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Effective Compactness and Sigma-Compactness

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Abstract—Using the Gandy–Harrington topology and other methods of effective descriptive set theory, we prove several theorems about compact and σ-compact sets. In particular, it is proved that any Δ^1_1-set A in the Baire space $\mathscr{N}$ either is an at most countable union of compact Δ^1_1-sets (and hence is σ-compact) or contains a relatively closed subset homeomorphic to $\mathscr{N}$ (in this case, of course, A cannot be σ-compact).

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1. INTRODUCTION

Effective descriptive set theory emerged in the 1950s mainly as a tool for improving and simplifying constructions of classical descriptive theory and as a mechanism for applying some theorems of recursion theory to problems of descriptive set theory. However, it soon turned out that effective descriptive set theory itself leads to setting and solving problems which have no counterparts in classical descriptive theory. An example is the following *basis theorem*: Any countable Δ^1_1-set A in the Baire space $\mathscr{N} = \mathbb{N}^{\mathbb{N}}$ consists of Δ^1_1-points. A vague analog of this theorem in classical descriptive theory is the Luzin–Novikov splitting theorem for plane Borel sets with countable sections.

In this paper, we apply methods of effective descriptive set theory to study the compactness and σ-compactness of sets $A \subseteq \mathscr{N}$. Our main results are as follows.

Theorem 1.1. *For any Δ^1_1-set $A \subseteq \mathscr{N}$, precisely one of the following two conditions holds*:

I) *A coincides with the union U of all sets of the form $[T]$, where $T \subseteq \mathbb{N}^{<\omega}$ is a compact Δ^1_1-tree and $[T] \subseteq A$;*[1]

II) *there is a set $Y \subseteq A$ homeomorphic to $\mathscr{N}$ and closed in A.*

Moreover, in case (I)*, if $A \neq \varnothing$, then A contains a Δ^1_1-point.*

Obviously, conditions (I) and (II) in this theorem contradict each other, because any relatively closed subset of a σ-compact set is itself σ-compact, while the space $\mathscr{N}$ and, therefore, the set Y in (II), which is homeomorphic to $\mathscr{N}$, are not σ-compact.

Theorem 1.2. *For any Σ^1_1-set $A \subseteq \mathscr{N}$, precisely one of the following two conditions holds*:

I) *A is contained in the union U of all sets of the form $[T]$, where $T \subseteq \mathbb{N}^{<\omega}$ is a compact Δ^1_1-tree;*[2]

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[1] In this case, the set A is σ-compact, because each effective class (in particular, Δ^1_1) is at most countable.
[2] The set U is σ-compact for the same reason as in condition (I) in Theorem 1.1.

II) *there is a set $Y \subseteq A$ closed in $\mathscr{N}$ and homeomorphic to $\mathscr{N}$.*

Conditions (I) and (II) in this theorem also contradict each other; otherwise, Y would be a closed subset of the σ-compact set U, which is impossible, as mentioned above.

Note that condition (I) in Theorem 1.2 is weaker than condition (I) in Theorem 1.1, and condition (II) in Theorem 1.2 is stronger than condition (II) in Theorem 1.1. Theorem 1.1 in which (II) is replaced by the stronger condition (II) of Theorem 1.2 is no longer valid. Indeed, suppose that A consists of all points $a \in 2^{\mathbb{N}}$ for which the set $\{k : a(k) = 1\}$ is infinite. Then A is not a σ-compact Δ^1_1-set, but it is covered by the compact set $2^{\mathbb{N}}$; hence there are no sets $X \subseteq A$ closed in $\mathscr{N}$ and homeomorphic to $\mathscr{N}$. Thus, A satisfies neither condition (I) in Theorem 1.1 nor condition (II) in Theorem 1.2.

On the other hand, Theorem 1.1 in which condition (I) is replaced by the weaker condition (I) of Theorem 1.2 becomes false as well; indeed, there exists a set A (the same as in the preceding paragraph) which satisfies both condition (I) in Theorem 1.2 and condition (II) in Theorem 1.1. This set itself is homeomorphic to $\mathscr{N}$ by means of the map taking each point $x \in \mathscr{N}$ to the point $a = f(x) \in A$ chosen so that, for any n, $x(n)$ is equal to the number of zeros between the nth and the $(n+1)$th 1 in the sequence a.

Corollary 1.3. *Any σ-compact Δ^1_1-set $A \subseteq \mathscr{N}$ is a countable union of compact Δ^1_1-sets; if such a set is nonempty, then it contains a Δ^1_1-point.*

Corollary 1.4. *If a Δ^1_1-set $A \subseteq \mathscr{N}$ is not σ-compact, then it contains a subset homeomorphic to $\mathscr{N}$ and closed in A. If a Σ^1_1-set $A \subseteq \mathscr{N}$ is not covered by a σ-compact set, then it contains a subset homeomorphic to $\mathscr{N}$ and closed in $\mathscr{N}$.*

As usual, the theorems and corollaries remain valid in relativized form, i.e., when the classes Δ^1_1 and Σ^1_1 are replaced by classes of the form $\Delta^1_1(p)$ and $\Sigma^1_1(p)$, respectively, where $p \in \mathscr{N}$ is any fixed parameter; the proofs of the relativized statements are the same. We do not dwell on this generalization.

The proofs of both theorems are based on methods of effective descriptive set theory; namely, they use the Gandy–Harrington topology, Kreisel's choice theorem, and an effective enumeration of Δ^1_1-sets.

2. BASIC DEFINITIONS AND RESULTS

We use the standard notations Σ^1_1, Π^1_1, and Δ^1_1 for the effective projective hierarchy classes of points and subsets of the Baire space $\mathscr{N}$ and denote the corresponding projective classes by $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$, and $\boldsymbol{\Delta}^1_1$; for details, see, e.g., [1, Chap. 8] and [2]–[4].

By $\mathbb{N}^{<\omega}$ we denote the set of all finite tuples (sequences) of positive integers, including the empty tuple Λ. The length of a tuple $s \in \mathbb{N}^{<\omega}$ is denoted by $\operatorname{lh} s$. The concatenation of tuples $s \in \mathbb{N}^{<\omega}$ and $n \in \mathbb{N}$ is denoted by $s^\wedge n$. For $s \in \mathbb{N}^{<\omega}$, we set

$$\mathscr{N}_s = \{x \in \mathscr{N} : s \subset x\} \qquad \text{(this is a \textit{Baire interval} in } \mathscr{N}).$$

A set $T \subseteq \mathbb{N}^{<\omega}$ is called a *tree* if $s \in T$ whenever $s^\wedge n \in T$ for at least one n. Clearly, any nonempty tree contains Λ. A tree $T \subseteq \mathbb{N}^{<\omega}$ is said to be *compact* if

1) it has no *pendant vertices*, i.e., if $s \in T$, then $s^\wedge n \in T$ for at least one n;

2) it is *finitely branched*, i.e., if $s \in T$, then $s^\wedge n \in T$ only for finitely many n.

A tree T is compact if and only if so is the set

$$[T] = \{x \in \mathscr{N} : \forall m (x \restriction m \in T)\}.$$

Given any sets $\mathbb{X}$, $\mathbb{Y}$, and $P \subseteq \mathbb{X} \times \mathbb{Y}$, we define the *projection* of P on $\mathbb{X}$ and the *x-section* of P for $x \in \mathbb{X}$ as

$$\operatorname{pr} P = \{x \in \mathbb{X} : \exists y(\langle x, y\rangle \in P)\} \qquad \text{and} \qquad (P)_x = \{y \in \mathbb{Y} : \langle x, y\rangle \in P\},$$

respectively. We say that a set $P \subseteq \mathbb{X} \times \mathbb{Y}$ is *single-valued* if each of its sections $(P)_x$ (where $x \in \mathbb{X}$) contains at most one point; we say that P is *countable-valued* if each section $(P)_x$ is at most countable.

The proofs of our main theorems use the following well-known results of effective descriptive set theory (see [5] and [6] for details).

Theorem 2.1 (on separation). *If Σ^1_1-sets $X, Y \subseteq \mathscr{N}$ are disjoint, then there exists a Δ^1_1-set $Z \subseteq \mathscr{N}$ such that $X \subseteq Z$ and $Y \cap Z = \varnothing$.*

Theorem 2.2 (Kreisel's choice theorem). *If $P \subseteq \mathscr{N} \times \mathbb{N}$ is a Π^1_1-set and its projection $\operatorname{pr} P$ is a Δ^1_1-set, then there exists a Δ^1_1-function $f\colon \operatorname{pr} P \to \mathbb{N}$ such that*

$$\langle x, f(x)\rangle \in P \qquad \text{for all} \quad x \in \operatorname{pr} P.$$ [3]

The following theorem provides a convenient effective enumeration of all Δ^1_1-sets $X \subseteq \mathbb{N}^{<\omega}$. Actually, the set $\mathbb{N}^{<\omega}$ of all (finite) tuples of positive integers can be identified with $\mathbb{N}$ by means of any recursive bijection between $\mathbb{N}$ and $\mathbb{N}^{<\omega}$.

Theorem 2.3 (on enumeration of Δ^1_1-sets). *There exist Π^1_1-sets $E \subseteq \mathbb{N}$ and $W \subseteq \mathbb{N} \times \mathbb{N}^{<\omega}$ and a Σ^1_1-set $W' \subseteq \mathbb{N} \times \mathbb{N}^{<\omega}$ satisfying the following conditions*:

i) *if $e \in E$, then the sets*

$$(W)_e = \{s \in \mathbb{N}^{<\omega} : \langle e, s\rangle \in W\} \qquad \text{and} \qquad (W')_e$$

coincide;

ii) *$T \subseteq \mathbb{N}^{<\omega}$ is a Δ^1_1-set if and only if there exists a number $e \in E$ for which*

$$T = (W)_e = (W')_e.$$

Corollary 2.4. *The sets*

$$D = \{T \subseteq \mathbb{N}^{<\omega} : T \text{ is } \Delta^1_1\}$$

and

$$\{\langle p, T\rangle : p \in \mathscr{N} \wedge T \subseteq \mathbb{N}^{<\omega} \wedge T \text{ is } \Delta^1_1(p)\}$$

belong to the class Π^1_1.

Proof. Take the same E, W, and W' as in Theorem 2.3. We have

$$\begin{aligned} T \in D &\iff \exists e(e \in E \wedge T = (W)_e) \\ &\iff \exists e\big(e \in E \wedge \forall s \in \mathbb{N}^{<\omega}(s \in (W')_e \implies s \in T \implies s \in (W)_e)\big), \end{aligned}$$

as required. □

Theorem 2.5. *If a Δ^1_1-set $P \subseteq \mathscr{N} \times \mathscr{N}$ is countable-valued, then*

i) $\operatorname{pr} P$ *is a Δ^1_1-set*;

ii) *P is a countable union of single-valued Δ^1_1-sets*;

iii) *P is uniformized by a Δ^1_1-set.*

Theorems 2.1–2.3 and 2.5 and Corollary 2.4 remain true under the replacement in their statements of the classes Σ^1_1, Π^1_1, and Δ^1_1 by the relativized classes $\Sigma^1_1(p)$, $\Pi^1_1(p)$, and $\Delta^1_1(p)$, where $p \in \mathscr{N}$ is any parameter. Therefore, Theorems 2.1, 2.2, and 2.5 are valid for the projective classes $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$, and $\boldsymbol{\Delta}^1_1$.

[3] In this situation, we say that the set P is *uniformized* by the function f.

3. THE GANDY–HARRINGTON TOPOLOGY

The Gandy–Harrington topology on the Baire space $\mathscr{N}$ consists of all unions of Σ^1_1-sets $S \subseteq \mathscr{N}$. This topology contains the Polish topology of $\mathscr{N}$, but it is neither Polish nor even metrizable.

Indeed, there exists a Π^1_1-set $P \subseteq \mathscr{N}$ which is not a $\mathbf{\Sigma}^1_1$-set. By definition, P is closed in the Gandy–Harrington topology. Suppose that this topology is metrizable. In metric spaces, all closed subsets are of type $\mathbf{G}_\delta$. Hence

$$P = \bigcap_n \bigcup_m S_{mn},$$

where all $S_{mn} \subseteq \mathscr{N}$ are Σ^1_1-sets (i.e., base open sets in the Gandy–Harrington topology). But the class $\mathbf{\Sigma}^1_1$ is closed with respect to countable unions and intersections. Thus, P turns out to be a $\mathbf{\Sigma}^1_1$-set, which contradicts the assumption.

Nevertheless, although the Gandy–Harrington topology itself is not Polish, it has a property which makes it possible to obtain results typical of Polish spaces.

Definition 3.1. A *Polish net* for a family of sets $\mathscr{F}$ is a family $\{\mathscr{D}_n : n \in \mathbb{N}\}$ of open dense sets $\mathscr{D}_n \subseteq \mathscr{F}$ such that $\bigcap_n F_n \neq \varnothing$ for any sequence of sets $F_n \in \mathscr{D}_n$ with the finite intersection property, i.e., satisfying the condition

$$\bigcap_{k \le n} F_k \neq \varnothing \qquad \text{for each } n.$$

Recall that a set $\mathscr{D} \subseteq \mathscr{F}$ is said to be *open dense* if

$$\forall F \in \mathscr{F}\ \exists D \in \mathscr{D}\ (D \subseteq F)$$

and

$$\forall F \in \mathscr{F}\ \forall D \in \mathscr{D}\ (F \subseteq D \implies F \in \mathscr{D}).[4]$$

For example, the family of all nonempty closed subsets in any complete metric space $\mathbb{X}$ has a Polish net: for $\mathscr{D}_n$ one can take the set of all nonempty closed sets of diameter $\le n^{-1}$ in $\mathbb{X}$. The following theorem is less elementary. Both the theorem and its corollary are well known; see, e.g., [7], [8], or [9].

Theorem 3.2. *The family $\mathbb{P}$ of all nonempty Σ^1_1-sets in the space $\mathscr{N}$ has a Polish net.*

Corollary 3.3. *The set $\mathscr{N}$ with the Gandy–Harrington topology has the Baire property, i.e., all comeager*[5] *sets are dense in this space.*

Proof of Theorem 3.2 (sketch). Let $P \subseteq \mathscr{N} \times \mathscr{N}$. We set

$$\operatorname{pr} P = \{a : \exists b\, P(a,b)\}$$

(this is the projection of P) and

$$P_{st} = \{\langle a,b\rangle \in P : s \subset a \wedge t \subset b\} \qquad \text{for} \quad s,t \in \mathbb{N}^{<\omega}.$$

By $\mathscr{D}(P,s,t)$ we denote the family of all nonempty Σ^1_1-sets $X \subseteq \mathscr{N}$ such that

$$\text{either} \quad X \cap \operatorname{pr} P_{st} = \varnothing \quad \text{or} \quad X \subseteq \operatorname{pr} P_{s^\wedge i, t^\wedge j} \qquad \text{for some} \quad i,j \in \mathbb{N}.$$

(Note that, in the latter case, the number i is unique, but j may be nonunique.) Since the effective class Π^0_1 is countable, we can choose an enumeration $\{\mathscr{D}_n : n \in \mathbb{N}\}$ of all families of the form $\mathscr{D}(P,s,t)$, where $P \subseteq \mathscr{N} \times \mathscr{N}$ is a Π^0_1-set. Let us prove that all $\mathscr{D}_n$ form a Polish net for $\mathbb{P}$.

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[4] A set $\mathscr{D}$ satisfying only the first condition is said to be *dense*. If a set $\mathscr{D} \subseteq \mathscr{F}$ is dense, then the set defined as $\mathscr{D}' = \{F \in \mathscr{F} : \exists D \in \mathscr{D}(F \subseteq D)\}$ is *open dense*. The openness and density defined as above can be associated with a certain topology on $\mathscr{F}$, but we regard them as purely combinatorial properties here.

[5] We use the term *comeager* for sets complementary to meager (first-category) sets in a given topological space.

It is easy to show that $\mathscr{D}(P,s,t)$ is an open dense set in $\mathbb{P}$ in the sense of Definition 3.1, whatever the Π^0_1-set $P \subseteq \mathscr{N} \times \mathscr{N}$. Thus, all of the sets $\mathscr{D}_n$ are open and dense in $\mathbb{P}$.

Consider a sequence of nonempty Σ^1_1-sets $X_n \in \mathscr{D}_n$ with the finite intersection property. Let us prove that $\bigcap_n X_n \neq \varnothing$. The sets

$$Y_n = \bigcap_{k \le n} X_k$$

are nonempty; obviously, they belong to Σ^1_1 (as well as the sets X_k) and satisfy the condition $Y_n \subseteq X_n$. Therefore, we have $Y_n \in \mathscr{D}_n$, because the $\mathscr{D}_n$ are open dense. Finally, the sets Y_n not only have the finite intersection property but also are contained in each other, i.e., $Y_{n+1} \subseteq Y_n$ for all n. It suffices to prove that $\bigcap_n Y_n \neq \varnothing$.

We say that a set $X \subseteq \mathscr{N}$ is *positive* if there exists a subscript m for which $Y_m \subseteq X$. For every n, we fix a Π^0_1-set

$$P^n \subseteq \mathscr{N} \times \mathscr{N} \qquad \text{for which} \quad Y_n = \operatorname{pr} P^n.$$

If $s,t \in \mathbb{N}^{<\omega}$ and the projection $\operatorname{pr} P^n_{st}$ is positive, then, by the construction of the sets Y_n, there exists a unique i and some j for which the projection $\operatorname{pr} P^n_{s^\wedge i, t^\wedge j}$ is positive too. Indeed, suppose that $Y_m \subseteq \operatorname{pr} P^n_{st}$ (the existence of such an m follows from positivity). The family $\mathscr{D}(P^n,s,t)$ coincides with some $\mathscr{D}_k$; hence there exists a subscript k for which $Y_k \in \mathscr{D}(P^n,s,t)$. Thus,

$$\text{either} \quad Y_k \cap \operatorname{pr} P^n_{st} = \varnothing, \qquad \text{or} \quad Y_k \subseteq \operatorname{pr} P^n_{s^\wedge i, t^\wedge j} \quad \text{for some} \quad i,j \in \mathbb{N}.$$

The former case cannot occur, because $Y_m \subseteq \operatorname{pr} P^n_{st}$ and the sets Y_j are contained in each other. Therefore, $Y_k \subseteq \operatorname{pr} P^n_{s^\wedge i, t^\wedge j}$ for some $i,j \in \mathbb{N}$, and the projection $P^n_{s^\wedge i, t^\wedge j}$ is positive. It follows that there exists a unique point $a = a_n \in \mathscr{N}$ and a (not necessarily unique) point $b = b_n \in \mathscr{N}$ for which the projections $\operatorname{pr} P^n_{a\restriction k, b \restriction k}$ are positive for all k. The the fact that the sets P^n are closed implies $\langle a_n, b_n \rangle \in P^n$; hence $a_n \in X_n$ for each n.

It remains to prove that the points a_n coincide, i.e., $a_m = a_n$ even if $m \neq n$. To this end, we simply note that if projections $\operatorname{pr} P_{st}$ and $\operatorname{pr} Q_{s't'}$ are both positive (even for two different sets P and Q!), then, by virtue of the same finite intersection property, we have either $s \subseteq s'$ or $s' \subseteq s$. $\square$

4. COMPACT Δ^1_1-SETS

Clearly, if $T \subseteq \mathbb{N}^{<\omega}$ is a Δ^1_1-tree, then

$$[T] = \{a \in \mathscr{N} : \forall m (a \restriction m \in T)\}$$

is a Δ^1_1-set, because

$$x \in [T] \iff \forall m (x \restriction m \in T).$$

Lemma 4.1. *If $F \subseteq \mathscr{N}$ is a closed Δ^1_1-set and $A \subseteq F$ is a compact Σ^1_1-set, then there exists a compact Δ^1_1-tree $T \subseteq \mathbb{N}^{<\omega}$ for which $A \subseteq [T] \subseteq F$. In the special case $A = F$, the compactness of the Δ^1_1-set $A \subseteq \mathscr{N}$ implies the existence of a compact Δ^1_1-tree $T \subseteq \mathbb{N}^{<\omega}$ for which $A = [T]$.*

Proof. *Part* 1. First, let us try to find a Δ^1_1-tree S (not necessarily compact) for which $F = [S]$. Since the complementary Δ^1_1-set $G = \mathscr{N} \setminus F$ is open, it follows that, for

$$P = \{\langle x,t \rangle : x \in G \wedge t \in \mathbb{N}^{<\omega} \wedge x \in \mathscr{N}_t \subseteq G\} \subseteq \mathscr{N} \times \mathbb{N}^{<\omega},$$

we have $\operatorname{dom} P = G$. Moreover, P is a Π^1_1-set, because the key relation $\mathscr{N}_t \subseteq G$ can be expressed by the Π^1_1-formula

$$\forall x (t \subset x \implies x \in G).$$

Therefore, Theorem 2.2 gives a Δ^1_1-function $f : G \to \mathbb{N}^{<\omega}$ such that $x \in \mathscr{N}_{f(x)} \subseteq G$ for all $x \in G$. The set

$$U = \{f(x) : x \in G\} = \{t \in \mathbb{N}^{<\omega} : \exists x \in G\,(f(x) = t)\}$$

belongs to Σ^1_1 and satisfies the condition $G = \bigcup_{t\in U} \mathscr{N}_t$. However,

$$V = \{t \in \mathbb{N}^{<\omega} : \mathscr{N}_t \subseteq G\} = \{t \in \mathbb{N}^{<\omega} : \forall x(x \in \mathscr{N}_t \implies x \in G)\}$$

is a Π^1_1-set; moreover, $U \subseteq V$ and $G = \bigcup_{t\in V} \mathscr{N}_t$. By Theorem 2.1 (on separation), there exists a Δ^1_1-set W for which $U \subseteq W \subseteq V$; we then still have $G = \bigcup_{t\in W} \mathscr{N}_t$. Let

$$S = \{s \in \mathbb{N}^{<\omega} : \forall t\,(t \in W \implies t \not\subseteq s)\}.$$

It is easy to see that S is a Δ^1_1-tree (which may have pendant vertices) and $[S] = F$; this completes the first part of the proof.

Part 2. Note that the set P of all pairs $\langle s, u\rangle$ for which $s \in \mathbb{N}^{<\omega}$, $u \subseteq \mathbb{N}$ is finite and nonempty, and

$$\forall x \in \mathscr{N}\big((x \in A \wedge s \subset x) \implies \exists k \in u(s^\wedge k \subset x) \wedge \forall k \in u(s^\wedge k \in S)\big)$$

is a Π^1_1-set in the space $\mathbb{N} \times \mathscr{P}_{\text{fin}}\mathbb{N}$, where the second multiplier (the set of all finite $u \subseteq \mathbb{N}$) is identified with $\mathbb{N}$ by means of a suitable recursive bijection. Moreover, $\operatorname{dom} P = \mathbb{N}^{<\omega}$. (Indeed, if there is no $x \in A$ for which $s \subset x$, then $\langle s, u\rangle \in P$ for any finite u, and if such a point x exists, then the set

$$u = \{x(n) : s \subset x \in A\}, \qquad \text{where} \quad n = \operatorname{lh} s,$$

is finite by compactness, and we take this set.) Thus, we again have a Δ^1_1-function $f\colon \mathbb{N}^{<\omega} \to \mathscr{P}_{\text{fin}}\mathbb{N}$ such that $\langle s, f(s)\rangle \in P$ for all $s \in \mathbb{N}^{<\omega}$. It remains to set

$$T = \{s \in \mathbb{N}^{<\omega} : \forall n < \operatorname{lh} s(s(n) \in f(s \restriction n))\}. \quad \square$$

Corollary 4.2. *Any compact Δ^1_1-set $A \subseteq \mathscr{N}$ contains a Δ^1_1-point $x \in A$.*

Proof. By Lemma 4.1, we have $A = [T]$ for a suitable compact Δ^1_1-tree $T \subseteq \mathbb{N}^{<\omega}$. For x we take the lexicographically leftmost branch of the tree T, so that, for each n, we have

$$x(n) = \min\big\{s(n) : s \in T \wedge n < \operatorname{lh} s \wedge s \restriction n = \langle x(0), \dots, x(n-1)\rangle\big\}. \quad \square$$

5. PROOF OF THE FIRST MAIN THEOREM

We start the proof of Theorem 1.1 by showing that the set U in condition (I) of Theorem 1.1 (i.e., the union of all compact Δ^1_1-sets $K \subseteq A$) is of class Π^1_1. Take the same Π^1_1-sets $E \subseteq \mathbb{N}$ and $W \subseteq \mathbb{N} \times \mathbb{N}^{<\omega}$ and the Σ^1_1-set $W' \subseteq \mathbb{N} \times \mathbb{N}^{<\omega}$ as in Theorem 2.3. We have

$$x \in U \iff \exists e\big(e \in E \wedge (W)_e \text{ is a compact tree} \wedge x \in [(W)_e] \subseteq A\big).$$

The property of being a compact tree is expressed by a boringly long arithmetic formula; it would occupy several lines here, and we do not write it out. The inclusion $[(W)_e] \subseteq A$ is expressed by the Π^1_1-formula

$$\forall y(\forall n(y \restriction n \in (W)_e) \implies y \in A),$$

where $y \restriction n \in (W)_e$ can be replaced by $\langle e, y \restriction n\rangle \in W$. We have a Π^1_1-formula prefixed by the arithmetic quantifier $\exists e$, which is again a Π^1_1-formula. Thus, $U \in \Pi^1_1$. This means that $A' = A \setminus U$ is a Σ^1_1-set. It remains to prove that if the set A' is nonempty, then it contains a set $Y \subseteq A'$ homeomorphic to $\mathscr{N}$ and closed in A'.

Lemma 5.1. *The set A' has no nonempty closed σ-compact Σ^1_1-subsets.*

Proof. First, let us prove that A' has no nonempty compact Σ^1_1-subsets. Suppose that, on the contrary, $\varnothing \ne Z \subseteq A'$ is a compact Σ^1_1-set. If F is a closed Δ^1_1-set for which $Z \subseteq F \subseteq A$, then, by Lemma 4.1, there exists a compact Δ^1_1-tree $T \subseteq \mathbb{N}^{<\omega}$ satisfying the condition

$$Z \subseteq [T] \subseteq F \subseteq A.$$

We have $[T] \subseteq U$ by the definition of U, and hence $Z \subseteq U$, which is impossible.

Since the complement $C = \mathscr{N} \setminus Z$ is open, it follows that the set

$$H = \{\langle x, s\rangle : x \in C \cap \mathscr{N}_s \wedge \mathscr{N}_s \cap Z = \varnothing\}$$

belongs to the class Π^1_1 and satisfies the condition $\operatorname{pr} H = C$. In particular, the Δ^1_1-set $D = \mathscr{N} \setminus A$ is contained in $\operatorname{pr} H$. This means that the narrower Π^1_1-set

$$H' = \{\langle x, s\rangle \in H : x \in D\}$$

satisfies the condition $\operatorname{pr} H' = D$. By Theorem 2.2, there exists a Δ^1_1-function $f\colon D \to \mathbb{N}^{<\omega}$ for which

$$x \in D \implies \langle x, f(x)\rangle \in H;$$

in other words, we have

$$x \in \mathscr{N}_{f(x)} \subseteq C \qquad \text{for all} \quad x \in D.$$

Thus,

$$\Sigma = \operatorname{ran} f = \{f(x) : x \in D\} \subseteq \mathbb{N}^{<\omega}$$

is a Σ^1_1-set, and

$$D \subseteq \bigcup_{s\in\Sigma} \mathscr{N}_s \subseteq C.$$

However,

$$\Pi = \{s \in \mathbb{N}^{<\omega} : \mathscr{N}_s \subseteq C\}$$

is a Π^1_1-set, and $\Sigma \subseteq \Pi$. Therefore, by the Σ^1_1-separation theorem (Theorem 2.1), there exists a Δ^1_1-set Δ for which $\Sigma \subseteq \Delta \subseteq \Pi$. We still have $D \subseteq \bigcup_{s\in\Delta} \mathscr{N}_s \subseteq C$, so that the closed set

$$F = \mathscr{N} \setminus \bigcup_{s\in\Delta} \mathscr{N}_s$$

satisfies the relations $Z \subseteq F \subseteq A$. However, $x \in F$ is equivalent to

$$\forall s (s \in \Delta \implies x \notin \mathscr{N}_s);$$

hence F, as well as Δ, is a Δ^1_1-set, as required.

Now, let us complete the proof of the lemma. Suppose that, on the contrary, a nonempty closed Σ^1_1-set $F = \bigcup_n F_n \subseteq A'$ is σ-compact and all of the F_n are compact. Then there exists a base clopen set $U \subseteq \mathscr{N}$ for which the intersection $Z = U \cap F$ is nonempty and is entirely contained in one of the sets F_n. Thus, Z is a nonempty compact Σ^1_1-set, which contradicts the above considerations. This completes the proof of the lemma. □

Returning to the theorem, we suppose that condition (I) in Theorem 1.1 does not hold, so that the Σ^1_1-set $A' \subseteq A$ is nonempty, and derive condition (II) in Theorem 1.1 from this assumption. The following two cases are possible.

Case 1. There exists a nonempty closed Σ^1_1-set $F \subseteq A'$. It is not σ-compact by Lemma 5.1, and a simple construction yields at once a closed set $Y \subseteq F$ homeomorphic to $\mathscr{N}$.

Case 2. There are no nonempty closed Σ^1_1-sets $F \subseteq A'$. To obtain the required *relatively* closed set $Y \subseteq A$ homeomorphic to $\mathscr{N}$ under this assumption, we construct a system of nonempty Σ^1_1-sets $Y_s \subseteq A'$ satisfying the following technical conditions:

1) $Y_{s^\wedge i} \subseteq Y_s$ whenever $s \in \mathbb{N}^{<\omega}$ and $i \in \mathbb{N}$;

2) the diameter of each Y_s is at most $2^{-\operatorname{lh} s}$;

3) $Y_{s^\wedge k} \cap Y_{s^\wedge n} = \varnothing$ for all s and $k \neq n$; moreover, each $Y_{s^\wedge k}$ is contained in an open set $Y'_{s^\wedge k}$ disjoint from $\bigcup_{n\neq k} Y_{s^\wedge n}$;

4) $Y_s \in \mathscr{D}_{\operatorname{lh} s}$, where, by Theorem 3.2, $\{\mathscr{D}_n : n \in \mathbb{N}\}$ is a Polish net for the family $\mathbb{P}$ of all nonempty Σ^1_1-sets $Y \subseteq \mathscr{N}$;

5) if $s \in \mathbb{N}^{<\omega}$ and $x_k \in Y_{s^\wedge k}$ for all $k \in \mathbb{N}$, then the sequence of points x_k converges to a point of $\mathscr{N} \setminus A'$.

If such a system is constructed, then condition (4) ensures the nonemptiness of each intersection $\bigcap_m Y_{a\restriction m}$ for $a \in \mathscr{N}$ by the definition of a Polish net. According to (2), any such intersection contains precisely one point, which we denote by $f(a)$; it is easy to see that the map

$$f\colon \mathscr{N} \xrightarrow{\text{on}} Y = \operatorname{ran} f = \{f(a) : a \in \mathscr{N}\}$$

is a homeomorphism. Note that

$$Y = \bigcup_{a\in\mathscr{N}} \bigcap_m Y_{a\restriction m}.$$

To prove that Y is closed in A', take $y \in \overline{Y}$ ($\overline{Y}$ is the closure of Y). We have $y \in \overline{Y}_\Lambda$, because $Y \subseteq Y_\Lambda$. Suppose that $s \in \mathbb{N}^{<\omega}$ and $y \in \overline{Y}_s$. By assumption, we have

$$y \in \overline{\bigcup_k Y_{s^\wedge k}}.$$

Therefore, according to (5), either $y \notin A'$ or $y \in \overline{Y_{s^\wedge k}}$ for some (in fact, unique) k. If follows that if $y \in \overline{Y} \cap A'$, then there exists a point $a \in \mathscr{N}$ such that $y \in \overline{Y_{a\restriction m}}$ for all m. Condition (2) implies $y = f(a)$, so that $y \in Y$, as required.

Finally, we describe the construction of the sets Y_s.

If we already have a nonempty Σ^1_1-set $Y_s \subseteq \mathscr{N}$, then its closure $\overline{Y_s}$ is a Σ^1_1-set as well, which implies $\overline{Y_s} \not\subseteq A'$ in case 2. Take any point $y \in \overline{Y_s} \setminus A'$ and a sequence of pairwise different points $y_n \in Y_s$ converging to y. For each n, let U_n be a base neighborhood (i.e., a Baire interval) of the point y_n of diameter not exceeding one third of the least distance from y_n to a point y_k with $k \neq n$. We set $Y_{s^\wedge n} = Y_s \cap U_n$; then, we replace the sets $Y_{s^\wedge n}$ by smaller Baire intervals so as to satisfy conditions (2) and (4). This completes the proof of the theorem.

6. PROOF OF THE SECOND MAIN THEOREM

In this section, we prove Theorem 1.2. The set U in (I) is of class Π^1_1; this is easy to see from Corollary 2.4. Thus, the difference $A \setminus U$ is a Σ^1_1-set.

Lemma 6.1. *Under the assumptions of Theorem* 1.2*, the topological closure in* $\mathscr{N}$ *of any nonempty* Σ^1_1*-set* $Y \subseteq A \setminus U$ *is noncompact, i.e., the tree*

$$T(Y) = \{y \restriction n : y \in Y \wedge n \in \mathbb{N}\}$$

has at least one infinite branching.

Proof. Assume the contrary. The set

$$H = \{\langle t, n\rangle : t \in \mathbb{N}^{<\omega} \wedge n \in \mathbb{N} \wedge \forall k(t^\wedge k \in T(Y) \implies k \le n)\}$$

is of class Π^1_1 (because the Σ^1_1-fragment of $T(Y)$ is on the left of the implication sign), and $\mathrm{dom} H = \mathbb{N}^{<\omega}$. Theorem 2.2 implies the existence of a Δ^1_1-function $f\colon \mathbb{N}^{<\omega} \to \mathbb{N}$ such that

$$\langle t, f(t)\rangle \in H \qquad \text{for all} \quad t \in \mathbb{N}^{<\omega}.$$

By the definition of H, we have

$$y(n) \le f(y \restriction n) \qquad \text{for all} \quad y \in Y \text{ and } n,$$

so that $Y \subseteq [T']$, where T' is the tree of all tuples $t \in \mathbb{N}^{<\omega}$ satisfying the condition

$$t(n) \le f(t \restriction n) \qquad \text{for each} \quad n < \mathrm{lh}\, t.$$

The tree T' is compact, and it is easy to see that T', as well as f, is of class Δ^1_1. Therefore, $[T'] \subseteq U$, which is impossible. $\square$

Returning to the proof of Theorem 1.2, suppose that (I) does not hold, i.e., the set $A \setminus U$ is nonempty. Under this assumption, there exists a system nonempty Σ^1_1-sets $Y_s \subseteq A \setminus U$ satisfying the technical conditions (1)–(4) in Sec. 5 and the following condition instead of (5):

(5′) if $s \in \mathbb{N}^{<\omega}$ and $x_k \in Y_{s^\wedge k}$ for all $k \in \mathbb{N}$, then the sequence of x_k has no convergent subsequences in $\mathscr{N}$.

If such a system is constructed, then, for the same reasons as in the proof of Theorem 1.1, the associated function $f\colon \mathscr{N} \to A \setminus U$ is one-to-one; moreover, this is a homeomorphism from $\mathscr{N}$ onto its complete image

$$Y = \mathrm{ran}\, f = \{f(a) : a \in \mathscr{N}\} \subseteq A \setminus U.$$

The verification of the fact that Y is absolutely closed in $\mathscr{N}$ is similar to that of the relative closedness in the proof of Theorem 1.1; note that condition (5′) is stronger. Thus, we obtain (II).

The sets Y_s are constructed as follows. If a Σ^1_1-set $Y_s \subseteq A \setminus U$ is already constructed, then, according to Lemma 6.1, we can choose $t \in T(Y_s)$ so that $t^\wedge k \in T(Y_s)$ for all k from an infinite set $K_s \subseteq \mathbb{N}$. Thus, there is a sequence of pairwise different points $y_k \in Y_s$, where $k \in \mathbb{N}$, which has no convergent subsequences. We cover these points by sufficiently small Baire intervals U_k so that the resulting Σ^1_1-sets $Y_{s^\wedge i} = Y_s \cap U_s$ satisfy (5′); then, if necessary, we shrink these sets to satisfy conditions (2) and (4). This completes the proof of the theorem.

7. REMARKS

The main results of this paper are parallel to the following theorems of classical descriptive set theory.

Theorem 7.1 (Hurewicz [10]). *Let $\mathbb{X}$ be a Polish space. Then any non-σ-compact $\boldsymbol{\Sigma}^1_1$-set $A \subseteq \mathbb{X}$ has a subset homeomorphic to $\mathscr{N}$ and closed in A.*

Theorem 7.2 (Saint-Raymond [11]; see also 21.23 in [12]). *Let $\mathbb{X}$ be a Polish space, and let $A \subseteq \mathbb{X}$ be a $\boldsymbol{\Sigma}^1_1$-set not contained in a σ-compact set $Z \subseteq \mathbb{X}$. Then there exists a set $P \subseteq A$ homeomorphic to $\mathscr{N}$ and closed in $\mathbb{X}$.*

Considerations of [12] show that each of these theorems can be proved in the case $\mathbb{X} = \mathscr{N}$ and then extended to sets in any Polish space $\mathbb{X}$ by purely topological methods. The $\mathbb{X} = \mathscr{N}$ version of Theorem 7.2 follows directly from our Theorem 1.2 in relativized form (i.e., for the classes $\Sigma^1_1(p)$, where $p \in \mathscr{N}$ is any parameter). Theorem 7.1 follows from the relativized form of Theorem 1.1 only for sets A of class $\boldsymbol{\Delta}^1_1$ (i.e., for Borel sets). An interesting problem is to appropriately extend Theorem 1.1 to the class Σ^1_1. Such an extension may require relaxing condition (I) in Theorem 1.1, provided that condition (II) remains the same; for example, it may involve trees not belonging to the class Δ^1_1.

Theorem 1.1 readily implies several known results, which are collected in the following theorem.

Theorem 7.3 (cf. Theorem 2.5). *Let $P \subseteq \mathscr{N} \times \mathscr{N}$ be a Δ^1_1-set for which all sections*

$$(P)_x = \{y : \langle x, y\rangle \in P\}, \qquad x \in \mathbb{X},$$

are σ-compact. Then

i) pr P *is a* Δ^1_1-*set*;

ii) *P is a countable union of Δ^1_1-sets with compact sections*;

iii) *P is uniformized by a Δ^1_1-set.*

Proof. (i) Consider the set H of all pairs $\langle x, T\rangle$ such that $x \in \mathscr{N}$, $T \subseteq \mathbb{N}^{<\omega}$ is a compact $\Delta^1_1(x)$-tree, and $[T] \subseteq (P)_x$. According to Corollary 2.4, H is a Π^1_1-set. By Theorem 1.1, if $\langle x, y\rangle \in P$, then there exists a tree T for which $\langle x, T\rangle \in H$ and $y \in [T]$. Therefore, the Π^1_1-set

$$E = \{\langle x, y, T\rangle : \langle x, y\rangle \in P \wedge \langle x, T\rangle \in H \wedge y \in [T]\} \subseteq \mathscr{N} \times \mathscr{N} \times 2^{(\mathbb{N}^{<\omega})}$$

satisfies the condition $\mathrm{pr}_{xy}\, E = P$, i.e., $\langle x, y\rangle \in P$ implies the existence of a tree T for which we have $\langle x, y, T\rangle \in E$. Suppose that E is uniformized by a Π^1_1-set $U \subseteq E$. This means that if $\langle x, y\rangle \in P$, then there exists a unique T for which $\langle x, y, T\rangle \in U$. But U is a Σ^1_1-set, because the relation $\langle x, y, T\rangle \in U$ is described by the formula

$$\langle x, y\rangle \in P \wedge y \in [T] \wedge \forall T' \in \Delta^1_1(x) \qquad (\langle x, y, T'\rangle \in U \implies T = T'),$$

and quantifiers of the form $\forall x \in \Delta^1_1(y)$ preserve the class Σ^1_1. Thus, the Σ^1_1-set

$$F = \{\langle x, T\rangle : \exists y(\langle x, y, T\rangle \in U)\}$$

is contained in the Π^1_1-set H. By the separation theorem (Theorem 2.1), there exists a Δ^1_1-set V for which $F \subseteq V \subseteq H$. By construction, we have

$$\langle x, y\rangle \in P \Longleftrightarrow \exists T(\langle x, T\rangle \in V \wedge y \in [T]).$$

Finally, the set V is countable-valued; indeed, if $\langle x, T\rangle \in V$, then $T \in \Delta^1_1(x)$ (because $V \subseteq H$). Note at once that $\mathrm{pr}\, P = \mathrm{pr}\, V$, and hence the projection $D = \mathrm{pr}\, P$ is a Δ^1_1-set by Theorem 2.5.

(ii) According to Theorem 2.5, the set V is the union $\bigcup_n V_n$ of single-valued Δ^1_1-sets V_n, and each projection $D_n = \mathrm{pr}\, V_n \subseteq D$ is a Δ^1_1-set. Each V_n is essentially the graph of a Δ^1_1-function $\tau_n \colon D_n \to$ *compact trees*, and

$$(P)_x = \bigcup_{x \in D_n} [\tau_n(x)].$$

Setting

$$P_n = \{\langle x, y\rangle : x \in D_n \wedge y \in [\tau_n(x)]\})\}$$

for each n, we see from the above considerations that $P = \bigcup_n P_n$, all sets P_n have only compact sections, and each P_n is a Δ^1_1-set, because all of the sets D_n and the function τ_n belong to this class.

(iii) Again applying Theorem 2.5 (iii), we see that the set V is uniformized by a single-valued Δ^1_1-set; i.e., there exists a Δ^1_1-function

$$\tau \colon D_n \to \textit{compact trees}$$

such that $\langle x, \tau(x)\rangle \in V$ for all $x \in D$. Consider the set Q of all pairs $\langle x, y\rangle \in P$ for which y is the lexicographically leftmost point in the compact set $[\tau(x)]$. It is easy to show that Q is a Δ^1_1-set (see the proof of Corollary 4.2) uniformizing the given set P. ☐

Theorem 7.3 remains true for the class $\Delta^1_1(p)$ with any fixed parameter $p \in \mathscr{N}$ (the proof is the same), which, certainly, implies the validity of this theorem for the projective class $\mathbf{\Delta}^1_1$ (of all Borel sets). Thus, we obtain the following corollary, which contains several classical results of descriptive set theory (due to Arsenin, Kunugui, Shchegol'kov, and Saint-Raymond; see [3, Sec. 4] for references).

Corollary 7.4. *Let $\mathbb{X}$ and $\mathbb{Y}$ be Polish spaces, and let $P \subseteq \mathbb{X} \times \mathbb{Y}$ be a $\mathbf{\Delta}^1_1$-set for which all sections $(P)_x = \{y : \langle x, y\rangle \in P\}$, $x \in \mathbb{X}$, are σ-compact. Then*

i) $\mathrm{pr}\, P$ *is a* $\mathbf{\Delta}^1_1$*-set*;

ii) P *is a countable union of* $\mathbf{\Delta}^1_1$*-sets with compact sections*;

iii) P *is uniformized by a* $\mathbf{\Delta}^1_1$*-set.*

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