ISSN 0001-4346, Mathematical Notes, 2015, Vol. 98, No. 2, pp. 273–282. © Pleiades Publishing, Ltd., 2015.
Original Russian Text © V. G. Kanovei, V. A. Lyubetsky, 2015, published in Matematicheskie Zametki, 2015, Vol. 98, No. 2, pp. 247–257.

On Effective σ-Boundedness and σ-Compactness in Solovay's Model

V. G. Kanovei[1,2*] **and V. A. Lyubetsky**[1**]

[1] *Kharkevich Institute for Problems of Data Transmission, Moscow, Russia*

[2] *Moscow State University of Railway Engineering (MIIT), Moscow, Russia*

Received September 27, 2013; in final form, March 3, 2015

Abstract—Two dichotomy theorems on the effective σ-boundedness and effective σ-compactness of ordinal definable point sets in Solovay's model are proved.

DOI: 10.1134/S0001434615070299

Keywords: *Solovay's model, effective σ-boundedness, effective σ-compactness, descriptive set theory.*

1. INTRODUCTION

Effective descriptive set theory appeared in the middle of the twentieth century as a set of technical tools and methods for refining and simplifying constructions and arguments of classical descriptive set theory and, to lesser degree, as a mechanism for applying theorems of recursion theory to problems of descriptive set theory. However, it had soon turned out that effective descriptive set theory itself leads to problems having no counterparts in classical descriptive theory, in particular, to problems related with the *effectiveness* of certain properties of sets under consideration.

This category includes the following two theorems (see the paper [1], the book [2, Secs. 10.6, 10.7], and the papers [3] and [4]) about effective versions of σ-boundedness[1] and σ-compactness of sets in the Baire space $\mathcal{N} = \omega^\omega$.

Theorem 1. *If $A \subseteq \mathcal{N}$ is a Σ^1_1 set, then precisely one of the following two assertions holds*:

(I) *A is Δ^1_1-effectively σ-bounded, i.e., there exists a Δ^1_1-sequence $\{T_n\}_{n\in\omega}$ of compact trees $T_n \subseteq \omega^{<\omega}$ for which $A \subseteq \bigcup_n [T_n]$*;

(II) *there exists a superperfect $\mathbf{P}$-set $Y \subseteq A$.*

Theorem 2. *If $A \subseteq \mathcal{N}$ is a Δ^1_1 set, then precisely one of the following two assertions holds*:

(I) *A is Δ^1_1-effectively σ-compact, i.e., there exists a Δ^1_1-sequence $\{T_n\}_{n\in\omega}$ of compact trees $T_n \subseteq \omega^{<\omega}$ for which $A = \bigcup_n [T_n]$*;

(II) *there exists a set $Y \subseteq A$ homeomorphic to the entire space $\mathcal{N}$ and relatively closed in A.*

*E-mail: kanovei@iitp.ru
**E-mail: lyubetsk@iitp.ru

[1] A set X in the Baire space $\mathcal{N}$ is *σ-bounded* if it can be covered by a σ-compact set in $\mathcal{N}$.

The effectiveness of the existence statements in these two theorems is in that the sequences of compact trees in (I) belong to the effective class $\varDelta^1_1$. The noneffective (coarser) assertion corresponding to Theorem 1 is that any $\mathbf{\Sigma}^1_1$ set $A \subseteq \mathcal{N}$ either is covered by a σ-compact set or contains a superperfect subset; this was proved in [5]. The noneffective result corresponding to Theorem 2 was obtained in Hurewicz' old paper [6].

All results mentioned above are of the type of *dichotomy theorems*, which classify point sets into "small" (of type (I) in both theorems) and "large" (of type (II)) according to various criteria. Such theorems attract much interest in modern descriptive set theory; see, e.g., the books [7]–[9]. Small-type sets are characterized by structural properties of point sets, and large-type sets, simply by the presence of a subset which is a canonical counterexample to the "smallness" property under consideration; thus, the Baire space $\mathcal{N} = \omega^\omega$ can be regarded as a canonical example of a non-σ-compact set. The main results of this paper, Theorems 3 and 4, are classified with such dichotomy theorems.

In our later paper [10], we proved that Theorem 2 is not valid in the case where the set A belongs to the larger class $\mathbf{\Sigma}^1_1$ (as in Theorem 1); however, for $\mathbf{\Sigma}^1_1$ sets, a somewhat weaker assertion holds. In the same paper, we obtained a far-reaching generalization of Theorem 1, in which the σ-boundedness assumption of part (I) of the theorem was relaxed to the $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-$\sigma$-boundedness assumption, where $\mathsf{F}_1, \dots, \mathsf{F}_n$ are given equivalence relations of class $\varDelta^1_1$ and the $\{\mathsf{F}_1, \dots, \mathsf{F}_n\}$-$\sigma$-boundedness of a set means that this set can be covered by a σ-bounded set and a countable union of equivalence classes of the relations $\mathsf{F}_1, \dots, \mathsf{F}_n$. Accordingly, condition (II) is strengthened by the requirement that there exists a superperfect set pairwise F_i-nonequivalent for each $i = 1, \dots, n$.

The proofs of the results mentioned above are very typical of the first projective level, and they cannot be generalized to higher levels of the projective hierarchy (e.g., to the case of $\varSigma^1_2$ and $\varDelta^1_2$ sets A); as shown in [10], at these levels, the direct generalizations of the theorems themselves are false. Correct generalizations of Theorem 1 to $\varSigma^1_2$ sets were obtained in [3] and of its more complicated version with equivalence classes, in [10]; in part (I), uncountable unions are perforce allowed.

As usual, for the third and higher projective levels, results similar to Theorems 1 and 2 cannot be obtained. In this case, it is common practice to solve the arising problems in the context of consistency of certain statements with the Zermelo–Fraenkel axioms **ZFC** or, which is essentially equivalent, investigate the *status quo* in particular models of **ZFC**. A special role is played by the *Lévy–Solovay model*, which was first used in [11] to prove the consistency of the conjecture that all projective and even all real-ordinal definable (**ROD**) sets of real numbers are measurable; see our book [12, Chap. 13] for details.

In this paper, we prove that, in the Lévy–Solovay model, Theorems 1 and 2 admit natural generalizations valid for sets A in a very large (but still effective enough) class OD of *ordinal definable* point sets, which naturally includes all of the classes $\varSigma^1_n$, $\varPi^1_n$, and $\varDelta^1_n$ of the effective projective hierarchy. The main results are as follows.

Theorem 3 (in Solovay's model). *If* $A \subseteq \mathcal{N}$ *is an* OD *set, then precisely one of the following conditions holds*:

(I) *A is* OD-*effectively σ-bounded in the sense that there exists an* OD-*sequence* $\{T_\xi\}_{\xi<\omega_1^{\mathbf{L}}}$ *of compact trees* $T_\xi \subseteq \omega^{<\omega}$ *for which*

$$A \subseteq \bigcup_{\xi<\omega_1^{\mathbf{L}}} [T_\xi];$$

(II) *there exists a superperfect* OD *set* $Y \subseteq A$.

Theorem 4 (in Solovay's model). *If* $A \subseteq \mathcal{N}$ *is an* OD *set, then precisely one of the following conditions holds*:

(I) *A is* OD-*effectively σ-compact in the sense that there exists an* OD-*sequence* $\{T_\xi\}_{\xi<\omega_1^{\mathbf{L}}}$ *of compact trees* $T_\xi \subseteq \omega^{<\omega}$ *for which*

$$A = \bigcup_{\xi<\omega_1^{\mathbf{L}}} [T_\xi];$$

(II) *there exists an* OD *set* $Y \subseteq A$ *homeomorphic to the entire space* $\mathcal{N}$ *and relatively closed in* A.

Note that, in Solovay's model, the ordinal $\omega_1^{\mathbf{L}}$ (i.e., the first uncountable cardinal in the constructive universe $\mathbf{L}$) is countable in the universe of all sets (see Lemma 4.1 below); therefore, the unions in conditions (I) in both theorems are countable, although not indexed by positive integers (and they cannot be indexed with the preservation of ordinal definability; see Remark 6.1). Note also that Theorems 3 and 4 also contain an effectiveness condition at the OD level in parts (II), in contrast to Theorems 1 and 2, where effectiveness at the levels of Δ^1_1 and Σ^1_1 in parts (II) cannot be achieved.

The proofs of Theorems 3 and 4 are given in Secs. 6 and 7, after a technical introduction in Sec. 3 and a review of properties of the Lévy–Solovay model in Secs. 4 and 5.

2. REMARK

Theorems 3 and 4 are based on properties of Solovay's model and may be false in other models of **ZFC**. For example, in Gödel's constructive model $\mathbf{L}$ (the class of all constructive sets), any OD set $X \subseteq \mathcal{N}$ satisfies condition (I) in both theorems, so that strict dichotomy is already impossible.

On the other hand, there also exist models in which some OD point sets satisfy the disjunction (I) $\vee$ (II) (in any one of the theorems). Namely, there exist models of **ZFC** in which

(a) the continuum hypothesis is false, i.e., $\omega_1 < 2^{\aleph_0}$, and

(b) there exists an OD well-order $\prec$ on the space $\mathcal{N}$; to be more precise, this order even belongs to one of the projective classes Δ^1_n;

see [13] and [14]. A certain modification of a construction of [13] yields a model in which the following additional condition holds:

(c) $\omega_1^{\mathbf{L}} < \omega_1$.

In this model, we perform the well-known construction of a *Bernstein set* (i.e., a set $A \subseteq \mathcal{N}$ such that neither A not its complement $A' = \mathcal{N} \setminus A$ contains perfect subsets), replacing the abstract axiom of choice at certain places by the choice of a $\prec$-least point (where $\prec$ is the well-order in condition (b)). As a result, we obtain a Bernstein OD set A. According to the Alexandroff–Hausdorff theorem, this set cannot contain uncountable Borel subsets, so that, in both theorems, it does not satisfy condition (II). But condition (I) cannot hold either. Indeed, if A satisfies condition (I) in Theorem 3 (which is the weakest among the two conditions (I)), then, according to (c), A is meager in $\mathcal{N}$, and its complement $A' = \mathcal{N} \setminus A$ surely contains a perfect subset in contradiction to the choice of A.

Constructing a model containing a similar counterexample in which $\omega_1^{\mathbf{L}} = \omega_1$ is an interesting unsolved problem.

3. TECHNICAL INTRODUCTION

We use the standard notation Σ^1_1, Π^1_1, and Δ^1_1 for effective projective classes in the Baire space $\mathcal{N}$ and also $\mathbf{\Sigma}^1_1$, $\mathbf{\Pi}^1_1$, and $\mathbf{\Delta}^1_1$ for the corresponding ineffective classes; see [2], [7], [12], [15], and [16].

By $\omega^{<\omega}$ we denote set of all tuples (finite sequences) of positive integers, including the empty tuple Λ. Given $u, v \in \omega^{<\omega}$, $\operatorname{lh} u$ denotes the *length* of u and $u \subset v$ means that v is a *proper extension* of u. For $s \in \omega^{<\omega}$ and $n \in \omega$, by $s^\wedge n$ we denote the tuple obtained by attaching n to s on the right. Let

$$\mathcal{N}_s = \{x \in \mathcal{N} \mid s \subset x\} \quad \text{a (\textit{Baire interval} in } \mathcal{N} = \omega^\omega)$$

for $s \in \omega^{<\omega}$. If the set $X \subseteq \mathcal{N}$ contains at least two points, then there exists a maximal tuple $s = s_X$ for which $X \subseteq \mathcal{N}_s$. In this case, we set $\operatorname{diam} X = 1/(1 + \operatorname{lh} s)$; if X contains at most one point, then we set $\operatorname{diam} X = 0$.

A set $T \subseteq \omega^{<\omega}$ is called a *tree* if $u \in T$ whenever $u^\wedge n \in T$ for at least one n. The elements $u \in T$ of a tree T are called the *vertices* of this tree. A vertex $u \in T$ is said to be *terminal* if there is no such n for which $u^\wedge n \in T$. Any nonempty tree contains the empty tuple Λ. A vertex $u \in T$ is a *branching point* in T if there exists a number $k \neq n$ for which $u^\wedge k \in T$ and $u^\wedge n \in T$; the notation $\operatorname{bran} T$ is used for the set of all branching points in T.

A tree $T \subseteq \omega^{<\omega}$ without terminal vertices is said to be *compact* if it is *finitely branching*, i.e., whenever $u \in \operatorname{bran} T$, we have $u^\wedge n \in T$ for only finitely many n. In this case, the subset

$$[T] = \{x \in \mathcal{N} \mid \forall\, m\, (x \restriction m \in T)\}$$

of the space $\mathcal{N}$ is compact. Conversely, if $X \subseteq \mathcal{N}$ is compact, then

$$T = \operatorname{tree}(X) = \{x \restriction n \mid x \in X \wedge n \in \omega\}$$

is a compact tree and $X = [T]$.

A tree $T \subseteq \omega^{<\omega}$ without terminal vertices is said to be *perfect* if, for any vertex $u \in T$, there exists a branching point $v \in \operatorname{bran} T$ such that $u \subset v$. In this case, the set $[T]$ is perfect. A perfect tree T is said to be *superperfect* if, for each branching point $u \in \operatorname{bran} T$, there exist infinite many numbers n such that $u^\wedge n \in T$. In this case, the set $[T]$ is superperfect. Conversely, if $X \subseteq \mathcal{N}$ is a perfect set, then the tree $\operatorname{tree}(X)$ is perfect, and, for any superperfect set $X \subseteq \mathcal{N}$, there exists a superperfect tree $T \subseteq \operatorname{tree}(X)$. Recall that a set $X \subseteq \mathcal{N}$ is said to be

- *perfect* if it has no isolated points;
- *superperfect* if it has no nonempty open-and-closed σ-compact subsets.

4. ON SOLOVAY'S MODEL

The proofs of our main results, Theorems 3 and 4, will be given later on; in this section, we describe the those properties of Solovay's model which are used in these proofs.

First, by *Solovay's model* we mean the model of **ZFC** constructed in [11] in which all projective sets of real numbers are Lebesgue measurable, rather than a narrower model, also constructed in [11], in which only the axioms **ZF** + DC hold (i.e., the axiom of choice is replaced by the axiom DC of dependent choice), but all sets of real numbers in general are measurable. These models were considered in more detail in the book [12, Chap. 13] and the paper [17, Sec. 4].

Definition 4.1. Let Ω be any ordinal. By Ω-SM we denote the conjunction of the following three conjectures (A), (B), and (C):

(A) $\Omega = \omega_1$;

(B) in the class $\mathbf{L}$ (Gödel's universe of constructive sets), the cardinal Ω is strongly inaccessible;

(C) the set-theoretic universe $\mathbf{V}$ is a generic extension of the class $\mathbf{L}$ obtained by using the collapse forcing poset $\mathscr{P} = \operatorname{Coll}(\omega, <\Omega)$ as in [12, Sec. 13.6].

Thus, Ω-SM is the hypothesis that the universe $\mathbf{V}$ is Solovay's model over the initial model $\mathbf{L}$ with key cardinal Ω.

Definition 4.2. The class OD is the class of all ordinal definable sets. In other words, a set X belongs to OD if it can be defined by a **ZFC** formula in which all parameters are ordinals.[2]

Lemma 4.1. *Under* Ω-SM*, the following assertions hold*:

(i) *if X is a countable* OD *set, then there exists an ordinal $\lambda < \Omega$ and a one-to-one* OD *map $f\colon \lambda \xrightarrow{\text{on}} X$;*

(ii) *if $\xi < \Omega$ and $a \in \mathcal{N}$, then $\omega_\xi^{\mathbf{L}[a]} < \Omega$; in particular, $\omega_\xi^{\mathbf{L}} < \Omega$;*

[2]Details related to this definition can be found in [12, Sec. 3.5] and [18, Chap. 14].

(iii) *if* $X \subseteq Y \in \mathbf{L}$*, then* $X \in \mathbf{L}$ *if and only if* $X \in \mathrm{OD}$.

Proof. (i) Consider the canonical OD map $F\colon \mathrm{Ord} \xrightarrow{\text{onto}} \mathrm{OD}$ (see Statement 1 in [18, Chap. 14]). The relations $F(\xi) \in X$ and $F(\xi) = F(\eta)$ (with arguments ξ and η) belong to OD as well. The rest is obvious.

For the proof of (ii) and (iii), see Lemmas 13.6.5 and 13.6.7 in the book [12]. □

The following lemma describes the key properties of Solovay's model.

Lemma 4.2. *Under* Ω-SM*, the following statements hold*:

(i) *if* $\lambda < \Omega$ *and* $f : \omega \xrightarrow{\text{onto}} \lambda$*, then the universe is a* $\mathscr{P}$*-generic extension of the class* $\mathbf{L}[f]$;

(ii) *given any* $\in$*-formula* $\varphi(x)$*, there exists a* $\in$*-formula* $\overline{\varphi}(\lambda, x)$ *such that*

$$\varphi(f) \quad \Longleftrightarrow \quad \overline{\varphi}(\lambda, f) \text{ in } \mathbf{L}[f]$$

for any $\lambda < \Omega$ *and any function* $f : \omega \xrightarrow{\text{on}} \lambda$.

Proof. (i) For the case $\lambda = \omega$, where $f \in \mathcal{N}$ (even without the assumption $\operatorname{ran} f = \omega$), the proof of the required assertion is contained in that of Lemma 13.6.6 in [12]. The case of any λ is reduced to this special case by considering the map taking each function $f : \omega \xrightarrow{\text{on}} \lambda$ to a point $f' \in \mathcal{N}$ defined by setting $f'(n) = 1$ if $n = 2^m \cdot 3^k \in \omega$ and $f(k) < f(m)$ and $f'(n) = 0$ otherwise. We have $\mathbf{L}[f'] = \mathbf{L}[f]$, and the required assertion for f follows from that for f'.

(ii) Lemma 13.6.7 (A) with $w = \varnothing$ in the book [12] contains the required assertion for the case $\lambda = \omega$: the formula $\overline{\varphi}(\lambda, f)$ expresses the forcing of $\varphi(f)$ over $\mathbf{L}[f]$. The general case is proved by using the same transformation as in (i). □

Assertion (ii) of Lemma 4.2 is important because it reduces the truth of a formula $\varphi(f)$ in Solovay's model to that of another formula $\overline{\varphi}(f)$ in the class $\mathbf{L}[f]$ of all sets constructive with respect to f. This reduction is used in the next section.

5. FORCING BY OD-SETS IN SOLOVAY'S MODEL

Definition 5.1 (Ω-SM)**.** Let $\mathbf{P}$ denote the set of all nonempty OD sets $Y \subseteq \mathcal{N}$. This A set $\mathbf{P}$ is regarded as a forcing poset; thus, we refer to its elements as *(forcing)* "*conditions*," and subsets of $\mathbf{P}$ which are smaller (with respect to inclusion) are considered *stronger* conditions. A set of conditions $W \subseteq \mathbf{P}$ is said to be

- *dense* if, for each $Y \in \mathbf{P}$, there exists a condition $Z \in W$ for which $Z \subseteq Y$;
- $\mathbf{P}$-*generic* if it satisfies the conditions
  (1) if $X, Y \in W$, then $X \cap Y \in W$ and
  (2) if $D \subseteq \mathbf{P}$ belongs to OD and is dense, then $W \cap D \neq \varnothing$.

Proposition 5.1. *Under* Ω-SM*, if a set* $G \subseteq \mathbf{P}$ *is* $\mathbf{P}$*-generic, then the intersection* $\bigcap G$ *contains a unique point*.

Proof. See Lemma 14 in [19]. □

The set $\mathbf{P}$ is uncountable; therefore, the existence of $\mathbf{P}$-generic sets does not directly follow from Ω-SM. However, fortunately, $\mathbf{P}$ turns out to be *locally countable* in a certain sense.

Definition 5.2 (Ω-SM)**.** We say that a call set $X \in \mathrm{OD}$ is *regular* if the OD part

$$\mathscr{P}^{\mathrm{OD}}(X) = \mathscr{P}(X) \cap \mathrm{OD}$$

of its power set $\mathscr{P}(X)$ is at most countable. By $\mathbf{P}^*$ we denote the set of all regular $X \in \mathbf{P}$.

For example, under Ω-SM, the set $X = \mathcal{N} \cap \mathrm{OD} = \mathcal{N} \cap \mathbf{L}$ of all OD points in the Baire space belongs to $\mathbf{P}^*$. Indeed, we have

$$\mathscr{P}^{OD}(X) = \mathscr{P}(X) \cap \mathrm{OD} = \mathscr{P}(X) \cap \mathbf{L};$$

therefore, $\mathscr{P}^{\mathrm{OD}}(X)$ admits an OD bijection onto the ordinal $\omega_2^{\mathbf{L}}$, and $\omega_2^{\mathbf{L}} < \Omega$ by Lemma 4.1 (ii).

Recall that, for $\lambda \in \mathrm{Ord}$, $\mathrm{Coll}(\omega, \lambda) = \lambda^{<\omega}$ is the collapse forcing poset for the collapse ordinal λ. It consists of all (finite) tuples of ordinals $\alpha < \lambda$ and generates a generic function $f\colon \omega \xrightarrow{\text{on}} \lambda$; see [12, Sec. 9.7].

Lemma 5.1. *Under* Ω-SM*, if* $\lambda < \Omega$*, then the set* Coh_λ *of all functions* $f \in \lambda^\omega$ $\mathrm{Coll}(\omega, \lambda)$*-generic over* $\mathbf{L}$ *is regular.*

Proof. First, we have $\mathrm{Coh}_\lambda \in \mathbf{P}$ from obvious considerations. Next, consider any set $Y \subseteq \mathrm{Coh}_\lambda$, $Y = \{f \in \mathrm{Coh}_\lambda \mid \varphi(f)\} \in \mathrm{OD}$, where φ is a formula in which all parameters are ordinals. By Lemma 4.2 (ii), we have

$$Y = \{f \in \mathrm{Coh}_\lambda \mid \overline{\varphi}(f) \text{ in } \mathbf{L}[f]\}$$

for some other formula $\overline{\varphi}(f)$ whose parameters are ordinals. Therefore,

$$Y = \mathrm{Coh}_\lambda \cap \bigcup_{p \in S} \{f \in \lambda^\omega \mid p \subset f\},$$

where S consists of all conditions $p \in \mathrm{Coll}(\omega, \lambda)$ forcing $\overline{\varphi}(\dot{f})$, and $\dot{f}$ is the name of a $\mathrm{Coll}(\omega, \lambda)$-generic element. But the family $\mathscr{S}$ of all such sets S belongs to $\mathbf{L}$ (because forcing over $\mathbf{L}$ is expressible in $\mathbf{L}$) and has cardinality $\aleph_{\lambda+1}^{\mathbf{L}}$ in $\mathbf{L}$. Therefore, again according to Lemma 4.1 (ii), $\mathscr{S}$ is countable under Ω-SM. However, it follows from the above considerations that each set $Y \subseteq \mathrm{Coh}$, $Y \in \mathrm{OD}$, is uniquely determined by suitable $S \in \mathscr{S}$. □

Lemma 5.2 ([19] (Ω-SM))**.** *The set* $\mathbf{P}^*$ *is dense in* $\mathbf{P}$; *i.e., for any* $X \in \mathbf{P}$*, there exists a condition* $Y \in \mathbf{P}^*$ *such that* $Y \subseteq X$*.*

Proof. Take any condition $X \in \mathbf{P}$. By definition, we have $X \neq \varnothing$; take any point $x \in X$. Under Ω-SM, according to Lemma 13.6.5 in [12], x belongs to a certain subclass $\mathbf{L}[\mathscr{G}_{\le\lambda}]$ of Solovay's entire model, where $\lambda < \omega_1 = \Omega$, and this subclass itself is a $\mathrm{Coll}(\omega, \lambda)$-generic extension of $\mathbf{L}$, i.e., $\mathbf{L}[\mathscr{G}_{\le\lambda}] = \mathbf{L}[f]$, where $f \in \mathrm{Coh}_\lambda$ (see the proof of Lemma 13.6.5 in [12]). This implies the existence of a OD map $H\colon \lambda^\omega \to \mathcal{N}$ for which $x = H(f)$. The set

$$P = \{f' \in \mathrm{Coh}_\lambda \mid H(f') \in X\}$$

belongs to OD as well, and it is nonempty (contains f); the same is true for its image

$$Y = \{H(f') \mid f' \in P\} \subseteq X$$

(it contains x). Finally, the set Coh_λ is regular by Lemma 5.1, which implies the regularity of the set Y. Thus, $Y \in \mathbf{P}^*$. □

6. PROOF OF THE EFFECTIVE σ-BOUNDEDNESS THEOREM

In this section, we prove Theorem 3.

We argue in Solovay's model, i.e., assume Ω-SM.

Lemma 6.1. *Conditions* (I) *and* (II) *in Theorem* 3 *are inconsistent.*

Proof. In Solovay's model, the set $S = \bigcup_{\xi<\omega_1^{\mathbf{L}}}[T_\xi]$ in (I) is a countable union of compact sets; therefore, it is σ-compact. Hence if $Y \subseteq A$ is superperfect, as in (II), then it is covered by σ-compact set, which is impossible. □

Now, consider any OD set $A \subseteq \mathcal{N}$. Let U denote the union of all sets of the form $[T]$, where $T \subseteq \omega^{<\omega}$ is a compact tree in OD. Obviously, the sets U and $A' = A \setminus U$ belong to OD.

Lemma 6.2. *Under the assumptions of Theorem* 3*, the topological closure in* $\mathcal{N}$ *of any* OD *set* $\varnothing \neq Y \subseteq A'$ *is noncompact.*

Proof. If the closure $\overline{Y}$ of such Y is compact, then the tree $T = \operatorname{tree}(Y)$ is compact as well and belongs to OD; therefore, $Y \subseteq \overline{Y} = [T] \subseteq U$, which contradicts the assumption $Y \subseteq A'$.

There are two possible cases.

case 1: $A' = \varnothing$, i.e., $A \subseteq U$. In this case, condition (I) in Theorem 3 holds. Indeed, it suffices to note that, under Ω-SM, the OD points of the space $\mathcal{N}$ are the same thing as constructive points (of $\mathbf{L}$), and hence there exists an OD-enumeration of all OD trees by the ordinals $\xi < \omega_1^{\mathbf{L}}$.

Case 2: the set $A' = A \setminus U$ is nonempty. According to Lemma 5.2, there exists a condition $A'' \subseteq A'$ such that $A'' \in \mathbf{P}^*$. The set $P = \mathcal{P}^{\mathrm{OD}}(A'') = \mathcal{P}(A'') \cap \mathrm{OD}$ is at most countable. By Lemma 4.1, there exists an ordinal $\lambda < \Omega$ and an OD map $f\colon \lambda \xrightarrow{\text{on}} P$. But the set $\mathcal{P}^{\mathrm{OD}}(\lambda)$ is countable; therefore, so is $\mathcal{P}^{\mathrm{OD}}(P)$ (because of the presence of the map f). Let us fix any enumeration $\{\mathscr{D}_n\}_{n\in\omega}$ of all OD sets $\mathscr{D} \subseteq P = \mathcal{P}^{\mathrm{OD}}(A'')$ dense in $\mathbf{P}^*$ below A''[3].

We claim that there exists a system of conditions $Y_s \in \mathbf{P}^*$, $Y_s \subseteq A''$, indexed by the tuples $s \in \omega^{<\omega}$, and satisfying the following requirements:

(1) if $s \in \omega^{<\omega}$ and $i \in \omega$, then $Y_{s^\wedge i} \subseteq Y_s$;

(2) $\operatorname{diam} Y_s \le 2^{-\operatorname{lh} s}$;

(3) if $s \in \omega^{<\omega}$ and $k \neq n$, then $Y_{s^\wedge k} \cap Y_{s^\wedge n} = \varnothing$ and, moreover, the conditions $Y_{s^\wedge k}$ can be covered by pairwise disjoint (open-and-closed) Baire intervals $J_{s^\wedge k}$;

(4) if $s \in \omega^{<\omega}$, then $Y_s \in \mathscr{D}_{\operatorname{lh} s}$, where the sets $\mathscr{D}_n$ are defined as above;

(5) if $s \in \omega^{<\omega}$ and $x_k \in Y_{s^\wedge k}$ for all $k \in \omega$, then the sequence of x_k has no convergent subsequences in $\mathcal{N}$.

To construct the initial condition Y_Λ, note that the density of $\mathscr{D}_0$ implies the existence of a condition $Z \subseteq A''$ in $\mathscr{D}_0$. To satisfy (2), we set Y_Λ equal to the intersection Z with an appropriate Baire interval of sufficiently small diameter. Then, to satisfy (4), we again narrow the condition thus obtained, using the density of $\mathscr{D}_0$.

Now, arguing by induction on the length of tuples, suppose that $s \in \omega^{<\omega}$ and the condition $Y_s \in \mathbf{P}^*$, $Y_s \subseteq A''$, is already constructed. According to Lemma 6.2, there exists a tuple $\tau \in \operatorname{tree}(Y_s)$ for which the set

$$K_s = \{k \in \omega \mid \tau^\wedge k \in \operatorname{tree}(Y_s)\}$$

[3] A set $\mathscr{D} \subseteq P = \mathcal{P}^{\mathrm{OD}}(A'')$ is dense in $\mathbf{P}^*$ *below* A'' if the completed set $\mathscr{D}^+ = \mathscr{D} \cup \{Y \in \mathbf{P}^* \mid Y \cap Y'' = \varnothing\}$ is dense in $\mathbf{P}^*$ in the sense of Definition 5.1. In this case, according to Lemma 5.2, $\mathscr{D}^+$ is also dense in $\mathbf{P}$.

is infinite. This allows us to define a sequence of pairwise different points $y_k \in Y_s$, $k \in \omega$, containing no convergent subsequences. We cover these points by Baire intervals U_k sufficiently small for the OD sets $Y_{s^\wedge i} = Y_s \cap U_i$ to meet requirement (5) and then narrow these sets so that (2) and (4) hold; to satisfy (4), we use the density of the sets $\mathscr{D}_n$. This completes the inductive step of the construction of the conditions Y_s.

After the construction is completed, we note that, for any point $a \in \mathcal{N}$, the intersection $\bigcap_m Y_{a\restriction m}$ contains a unique point by Proposition 5.1, because condition (4) ensures the required genericity of the set

$$\{Y_{a\restriction m} \mid m \in \omega\}.$$

Let $\bigcap_m Y_{a\restriction m} = \{f(a)\}$. The map

$$f \colon \mathcal{N} \xrightarrow{\text{on}} Y = \{f(a) \mid a \in \mathcal{N}\}$$

is a homeomorphism from fairly obvious considerations.

Let us verify that Y is closed in $\mathcal{N}$. Consider any sequence of points $a_n \in \mathcal{N}$ and suppose that the corresponding sequence points $y_n = f(a_n) \in Y$ converges to $y \in \mathcal{N}$; we must prove that $y \in Y$.

We claim that the sequence $\{a_n\}_{n\in\omega}$ contains a convergent subsequence. Indeed, otherwise, the sequence $\{a_n\}_{n\in\omega}$ cannot be covered by compact set. This implies the existence of a tuple $u \in \omega^{<\omega}$, an infinite set $K \subseteq \omega$, and numbers $n(k)$, $k \in K$, for which $u^\wedge k \subset a_{n(k)}$. By construction, we have $y_{n(k)} \in Y_{u^\wedge k}$. Therefore, (5) implies the divergence of the sequence $\{y_{n(k)}\}_{k\in\omega}$, which is a contradiction.

Thus, the sequence $\{a_n\}_{n\in\omega}$ contains a subsequence $b_k = a_{n(k)}$ converging to some point $b \in \mathcal{N}$. The sequence $z_k = f(b_k)$ (which is a subsequence of $\{y_n\}_{n\in\omega}$) converges to $z = f(b) \in Y$, as required.

Thus, the set Y is closed and, therefore, condition (II) in Theorem 3 holds for the set A. □

Remark 6.1. Condition (I) in Theorem 3 proved above cannot be strengthened to the condition that *there exists an* OD *sequence* $\{T_n\}_{n\in\omega}$ *of compact trees* $T_n \subseteq \omega^{<\omega}$ *for which* $A \subseteq \bigcup_n [T_n]$. A counterexample is $A = \mathcal{N} \cap \mathbf{L}$ (all constructive points in $\mathcal{N}$). This is a countable set in Solovay's model admitting an OD bijection onto the ordinal $\omega_1^{\mathbf{L}}$. Therefore, condition (I) in Theorem 3 holds (while (II) does not); but the existence of a sequence $\{T_n\}_{n\in\omega}$ of compact trees in the class OD (which must be constructive) is, obviously, impossible.

7. PROOF OF THE EFFECTIVE σ-COMPACTNESS THEOREM

In this section, we prove Theorem 4. We argue in Solovay's model, i.e., assume Ω-SM.

Lemma 7.1. *Conditions* (I) *and* (II) *in Theorem* 4 *are inconsistent.*

Proof. As in the proof of Lemma 6.1, the set $A = \bigcup_{\xi<\omega_1^{\mathbf{L}}} [T_\xi]$ in (I) is σ-compact and, therefore, cannot contain relatively closed subsets homeomorphic to the Baire space. □

Now, consider any OD set $A \subseteq \mathcal{N}$. Let U denote the union of all sets of the form $[T]$, where $T \subseteq \omega^{<\omega}$ is a compact OD tree and $[T] \subseteq A$. Obviously, the set U and the complement $A' = A \setminus U$ of A belong to OD.

By Theorem 3 proved above, we can assume without loss of generality that A is σ-bounded (i.e., covered by a σ-compact set); therefore, each closed set $F \subseteq A$ is σ-compact as well.

Lemma 7.2. *If* $\varnothing \neq F \subseteq A'$ *is a nonempty* OD *set, then* $\overline{F} \not\subseteq A$.

Proof. Suppose that, on the contrary, $\overline{F} \subseteq A$. According to the above assumption, which does not limit generality, the set $\overline{F}$ is σ-compact, i.e., $\overline{F} = \bigcup_n F_n$, where all sets F_n are compact. Clearly, there exists a Baire interval $\mathcal{N}_s$ such that the set $X = \mathcal{N}_s \cap \overline{F}$ is nonempty and $X \subseteq F_n$ for some n. We have $X \subseteq A$, and X is a nonempty compact OD set. Therefore, by definition, $X \subseteq U$ and $A' \cap X = \varnothing$. In other words, $\mathcal{N}_s \cap \overline{F} \cap A' = \varnothing$. It follows that $\mathcal{N}_s \cap F = \varnothing$ (because $F \subseteq A'$), which contradicts the relation $X = \mathcal{N}_s \cap \overline{F} \neq \varnothing$.

Case 1: $A' = \varnothing$, i.e., $A = U$. This implies condition (I) in the theorem.

Case 2: $A' \neq \varnothing$. As in the proof of Theorem 3, we take any set $A'' \subseteq A'$, $A'' \in \mathbf{P}^*$, and fix an enumeration $\{\mathscr{D}_n\}_{n\in\omega}$ of all OD-sets $\mathscr{D} \subseteq P = \mathcal{P}^{\mathrm{OD}}(A'')$ dense in $\mathbf{P}^*$ below A''. To obtain a set $Y \subseteq A''$ *relatively* closed in A and homeomorphic to $\mathcal{N}$, we use a system of conditions $Y_s \in \mathbf{P}^*$, $Y_s \subseteq A''$, satisfying requirements (1)–(4) of Sec. 6 and the following requirement instead of (5):

(5′) if $s \in \omega^{<\omega}$, then there exists a point $y_s \in \overline{Y_s} \setminus A$ such that any sequence of points $x_k \in Y_{s^\wedge k}$, $k \in \omega$, converges to y_s.

The construction is similar to the corresponding construction of Sec. 6. Namely, suppose that $s \in \omega^{<\omega}$ and the condition $Y_s \subseteq A''$ is already constructed. Then, for its closure $\overline{Y_s}$, we have $\overline{Y_s} \not\subseteq A$ by Lemma 7.2. Hence there exists a sequence of pairwise different points $x_n \in Y_s$ converging to a point $y_s \in \overline{Y_s} \setminus A$. Let U_n be a Baire interval (containing x_n) of diameter smaller than one third of the least distance from x_n to x_k, $k \neq n$. We set $Y_{s^\wedge n} = Y_s \cap U_n$ for each n and narrow the set $Y_{s^\wedge n}$ in order to satisfy (2) and (4). This completes the inductive step of the construction.

Having the system of sets Y_s, we obtain (see Sec. 6) a homeomorphism

$$f\colon \mathcal{N} \xrightarrow{\text{onto}} Y = \operatorname{ran} f = \{f(a) \mid a \in \mathcal{N}\} \subseteq A''$$

such that $\bigcap_m Y_{a\restriction m} = \{f(a)\}$ for all points $a \in \mathcal{N}$.

It remains to show that Y is relatively closed in A.

Consider a sequence of points $a_n \in \mathcal{N}$ for which the corresponding sequence of images

$$y_n = f(a_n) \in Y$$

converges to a point $y \in \mathcal{N}$; we must prove that $y \in Y$ or $y \notin A$. If the sequence $\{a_n\}$ contains a subsequence converging to some point $b \in \mathcal{N}$, then, as in the proof of Theorem 3, the sequence $\{y_n\}$ converges to the point $f(b) \in Y$. If $\{a_n\}$ has no convergent subsequences, then there exist a tuple $s \in \omega^{<\omega}$, an infinite set $K \subseteq \omega$, and numbers $n(k)$, $k \in K$, for which $s^\wedge k \subset a_{n(k)}$. In this case, $y_{n(k)} \in Y_{s^\wedge k}$ by construction. Therefore, by (5′), the subsequence $\{y_{n(k)}\}_{k\in\omega}$ converges to a point $y_s \notin A$, as required. □

ACKNOWLEDGMENTS

The work of the first author was supported by the Russian Foundation for Basic Research under grant 13-01-00006 and that of the second author, by the Russian Science Foundation under grant 14-50-00150.

REFERENCES

1. V. G. Kanovei and V. A. Lyubetsky, "Effective compactness and sigma-compactness," Mat. Zametki **91** (6), 840–852 (2012) [Math. Notes **91** (6), 789–799 (2012)].
2. V. G. Kanovei and V. A. Lyubetsky, *Modern Set Theory*: *Borel and Projective Sets* (MTsNMO, Moscow, 2010) [in Russian].
3. A. S. Kechris, "On a notion of smallness for subsets of the Baire space," Trans. Amer. Math. Soc. **229**, 191–207 (1977).
4. A. Louveau and J. Saint-Raymond, "Borel classes and closed games: Wadge-type and Hurewicz-type results," Trans. Amer. Math. Soc. **304**, 431–467 (1987).
5. J. Saint-Raymond, "Approximation des sous-ensembles analytiques par l'interior," C. R. Acad. Sci. Paris Sér. A **281**, 85–87 (1975).

6. W. Hurewicz, "Relativ perfekte Teile von Punktmengen und Mengen (A)," Fundam. Math. **12**, 78–109 (1928).
7. V. Kanovei, *Borel Equivalence Relations. Structure and Classification*, in *Univ. Lecture Ser.* (Amer. Math. Soc., Providence, RI, 2008), Vol. 44.
8. Su Gao, *Invariant Descriptive Set Theory*, in *Pure Appl. Math. (Boca Raton)* (CRC Press, Boca Raton, FL, 2009), Vol. 293.
9. V. Kanovei, M. Sabok, and J. Zapletal, *Canonical Ramsey Theory on Polish Spaces*, in *Cambridge Tracts in Math.* (Cambridge Univ. Press, Cambridge, 2013), Vol. 202.
10. V. Kanovei and V. Lyubetsky, "On effective σ-boundedness and σ-compactness," MLQ Math. Log. Q. **59** (3), 147–166 (2013).
11. R. M. Solovay, "A model of set theory in which every set of reals is Lebesgue measurable," Ann. of Math. (2) **92**, 1–56 (1970).
12. V. G. Kanovei and V. A. Lyubetsky, *Modern Set Theory: Absolutely Unsolvable Classical Problems* (MTsNMO, Moscow, 2013) [in Russian].
13. L. Harrington, "Long projective wellorderings," Ann. Math. Logic **12** (1), 1–24 (1977).
14. V. G. Kanovei, "Descriptive forms of the countable axiom of choice," in *Studies in Nonclassical Logics and Set Theory* (Nauka, Moscow, 1979), pp. 3–136 [in Russian].
15. V. G. Kanovei and V. A. Lyubetsky, "On some classical problems of descriptive set theory," Uspekhi Mat. Nauk **58** (5), 3–88 (2003) [Russian Math. Surveys **58** (5), 839–927 (2003)].
16. A. S. Kechris, *Classical Descriptive Set Theory*, in *Grad. Texts in Math.* (Springer-Verlag, New York, 1995), Vol. 156.
17. V. G. Kanovei, "The projective hierarchy of N. N. Luzin: The current state-of-the-art in the theory," in *Handbook of Mathematical Logic*, Part II: *Set Theory*, Supplement to the Russian translation (Nauka, Moscow, 1982), pp. 273–364 [in Russian].
18. T. J. Jech, *Lectures in Set Theory with Particular Emphasis on the Method of Forcing* (Springer-Verlag, Berlin–Heidelberg–New York, 1971; Mir, Moscow, 1973).
19. V. Kanovei, "An Ulm-type classification theorem for equivalence relations in Solovay model," J. Symbolic Logic **62** (4), 1333–1351 (1997).