Mathematical Notes, Vol. 66, No. 2, 1999

Extension of Standard Models of ZFC to Models of Nelson's Nonstandard Set Theory IST

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UDC 510.223

ABSTRACT. A characterization of the standard models of **ZFC** set theory that are embeddable as the class of standard sets in models of the internal set theory **IST** and some of its versions is proposed.

KEY WORDS: Zermelo–Fraenkel set theory, nonstandard set theory, internal sets, models of axiomatic systems, conservative theories, well-ordering, ultrafilter, metamathematics.

Introduction

The internal set theory **IST** was introduced by Nelson [1] as a unified axiomatic foundation of "nonstandard" mathematics. The **IST** theory describes the universe **V** of all sets in such a way that, in addition to "standard" sets (which are identified with regular objects of "standard" mathematics and obey the axioms of Zermelo–Fraenkel theory **ZFC**), there are objects such as infinitely large and infinitely small numbers, etc., incompatible with the present-day "standard" system of foundations of mathematics. In spite of the fact that other nonstandard set theories, more perfect and, in particular, eliminating certain shortcomings of the **IST** system, were proposed later (see, for instance, [2–7]), **IST** remains the most frequently used nonstandard axiomatic theory.

The axioms of **IST** are given below.

One of the most important features of Nelson's theory is its *conservativeness*: **IST** proves those and only those propositions about the class **S** of standard sets that the "standard" **ZFC** theory proves about all sets. However, the interconnection between these two theories is much more complex than it may appear by this result [1]. For instance, as shown in [2], not every $\in$-model is **ZFC** extendable (i.e., embeds as the class **S** of all standard sets) to the **IST** model: in particular, the *minimal* **ZFC** model does not permit such an extension.

This leads to the problem of describing the transitive $\in$-models of **ZFC** that can be extended to a model of **IST**. The present note is devoted to this problem. We shall derive convenient sufficient conditions for the existence of such an extension, which at the same time are necessary for the $\mathbf{IST}^+$ theory, obtained by adding a certain natural form of the axiom of choice to **IST**.

§1. Internal set theory

The **IST** theory is formulated in the $\mathcal{L}_{\in,\mathrm{st}}$ language, i.e., the language with two atomic predicates, membership $\in$ (a dyadic one) and standardness st (a monadic one; $\mathrm{st}\,x$ means "x is a standard set"). The class of all standard sets is denoted by $\mathbf{S} = \{x : \mathrm{st}\,x\}$. The list of the axioms of **IST** includes all the axioms of **ZFC** (in the $\in$-language) and three additional "principles":

(a) (*carry-over*) $\exists x\ \Phi(x) \implies \exists^{\mathrm{st}} x\ \Phi(x)$ for all $\in$-formulas $\Phi(x)$ with standard parameters;

(b) (*idealization*) $\forall^{\mathrm{stfin}} A\ \exists x\ \forall a \in A\ \Phi(a,x) \iff \exists x\ \forall^{\mathrm{st}} a\ \Phi(a,x)$ for all $\in$-formulas $\Phi(x)$ with arbitrary parameters;

(c) (*standardization*) $\forall^{\mathrm{st}} X\ \exists^{\mathrm{st}} Y\ \forall^{\mathrm{st}} x\ (x \in Y \iff x \in X \wedge \Phi(x))$ for all st-$\in$-formulas $\Phi(x)$ with arbitrary parameters.

The quantifiers $\exists^{\mathrm{st}} x$ and $\forall^{\mathrm{st}} x$ are understood in the natural way, as "there exists a standard x" and "for any standard x"; the quantifier $\forall^{\mathrm{stfin}} A$ means "for all standard finite sets A."

Translated from *Matematicheskie Zametki*, Vol. 66, No. 2, pp. 202–210, August, 1999.
Original article submitted July 16, 1998.

 0001-4346/1999/6612-0160 $22.00

§2. The axiom of standard well-ordering

Consider the extension $\mathbf{IST}^+$ of the $\mathbf{IST}$ theory obtained by

(i) adding the dyadic predicate symbol $<$ to the language of $\mathbf{IST}$ and the axiom that $<$ is a well-ordering on the class $\mathbf{S}$ of all standard sets, in particular, that each standard $X \neq \varnothing$ has the $<$-smallest standard element, to the axioms of $\mathbf{IST}$;

(ii) permission to use $<$ in the formula Φ of the standardization scheme (but not in the carry-over, idealization, separation, and substitution schemes).

The interconnection between $\mathbf{IST}^+$ and $\mathbf{IST}$ is not quite clear. On the one hand, $\mathbf{IST}^+$ is ensured by adjoining any "standard" set-theoretic axiom providing a well-ordering of the universe (for instance, the Gödel constructibility axiom) to $\mathbf{IST}$ and holds in ordinary models of $\mathbf{IST}$ such as ultraproducts; also, it is guaranteed by certain "nonstandard" axioms in the language of $\mathbf{IST}$ (see §8 below). On the other hand, whether $\mathbf{IST}^+$ is deducible in $\mathbf{IST}$ [2, Problem 6] remains an open question.

To formulate the main result of this note, we introduce some additional terminology. Suppose that M is a transitive $\in$-model of the Zermelo system $\mathbf{Z}$. We say that a family of sets $P_1, \dots, P_n$ (where $P_i \subseteq M^{p_i}$, $i = 1, \dots, n$) *preserves separation in* M if the separation scheme in the language $\mathcal{L}_{\in, P_1, \dots, P_n}$ is true in $\langle M; \in, P_1, \dots, P_n\rangle$. The phrase "*preserves substitution in* M" has a similar meaning.

Note that $\in$-formulas with parameters from M can be considered to be elements of M. Let $\mathrm{Truth}_{\in}^{M}$ be the set of all closed $\in$-formulas with the parameters from M valid in $\langle M; \in\rangle$.

Theorem 1. *Let M be a transitive $\in$-model of* $\mathbf{ZFC}$. *Then the existence of a well-ordering $<$ on the set M such that the pair of sets $\mathrm{Truth}_{\in}^{M}$ and $<$ preserves separation in M is necessary and sufficient for M to be embeddable as the class of all standard sets in a model of* $\mathbf{IST}^+$.

Corollary. *Under the assumptions of Theorem* 1, *the existence of the well-ordering in question suffices for M to be embeddable as the class of all standard sets in a model of* $\mathbf{IST}$.

It appears that this result substantially improves the one obtained by Nelson [1] (which establishes that it suffices that M be a model of $\mathbf{ZFC}$ of the form $M = V_\varkappa$, where $\varkappa$ is an uncountable cardinal).

Below we prove the necessity (§3) and sufficiency (§§4–6) in Theorem 1. Along with the proof of sufficiency, we shall establish some metamathematical traits of the theory $\mathbf{IST}^+$, in particular, the fact that it shares with $\mathbf{IST}$ the property of *conservativeness* with respect to $\mathbf{ZFC}$ mentioned in the introduction.

§3. Necessity

Consider a model $\mathbf{I} = \langle \mathbf{I}; \varepsilon, \mathrm{st}, \lhd\rangle$ of the theory $\mathbf{IST}^+$. Suppose that $M = \mathbf{S}$ (the set of all standard elements of $\mathbf{I}$) is a transitive set and that $\in \restriction M = \varepsilon \restriction M$. Then $\langle M; \in\rangle \models \mathbf{ZFC}$, and so it remains to find a well-ordering $\prec$ of the set M such that the sets $\mathrm{Truth}_{\in}^{M}$ and $\prec$ preserve separation in M.

Recall that the set-theoretic *rank*, $\mathrm{rk}\, x$, of a set x is the smallest ordinal α such that x belongs to the αth level V_α of the von Neumann hierarchy. By carry-over, for any set $x \in M = \mathbf{S}$, the rank $\mathrm{rk}\, x$ defined in $\langle \mathbf{I}; \varepsilon\rangle$ coincides with the rank $\mathrm{rk}\, x$ defined in M, and thus, with the rank $\mathrm{rk}\, x$ in the sense of the basic universe of $\mathbf{ZFC}$.

Now we set $x \prec y$ for $x, y \in M$ if either $\mathrm{rk}\, x < \mathrm{rk}\, y$ or $\mathrm{rk}\, x = \mathrm{rk}\, y$ and $x \lhd y$. It follows from the remarks above that the relation $\prec$ is definable in the structure $\langle \mathbf{I}; \varepsilon, \mathrm{st}, \lhd\rangle$.

On the other hand, by the following result, the set $\mathrm{Truth}_{\in}^{M}$ is also definable in the structure $\langle \mathbf{I}; \varepsilon, \mathrm{st}\rangle$.

Lemma 1 [2]. *There exists a formula $\tau(x)$ of the language $\mathcal{L}_{\in,\mathrm{st}}$ such that the following sentence is a theorem in* $\mathbf{IST}$ *for any $\in$-formula $\varphi(x_1, \dots, x_n)$*:

$$\forall^{\mathrm{st}} x_1 \ \dots \ \forall^{\mathrm{st}} x_n \ \bigl(\varphi^{\mathrm{st}}(x_1, \dots, x_n) \iff \tau(\ulcorner \varphi(x_1, \dots, x_n)\urcorner)\bigr).$$

Here $\ulcorner\psi\urcorner$ denotes the formula ψ regarded as a finite sequence of symbols of the $\in$-language and the sets that appear in ψ as parameters.

Since the formula Φ in the standardization scheme in $\mathbf{IST}^+$ can involve $<$ (together with $\in$ and st), it follows that the pair of sets $\mathrm{Truth}_{\in}^{M}$ and $\prec$ preserves separation in M.

It remains to check that $\prec$ is a well-ordering of M (from the standpoint of the universe of all sets). Note that since any initial segment of M in the sense of $\prec$, by the construction, is covered by a set from M, it will suffice to ascertain that $\prec$ well orders any $X \in M$ in the universe; and since "to be a well-ordering" is an absolute property for transitive models of **ZFC**, it remains to verify that any nonempty set $Y \in M$ has the $\prec$-minimal element. But this is ensured by the choice of $\lhd$ and the definition of $\prec$ from $\lhd$.

§4. Sufficiency: transforming the order

Let us fix a transitive set $M \models \mathbf{ZFC}$ well-ordered by the relation $\lhd$ in such a way that the pair of sets $T = \mathrm{Truth}^M_{\in}$ and $\lhd$ preserves separation in M. The argument from §3 shows that without loss of generality we can assume that *any proper initial segment of M in the sense of $\lhd$ belongs to M.*

The goal of this section is to somewhat improve the ordering $\lhd$. Namely, under our assumptions, we shall prove the following fact.

Theorem 2. *There exists a well-ordering $\prec$ of the set M such that any initial segment of M in the sense of $\prec$ belongs to M and, in addition, $\prec$ preserves substitution in M, and the pair consisting of $\prec$ and the set $T' = \mathrm{Truth}^M_{\in,\prec}$ of all the formulas in the language $\mathcal{L}_{\in,<}$ with parameters from M valid in $\langle M\,;\in,\prec\rangle$ preserves separation in M.*

We begin the proof with a number of definitions.

The set of all structures of the form $\sigma = \langle X\,;<\rangle$, where $X \in M$ is a transitive set of the form $X = M_\alpha = V_\alpha \cap M$ for a certain ordinal $\alpha \in M$ and $<\in M$ is a well-ordering of X, will be denoted by Σ.

A structure $\sigma' = \langle X'\,;<'\rangle$ *extends* $\sigma = \langle X\,;<\rangle$ if $X \subseteq X'$ and $<'$ is a final extension of $<$ (i.e., $<$ coincides with $<' \restriction X$ and X is an initial segment of X' in the sense of $<'$).

We define a relation $\sigma\,\mathrm{forc}\,\Phi(x_1,\dots,x_n)$, where $\sigma \in \Sigma$, Φ is a $\mathcal{L}_{\in,<}$-formula, and $x_1,\dots,x_n \in X$, by induction on the complexity of Φ:

(1) if Φ is an elementary formula of the language $\mathcal{L}_{\in,<}$, i.e., the formula $x < y$, $x = y$, or $x \in y$, then $\sigma\,\mathrm{forc}\,\Phi$ whenever Φ is true in σ (here and subsequently the notation $\sigma = \langle X\,;<\rangle \in \Sigma$ is understood as $\langle X\,;\in,<\rangle$);
(2) $\sigma\,\mathrm{forc}(\Phi \wedge \Psi)$ if $\sigma\,\mathrm{forc}\,\Phi$ and $\sigma\,\mathrm{forc}\,\Psi$;
(3) $\sigma\,\mathrm{forc}(\neg\Phi)$ if none of the structures $\sigma' \in \Sigma$ extending σ satisfies the relation $\sigma'\,\mathrm{forc}\,\Phi$;
(4) $\sigma\,\mathrm{forc}\,\exists x\,\Phi(x)$ if there exists an $x \in X$ such that $\sigma\,\mathrm{forc}\,\Phi(x)$.

Let Φ be a formula of the language $\mathcal{L}_{\in,<}$. We say that a structure $\sigma = \langle X\,;<\rangle \in \Sigma$ is *Φ-complete* if for any subformula $\Psi(x_1,\dots,x_n)$ of the formula Φ and any parameters $x_1,\dots,x_n \in X$ we have $\sigma\,\mathrm{forc}\,\Psi(x_1,\dots,x_n)$ or $\sigma\,\mathrm{forc}\,\neg\Psi(x_1,\dots,x_n)$.

Proposition 1. *If Φ is a closed formula of the language $\mathcal{L}_{\in,<}$ with parameters from X and the structure $\sigma = \langle X\,;<\rangle \in \Sigma$ is Φ-complete, then $\sigma\,\mathrm{forc}\,\Phi$ and $\sigma \models \Phi$ are equivalent.*

Proof. The proof is carried out by induction on the complexity of Φ. $\square$

Our nearest goal is to construct an increasing sequence of structures $\sigma_\gamma = \langle X_\gamma\,;<_\gamma\rangle \in \Sigma$, $\gamma < \lambda$, such that $M = \bigcup_{\gamma<\lambda} X_\gamma$, that is, the relation $< = \bigcup_{\gamma<\lambda} <_\gamma$ is a well-ordering of M. The structures σ_γ will be sufficiently "complete" (in the sense indicated above) to ensure the desired properties of $<$.

We say that a structure $\sigma \in \Sigma$ is *totally complete* if it is Φ-complete for any formula Φ of the language $\mathcal{L}_{\in,<}$. The construction depends on the frequency of totally complete structures in Σ:

(1) any $\sigma \in \Sigma$ is extendable to a totally complete $\sigma' \in \Sigma$;
(2) this is not the case.

In case (1), we construct a sequence of structures $\sigma_\gamma = \langle X_\gamma\,;<_\gamma\rangle \in \Sigma$, where $\gamma < \lambda$, so that $X_\delta = \bigcup_{\gamma<\delta} X_\gamma$, $<_\delta = \bigcup_{\gamma<\delta} <_\gamma$ for all limiting ordinals $\delta < \lambda$, and $\sigma_{\gamma+1}$ is the $\lhd$-minimal totally complete structure in Σ that properly extends σ_γ. Here λ is the greatest ordinal such that σ_γ is defined (and belongs to Σ, hence, to M) for all $\gamma < \lambda$; clearly, λ is not greater than the smallest of ordinals not in M.

In case (2), we fix a recursive enumeration $\{\Phi_n : n \in \omega\}$ of all formulas of the language $\mathcal{L}_{\in,<}$. Let us say that a structure $\sigma \in \Sigma$ is *n-complete* if it is Φ_k-complete for any $k \le n$. We set $\lambda = \omega$, choose an arbitrary structure $\sigma_0 \in \Sigma$ not extendable to a totally complete structure, and define a sequence of structures $\sigma_n = \langle X_n ; <_n \rangle \in \Sigma$ so that for any $n \in \omega$, σ_{n+1} is the $\triangleleft$-minimal n-complete structure in Σ properly extending σ_n.

In either of the cases considered above, $\langle \sigma_\gamma : \gamma < \lambda \rangle$ is a sequence of elements of M definable in $\langle M ; \in, \triangleleft, T \rangle$ by construction (recall that $T = \mathrm{Truth}^M_\in$).

Lemma 2. *λ is a limiting ordinal and $\bigcup_{\gamma<\lambda} X_\gamma = M$.*

Proof. Recall that $\lambda = \omega$ in case (2). If we had $\lambda = \gamma + 1$ in case (1), then we could define σ_λ. Thus, λ is a limiting ordinal and the relation $\prec = \bigcup_{\gamma<\lambda} <_\gamma$ is a well-ordering of X.

Suppose that $X = \bigcup_{\gamma<\lambda} X_\gamma \neq M$. Then $X \in M$, because all the sets X_γ are of the form $M_\alpha = V_\alpha \cap M$. It follows that $\prec$ belongs to M, because $\triangleleft$ and T preserve separation in M. Therefore, $\sigma = \langle X ; \prec \rangle \in \Sigma$. Furthermore, σ is totally complete (as the limit of an increasing sequence of totally complete structures in case(1) and for a similar reason in case (2)). In case (2), this immediately comes in contradiction with the choice of σ_0; in case (1), this adds one more term to the sequence, which contradicts the choice of λ. □

Proof of Theorem 2. Thus, $\prec = \bigcup_{\gamma<\lambda} <_\gamma$ is a well-ordering of M. Let us verify that the pair of sets $\prec$ and $T' = \mathrm{Truth}^M_{\in,\prec}$ preserves separation in M. It will suffice to ascertain that $\prec$ and T' are definable in the structure $\langle M ; \in, \triangleleft, T \rangle$, where $T = \mathrm{Truth}^M_\in$.

The fact that the ordering $\prec$ is definable in $\langle M ; \in, \triangleleft, T \rangle$ is obvious from the construction. Let us examine the set T'.

Consider a closed formula $\Phi(p_1, \ldots, p_k)$ of the language $\mathcal{L}_{\in,<}$ with parameters $p_1, \ldots, p_k \in M$. Let n be the number of the formula $\Phi(x_1, \ldots, x_k)$ (see the case (2) above). Consider the smallest ordinal $\gamma < \lambda$ such that $p_1, \ldots, p_k \in X_\gamma$ and, in the case (2), $\gamma \geq n$. By using Proposition 1, it is readily verified that σ_γ is an elementary substructure of $\langle M ; \in, \prec \rangle$ with respect to the formula Φ; in particular, $\Phi(p_1, \ldots, p_k)$ is true (or false) in σ_γ and $\langle M ; \in, \prec \rangle$ simultaneously. However, the sequence of structures σ_γ is definable in $\langle M ; \in, \triangleleft, T \rangle$.

The same argument (i.e., the fact that the model $\langle M ; \in, \prec \rangle$ has elementary substructures of the form σ_γ) shows that the relation $\prec$ preserves substitution in M. This completes the proof of Theorem 2. □

§5. The ultrafilter

We proceed with the proof of sufficiency in Theorem 1. In what follows, let us fix the well-ordering $\prec$ on the set M provided by Theorem 2. In particular, we assume that

(1) all proper initial segments of M in the sense of $\prec$ belong to M;

(2) the pair of sets $\prec$ and $T' = \mathrm{Truth}^M_{\in,\prec}$ preserves separation in M;

but in order to make the argument applicable to the metamathematical analysis of the theory $\mathbf{IST}^+$ in §7, for the time being, we shall not assume that $\langle M ; \in \rangle$ satisfies the substitution scheme.

We must construct a model $\mathbf{I} \models \mathbf{IST}^+$ with M as the class of all standard sets. This model will be obtained as the Nelson *adequate ultraproduct* of M [1] (in the modification of [2]). Let us choose a suitable ultrafilter.

Denote by $\mathrm{Def}^M_{\in,\prec}$ the family of all sets $X \subseteq M$ definable in $\langle M ; \in, \prec \rangle$ by a formula of the language $\mathcal{L}_{\in,<}$ with parameters from M. Set $I = \mathcal{P}_{\mathrm{fin}}(M) = \{i \subseteq M : i \text{ is finite}\}$ (this is a proper class in M), and consider the algebra $\mathcal{A}$ of all sets $X \subseteq I$, $X \in \mathrm{Def}^M_{\in,\prec}$.

Lemma 3. *There exists an ultrafilter $U \subseteq \mathcal{A}$ such that*

(a) *if $a \in M$, then the set $\{i \in I : a \in i\}$ belongs to U;*

(b) *if $P \subseteq M \times I$, $P \in \mathrm{Def}^M_{\in,\prec}$, then the set*

$$\{x \in M : \text{the section } P_x = \{i : \langle x, i \rangle \in P\} \text{ belongs to } U\}$$

belongs to $\mathrm{Def}^M_{\in,\prec}$;

(c) *there exists a set $\mathcal{U} \subseteq M$ definable in $\langle M ; \in, \prec, T \rangle'$ such that $U = \{\mathcal{U}_x : x \in M\}$, where $\mathcal{U}_x = \{i \in I : \langle x, i \rangle \in \mathcal{U}\}$ for all x.*

Proof. The family U_0 of all the sets of the form

$$I_{a_1 \dots a_m} = \{i \in I : a_1, \dots, a_m \in i\},$$

where $a_1, \dots, a_m \in M$, obviously satisfies FIP (the *Finite Intersection Property*), i.e., any intersection of finitely many sets from U_0 is nonempty.

Suppose that the FIP-family $U_n \subseteq \mathcal{A}$ has already been defined. Assuming that a recursive enumeration of all the formulas of $\mathcal{L}_{\in,<}$ with two free variables has been fixed in advance, denote by $\chi_n(x, i)$ the nth formula in this enumeration.

Set $U_{n+1} = U_n \cup \{B_x : x \in M\}$, where B_x is the set

$$A_x = \{i \in I : \langle M\,; \in, \prec\rangle \models \chi_n(x, i)\}$$

if the family $U_n \cup \{B_y : y \prec x\} \cup A_x$ still satisfies FIP, and $B_x = I \setminus A_x$ otherwise.

It is readily seen that $U = \bigcup_n U_n$ is the desired ultrafilter; condition (c) is ensured by carrying out the entire construction in $\langle M\,; \in, \prec, T'\rangle$. $\square$

In what follows, we fix the ultrafilter $U \subseteq \mathcal{A}$ from the last lemma.

The phrase "the set $\{i \in I : \Phi(i)\}$ belongs to U" will be written as $\mathsf{U} i\, \Phi(i)$ (the quantifier U means "there exist U-many"). Then by the choice of U we see that

(a) $\mathsf{U} i\, (a \in i)$ for all $a \in M$;

(b) whatever is the relation $P(i, \dots)$ in $\mathrm{Def}^M_{\in,\prec}$, the relation $\mathsf{U} i\, P(i, \dots)$ belongs to $\mathrm{Def}^M_{\in,\prec}$ as well.

§6. The model

For $r \geq 1$, define $I^r = I \times \dots \times I$ (r factors I) and

$$F_r = \{f \in \mathrm{Def}^M_{\in,\prec} : f \text{ maps } I^r \text{ to } M\};$$

for $r = 0$, set $I^0 = \{0\}$ and $F_0 = \big\{\{\langle 0, x\rangle\} : x \in M\big\}$. Finally, let $F_\infty = \bigcup_{r \in \omega} F_r$ and for $f \in F_\infty$, denote by $r(f)$ the unique r such that $f \in F_r$.

Suppose that $f \in F_\infty$, $q \geq r = r(f)$, and $\boldsymbol{i} = \langle i_1, \dots, i_r, \dots, i_q\rangle \in I^q$. In this notation, set $f[\boldsymbol{i}] = f(\langle i_1, \dots, i_r\rangle)$. In particular, $f[\boldsymbol{i}] = f(\boldsymbol{i})$ for $r = q$. Also, $f[\boldsymbol{i}] = x$ for all $\boldsymbol{i}$ if $f = \{\langle 0, x\rangle\} \in F_0$.

Let $f, g \in F_\infty$, and let $r = \max\{r(f), r(g)\}$. We write $f \,{}^*\!\!= g$ if

$$\mathsf{U} i_r\, \mathsf{U} i_{r-1} \dots \mathsf{U} i_1\, (f[\boldsymbol{i}] = g[\boldsymbol{i}]) \tag{1}$$

where $\boldsymbol{i} = \langle i_1, \dots, i_r\rangle$; the order of quantifiers in formula (1) is essential. The relations $f \,{}^*\!\!\in g$ and $f \,{}^*\!\!\prec g$ are defined similarly. The next statement is readily verified.

Proposition 2. *The relation ${}^*\!\!=$ is an equivalence on F_∞. The relations ${}^*\!\!\in$ and ${}^*\!\!\prec$ on F_∞ are ${}^*\!\!=$-invariant in each argument.*

Let $[f] = \{g \in F_\infty : f \,{}^*\!\!= g\}$. Consider the quotient set $\mathbf{I} = \{[f] : f \in F_\infty\}$. For $[f], [g] \in \mathbf{I}$, we write $[f] \,{}^*\!\!\in [g]$ if $f \,{}^*\!\!\in g$; the relation $[f] \,{}^*\!\!\prec [g]$ is defined similarly. For $x \in M$, let ${}^*x = [\{\langle 0, x\rangle\}]$ be the image of x in $\mathbf{I}$. Let $\mathbf{S} = \{{}^*x : x \in M\}$.

Finally, we set $\mathrm{st}[f]$ if $[f] = {}^*x$ for a certain $x \in M$.

Theorem 3. *The map $x \mapsto {}^*x$ is a one-to-one function from M onto $\mathbf{S}$ that takes $\in$ and $\prec$ to ${}^*\!\!\in$ and ${}^*\!\!\prec$, respectively, and the elementary embedding of $\langle M\,; \in, \prec\rangle$ to $\langle \mathbf{I}\,; {}^*\!\!\in, {}^*\!\!\prec\rangle$. In addition, $\langle \mathbf{I}\,; {}^*\!\!\in, {}^*\!\!\prec, \mathrm{st}\rangle$ satisfies the idealization and standardization schemes, and the formula Φ in each of these schemes can include the predicate ${}^*\!\!\prec$ together with ${}^*\!\!\in$.*

This theorem yields the sufficiency in Theorem 1. Indeed, suppose that in addition to the assumptions made at the beginning of §4, M is a model of **ZFC**. Then, by Theorem 3, $\langle \mathbf{I}\,; {}^*\!\!\in, \mathrm{st}\rangle$ is a model of **IST**, and with the restriction ${}^*\!\!\prec \restriction \mathbf{S}$ of the ordering ${}^*\!\!\prec$, even a model of $\mathbf{IST}^+$ whose class $\mathbf{S}$ of standard sets is isomorphic to M.

We begin the proof of Theorem 3 with several definitions.

Let $\Phi(f_1, \dots, f_m)$ be a formula of the language $\mathcal{L}_{\in,<}$ with functions $f_1, \dots, f_m \in F$ as parameters. Set $r(\Phi) = \max\{r(f_1), \dots, r(f_m)\}$. If $r \leq q$ and $\boldsymbol{i} \in I^q$, then we denote by $\Phi[\boldsymbol{i}]$ the formula $\Phi(f_1[\boldsymbol{i}], \dots, f_m[\boldsymbol{i}])$ (of the language $\mathcal{L}_{\in,<}$ with parameters from M), and by $[\Phi]$ the formula $\Phi([f_1], \dots, [f_m])$ (with parameters from $\mathbf{I}$).

Proposition 3 (Los' Theorem). *For any formula $\Phi = \Phi(f_1, \dots, f_m)$ of the language $\mathcal{L}_{\in,<}$ with parameters $f_1, \dots, f_m \in F$ and for $r = r(\Phi)$, we have*

$$\langle \mathbf{I}\,;\, {}^*\!\in, {}^*\!\prec \rangle \models [\Phi] \iff \mathsf{U} i_r\, \mathsf{U} i_{r-1} \,\dots\, \mathsf{U} i_1\, (\langle M\,;\, \in, \prec \rangle \models \Phi[\boldsymbol{i}])$$

(where $\boldsymbol{i}$ denotes $\langle i_1, \dots, i_r \rangle$).

Proof. The trick is that since the index set I is a proper class in M, we need a well-ordering of M for the usual inductive argument in the Los' Theorem to be correct. That is why we need $\mathcal{L}_{\in,<}$ rather than just the $\in$-language as the base language, and accordingly, the model $\langle M\,;\, \in, \prec \rangle$ rather than just $\langle M\,;\, \in \rangle$ as the initial structure. □

Proof of Theorem 3. In essence, it remains to verify the idealization and standardization schemes in $\langle \mathbf{I}\,;\, {}^*\!\in, {}^*\!\prec, \mathrm{st} \rangle$.

Idealization. Consider a $\mathcal{L}_{\in,<}$-formula $\Phi(a, x)$ with two free variables a, x and functions from F as parameters. We must prove that

$$\forall^{\mathrm{stfin}} A\, \exists x\, \forall a \in A\, [\Phi](a, x) \implies \exists x \quad \forall^{\mathrm{st}} a \quad [\Phi](a, x) \tag{2}$$

in $\mathbf{I}$ (it is known [1] that the implication "$\Longleftarrow$" in (2) follows from other axioms of **IST**). By Proposition 3, the left-hand side of (2) implies $\forall_{\mathrm{finite}} A \subseteq M\, \mathsf{U} i_r\, \mathsf{U} i_{r_1} \,\dots\, \mathsf{U} i_1\, \exists x\, \forall a \in A\, \Phi[\langle i_1, \dots, i_r \rangle](a, x)$ in M, where $r = r(\Phi)$. To simplify this formula, we note that the domain of the leftmost quantifier is I. Consequently, for the function $\alpha \in F_{r+1}$ defined by the equation $\alpha(i_1, \dots, i_r, i) = i$, we obtain $\forall i \in I\, \mathsf{U} i_r\, \mathsf{U} i_{r_1} \,\dots\, \mathsf{U} i_1\, (\exists x\, \forall a \in \alpha\, \Phi)[\langle i_1, \dots, i_r, i \rangle](a, x)$, whence by Proposition 3 it follows that $\exists x\, \forall a \in [\alpha]\, [\Phi](a, x)$ in $\mathbf{I}$. Now by the definition of the predicate st, it suffices to check that ${}^*x \,{}^*\!\in [\alpha]$ in $\mathbf{I}$ for any $x \in M$; in other words, that $\mathsf{U} i\, \mathsf{U} i_r \,\dots\, \mathsf{U} i_1\, (x \in i)$. But the latter is true by the choice of U.

Standardization. Recall that U is definable in $\langle M\,;\, \in, \prec, T' \rangle$ in the sense of condition (c) of Lemma 3. Therefore, the model $\langle \mathbf{I}\,;\, {}^*\!\in, \mathrm{st} \rangle$ is also definable in $\langle M\,;\, \in, \prec, T' \rangle$. Thus, it remains to check that for $x \in M$, any set $y \subseteq x$ definable in $\langle M\,;\, \in, \prec, T' \rangle$ (with parameters from M permitted) belongs to M. But this follows from the fact that the pair of sets $\prec$ and T' preserves separation in M. □

§7. Metamathematical properties of the $\mathbf{IST}^+$ theory

Let us show that the $\mathbf{IST}^+$ and **IST** theories are in much the same relation to **ZFC**, in particular, $\mathbf{IST}^+$ and **ZFC** are equally consistent.

Theorem 4. *The $\mathbf{IST}^+$ theory is a conservative extension of the* **ZFC** *theory in the sense that any closed $\in$-formula is deducible in $\mathbf{IST}^+$ if and only if it is deducible in* **ZFC**.

Proof. Suppose that a sentence φ in the $\in$-language is a theorem in $\mathbf{IST}^+$. Let us prove that φ is deducible in **ZFC** as well. Let Φ be the finite fragment of $\mathbf{IST}^+$ involved in the derivation of φ. The following argument deduces φ in **ZFC**.

We fix a limiting ordinal λ such that $M = V_\lambda$ satisfies all the instances of substitution in Φ, and additionally is an elementary submodel of the universe with respect to our formula φ. Let $\prec$ be any well-ordering of M such that any initial segment of M in the sense of $\prec$ belongs to M. Since M contains all subsets of sets from M, in this case we are in the situation described at the beginning of §5. This means that there exists a structure $\langle \mathbf{I}\,;\, {}^*\!\in, {}^*\!\prec, \mathrm{st} \rangle$ and an embedding $x \mapsto {}^*x$ from M to $\mathbf{I}$ satisfying Theorem 3.

Then the structure $\langle \mathbf{I}\,;\, {}^*\!\in, {}^*\!\prec \restriction, \mathrm{st} \rangle$ is a model of Φ, i.e., the formula φ is true in $\langle \mathbf{I}\,;\, {}^*\!\in \rangle$. It follows that φ is true in M as well, and hence, by the choice of M, in the universe. □

§8. On some other extensions of IST

The method of extension of **IST** we used for the construction of the theory $\mathbf{IST}^+$ provides tools minimally sufficient for the argument in §6. However, the sufficiency part of Theorem 1 applies to some stronger and, perhaps, more natural extensions of **IST** as well. We shall consider two of them.

Let us define $\mathbf{IST}'$ as the extension of $\mathbf{IST}$ in the $\mathcal{L}_{\in,\mathrm{st}}$ language by means of a single axiom:

> there exist a set D and a well-ordering $<$ on D such that $\mathbf{S} \subseteq D$ and the $<$-smallest element of any standard set is standard.

It is clear that $\mathbf{IST}'$ implies $\mathbf{IST}^+$, i.e., the necessity in Theorem 1 is valid for $\mathbf{IST}'$ as well. To study the question of sufficiency, consider one more theory.

Recall that $\mathbf{ZF}\,\mathrm{GC}$ (or $\mathbf{ZF}$ plus the *Global Choice*) is an extension of $\mathbf{ZFC}$ in the language $\mathcal{L}_{\in,<}$ by the following axiom:

> the relation $<$ is a well-ordering of the class of all sets and any initial segment in the sense that $<$ is a set.

In addition, $<$ can be involved in the separation and substitution schemes. Denote by $\mathbf{IST}\,\mathrm{GC}$ the theory that can be called an $\mathbf{IST}$-*extension* of $\mathbf{ZF}\,\mathrm{GC}$, i.e., the theory in the language $\mathcal{L}_{\in,<,\mathrm{st}}$ that includes $\mathbf{ZF}\,\mathrm{GC}$ (in the language $\mathcal{L}_{\in,<}$) and idealization, standardization, and carry-over schemes, in each of which Φ can be a formula of $\mathcal{L}_{\in,<}$ (cf. the definition of $\mathbf{IST}$ in §1).

On the one hand, $\mathbf{IST}\,\mathrm{GC}$ implies $\mathbf{IST}'$, and hence, $\mathbf{IST}^+$. On the other hand, an obvious modification of the reasoning in §6 shows that if in the situation considered there it is additionally known that $\langle M\,;\in,\prec\rangle$ is a model of $\mathbf{ZF}\,\mathrm{GC}$, i.e., that $\prec$ preserves substitution in M, then $\langle \mathbf{I};\,{}^*\!\in,{}^*\!\prec,\mathrm{st}\rangle$ is a model of $\mathbf{IST}\,\mathrm{GC}$. Hence it follows that the sufficiency in Theorem 1 extends to the $\mathbf{IST}\,\mathrm{GC}$ theory, and thus, to $\mathbf{IST}'$ as well.

The research of the first author was supported by the Russian Foundation for Basic Research under grants No. 98-01-00045 and No. 98-01-00790 and by the DFG Foundation under grant No. 436-RUS-17/66/97.

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