omega \to W_\varkappa \cup [W_\lambda],\ \text{then } g \in \mathcal{N}. \tag{6}$$

We first prove statement (5). Carrying out our arguments in $\mathcal{M}$, we define

$$Q_n = \{p \in P\colon \exists x\, (p \Vdash g(n) = x)\}\ (n \in \omega),$$

where $\Vdash$ is the restriction corresponding to the m.v.u. P and a generic extension of type $\mathcal{M}[G]$ (to avoid excessive formalism, in our notation we do not differentiate between sets in the generic extension and the corresponding terms of the special language used to write the restriction formulas). In every Q_n we choose a maximal antichain $A_n \subseteq Q_u$; then c.c.c. implies that A_n is countable, and a set $u = \bigcup_{n\in\omega}\bigcup_{p\in A_n}\|p\|$, where $\|p\| = \{\xi\colon \exists k\, (\langle \xi, k\rangle \in \operatorname{dom} p)\}$, is also countable (in $\mathcal{M}$). The maximality of A_n implies that

$$g(n) = x \leftrightarrow \exists p \in G\, (\|p\| \subseteq u \wedge p \Vdash g(n) = x). \tag{7}$$

However, if $\|p\| \subseteq u$ then we have an equivalence

$$p \in G \leftrightarrow \forall \langle \xi, k\rangle \in \operatorname{dom} p\, (p(\xi, k) = a_\xi(k)),$$

Using this together with (7) and the fact that a restriction on $\mathcal{M}$ can be written in $\mathcal{M}$ and is ordinally definable in the extension $\mathcal{M}[G]$, we see that $g \in HOD(\varnothing e \mid u)$ in $\mathcal{M}[G]$, where $e \in B$ is the identity bijection defined by $e(\xi) = \xi$ for all ξ.

Now we prove statement (6). For simplicity, let $g\colon \omega \to W_\varkappa$ and

$$g(n) = z_n b_n \mid u_n,\ u_n \in C_\varkappa,\ b_n \in B,\ z_n \in Z$$

for all n. A set $v = \bigcup_{n\in\omega}(b_n'' u_n) \subseteq \lambda$ has cardinality $< \varkappa$ in $\mathcal{M}[G]$, since the noncountable cofinality of $\varkappa$ is preserved in $\mathcal{M}[G]$ by (2). Therefore, using (3) we obtain a set $u \in C_\varkappa$ such that $v \subseteq u$. It is now easy to see that $g \in HOD$ in $\mathcal{M}[G]$ with respect to $\varnothing e \mid u\ (\in W_\varkappa)$ and sequences $\langle u_n\colon n \in \omega\rangle$, $\langle b_n\colon n \in \omega\rangle$, $\langle z_n\colon n \in \omega\rangle$, which belong to $\mathcal{N}$ by (5).

LEMMA 2. $H(\mathfrak{c}) = \varkappa$ in $\mathcal{N}$.

Proof. Inequality $\geqslant$ is easily obtained from the construction of the set $W_\varkappa$. Namely, choose an arbitrary cardinal $\theta < \varkappa$ and a set $u \in C_\varkappa$ of cardinality θ (in $\mathcal{M}$, so therefore also in $\mathcal{N}$), and define a function $f = \varnothing e \mid u \in \mathcal{N}$. This mapping maps u into $\mathcal{P}(\omega)$, and is one-to-one by (4).

To derive the opposite inequality, we let $h \in \mathcal{N}$, $h\colon \varkappa \to \mathcal{P}(\omega)$ be an arbitrary function. We have to prove that the domain of its values $\operatorname{ran} h = h''\varkappa$ has cardinality $< \varkappa$ in $\mathcal{N}$; this would then imply the desired inequality. We note that

$$n \in h(\eta) \leftrightarrow \varphi(\eta, n) \text{ holds in } \mathcal{M}[G]$$

for some formula φ, whose variables are only ordinals, sets $W_\varkappa$ and $[W_\lambda]$, and their elements. Let

$$v = \bigcup \{b''u\colon \exists z\, (\text{variable } zb \mid u \text{ is contained in } \varphi)\}$$

then $v \in C_\varkappa$, We prove the following equivalence:

$$n \in h(\eta) \leftrightarrow \exists p \in G\, (\|p\| \subseteq v \wedge p \Vdash \varphi(\eta, n)). \tag{8}$$

If we assume that this relation is already proven, then every set $h(\eta)$, $\eta < \varkappa$, for a known G is determined by a countable sequence $\langle A_{\eta n}\colon n \in \omega\rangle \in \mathcal{M}$ of maximal antichains $A_{\eta n}$ contained in sets

$$Q_{\eta n} = \{p \in P\colon \|p\| \subseteq v \wedge p \Vdash \varphi(\eta, n)\}.$$

On the other hand, the cardinality of a set $\{p \in P: \|p\| \subseteq v\}$ is equal to the cardinality of v, i.e., is less than $\varkappa$ in $\mathcal{N}$. Therefore, using a condition $\varkappa \neq \theta^+$ for cardinals θ of countable cofinality in $\mathcal{M}$ (see the statement of the theorem), we conclude that the family of all sequences of antichains of this form has cardinality $<\varkappa$ in $\mathcal{N}$. Therefore, the set ran h also has cardinality $<\varkappa$ in $\mathcal{N}$.

Thus, equivalence (8) implies Lemma 2. The implication from left to right in the equivalence itself is nontrivial, so let $n \in h(\eta)$. There exists a condition $p_1 \in G$, restricting $\varphi(\eta, n)$, that does not necessarily satisfy $\|p_1\| \subseteq v$. However, assume that

$$p_0 = p_1 \mid v = \{\langle \xi, k, i\rangle \in p_1: \xi \in v\};$$

Then $p_0 \in G$ (since $p_0 \subseteq p_1 \in G$) and $\|p_0\| \subseteq v$, i.e., it remains to show that $p_0 \Vdash \varphi(\eta, n)$.

To do this, we assume the opposite, i.e., that this restriction does not hold. Then there exists a condition $p_2 \in P$, $p_0 \subseteq p_2$, restricting $\neg\varphi(\eta, n)$. The contradiction between p_1 and p_2 localizes on a set

$$\zeta = \{\langle \xi, k\rangle \in \operatorname{dom} p_1 \cap \operatorname{dom} p_2: p_1(\xi, k) \neq p_2(\xi, k)\},$$

It defines an order automorphism of the set P such that $\operatorname{dom}\zeta p = \operatorname{dom} p$ and

$$\zeta p(\xi, k) = \begin{cases} p(\xi, k) & \text{for } \langle \xi, k\rangle \notin \zeta, \\ 1 - p(\xi, k) & \text{for } \langle \xi, k\rangle \in \zeta, \end{cases}$$

and also a change of variables in $W_\varkappa$ and $[W_\lambda]$ as follows:

$$\zeta(zb \mid u) = z'b \mid u; \quad \zeta[zb \mid u] = [z'b \mid u];$$

where $z' = (z - \zeta) \cup (\zeta - z)$, and the sets $W_\varkappa$ and $[W_\lambda]$ are mapped to themselves. Let $\zeta\varphi$ be a formula obtained from φ under this change of variables.

We use the standard technique of symmetric models to conclude that $\zeta p_2 \Vdash \neg\zeta\varphi(\eta, n)$ assuming that p_2 restricts $\neg\varphi(\eta, n)$. Meanwhile, by construction there are no pairs $\langle \xi, k\rangle \in \zeta$ such that $\xi \in v$. Therefore, the definition of v implies that every variable of form zb|u contained in v, satisfies $\zeta(zb|u) = zb|u$. On the other hand, variables [xb|u] do not depend on z at all and therefore also remain the same. Thus, formulas zb and u, are identical, and we have $\zeta p_2 \Vdash \neg\varphi(\eta, n)$.

Finally, by definition of ζ, conditions p_1 and ζp_2 are compatible in P, which is the desired contradiction with the choice of p_1. This concludes the proof of relation (8) and Lemma 2.

<u>LEMMA 3.</u> $H(\mathfrak{m}) = \lambda$ in $\mathcal{N}$.

<u>Proof.</u> As before, it suffices to prove the direction $\leqslant$. Let $h \in \mathcal{N}$, $h: \lambda \to M$; be an arbitrary function, i.e., the values of h are Vitali classes. Furthermore,

$$x \in h(\eta) \leftrightarrow \varphi(\eta, x) \text{ holds in } \mathcal{M}[G]$$

for some formula φ, whose variables are only ordinals, sets $W_\varkappa$ and $[W_\lambda]$, and their elements. Let $v = \bigcup\{b''u: \exists z$ (either zb|u or [zb|u] is contained in $zb \mid u$ Thus, $[zb \mid u]$. We give the following auxiliary definition.

Suppose that $v \in C_\lambda$, A w-term is an indexed set $t = \langle A_{ni}: n \in \omega$ such that $i \in \{0, 1\}\rangle \in \mathcal{M}$

1) every set $A_{n0} \cup A_{n1}$ is a maximal antichain in P,

2) $A_{n0} \cap A_{n1} = \varnothing$; and

3) a set $\|t\| = \bigcup_{n\in\omega} \bigcup_i \bigcup_{p\in A_{ni}} \|p\|$ is contained in w.

Every term of this form induces in the generic extension $\mathcal{M}[G]$ a function $a = G(t)$, $a: \omega \to \{0, 1\}$ as follows: a(n) = i if $G \cap A_{ni} \neq \varnothing$.

Now we fix a set $u_1 \subseteq \lambda - v$, $u_1 \in \mathcal{M}$, that is countable in $\mathcal{M}$. The key result in the proof of Lemma 3 is as follows.

> If $\eta < \lambda$ then there exists a $(v \cup u_1)$-term t such that $G(t) \in h(\eta)$ [taking into account the identification of a function a = G(t) with the corresponding proper fraction defined above]. (9)

We note that the set of all $(v \cup u_1)$-terms has cardinality $<\lambda$ in $\mathcal{M}$, since in the statement of theorem we require that $\lambda \neq \theta^+$ for cardinals θ that have countable cofinality in $\mathcal{M}$.

Thus, assuming that statement (9) is already proven, the domain of values ran h of the function h has cardinality $<\lambda$ in $\mathcal{M}$, and the arbitrariness of h implies that $H(m) \leqslant \lambda$.

To prove (9), we assume that $\eta < \lambda$ and $x \in h(\eta)$. Assuming that x is a proper fraction, we study the corresponding (under the defined identification) function $a: \omega \to \{0, 1\}$ and define a term $t' \in \mathcal{M}$ using maximal antichains A'_{ni} chosen in $\mathcal{M}$ in sets

$$Q_{ni} = \{p \in P: p \Vdash a(n) = i\}.$$

The c.c.c. for P implies the countableness in $\mathcal{M}$ of a set $\|t'\|$; thus, a set $u_2 = \|t'\| - v$ is also countable in $\mathcal{M}$ (or finite), and by construction t' is a $(v \cup u_2)$-term.

Furthermore, G(t') = a, so the choice of x shows that there exists a condition $p_2 \in G$, that implies that $G(t') \in h(\eta)$, i.e., it restricts a formula $\varphi(\eta, G(t'))$.

Furthermore, since u_1 is countable, u_2 is no more than countable, and G is generic, there exists a bijection $\beta \in B$ such that $\beta'' u_2 \subseteq u_1$, $\beta(\xi) = \xi$ for all $\xi \in v$, and $\beta p_2 = p_1 \in G$, where

$$\beta p = \{\langle \beta(\xi), k, i\rangle : \langle \xi, k, i\rangle \in p\} \text{ for all } p \in P.$$

The bijection β not only defines an order automorphism of the set P with the above equation, but also a transformation of formulas for which zb|u and [zb|u] become

$$\beta(zb \mid u) = (\beta z)(\beta b) \mid u \text{ and } \beta [zb \mid u] = [(\beta z)(\beta b) \mid u],$$

respectively, where $\beta z = \{\langle \beta(\xi), k\rangle : \langle \xi, k\rangle \in z\}$ and $(\beta b)(\xi) = \beta(b(\xi))$ for all $\xi < \lambda$. The formula $\beta\varphi$, obtained from φ under this transformation is equal to φ by the construction of v, and a term

$$t = \langle A_{ni}: n \in \omega, i \in \{0, 1\}\rangle, \text{ where } A_{ni} = \{\beta p: p \in A'_{ni}\},$$

is a $(v \cup u_1)$-term by the choice of β. Thus, applying the transformation β to a relation $p_2 \Vdash \varphi(\eta, G(t'))$, we obtain $p_1 \Vdash \varphi(\eta, G(t))$. Therefore, $G(t) \in h(\eta)$, since $p \in G$. This proves relation (9), Lemma 3, and the theorem.

<u>Application.</u> A proof of Sierpinski's theorem (we cite it because of the beauty and the unexpectedness of this theorem and the fact that it has received very little attention in recent papers). Suppose that a set M is linearly ordered by a relation $\prec$. Decompose the plane R^2 into three sets as follows:

$$P = \{\langle x, y\rangle : [x] \prec [y]\}, \quad Q = \{\langle x, y\rangle : [y] \prec [x]\}$$

and $E = \{\langle x, y\rangle : [x] = [y]\}$. If we assume that every set $X \subseteq R$ is Lebesgue measurable then all plane sets are also measurable. In particular, the set E is measurable and has measure zero by Fubini's theorem, since every vertical section of E is countable.

Furthermore, the set P by its construction is invariant with respect to vertical and horizontal rational translations, so therefore (since P is measurable) either P has measure zero or its complement has measure zero. Both cases immediately lead to a contradiction, for P and Q have the same measure, which can be seen from the fact that $P = \{\langle x, y\rangle : \langle y, x\rangle \in Q\}$.

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