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# Reducibility of Monadic Equivalence Relations

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**Abstract**—Each additive cut in the nonstandard natural numbers $^*\mathbb{N}$ induces the equivalence relation $\mathrm{M}_U$ on $^*\mathbb{N}$ defined as $x\,\mathrm{M}_U\,y$ if $|x-y| \in U$. Such equivalence relations are said to be monadic. Reducibility between monadic equivalence relations is studied. The main result (Theorem 3.1) is that reducibility can be defined in terms of cofinality (or coinitiality) and a special parameter of a cut, called its width. Smoothness and the existence of transversals are also considered. The results obtained are similar to theorems of modern descriptive set theory on the reducibility of Borel equivalence relations.

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## INTRODUCTION

Classical descriptive set theory mainly considers subsets of Polish (i.e., complete separable metric) spaces. However, as early as in the 1980s, it was discovered that ideas of descriptive set theory can be carried over to nonstandard analysis, where Polish spaces are replaced by internal (e.g., hyperfinite) sets of a certain nonstandard structure. This theory was called *hyperfinite* (or *nonstandard*) descriptive set theory; see, e.g., [1]. It considers structures which resemble those arising in Polish spaces in some respects and strongly differ from them in other respects. Proofs are usually based on combinatorial ideas (and use saturation). Nonstandard descriptive set theory also considers objects which have no analogs at all in Polish spaces, such as determined sets.

In this paper, we consider the equivalence relations $\mathrm{M}_U$ on the set $^*\mathbb{N}$ of all positive hyperintegers that are induced by additive cuts (initial intervals) $U \subseteq {}^*\mathbb{N}$ in the sense that $x\,\mathrm{M}_U\,y$ if and only if $|x-y| \in U$. Their equivalence classes are called *$U$-monads*. Such *monadic* equivalence relations (or partitions) have been considered in connection with various questions of nonstandard analysis, starting in the 1980s (see [1], [2]). In [3], the existence problem for countably determined transversals (i.e., sets choosing one element in each monad) for relations of the form $\mathrm{M}_U$ was solved. A number of other problems concerning countably determined equivalence relations (such as dichotomy theorems and smoothness and reducibility problems) were solved in [4]. In particular, in was proved in [4] that each of the two natural families of countably determined monadic relations (namely, countably cofinal and countably coinitial relations) is linearly ordered by the relation of countably determined reducibility. The direction of the order depends on a parameter (here called the *width* of a cut) characterizing the final rate of the increasing cofinal or decreasing coinitial sequence determining the cut under consideration.

In a somewhat different situation (in the framework of nonstandard set theory), similar results were obtained in the monograph [5, Ch. 9].

In this paper, we study monadic equivalence relations induced by cuts of any cofinality and coinitiality. To be more precise, assuming that the original nonstandard universe is $\kappa^+$-saturated, where $\kappa$ is a given infinite cardinal, we consider additive cuts which belong to the type of $\kappa$-determined sets; each of them

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is either $\leq \kappa$-cofinal or $\leq \kappa$-coinitial. Accordingly, we consider the $\kappa$-determined reducibility of $\mathrm{M}_U$ to $\mathrm{M}_V$, which means the existence of a $\kappa$-determined map from ${}^*\mathbb{N}$ to ${}^*\mathbb{N}$ inducing an embedding of the coset space ${}^*\mathbb{N}/\mathrm{M}_U$ into ${}^*\mathbb{N}/\mathrm{M}_V$. The main result, Theorem 3.1, characterizes such a reducibility in terms of the cofinality, coinitiality, and width of the cuts $U$ and $V$. As a byproduct, we obtain results on $\kappa$-smoothness (i.e., $\kappa$-determined reducibility to an equality) and the existence of $\kappa$-determined transversals.

Similar questions (on Borel equivalence relation and Borel reducibility) have been extensively studied in modern descriptive sets theory (in Polish spaces); see, e.g., [6], [7].

We assume that the reader is familiar with the basics of nonstandard analysis (see [8]–[11]) and of hyperfinite descriptive set theory at the level of, e.g., the introductory sections of [1].

## 1. DETERMINED SETS

All "nonstandard" objects considered below (such as ${}^*\mathbb{N}$) are assumed to belong to a fixed nonstandard universe (e.g., to a nonstandard superstructure, as in [10]), whose elements are called *nonstandard* (internal or external) sets. The degree of saturation of this nonstandard universe will be specified in each particular case, but we always assume that at least ordinary $\aleph_1$-saturation takes place. (See [12] on saturation.)

By $\mathscr{P}_{\mathrm{int}}X$ we denote the set of all internal subsets of a nonstandard set $X$. The notation $\#(X) \in {}^*\mathbb{N}$ is used for the number of all elements in a hyperfinite set $X$, and $\operatorname{card} X$ denotes the cardinality of $X$ in the universe of all sets. Finally, $\mathscr{P}_{\mathrm{fin}}(X) = \{Y \subseteq X : Y \text{ finite}\}$.

The notion of a *determined set* goes back to the works of Kolmogorov and Hausdorff on $\delta s$-operations in the 1920s (see the historical comments in [13]); in the context of nonstandard analysis, it was introduced in [14], [15]. Let $\kappa$ be an infinite cardinal. We consider subsets of some fixed internal (in a given nonstandard universe) set $I$.

We use the following two definitions of $\kappa$-determinacy.

(1) A set $X \subseteq I$ is said to be *$\kappa$-determined* if it has the form

$$\begin{gathered} X = \{x \in I : b(x) \in B\}, \qquad \text{where} \quad B \subseteq \mathscr{P}(\kappa), \\ b(x) = \{\xi < \kappa : x \in X_\xi\}, \qquad \text{and} \qquad X_\xi \in \mathscr{P}_{\mathrm{int}}(I) \quad \text{for all } \ \xi < \kappa; \end{gathered} \tag{1}$$

(2) A set $X \subseteq I$ is said to be *$\kappa$-determined* if it has the form

$$X = \bigcup_{b \in B} \bigcap_{\xi \in b} X_\xi, \qquad \text{where} \quad B \subseteq \mathscr{P}(\kappa), \quad X_\xi \in \mathscr{P}_{\mathrm{int}}(I) \quad \text{for all } \ \xi < \kappa. \tag{2}$$

Definitions (1) and (2) are equivalent. Indeed, if $X$ is defined as in (2), then, setting

$$B' = \bigcup_{b \in B} \{b' \subseteq \kappa : b \subseteq b'\},$$

we obtain an expression of type (1). Conversely, if $X$ is defined as in (1), then, setting $X'_{2\xi} = X_\xi$ and $X'_{2\xi+1} = I \setminus X_\xi$ and taking the set of all $b' \subseteq \kappa$ such that $b = \{\xi : 2\xi \in b'\} \in B$ and $\{\xi : 2\xi + 1 \in b'\} = \kappa \setminus b$ for $B'$, we obtain an expression of type (2)[1].

**Lemma 1.1.** *If $\kappa$ is an infinite cardinal and $I$ is an internal set, then the family of all $\kappa$-determined subsets of $I$ is closed under complements and under the union and intersection of at most $\kappa$ sets.*

**Proof.** We use definition (1). To obtain a complement, we simply set $B' = \mathscr{P}(\kappa) \setminus B$. Consider the union or intersection of at most $\kappa$ sets. Obviously, we can assume that all of the $\leq \kappa$ sets are determined in the sense of the same indexed family $\{X_\xi\}_{\xi<\kappa}$ of sets $X_\xi \in \mathscr{P}_{\mathrm{int}}(I)$; this reduces the problem to the union or intersection of the corresponding sets $B \subseteq \mathscr{P}(\kappa)$. $\square$

[1] We define $2\xi$ as the sum of $\xi$ copies of the ordinal 2; thus, $2\omega = \omega \neq \omega 2 = \omega + \omega$, and $2(\omega + 3) = 2\omega + 2 \cdot 3 = \omega + 6$.

## 2. CUTS IN POSITIVE HYPERINTEGERS

Initial intervals $U \subseteq {}^*\mathbb{N}$ are usually called *cuts*. A cut $U$ is *additive* if $x \in U$ implies $2x \in U$. Examples of additive cuts are

$$h\mathbb{N} = \{x \in {}^*\mathbb{N} : \exists\, n \in \mathbb{N}\ (x < nh)\} \qquad \text{and} \qquad h/\mathbb{N} = \left\{x \in {}^*\mathbb{N} : \forall\, n \in \mathbb{N}\ \left(x < \frac{h}{n}\right)\right\},$$

where $h \in {}^*\mathbb{N}$. The *cofinality* $\operatorname{cof} U$ of a cut $U \subseteq {}^*\mathbb{N}$ is defined as the minimal cardinal $\vartheta$ for which $U$ has an increasing cofinal subsequence of type $\vartheta$. The *coinitiality* $\operatorname{coi} U$ of a cut $U \subseteq {}^*\mathbb{N}$ is defined as the minimal cardinal $\vartheta$ for which ${}^*\mathbb{N} \setminus U$ has a coinitial decreasing subsequence of type $\vartheta$. Note that if $U$ has no greatest element, then $\operatorname{cof} U$ and $\operatorname{coi} U$ are regular infinite cardinals. (Moreover, in an additive cut, only $\{0\}$ can be a greatest element.)

**Lemma 2.1.** *Suppose that $\kappa$ is an infinite cardinal and the nonstandard universe is $\kappa^+$-saturated. Then any $\kappa$-determined cut $\varnothing \neq U \subsetneqq {}^*\mathbb{N}$ satisfies the condition $\operatorname{cof} U \le \kappa$ or $\operatorname{coi} U \le \kappa$, and if both of these inequalities hold, then $U$ contains a greatest element* (*and is an internal set*).

**Proof.** Let

$$U = \bigcup_{b \in B} \bigcap_{\xi \in b} X_\xi,$$

where $B \subseteq \mathscr{P}(\kappa)$ and the sets $X_\xi \subseteq {}^*\mathbb{N}$ are the same as in (2). For any $a \subseteq \kappa$, we set

$$X_a = \bigcap_{\xi \in a} X_\xi \qquad \text{and} \qquad U_a = \{y \in {}^*\mathbb{N} : \exists\, x \in X_a\ (y \le x)\}.$$

If $a$ is finite, then the sets $X_a$ and $U_a$ are internal and $U_a$ is a cut in ${}^*\mathbb{N}$. Therefore $U_a = [0, \mu_a)$, where $\mu_a = \max U_a + 1$ or, conventionally, $\mu_a = \infty$ (if $U_a = {}^*\mathbb{N}$). Moreover,

$$U = \bigcup_{b \in B} U_b,$$

and $\kappa^+$-saturation implies

$$U_b = \bigcap_{a \in \mathscr{P}_{\text{fin}}(b)} U_a.$$

*Case* 1: $U_b \subsetneqq U$ for all $b \in B$. Taking $h_b \in U \setminus U_b$, we obtain $\mu_a \le h_b$ for at least one $a = a(b)$ belonging to $\mathscr{P}_{\text{fin}}(b)$ (because $U_b = \bigcap_{a \in \mathscr{P}_{\text{fin}}(b)} U_a$); therefore, the set $\{\mu_{a(b)} : b \in B\}$ is cofinal in $U$.

*Case* 2: $U_b = U$ for some $b \in B$. If $U_b \subsetneqq U_a$ for all $a \in \mathscr{P}_{\text{fin}}(b)$, then the set

$$\{\mu_a \neq \infty : a \in \mathscr{P}_{\text{fin}}(b)\}$$

of cardinality $\le \kappa$ is coinitial in ${}^*\mathbb{N} \setminus U$, and if $U_b = U_a$ for some $a \in \mathscr{P}_{\text{fin}}(b)$, then $U = U_b = U_a$ is an internal set whose cofinality and coinitiality are equal to 1.

Finally, if a set $K \subseteq U$ of cardinality $\le \kappa$ is cofinal in $U$ and a set $L \subseteq {}^*\mathbb{N} \setminus U$ of cardinality $\le \kappa$ is coinitial in ${}^*\mathbb{N} \setminus U$, then $\kappa^+$-saturation implies the existence of an $x \in K$ and a $y \in L$ which are, respectively, the greatest element of $U$ and the least element of ${}^*\mathbb{N} \setminus U$; thus, $U$ is an internal set. $\square$

Conversely, any cut $U \subseteq {}^*\mathbb{N}$ satisfying the condition $\operatorname{cof} U \le \kappa$ or $\operatorname{coi} U \le \kappa$ is $\kappa$-determined. Indeed, if a sequence $\{u_\xi\}_{\xi<\kappa}$ is, say, cofinal in $U$, then

$$U = \bigcup_{\xi < \kappa} U_\xi,$$

where all sets $U_\xi = [0, u_\xi]$ are internal and, therefore, $\kappa$-determined, and it remains to apply Lemma 11.

## 3. MONADIC PARTITIONS

Any additive (i.e., such that $a \in U \Rightarrow 2a \in U$) cut $U \subseteq {}^*\mathbb{N}$ induces the equivalence relation $\mathrm{M}_U$ on ${}^*\mathbb{N}$ defined by $x \, \mathrm{M}_U \, y$ if $|x-y| \in U$. The equivalence classes

$$[x]_U = \{y : x \, \mathrm{M}_U \, y\} = \{y : |x-y| \in U\}$$

are called *$U$-monads*; they form the *coset space*

$${}^*\mathbb{N}/U = {}^*\mathbb{N}/\,\mathrm{M}_U = \{[x]_U : x \in {}^*\mathbb{N}\}.$$

The relations $\mathrm{M}_U$ themselves are called *monadic* equivalence relations, or simply *monadic partitions*.

Two degenerate examples are $U = \varnothing$, for which $\mathrm{M}_U$ coincides with the equality on ${}^*\mathbb{N}$ and is often denoted by $\mathrm{D}_{{}^*\mathbb{N}}$ (the *diagonal* of ${}^*\mathbb{N}$). If $U = {}^*\mathbb{N}$, then all $x \in {}^*\mathbb{N}$ are $\mathrm{M}_U$-equivalent to each other. A nondegenerate example is $U = \mathbb{N}$, in which $x \, \mathrm{M}_U \, y$ if and only if $|x-y|$ is finite.

Various monads are often encountered in works on nonstandard analysis and nonstandard models of Peano arithmetic. Monads induced by additive cuts in ${}^*\mathbb{N}$ were studied in, e.g., [2], [3]. In this paper, we consider monadic partitions and equivalences from the point of view of the existence of transversals, smoothness, and reducibility.

Recall that a *transversal* for an equivalence relation E is any set which has precisely one common element with each E-class. Let $\kappa$ be an infinite cardinal. An equivalence relation E on ${}^*\mathbb{N}$ is said to be *$\kappa$-smooth* if there exists a $\kappa$-determined (as a set of pairs) function $f : {}^*\mathbb{N} \to {}^*\mathbb{N}$ for which

$$x \, \mathrm{E} \, y \;\Leftrightarrow\; f(x) = f(y) \qquad \text{for all} \quad x, y \in {}^*\mathbb{N}.$$

Finally, if E and F are equivalence relations on ${}^*\mathbb{N}$, then $\mathrm{E} \le_\kappa \mathrm{F}$ (the *$\kappa$-determined reducibility* of E to F) is understood as the existence of a $\kappa$-determined set $R \subseteq {}^*\mathbb{N} \times {}^*\mathbb{N}$ such that

$$\begin{aligned} &\text{(a) the relation } R \text{ is invariant in the sense that } x \, \mathrm{E} \, x' \Leftrightarrow y \, \mathrm{F} \, y' \text{ for all pairs} \\ &\quad\; \langle x, y\rangle \text{ and } \langle x', y'\rangle \text{ from } R, \text{ and} \\ &\text{(b) } \operatorname{dom} R = {}^*\mathbb{N}. \end{aligned} \tag{3}$$

In this case, we can define an embedding $F : {}^*\mathbb{N}/\mathrm{E} \to {}^*\mathbb{N}/\mathrm{F}$ so that $F([x]_\mathrm{E}) = [y]_\mathrm{F}$ if there exists a $y' \in [y]_\mathrm{F}$ for which $\langle x, y'\rangle \in R$. Note that the $\kappa$-smoothness of E defined above is equivalent to $\mathrm{E} \le_\kappa \mathrm{D}_{{}^*\mathbb{N}}$.

According to the following theorem, the relation of $\kappa$-determined reducibility between monadic partitions is determined by their cofinalitites (coinitialities) and the parameter

$$\operatorname{wid} U = \bigcap_{u \in U,\, u' \in {}^*\mathbb{N} \setminus U} \left[0, \frac{u'}{u}\right) = \{h \in {}^*\mathbb{N} : \forall\, x \;\, (x \in U \Rightarrow hx \in U)\},$$

which we call the *width* of the cut $U \subsetneqq {}^*\mathbb{N}$.[2] It is easy to verify that $\operatorname{wid} U$ is a cut as well; moreover, it is not only additive but also multiplicative, i.e., $a \in \operatorname{wid} U \Rightarrow a^2 \in \operatorname{wid} U$.

Additive cuts of minimal possible width (except $\{0\}$) are the cuts $U = h\mathbb{N}$, where $h \in {}^*\mathbb{N}$, and $U = h/\mathbb{N}$, where $h \in {}^*\mathbb{N} \setminus \mathbb{N}$; for these cuts, $\operatorname{wid} U = \mathbb{N}$. We call them *slow* cuts. The other additive cuts (i.e., all cuts but those of the forms $h\mathbb{N}$ and $h/\mathbb{N}$) are said to be *fast*. Note that even $\aleph_1$-saturation prohibits the existence of cuts which can be written both as $c\mathbb{N}$ and as $c'/\mathbb{N}$.

**Theorem 3.1.** *Suppose that $\kappa$ is an uncountable cardinal and the given nonstandard universe is $\kappa^+$-saturated. Suppose also that $U$ and $V$ are additive $\kappa$-determined cuts in ${}^*\mathbb{N}$ different from $\varnothing$, $\{0\}$, and ${}^*\mathbb{N}$. Then*

(i) $\mathrm{D}_{{}^*\mathbb{N}} \le_\kappa \mathrm{M}_U$;

(ii) *the following three assertions are equivalent*:

[2] It is also called *thickness*.

(1) $\mathrm{M}_U$ *is* $\kappa$*-smooth*;

(2) $\mathrm{M}_U$ *has a* $\kappa$*-determined transversal*;

(3) *either* (a) $U = h\mathbb{N}$*, where* $h \in {}^*\mathbb{N}$*, or* (b) $U = h/\mathbb{N}$*, where* $h \in {}^*\mathbb{N} \setminus \mathbb{N}$*, and* $\mathfrak{c} \le \kappa$;[3]

(iii) *if* $U$ *is a cut of type* (ii) (3)*, then* $\mathrm{M}_U \le_\kappa \mathrm{M}_V$;

(iv) *if* $\operatorname{cof} U \le \kappa$, $\operatorname{cof} V \le \kappa$*, and* $U$ *is not of the form* $U = h\mathbb{N}$*, where* $h \in {}^*\mathbb{N}$*, then* $\mathrm{M}_U \le_\kappa \mathrm{M}_V$ *if and only if* $\operatorname{cof} U = \operatorname{cof} V$ *and* $\operatorname{wid} U \subseteq \operatorname{wid} V$;

(v) *if* $\operatorname{coi} U \le \kappa$, $\operatorname{coi} V \le \kappa$*, and* $U$ *is not of the form* $U = h/\mathbb{N}$*, where* $h \in {}^*\mathbb{N} \setminus \mathbb{N}$*, then* $\mathrm{M}_U \le_\kappa \mathrm{M}_V$ *if and only if* $\operatorname{coi} U = \operatorname{coi} V$ *and* $\operatorname{wid} U \subseteq \operatorname{wid} V$;

(vi) *if* $\operatorname{cof} U \le \kappa$ *but* $\operatorname{coi} V \le \kappa$ *or,* vice versa, $\operatorname{coi} U \le \kappa$ *but* $\operatorname{cof} V \le \kappa$*, then* $\mathrm{M}_U \not\le_\kappa \mathrm{M}_V$*, except for the case in which* $U$ *satisfies* (ii) (3).

Thus, monadic relations are linearly $\le_\kappa$-(pre)ordered inside each cofinality/coinitiality type; between these types, there are no relations except those induced by cuts of the forms $h\mathbb{N}$ and $h/\mathbb{N}$. There is an open question related to Theorem 3.1: Suppose that $\kappa < \mathfrak{c}$, $U = h/\mathbb{N}$ for some $h \in {}^*\mathbb{N}$, and $V$ is a $\kappa$-determined additive cut with $\omega < \operatorname{coi} U \le \kappa$; can the relation $\mathrm{M}_U \le_\kappa \mathrm{M}_V$ hold?

For $\kappa = \aleph_0$, Theorem 3.1 was proved in [4].

## 4. PRELIMINARY REMARKS AND THE BEGINNING OF THE PROOF

We start with the following definition. We say that an internal set $X \subseteq {}^*\mathbb{N}$ is *sparse* if there exists a number $s \in {}^*\mathbb{N} \setminus \mathbb{N}$ such that $\#(X \cap I)/s$ is infinitesimal for any interval $I$ of length $s$ in ${}^*\mathbb{N}$.

**Proposition 4.1.** *Under the conditions of Theorem* 3.1*,* ${}^*\mathbb{N}$ *is not a union of at most* $\kappa$ *sparse internal sets.*

**Proof.** By saturation, any cover by internal sets of cardinality $\le \kappa$ has a finite subcover; therefore, it suffices to prove the required assertion for finite unions. Let

$${}^*\mathbb{N} = \bigcup_{k \le n} X_k,$$

where $n \in \mathbb{N}$ and each $X_k$ is a sparse set with parameter $s_k \in {}^*\mathbb{N} \setminus \mathbb{N}$. We put $s = s_0 \cdot s_1 \cdots s_n$ and $I = [0, s)$. Then $\#(X_k \cap I)/s$ is infinitesimal for any $k$, which contradicts the $n$ being finite. $\square$

**Proof of Theorem 3.1.** We start with several simple assertions.

(i) Let $h \in {}^*\mathbb{N} \setminus U$. Then the map $x \mapsto [xh]$ witnesses $\mathrm{D}_{{}^*\mathbb{N}} \le_\kappa \mathrm{M}_U$.

(ii) If $\mathrm{M}_U$ has a $\kappa$-determined transversal, then it is $\kappa$-smooth. (For $x \in {}^*\mathbb{N}$, we set $f(x) = t_x$, where $t_x$ is the unique element of the transversal that is equivalent to $x$.)

Now, suppose that $\mathrm{M}_U$ is $\kappa$-smooth, i.e., $\mathrm{M}_U \le_\kappa \mathrm{D}_{{}^*\mathbb{N}}$. Then, by (i), $\mathrm{M}_U \le_\kappa \mathrm{M}_V$ for any other additive $\kappa$-determined cut; thus, by (vi), the cut $U$ can be only of type (ii) (3).

Finally, suppose that $U$ is a cut of type (ii) (3). For the case in which $U = h\mathbb{N}$ for $h \in {}^*\mathbb{N}$, the existence of even countably determined transversals was proved in [3] (see also [4]). Suppose that $U = h/\mathbb{N}$, where $h \in {}^*\mathbb{N} \setminus \mathbb{N}$, and $\mathfrak{c} \le \kappa$. In this case, $\mathrm{M}_U$ has no countably determined transversals ([3]; see also [4] or 9.7.14 in [5]). However, the inequality $\mathfrak{c} \le \kappa$ simplifies the situation. Indeed, obviously, the restricted relation $\mathrm{M}_U \restriction [0, h)$ has precisely $\mathfrak{c}$ equivalence classes, and since $\mathfrak{c} \le \kappa$, we can choose one element in each of them, thus obtaining a $\kappa$-determined transversal $T \subseteq [0, h)$ for $\mathrm{M}_U \restriction [0, h)$ (by Lemma 11). It remains to reproduce a copy of $T$ in each interval of the form $[ph, ph + h)$, where $p \in {}^*\mathbb{N}$, by applying an appropriate translation.

(iii) The cut $\mathrm{M}_U$ is $\kappa$-smooth by (ii). It remains to apply (i). $\square$

The other assertions of the theorem are proved in the following sections.

[3] By $\mathfrak{c} = 2^{\aleph_0}$ we denote the cardinality of the continuum.

## 5. COFINAL CUTS

In this section, we prove the implication $\Leftarrow$ in assertion (iv) of Theorem 3.1. Take increasing sequences $\{u_\xi\}_{\xi<\vartheta}$ and $\{v_\eta\}_{\eta<\tau}$ cofinal in $U$ and $V$, respectively; here $\vartheta = \operatorname{cof} U \le \kappa$ and $\tau = \operatorname{cof} V \le \kappa$ are regular infinite cardinals.

Thus, we assume that $\mathrm{M}_U \le_\kappa \mathrm{M}_V$. This implies the existence of a $\kappa$-determined set $R \subseteq {}^*\mathbb{N} \times {}^*\mathbb{N}$ satisfying the conditions $\operatorname{dom} R = {}^*\mathbb{N}$ and

$$xRy \wedge x'Ry' \;\Rightarrow\; (|x-x'| \in U \Leftrightarrow |y-y'| \in V)\,^4 \tag{4}$$

for all $x, y, x', y' \in {}^*\mathbb{N}$. By definition (2), we have $R = \bigcup_{b\in B} R_b$, where $B \subseteq \mathscr{P}(\kappa)$, $R_b = \bigcap_{\zeta<\kappa} R_\zeta$ for each $b \subseteq \kappa$, and all of the sets $R_\zeta \subseteq {}^*\mathbb{N} \times {}^*\mathbb{N}$ are internal.

*Part* 1. Let us prove that $\operatorname{wid} U \subseteq \operatorname{wid} V$. Suppose that, on the contrary, $\operatorname{wid} U \not\subseteq \operatorname{wid} V$, and let $e \in \operatorname{wid} U \setminus \operatorname{wid} V$. We claim that,

$$\begin{array}{l}\textit{for any } b \in B\textit{, there exists a finite set } a(b) \subseteq b \textit{ for which}\\ \textit{the set } D_b = \operatorname{dom} R_{a(b)} \textit{ is sparse in the sense of Proposition 4.1.}\end{array} \tag{5}$$

This contradicts 4.1, because ${}^*\mathbb{N} = \operatorname{dom} R = \bigcup_{b\in B} D_b$ (this follows from $R_b \subseteq R_{a(b)}$) and, on the other hand, the family of all finite sets $a \subseteq \kappa$ has cardinality $\kappa$.

To prove (5), take $b \in B$. We have $R_b \subseteq R$; thus, by (4),

$$(\forall\,\zeta \in b\;\; (xR_\zeta y \wedge x'R_\zeta y') \wedge \exists\,\eta < \tau(|y-y'| < v_\eta)) \;\Rightarrow\; (\exists\,\xi < \vartheta\;\; (|x-x'| < u_\xi))$$

for all $x, x', y, y' \in {}^*\mathbb{N}$. $\kappa^+$-Saturation implies

$$\begin{array}{l}\forall\,\eta < \tau \quad \exists^{\text{fin}}\, a \subseteq b \quad \exists\,\xi < \vartheta \quad \forall\, x, x', y, y' \in {}^*\mathbb{N}:\\ xR_a y \wedge \exists\, x'R_a y' \wedge |y-y'| < v_\eta \;\Rightarrow\; |x-x'| < u_\xi,\end{array} \tag{6}$$

where $\exists^{\text{fin}}\, a$ means the existence of a finite $a$. Using a similar argument, we obtain

$$\begin{array}{l}\forall\,\xi' < \vartheta \quad \exists^{\text{fin}}\, a' \subseteq b \quad \exists\,\eta' < \tau \quad \forall\, x, x', y, y' \in {}^*\mathbb{N}:\\ xR_{a'} y \wedge x'R_{a'} y' \wedge |x-x'| < u_{\xi'} \;\Rightarrow\; |y-y'| < v_{\eta'}.\end{array} \tag{7}$$

By the choice of $e \in \operatorname{wid} U \setminus \operatorname{wid} V$, there exists an index $\eta < \tau$ such that $ev_\eta \notin V$ and, at the same time, $eu_\xi \in U$ for any $\xi$. Choose $a$ and $\xi$ so that (6) holds for this $\eta$. Since $eu_\xi \in U$, it follows that there exists an index $\xi' > \xi$ for which $u_{\xi'}/u_\xi > e$, and since $U$ is a fast cut, we can assume, without loss of generality, that the relation $(u_{\xi'}/u_\xi) : e$ is infinitely large. Now, take $m'$ and $\eta'$ for which (7) holds. We can assume that $a \subseteq a'$ and $\eta \le \eta'$; otherwise, we take the union in the former case and a maximum in the latter. For all $\langle x, y\rangle$ and $\langle x', y'\rangle$ from the set $R_{a'}$, we have

$$|y-y'| < v_\eta \;\Rightarrow\; |x-x'| < u_\xi \qquad \text{and} \qquad |x-x'| < u_{\xi'} \;\Rightarrow\; |y-y'| < v_{\eta'}. \tag{8}$$

To prove the sparsity of the set $D_b = \operatorname{dom} R_{a'}$, note that any interval of length $v_{\eta'}$ in ${}^*\mathbb{N}$ consists of approximately $s = v'_\eta/v_\eta$ subintervals of length $v_\eta$. Accordingly, any interval of length $v_{\xi'}$ consists of approximately $t = u'_\xi/u_\xi$ subintervals of length $u_\xi$. The fraction $s/t$ is infinitesimal by the choice of $\xi'$. It follows from (8) that the fraction $\#(I \cap D_b)/\#(I)$ is infinitesimal for any interval $I$ of length $u_{\xi'}$ in ${}^*\mathbb{N}$. Therefore, $D_b$ is a sparse set, as required.

*Part* 2. Let us derive the equality $\operatorname{cof} U = \operatorname{cof} V$, i.e., prove that $\vartheta = \tau$. Suppose that, on the contrary, $\vartheta \ne \tau$; to be definite, let $\vartheta < \tau$. Again, it suffices to prove (5). Take $b \in B$. From cardinality considerations, there exists an index $\eta < \tau$ such that (7) holds for all $\xi' < \vartheta$ simultaneously; thus,

$$\begin{array}{l}\forall\,\xi' < \vartheta \quad \exists^{\text{fin}}\, a' \subseteq b \quad \forall\, x, x', y, y' \in {}^*\mathbb{N}:\\ xR_{a'} y \wedge x'R_{a'} y' \wedge |x-x'| < u_{\xi'} \;\Rightarrow\; |y-y'| < v_\eta.\end{array} \tag{9}$$

[4] If $R$ is a binary relation (a set of pairs), then $xRy$ means that $\langle x, y\rangle \in R$.

Choose an index $\xi < \vartheta$ and finite $a \subseteq b$ so that (6) holds for this $\eta$. Applying (9) to $\xi' > \xi$ for which the fraction $u_{\xi'}/u_\xi$ is infinitely large, we obtain a finite $a' \subseteq b$ such that $a \subseteq a'$ and, for all $x, x'$ belonging to $D_b = \operatorname{dom} R_{a'}$, we have

$$|x - x'| < u_{\xi'} \ \Rightarrow\ |x - x'| < u_\xi. \tag{10}$$

This implies the sparsity of $D_b$.

For $\tau < \vartheta$, the argument is somewhat different. Namely, there exists an ordinal $\xi < \vartheta$ such that (6) holds for all $\eta' < \tau$ simultaneously; thus,

$$\begin{gathered}\forall\, \eta' < \tau \quad \exists^{\text{fin}}\, a \subseteq b \quad \forall\, x, x', y, y' \in {}^*\mathbb{N}: \\ x R_a y \wedge x' R_a y' \wedge |y - y'| < v_{\eta'} \ \Rightarrow\ |x - x'| < u_\xi.\end{gathered} \tag{11}$$

Choose $\xi' > \xi$ so that the fraction $u_{\xi'}/u_\xi$ is infinitely large and take an ordinal $\eta' < \tau$ and a finite set $a' \subseteq b$ such that (7) holds for $\xi'$. Applying (11) to $\eta'$, we again obtain a finite set $a \subseteq b$ such that $a' \subseteq a$ and (10) holds for all $x, x' \in D_b = \operatorname{dom} R_a$, etc.

## 6. COINITIAL CUTS

Let us prove the implication $\Leftarrow$ in assertion (v) of Theorem 3.1. Calculations differ from those presented in Sec. 5 only in some fairly obvious details. We only sketch the proof of the inclusion $\operatorname{wid} U \subseteq \operatorname{wid} V$. We start with *decreasing* sequences $\{u_\xi\}_{\xi<\vartheta}$ and $\{v_\eta\}_{\eta<\tau}$ coinitial in $U$ and $V$, respectively.

Suppose that $e \in \operatorname{wid} U \setminus \operatorname{wid} V$, contrary to the required assertion. The same argument as that used in Sec. 5 proves the relations

$$\begin{gathered}\forall\, \eta < \tau \quad \exists^{\text{fin}}\, a \subseteq b \quad \xi < \vartheta \quad \forall\, x, x', y, y' \in {}^*\mathbb{N}: \\ x R_a y \wedge x' R_a y' \wedge |x - x'| < u_\xi \ \Rightarrow\ |y - y'| < v_\eta,\end{gathered} \tag{6'}$$

$$\begin{gathered}\forall \xi' < \vartheta \quad \exists^{\text{fin}}\, a' \subseteq b \quad \eta' < \tau \quad \forall\, x, x', y, y' \in {}^*\mathbb{N}: \\ x R_{a'} y \wedge x' R_{a'} y' \wedge |y - y'| < v_{\eta'} \ \Rightarrow\ |x - x'| < u_{\xi'},\end{gathered} \tag{7'}$$

yields two pairs of indices $\xi < \xi'$ and $\eta < \eta'$ for which $v_\eta/v_{\eta'} < e$ and $u_\xi/u_{\xi'} > e$, and reduces the key relation (8) to the form

$$|y - y'| < v_{\eta'} \ \Rightarrow\ |x - x'| < u_{\xi'} \qquad \text{and} \qquad |x - x'| < u_\xi \ \Rightarrow\ |y - y'| < v_\eta. \tag{8'}$$

Since $U$ is a fast cut, we can again choose $\xi'$ so that the fraction $(u_\xi/u_{\xi'}) : e$ is infinitely large and obtain a contradiction in the same way as above.

## 7. THE MIXED CASE

In this section, we prove assertion (vi) of Theorem 3.1. Suppose that, on the contrary, a $\kappa$-determined set $R \subseteq {}^*\mathbb{N} \times {}^*\mathbb{N}$ witnesses $\mathrm{M}_U \le_\kappa \mathrm{M}_V$, i.e., $\operatorname{dom} R = {}^*\mathbb{N}$ and (4) holds. As above, $R = \bigcup_{b\in B} R_b$, where $B \subseteq \mathscr{P}(\kappa)$, $R_b = \bigcap_{\zeta<\kappa} R_\zeta$, and all of the sets $R_\zeta \subseteq {}^*\mathbb{N} \times {}^*\mathbb{N}$ are internal.

*Case* 1: $\operatorname{cof} U \le \kappa$, $\operatorname{coi} V \le \kappa$, and $U$ is not of the form $h\mathbb{N}$. Take an increasing cofinal sequence $\{u_\xi\}_{\xi<\vartheta}$ in $U$ and a decreasing coinitial sequence $\{v_\eta\}_{\eta<\tau}$ in ${}^*\mathbb{N} \setminus V$; here $\vartheta = \operatorname{cof} U \le \kappa$ and $\tau = \operatorname{coi} V \le \kappa$ are regular infinite cardinals.

Let $b \in B$. Arguing as in Sec. 5, we obtain

$$\begin{gathered}\exists^{\text{fin}}\, a \subseteq b \quad \exists\, \xi < \vartheta \quad \exists\, \eta < \tau \quad \forall\, x, x', y, y' \in {}^*\mathbb{N}: \\ x R_a y \wedge x' R_a y' \wedge |y - y'| < v_\eta \ \Rightarrow\ |x - x'| < u_\xi\end{gathered} \tag{12}$$

and, conversely,

$$\begin{gathered}\forall\, \xi' < \vartheta \quad \forall\, \eta < \tau \quad \exists^{\text{fin}}\, a' \subseteq b \quad \forall\, x, x', y, y' \in {}^*\mathbb{N}: \\ x R_{a'} y \wedge x' R_{a'} y' \wedge |x - x'| < u_{\xi'} \ \Rightarrow\ |y - y'| < v_\eta.\end{gathered} \tag{13}$$

Take $\xi$, $\eta$, and $a$ as in (12), an ordinal $\xi' \ge \xi$ for which $u_{\xi'}/u_\xi$ is infinitely large, and, finally, a set $a'$ as in (13) (for $\xi'$) such that $a \subseteq a'$. For all $x$ and $x'$ from the set $D_b = \operatorname{dom} R_{a'}$, we have

$$|x - x'| < u_{\xi'} \ \Rightarrow \ |x - x'| < u_\xi;$$

thus, $D_b$ is sparse. This proves (5) and leads to the desired contradiction.

*Case* 2: $\operatorname{coi} U \le \kappa$, $\operatorname{cof} V \le \kappa$, and either $U$ is not of the form $h/\mathbb{N}$ or $U = h/\mathbb{N}$ and $\kappa < \mathfrak{c}$. Take a decreasing coinitial sequence $\{u_\xi\}_{\xi<\vartheta}$ in ${}^*\mathbb{N} \setminus U$ and an increasing cofinal sequence $\{v_\eta\}_{\eta<\tau}$ in $V$.

Let $b \in B$. Arguing as above, we obtain

$$\begin{aligned} &\exists^{\text{fin}}\, a \subseteq b \quad \exists\, \xi < \vartheta \quad \exists\, \eta < \tau \quad \forall\, x, x', y, y' \in {}^*\mathbb{N}: \\ &xR_a y \wedge x'R_a y' \wedge |x - x'| < u_\xi \ \Rightarrow \ |y - y'| < v_\eta \end{aligned} \tag{14}$$

and, conversely,

$$\begin{aligned} &\forall\, \xi' < \vartheta \quad \forall\, \eta < \tau \quad \exists^{\text{fin}}\, a' \subseteq b \quad \forall\, x, x', y, y' \in {}^*\mathbb{N}: \\ &xR_{a'} y \wedge x'R_{a'} y' \wedge |y - y'| < v_\eta \ \Rightarrow \ |x - x'| < u_{\xi'}. \end{aligned} \tag{15}$$

Take $\xi$, $\eta$, and $a$ as in (12) and a set $a' \supseteq a$ as in (13) with some $\xi' > \xi$. For all $x$ and $x'$ from the set $D_b = \operatorname{dom} R_{a'}$, we have

$$|x - x'| < u_\xi \ \Rightarrow \ |x - x'| < u_{\xi'}.$$

If $U \ne h/\mathbb{N}$, then $\xi'$ can be chosen so that $u'_\xi/u_\xi$ is infinitesimal. In this case, each set $D_b$ turns out to be sparse, and so on.

It remains to consider the case in which $U = h/\mathbb{N}$ for $h \in {}^*\mathbb{N} \setminus \mathbb{N}$ and $\kappa < \mathfrak{c}$. We can assume that $\vartheta = \mathbb{N}$ and $u_\xi = h/\xi$ for all $\xi \in \mathbb{N}$. In this case, sparse sets do not work, and we must use a different idea.

Take $b \in B$ and $\xi$, $\eta$, and $a = a(b)$ as in (14). Note that $|y - y'| < v_\eta$ implies $y \, \mathrm{M}_V \, y'$, which is, in turn, equivalent to $x \, \mathrm{M}_U \, x'$ provided that $xRy$ and $x'Ry'$. Thus, for all $x, x' \in D_b = \operatorname{dom} R_{a(b)}$ with $|x - x'| < u_\xi$, we have $x \, \mathrm{M}_U \, x'$. However, the interval $I = [0, h]$ in ${}^*\mathbb{N}$ is partitioned into $\xi$ subintervals of length $u_\xi$, and $\xi$ is finite. Therefore, $D_b \cap I$ intersects only finitely many $\mathrm{M}_U$-classes. But there are only $\le \kappa$ sets of the form $a(b)$ (because $\operatorname{card} \mathscr{P}_{\text{fin}}(\kappa) = \kappa$) and, hence, $\le \kappa$ sets of the form $D_b = \operatorname{dom} R_{a(b)}$, and their union covers ${}^*\mathbb{N}$. Thus, $I$ intersects $\le \kappa$ $\mathrm{M}_U$-classes, which contradicts the inequality $\kappa < \mathfrak{c} = \operatorname{card}(I/\mathrm{M}_U)$.

## 8. CONSTRUCTION OF REDUCTIONS

In this section, we prove the implication $\Rightarrow$ in the assertions (iv) and (v) of Theorem 3.1.

(iv) Suppose that $\operatorname{coi} U = \operatorname{coi} V = \vartheta \le \kappa$ (where $\vartheta$ is a regular infinite cardinal) and $\operatorname{wid} U \subseteq \operatorname{wid} V$. Let us prove that $\mathrm{M}_U \le_\kappa \mathrm{M}_V$. Choose increasing cofinal sequences $\{u_\xi\}_{\xi<\vartheta}$ and $\{v_\xi\}_{\xi<\vartheta}$ in $U$ and $V$, respectively. Since the cuts are additive, it follows that we can assume all terms $u_\xi$ and $v_\xi$ of these sequences to be hyperinteger powers of the number 2 in ${}^*\mathbb{N}$.

The relation $\operatorname{wid} U \subseteq \operatorname{wid} V$ means that

$$\forall\, \eta \quad \exists\, \xi \quad \forall\, \xi' > \xi \quad \exists\, \eta' > \eta \qquad \left(\frac{u_{\xi'}}{u_\xi} \le \frac{v_{\eta'}}{v_\eta}\right).$$

(Here and in the course of the proof, $\xi$, $\xi'$, $\eta$, $\eta'$, and $\zeta$ denote ordinals smaller than $\vartheta$.)

This allows us to separate out an unbounded subsequence in $\{u_\xi\}_{\xi<\vartheta}$ for which, after renumbering, we obtain

$$\forall\, \zeta \quad \forall\, \xi > \zeta \quad \exists\, \eta > \zeta \qquad \left(\frac{u_\xi}{u_\zeta} \le \frac{v_\eta}{v_\zeta}, \quad \text{i.e.,} \quad \frac{v_\zeta}{u_\zeta} \le \frac{v_\eta}{u_\xi}\right);$$

then, we separate out an unbounded subsequence in $\{v_\eta\}_{\eta<\vartheta}$ so that, after renumbering,

$$\forall\, \xi < \eta < \vartheta \qquad \left(\frac{v_\xi}{u_\xi} \le \frac{v_\eta}{u_\eta}, \quad \text{i.e.,} \quad \frac{u_\eta}{u_\xi} \le \frac{v_\eta}{v_\xi}\right). \tag{16}$$

Under the above assumptions, the function $f$ that maps each $u_\xi$ to $v_\xi$ satisfies the following conditions: $\operatorname{dom} f = \{u_\xi : \xi < \vartheta\}$ (this is a set of cardinality $\vartheta$ in ${}^*\mathbb{N}$), $\operatorname{dom} f$ and $\operatorname{ran} f$ consist of powers of the number 2, and $f(u)/u \le f(u')/u'$ for all $u < u'$ in $\operatorname{dom} f$ (by (16)). Using $\kappa^+$-saturation, it is easy to prove the existence of an internal function $\varphi$ such that

> $D = \operatorname{dom}\varphi$ is a hyperfinite subset of ${}^*\mathbb{N}$, $\operatorname{dom} f \subseteq \operatorname{dom}\varphi$, $\varphi(u_\xi) = v_\xi$ for all $\xi$, the sets $D$ and $Z = \operatorname{ran}\varphi$ consist of powers of the number 2, and $\varphi(d)/d \le \varphi(d')/d'$ for all $d < d'$ in $D$; as is easy to show, this implies that $\varphi$ is a bijection, the sets $D \cap U$ and $D \cap V$ are cofinal in $U$ and $V$, respectively, and the sets $D \setminus U$ and $D \setminus V$ are coinitial in ${}^*\mathbb{N} \setminus U$ and ${}^*\mathbb{N} \setminus V$, respectively. (17)

Suppose that $h = \#(D) = \#(Z)$ and the elements of $D = \{d_1, d_2, \dots, d_h\}$ and $Z = \{z_1, z_2, \dots, z_h\}$ are enumerated in increasing order in ${}^*\mathbb{N}$ so that $z_\nu = \varphi(d_\nu)$ for each $\nu = 1, \dots, h$. Since all terms $d_\nu$ and $z_\nu$ are powers of 2, it follows that the fractions $j_\nu = d_{\nu+1}/d_\nu$ and $k_\nu = z_{\nu+1}/z_\nu$ belong to ${}^*\mathbb{N}$, and, under the above assumptions, $j_\nu \le k_\nu$. Moreover, $d_1 \in U$ and $z_1 \in V$.

Each number $x \in {}^*\mathbb{N}$ admits a unique internal representation in the form $x = \alpha_0 + \sum_{\nu=1}^{h} \alpha_\nu d_\nu$, where $\alpha_\nu \in {}^*\mathbb{N}$, $\alpha_0 < d_1$, and $0 \le \alpha_\nu < j_\nu$ for all $\nu = 1, \dots, h-1$ (but the coefficient $\alpha_h$ is, of course, unbounded). We could take the map $\sigma(x) = \sum_{\nu=1}^{h} \alpha_\nu z_\nu$ for a reduction of $\mathrm{M}_U$ to $\mathrm{M}_V$, but this is not exactly what we need. Indeed, suppose that

$$x = \sum_{\nu=1}^{h} d_\nu \qquad \text{and} \qquad x' = \sum_{\nu=1}^{h-1} (j_\nu - 1)\, d_\nu;$$

then $x - x' = 1$, but the value of $|\sigma(x) - \sigma(x')|$ may be very large if, e.g., $k_\nu > j_\nu$ for all $\nu$. A certain modification is needed.

Suppose that $x = \alpha_0 + \sum_{\nu=1}^{h} \alpha_\nu d_\nu \in {}^*\mathbb{N}$, where, as above, $\alpha_0 < d_1$ and $0 \le \alpha_\nu < j_\nu$ for $1 \le \nu < h$. We say that $x$ is a number *of type* 1 if there exist indices $1 \le \nu' < \nu'' < h$ such that $d_{\nu'} \in U$, $d_{\nu''} \notin U$, and $\alpha_\nu = j_\nu - 1$ for each $\nu$ in the range $\nu' \le \nu \le \nu''$. In this case, we take maximal $\nu''$ and minimal $\nu'$ satisfying these conditions and define $\bar\alpha_\nu = a_\nu$ for all $\nu < \nu'$ and all $\nu > \nu''$; then, we put $\bar\alpha_\nu = 0$ for $\nu' \le \nu \le \nu''$ and $\bar\alpha_{\nu''+1} = \alpha_{\nu''+1} + 1$; finally, we set

$$\bar x = \sum_{\nu=1}^{h} \bar\alpha_\nu\, d_\nu.$$

Otherwise, we say that $x$ is a number *of type* 2 and set

$$\bar x = x.$$

Note that $\bar x - x = \alpha_0 + d_{\nu'} \in U$ if $x$ is of type 1.

Now, let us prove that *the map* $\rho(x) = \sigma(\bar x)$ *satisfies* (3), i.e.,

$$|x - x'| \in U \;\Leftrightarrow\; |\sigma(\bar x) - \sigma(\bar y)| \in V \qquad \text{for all} \quad x, x' \in {}^*\mathbb{N}.$$

Suppose that

$$x = \alpha_0 + \sum_{\nu=1}^{h} \alpha_\nu d_\nu \qquad \text{and} \qquad y = \gamma_0 + \sum_{\nu=1}^{h} \gamma_\nu d_\nu,$$

where $\alpha_0, \gamma_0 < d_1$ and $\alpha_\nu, \gamma_\nu < j_\nu$ for all $1 \le \nu < h$. Suppose also that $|x - y| \in U$. Then we have $|x - y| < d_\nu \in U$ for some $1 \le \nu < h$. To be definite, let $x < y$. We can assume that the numbers $x$ and $y$ are of type 2 and, therefore, $\alpha_0 = \gamma_0 = 0$. (Otherwise, we replace these numbers by $\bar x$ and $\bar y$.) Then there exist infinitely (but hyperfinitely) many indices $\nu' > \nu$ for which $\alpha_{\nu'} \ne j_{\nu'} - 1$. We have $\alpha_{\nu'} = \gamma_{\nu'}$ for any $\nu' \ge \nu$, because, by assumption, $|x - y| < d_\nu$. Hence $|\sigma(x) - \sigma(y)| \in V$ (because $j_{\nu'} \le k_{\nu'}$ for all $\nu'$), as required.

Now, let $x < y$ be as above (and of type 2) but $|x - y| \notin U$. The set $D'' = \{d_\nu \in D : \alpha_\nu \ne \gamma_\nu\}$ is internal; therefore, it has a greatest element $d_{\nu''} = \max D''$. Moreover, $d_{\nu''} \notin U$. (Otherwise,

$|x-y| \in U$; see above.) Thus, $\alpha_{\nu''} < \gamma_{\nu''}$ (because $x < y$). In this case, the only possibility for the element $|\sigma(x) - \sigma(y)| \in V$ is the existence of an index $\nu' < \nu''$ such that $z_{\nu'} \in V$ and $\gamma_\nu = 0$ and $\alpha_\nu = j_\nu - 1 = k_\nu - 1$ for all $\nu$ between $\nu'$ and $\nu''$. This contradicts the assumption that $x$ is of type 2. Thus, $|\sigma(x) - \sigma(y)| \notin V$, as required.

It remains to check that $\rho$ is a $\kappa$-determined map. Note that $\sigma$ is, obviously, even an internal function; so, it remains to consider the map $x \mapsto \bar{x}$. Of course, it is not internal, but the formulas expressing the statements that $x$ is a number of type 1 and $y = \bar{x}$ can be represented as propositional internal relations with $\le \kappa$ operations of union and intersection (because $U$ has a cofinal subsequence with $\le \kappa$ terms), after which the result follows from Lemma 11.

(v) Now, suppose that $\operatorname{coi} U = \operatorname{coi} V = \vartheta \le \kappa$ and $\operatorname{wid} U \subseteq \operatorname{wid} V$. Let us prove that $\mathrm{M}_U \le_\kappa \mathrm{M}_V$. Choose decreasing coinitial sequences $\{u_\xi\}_{\xi<\vartheta}$ and $\{v_\xi\}_{\xi<\vartheta}$ in ${}^*\mathbb{N} \setminus U$ and ${}^*\mathbb{N} \setminus V$, respectively. It can be assumed that all terms $u_\xi$ and $v_\xi$ in these sequences are hyperinteger powers of the number 2 in ${}^*\mathbb{N}$. The relation $\operatorname{wid} U \subseteq \operatorname{wid} V$ means in this case that

$$\forall \eta \quad \exists \xi \quad \forall \xi' > \xi \quad \exists \eta' > \eta \qquad \left(\frac{u_{\xi'}}{u_\xi} \ge \frac{v_{\eta'}}{v_\eta}\right).$$

As in the proof of (iv), we can separate out unbounded subsequences in $\{u_\xi\}_{\xi<\vartheta}$ and $\{v_\eta\}_{\eta<\vartheta}$ so that, after renumbering, we obtain

$$\forall \xi < \eta < \vartheta \qquad \left(\frac{v_\xi}{u_\xi} \ge \frac{v_\eta}{u_\eta}, \quad \text{i.e.,} \quad \frac{u_\eta}{u_\xi} \ge \frac{v_\eta}{v_\xi}\right). \tag{18}$$

Again, the function $f(u_\xi) = v_\xi$ can be extended to an internal function $\varphi$ satisfying (17), and so on (as above).

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