Изв.Акад.Наук СССР
Сер.Мат.Том 42(1978), Вып.3

Math. USSR Izvestija
Vol. 12 (1978), No. 3

ON THE NONEMPTINESS OF CLASSES IN AXIOMATIC SET THEORY

UDC 51.01.16

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Abstract. Theorems are proved on the consistency with ZF, for $n \geqslant 2$, of each of the following three propositions: (1) there exists an L-minimal (in particular, nonconstructive) $a \subseteq \omega$ such that $V = L[a]$ and $\{a\} \in \Pi^1_n$, but every $b \subseteq \omega$ of class Σ^1_n with constructive code is itself constructive; (2) there exist $a, b \subseteq \omega$ such that their L-degrees differ by a formula from Π^1_n, but not by formulas from Σ^1_n with constants from L (X and Y are said to differ by a formula $\varphi(x)$ if $\sim [(\exists x \in X)\varphi(x) \equiv (\exists y \in Y)\varphi(y)]$); (3) there exists an infinite, but Dedekind finite, set $X \in P(\omega)$ of class Π^1_n, whereas there are no such sets of class $\underset{\sim}{\Sigma}^1_n$. The proof uses Cohen's forcing method.

Bibliography: 17 titles.

§1. Introduction. Formulation of the theorems

As N. N. Luzin predicted (see the Conclusion to [1a], and [1b]), the classical methods of descriptive set theory are not successful in solving nontrivial problems concerning projective sets for levels beginning with the third, and sometimes for the second and even for the first level. For example, it is impossible to prove or to refute the assertion of the Lebesgue measurability of every set of real numbers of the class A_2. (This was established, on the one hand, by P. S. Novikov [2], and, on the other, by R. M. Solovay [6].) Thus, that assertion and its negation are each consistent with the Zermelo-Fraenkel axiomatic set theory ZF. Many other problems of this kind also allow of a solution only in terms of consistency with ZF or ZFC (ZF + Axiom of Choice).

The theorems proved in the present paper can be called "theorems on the consistency of the nonemptiness of differences". Theorems 1 and 2 have the following general form:

The proposition "In the class K_1 there exist elements satisfying a certain property Λ" is consistent (with respect to ZF or ZFC) with the proposition "In the class K_2 there do not exist elements satisfying the same property Λ."

Here, K_1 and K_2 are certain fixed classes of sets. For example, in Theorem 2 one takes as K_1 and K_2, respectively, the projective classes CA_n and A_n of N. N. Luzin ([1c], *Works* p. 586).

Thus, the difference $K_1 - K_2$ turns out to be "nonempty" (in the sense of consistency)

1980 *mathematics subject classification.* Primary 03E30; Secondary 03E35.

with respect to the property Λ. In Theorem 3, the "consistency of the nonemptiness" of the difference $K_1 - K_2$ is understood in a somewhat different way.

Before formulating the theorems, let us make some terminological remarks. By R we denote $P(\omega) = \{x : x \subseteq \omega\}$, the collection of all subsets of the natural number sequence ω. We shall use the standard notation (cf. [5], §§16.1 and 16.6) for classes of subsets of spaces of the form $\omega^m \times R^k$, as well as the notation for corresponding collections of formulas of second-order arithmetic with variables of type 1 over R ([5], §16.2). In particular, if $i < 2$ and $n \in \omega$, then Σ^i_n is the collection of all subsets of such spaces defined by Σ^i_n-formulas without parameters; $\Sigma^{i,x}_n$ is the same with the additional condition that we can use a fixed $x \subseteq \omega$ as a parameter of type 1. Similarly, Π^i_n and $\Pi^{i,x}_n$ are the collections of all subsets of the indicated spaces defined by Π^i_n-formulas without parameters and with the parameter x, respectively. Further, $\Delta^i_n = \Sigma^i_n \cap \Pi^i_n$ and $\Delta^{i,x}_n = \Sigma^{i,x}_n \cap \Pi^{i,x}_n$. In addition, one defines $\underline{\Sigma}^i_n = \bigcup_{x\subseteq\omega}\Sigma^{i,x}_n$ (coinciding with the class A_n for spaces of the form R^k) and $\underline{\Pi}^i_n = \bigcup_{x\subseteq\omega}\Pi^{i,x}_n$ (coinciding with CA_n). In this connection, in a Σ^1_n-prefix we do not permit a quantifier of type 0 to stand to the left of the quantifier of type 1; and the same for Π^1_n.

THEOREM 1. *Let $n \geqslant 2$. The proposition "In the class Δ^1_{n+1} there exists a nonconstructive subset of the set ω" is consistent with ZFC + "In the class Σ^1_n all subsets of ω are constructive".*

THEOREM 2. *Let $n \geqslant 2$. The proposition "In the class CA_n there exists an infinite D-finite subset of the set R" is consistent with ZF + "In the class A_n there are no infinite D-finite subsets of R."*

(A set is said to be *infinite* if it is not equinumerous with any natural number; that is, if it is not finite. A set is said to be *D-finite* if it is not equinumerous with any of its proper subsets (cf. [4], §5).)

Before formulating Theorem 3, let us introduce some more definitions. If U is any set, by $L(U)$ we denote the class of all sets constructible with respect to U (cf. [4], §11). For $a \subseteq \omega$, we define $[a] = \{x \subseteq \omega: L(x) = L(a)\}$, the collection of all subsets of ω which are "equiconstructible" with a; that is, the *L-degree* or the *degree of constructibility* of a. In addition, if $X, Y, Z \subseteq R$ and if $[X \cap Z = 0 \;\&\; Y \cap Z \neq 0] \vee [X \cap Z \neq 0 \;\&\; Y \cap Z = 0]$ is satisfied, then we say that Z *distinguishes* the sets X and Y.

THEOREM 3. *Let $n \geqslant 2$. The proposition "There exist nonconstructible $a, b \subseteq \omega$ such that there is in the class Π^1_n a set $Z \subseteq R$ which distinguishes $[a]$ and $[b]$, but in the class Σ^1_n there is no such set $Z \subseteq R$" is consistent with ZFC.*

§2. Comments on the theorems

2.1. *On Theorem* 1. The consistency of the assertion of the existence of a nonconstructible subset of ω was proved by Cohen [3]. The first nontrivial result on the position of such subsets in the analytic hierarchy is due to Shoenfield [10]:

PROPOSITION (A). *If $x, y \subseteq \omega$ and $y \in \Sigma_2^{1,x}$, then $y \in L(x)$, and $y \in \Sigma_2^{1,x}$ is true in $L(x)$.*

Proposition (A) follows easily from the well-known Absoluteness Principle:

ABSOLUTENESS PRINCIPLE [10]. *If M is a transitive class which is a model of ZF, if $\omega_1 \subseteq M$, and if φ is a closed Π_2^1-formula with parameters from M, then φ is true in M if and only if it is true in the universe V (that is, the class of all sets).*

(A) implies, in particular, the constructibility of every $x \subseteq \omega$ of the class Σ_2^1. Thus, Theorem 1 is not true for $n = 0, 1$.

In the present paper, we actually prove not Theorem 1, but rather the following stronger one.

THEOREM 1$'$. *Let $n \geqslant 2$. The proposition "There exists an L-minimal $a \subseteq \omega$ satisfying the condition $V = L(a)$ (i.e., all sets are constructible relative to a) and $\{a\} \in \Pi_n^1$" is consistent with ZFC + "If $x, y \subseteq \omega$, if $x \in L$, and if $y \in \Sigma_n^{1,x}$, then $y \in L$ and $L \models y \in \Sigma_n^{1,x}$."*

(A set $a \subseteq \omega$ is called *L-minimal* [8] if a is nonconstructible, and, for every $b \in L(a)$ such that $b \subseteq \omega$, from $b \notin L$ it follows that $a \in L(b)$.)

Theorem 1 follows from Theorem 1$'$ by virtue of the fact that every L-minimal subset of ω is nonconstructible by definition, and from $\{a\} \in \Pi_n^1$ it easily follows that $a \in \Delta_{n+1}^1$.

The consistency of the assertion of the existence of an L-minimal $a \subseteq \omega$ satisfying the condition $V = L(a)$ was proved by Sacks [8]. Later, Jensen [7] strengthened Sacks's result by the requirement $\{a\} \in \Pi_2^1$. In view of Proposition (A), Jensen's result coincides with the special case $n = 2$ of Theorem 1$'$.

Theorem 1$'$ was announced by the author in [16] and proved in [17] by a somewhat different method than in the present paper. Some other results about minimal subsets of ω were announced in [14].

2.2. *On Theorem* 2. The axiom of choice implies the equivalence of the two definitions of finiteness: Every set X is finite if and only if it is D-finite ([4], §5). Thus, the nonexistence of infinite D-finite sets is provable in *ZFC*. Let us also mention that any set X is D-finite if and only if it does not contain a subset equinumerous with the natural number sequence ω ([4], §5); the proof does not use the axiom of choice.

Although the existence of infinite D-finite sets is inconsistent with *ZFC*, nevertheless it is consistent with *ZF* ([3], Chapter 4, §9). Infinite D-finite sets can be projective ([15], Theorem T4) and even occur in Π_2^1 ([13], without proof). On the other hand, in *ZF* (without the axiom of choice!), the following holds.

PROPOSITION (B). *There does not exist an infinite D-finite set X such that $X \subseteq R$ and $X \in \underset{\sim}{\Sigma}_2^1$.*

PROOF. Assume $X \subseteq R$, $X \in \underset{\sim}{\Sigma}_2^1$, X infinite. Let us construct a set $Y \subseteq X$ which is equinumerous with ω. We define G as the collection of all pairs $(n, x) \in \omega \times R$ such that

$(x)_i \in X$ for arbitrary $i < n$ (where $(x)_i = \{k \in \omega: 2^i \cdot 3^k \in x\}$) and $(x)_i \neq (x)_j$ for $i, j < n, j \neq i$. It is clear that $G \in \underset{\sim}{\Sigma}^1_2$ and $\mathrm{dom}(G) = \omega$ (the latter from the infinity of X). By the Novikov-Kondo-Addison uniformization theorem ([5], §16.7, Theorem XLV), we find a function $F \subseteq G$ such that $\mathrm{dom}(F) = \mathrm{dom}(G) = \omega$. Now, $Y = \{(F(m))_i: i < m \in \omega\}$ is the desired subset of X equinumerous with ω, and Proposition (B) is proved.

Proposition (B) refutes Theorem 2 for $n = 0, 1$. It also implies the equivalence of the result of [13] mentioned above and the special case $n = 2$ of Theorem 2.

We also shall prove Theorem 2 in the following stronger form.

THEOREM 2′. *Let $n \geqslant 2$. The proposition "In the class Π^1_n there exists an infinite D-finite subset of the set R" is consistent with ZF + "In the class $\underset{\sim}{\Sigma}^1_n$ there are no infinite D-finite subsets of R."*

The derivation of Theorem 2 from Theorem 2′ is trivial by virtue of the fact that Π^1_n is a subclass of the class $CA_n = \underset{\sim}{\Pi}^1_n$. Theorem 2′ was announced by the author in [16].

2.3. *On Theorem* 3. Assume $X, Y \subseteq R$ and let K be any class. The sets X and Y are said to be *K-distinguishable* if there is a set $Z \in K$, $Z \subseteq R$, such that Z distinguishes X and Y. In the contrary case, we say that X and Y are *K-indistinguishable.*

The problem of the distinguishability of L-degrees of subsets of ω has not been considered in the literature. We note that, if one or both of a and b are constructible, then the situation becomes trivial. In fact, $[a] = [b]$ for $a, b \in L$; $\{0\}$ distinguishes $[a]$ and $[b]$ for $a \in L$ and $b \notin L$.

PROPOSITION (C). *If a and b are nonconstructible (for example, L-minimal) subsets of the set ω, then $[a]$ and $[b]$ are $\Sigma^{1,L}_2$-indistinguishable.*

(By $\Sigma^{1,L}_n$ we denote $\bigcup_{x \in L, x \subseteq \omega} \Sigma^{1,x}_n$; that is, the collection of all $\underset{\sim}{\Sigma}^1_n$-sets "with constructible coding".)

PROOF. Assume $Z \subseteq R$, $Z \in \Sigma^{1,L}_2$ and $Z \cap [a] \neq 0$. It is clear that $Z \nsubseteq L$ (since $a, b \notin L$). According to a theorem of [12], in this case there exists an R-perfect set $P \subseteq Z$ "with constructible coding". Obviously such a set has a nonempty intersection with every $[x], x \subseteq \omega$. Thus $Z \cap [b]$ is nonempty. Q. E. D.

From Proposition (C) it follows that Theorem 3 is not valid for $n = 0, 1$.

Theorem 3 also is proved in a stronger form.

THEOREM 3′. *Let $n \geqslant 2$. The proposition "There exist L-minimal $a, b \subseteq \omega$ such that the sets $R \cap L(a)$ and $R \cap L(b)$ are Π^1_n-distinguishable but $\Sigma^{1,L}_n$-indistinguishable" is consistent with ZFC.*

The derivation of Theorem 3 from Theorem 3′ is based upon the following lemma.

LEMMA. *Let $n \geqslant 2$. Assume that $a, b \subseteq \omega$ are L-minimal, and that the sets $X = R \cap L(a)$ and $Y = R \cap L(b)$ are Π^1_n-distinguishable but $\Sigma^{1,L}_n$-indistinguishable. Then the sets $[a]$ and $[b]$ are Π^1_n-distinguishable but $\Sigma^{1,L}_n$-indistinguishable.*

PROOF. We note two subsidiary facts:

(1) *$X = [a] \cup L^*$ and $Y = [b] \cup L^*$, where $L^* = L \cap R$.* (This follows from the definition of L-minimality.)

(2) $L^* \in \Sigma_2^1$ (cf., for example, [9]).

Assume now that $Z \subseteq R$, $Z \in \Pi_n^1$, and Z distinguishes X and Y. Then $Z' = Z - L^*$ distinguishes $[a]$ and $[b]$ by virtue of (1), and $Z' \in \Pi_n^1$ by virtue of (2) and the fact that $Z \in \Pi_n^1$, $n \geqslant 2$.

Conversely, assume $Z \subseteq R$, $Z \in \Sigma_n^{1,L}$, and Z distinguishes $[a]$ and $[b]$. Then Proposition (C) implies that $n \geqslant 3$. So, the set $Z' = Z - L^*$ belongs to $\Sigma_n^{1,L}$ by (2). On the other hand, Z' distinguishes X and Y by virtue of (1). This proves the lemma.

The derivation of Theorem 3 from Theorem 3′ and the lemma is trivial. Theorem 3′ was announced by the author in [16] in a somewhat different form.

2.4. *Formulation of the Fundamental Theorem (FT).* Theorems 1′, 2′ and 3′ are corollaries of the fundamental theorem FT that will be formulated below. Before formulating FT, let us introduce, for every n, Proposition $\mathfrak{B}_n(A)$ as the conjunction of the following nine propositions.

$\mathfrak{A}1(A)$: *$A \subseteq R$, $V = L(A)$, and the elements of A have pairwise distinct L-degrees.*

$\mathfrak{A}2(A)$: *A is infinite and D-finite.*

$\mathfrak{A}3(A)$: *The elements of A are L-minimal.*

$\mathfrak{A}4_n'(A)$: *A is a set of the class Π_n^1.*

$\mathfrak{A}4_n''(A)$: *In the class Σ_n^1 there are no infinite D-finite subsets of R.*

$\mathfrak{A}5_n'(A)$: *If $a \in A$, then $\{a\} \in \Pi_n^1$ is true in $L(a)$.*

$\mathfrak{A}5_n''(A)$: *If $a \in A$, $x \in R \cap L$, and $y \in R \cap L(a)$, and, moreover, $y \in \Sigma_n^{1,x}$ is true in $L(a)$, then $y \in L$, and $y \in \Sigma_n^{1,x}$ is true in L.*

$\mathfrak{A}6_n'(A)$: *If $a, b \in A$ and $a \neq b$, then the sets $L(a) \cap R$ and $L(b) \cap R$ are Π_n^1-distinguishable in $L(a, b)$.*

$\mathfrak{A}6_n''(A)$: *If $a, b \in A$, then the sets $L(a) \cap R$ and $L(b) \cap R$ are $\Sigma_n^{1,L}$-indistinguishable in $L(a, b)$.*

We shall prove that, for any $n \geqslant 2$, the proposition $\exists A\ \mathfrak{B}_n(A)$ is consistent with *ZF*, from which it is trivial to obtain Theorems 1′, 2′ and 3′. In fact, for Theorem 1′ one has to use parts $\mathfrak{A}1$, $\mathfrak{A}3$, $\mathfrak{A}5_n'$, and $\mathfrak{A}5_n''$ of Proposition $\mathfrak{B}_n$; for Theorem 2′, parts $\mathfrak{A}1$, $\mathfrak{A}2$, $\mathfrak{A}4_n'$, and $\mathfrak{A}4_n''$; for Theorem 3′, parts $\mathfrak{A}1$, $\mathfrak{A}3$, $\mathfrak{A}6_n'$, and $\mathfrak{A}6_n''$.

The consistency of $\exists A\ \mathfrak{B}_n(A)$ is proved in the following form.

FUNDAMENTAL THEOREM (FT). *Let $n \geqslant 2$ and assume that $(\omega_2)^L$ (constructible ω_2, [7]) is countable in the universe. Then there is a set A such that Proposition $\mathfrak{B}_n(A)$ is true in $L(A)$.*

The consistency of the premise of this theorem with *ZF* is well known (cf. [3], Chapter 4, §10); thus, this theorem implies the consistency with *ZF* of the proposition $\exists A\ \mathfrak{B}_n(A)$ for $n \geqslant 2$. Hence, FT implies Theorems 1′, 2′, 3′ (and 1, 2, 3).

2.5. *Plan of the proof.* In the remainder of the paper (§§3–7), we shall assume the countability of $(\omega_2)^L$. In §3, we shall consider a general construction of generic extensions $L(A)$ of the constructible universe L with the help of sets $A \subseteq R$. In this connection, as sets of forcing conditions (s.f.c.) we shall take sets of the form P^∞, where P is some constructible collection of perfect trees in ${}^{<\omega}2$. For the proof of FT it suffices to choose a set $P \in L$, $P \subseteq {}^{<\omega}2$, such that $\mathfrak{B}_n(A)$ is satisfied in all P-generic extensions of the form $L(A)$. The existence of such extensions is obtained in the usual way from the countability of $(\omega_2)^L$ in the universe. Properties $\mathfrak{A}1(A)$ and $\mathfrak{A}2(A)$ are of a trivial nature; their satisfaction in all extensions of the form $L(A)$ also is proved in §3 (Theorem 3.4).

In §4 we shall study properties of extensions of the form $L(A)$, under the assumption that the s.f.c. P has the form $P = \bigcup_{\alpha\in\omega_1} P_\alpha$ in L, where $(P_\alpha, \alpha \in \omega_1)$ is a sequence of the type constructed in [7] (a Jensen sequence, Definition 4.2). We shall show that if, in addition, A is a subset of R which is P-generic over L, then every $a \in A$ will be L-minimal (Theorem 4.6), and A itself will be the collection of all subsets of ω in $L(A)$ which are P-generic over L (4.4). If, in addition, the sequence P_α satisfies a suitable definability condition in L (in [7] such a condition follows from the construction), then in $L(A)$ the properties $\mathfrak{A}i'_n(A)$, $i = 4, 5, 6$, will be satisfied (4.5).

To ensure the truth in $L(A)$ of the "dual" properties $\mathfrak{A}i''_n(A)$ (Theorem 6.1), in §6 we impose on the sequence P_α the requirement of "elementary equivalence" of P and the collection P' of all constructible perfect trees ($P \subseteq P'$) relative to forcing of formulas of the class Π^1_{n-1} (Definition 6.1).

We note that, in the P'-generic extensions of L of the form $L(A)$, the properties $\mathfrak{A}i''_n(A)$, $i = 4, 5, 6$, are satisfied for any n; this stems from the existence of a sufficient number of order automorphisms of the set P'. However, we shall not deal with P'-generic extensions; instead of them we shall consider a suitable relation forc (5.4), to the study of which §5 is devoted.

Finally, in §7, for a fixed $n \geqslant 2$ we shall construct a sequence $(P_\alpha, \alpha \in \omega_1)$ satisfying the requirements of both §4 and §6; this concludes the proof of FT. That is the plan of the proof.

All the standard set-theoretic symbolism is taken from [4], except for the following change: cardinality is denoted by card(x), instead of $|x|$.

The author is deeply grateful to Professors V. A. Uspenskiĭ and V. N. Grišin for reading this paper and for their valuable assistance.

§3. Generic extensions used in the proof of FT

We presuppose that the reader is familiar with the general theory of generic extensions and the method of forcing, as well as with the elementary theory of the hierarchy of subsets of spaces of the form $\omega^k \times R^m$ and of formulas of second-order arithmetic ([3], Chapter 16). The construction of generic extensions of permutational (symmetric) type and the symmetry properties of such extensions are taken from [3], Chapter 4, §9. In the presentation of the connection between forcing and truth in generic extensions we follow [6]. Finally, from [7] we take the idea of replacing generic filters by generic subsets of ω, and the use of generic extensions of the constructible universe L, but not of a countable model of the theory $ZF + V = L$. Moreover, the existence of generic extensions is guaranteed by the introduction of the additional axiom on the countability of $(\omega_2)^L$ in the hypothesis of FT.

We shall begin with the definition of the forcing conditions–perfect trees in the set ${}^{<\omega}2$.

3.1. *Perfect trees.* By ${}^YX$ we denote the set $\{f : f \text{ is a function from } Y \text{ into } X\}$ ([4], p. 5). (This is not to be confused with the Cartesian power X^m; see below.) We introduce the set ${}^{<\omega}2 = \bigcup_{n\in\omega} {}^n2$, the collection of all finite sequences of zeros and ones (including the sequence 0 of length zero). For $e \in {}^{<\omega}2$ and $i \in 2$ we define $e \,\hat{}\, i = e \cup \{(\mathrm{dom}(e), i)\}$ (an "extension" of the sequence e) ([4], §18).

A nonempty $p \subseteq {}^{<\omega}2$ is called a *perfect tree* (in ${}^{<\omega}2$) if: (1) for $e \in p$ and $e' \in {}^{<\omega}2$, $e' \subseteq e$ implies $e' \in p$; (2) for arbitrary $e_0 \in p$ there exists $e \in p$ such that $e_0 \subseteq e$ and $e\hat{}0$ and $e\hat{}1$ belong to p. By Perf we denote the collection of all perfect trees.

For every $p \in \mathrm{Perf}$ one can define Cl_p as the collection of all $x \subseteq \omega$ such that, for each $n \in \omega$, there exists $e \in p \cap {}^n2$ satisfying $(\forall i < n)\ [i \in x \equiv e(i) = 1]$. Cl_p is a perfect subset of the space $R = \mathcal{P}(\omega)$.

If $e \in {}^{<\omega}2$, then by $\langle e\rangle$ we denote $\{e' \in {}^{<\omega}2\colon\ e' \subseteq e \text{ or } e \subseteq e'\}$; $\langle e\rangle \in \mathrm{Perf}$, and $\mathrm{Cl}_{\langle e\rangle}$ is clopen (closed and open).

Before any further definitions, let us agree that an (m-tuple) finite sequence $(s_0, \ldots, s_{m-1})$ coincides with 0 for $m = 0$ and with s_0 for $m = 1$. Then the Cartesian power X^m of every nonempty X is equal to $\{0\} = 1$ for $m = 0$ and is equal to X for $m = 1$.

Assume now that $p = (p_0, \ldots, p_{m-1}) \in \mathrm{Perf}^m$ (an m-tuple of perfect trees). We define $\mathrm{Cl}_p = \{(x_0, \ldots, x_{m-1})\colon\ x_i \in \mathrm{Cl}_{p_i} \text{ for every } i < m\}$ (if $m = 0$, then $\mathrm{Cl}_p = \mathrm{Cl}_0 = \{0\} = R^0$; if $m = 1$, then $\mathrm{Cl}_p = \mathrm{Cl}_{p_0}$). For the same p we define $\|p\| = m$ ("dimension"). If $k \leqslant m$, then we set $p \restriction k = (p_0, \ldots, p_{k-1})$ ($= 0$ for $k = 0$), and for $k \geqslant m$ we define

$$p \restriction k = (p_0, \ldots, p_{m-1}, \underbrace{{}^{<\omega}2, \ldots, {}^{<\omega}2}_{k-m \text{ times}}).$$

Similarly, if $a = (a_0, \ldots, a_{m-1}) \in R^m$, $k \leqslant m$, we set $a \restriction k = (a_0, \ldots, a_{k-1})$. If $a = (a_i, i \in \omega) \in {}^\omega R$ and $k \in \omega$, then by $a \restriction k$ we denote the finite sequence $(a_0, \ldots, a_{k-1})$ $(\in R^k)$.

3.2. *Generic sequences and finite sequences.* For every $P \subseteq \mathrm{Perf}$ we define $P^\infty = \bigcup_{m\in\omega} P^m$ (the collection of all finite sequences of elements of P). The set Perf^∞ is ordered by the relation $\leqslant$: $(p_0, \ldots, p_{k-1}) \leqslant (q_0, \ldots, q_{m-1})$ if and only if $k \leqslant m$ and, for every $i < k$, $q_i \subseteq p_i$ holds.

As sets of forcing conditions we shall consider sets of the form P^∞ (ordered by the relation $\leqslant$), where $P \subseteq \mathrm{Perf}$ satisfies the condition

$$\textit{if}\quad e \in p \in P, \quad \textit{then}\quad p \cap \langle e\rangle \in P.$$

The collection of all P satisfying this condition will be denoted by Spl ("splitting" sets).

We now introduce the important concept of generic sequences and finite sequences of subsets of ω. Let K be a transitive class which is a model of ZF (below, as a rule, $K = L$), and let $P \in L \cap \mathrm{Spl}$ be fixed. A sequence (function) $a \in {}^\omega R$ is said to be a *P-generic* *sequence over K* if the set $G_a = \{p \in P^\infty : a \upharpoonright \|p\| \in \mathrm{Cl}_p\}$ has a nonempty intersection with every set $Q \in K$ which is dense in P^∞. (Q is said to be *dense in U* ([6], I. 1.2) if $Q \subseteq U \subseteq \mathrm{Perf}^\infty$ and the following conditions hold: (1) $q \in Q \;\&\; u \in U \;\&\; u \geqslant q \longrightarrow u \in Q$, and (2) $(\forall u \in U)\, (\exists q \in Q)\, [q \geqslant u]$.) P-generic sequences over L will be called simply *P-generic sequences.*

The method of dealing with P-generic sequences instead of generic filters on P^∞ is taken from [7]. Moreover, membership of P in Spl implies that $K(a) = K(G_a)$, as well as the fact that G_a is a P-generic filter over K, for any P-generic sequence a over K (cf. [7], the reasoning on p. 124).

Similarly, if $a \in R^m$, we define $G_a = \{p \in P^m : a \in \mathrm{Cl}_p\}$; the finite sequence a is called a *P-generic m-tuple over K* if $G_a \cap Q$ is nonempty for every set $Q \in K$ which is dense in P^m. In particular, $a \subseteq \omega$ is a P-generic subset of ω over K if $(a) = a$ is a P-generic 1-tuple over K (this coincides with the definition in [7]). As above, P-generic m-tuples over L are simply called *P-generic m-tuples.*

We shall use the following properties of generic sequences and finite sequences over L. Assume $P \in$ Spl is constructible.

1. *If $(a_0, \ldots, a_m)$ is a P-generic $(m + 1)$-tuple, then $a_m \notin L(a_0, \ldots, a_{m-1})$.* In fact, by the "produce lemma" ([6], I. 2.3), a_m is P-generic over $K = L(a_0, \ldots, a_{m-1})$. From this and $P \in$ Spl it follows easily that $a_m \notin K$, Q.E.D.

2. *If $a = (a_i, i \in \omega)$ is a P-generic sequence (over L) and if $m \in \omega$, then $a_m \notin L(\{a_i : i \neq m\})$.* (The proof is similar.)

3. *If a is as above, $i, j \in \omega$, and a' is obtained from a by transposing a_i and a_j, then a' is also a P-generic sequence (over L).* (This follows easily from the definitions.)

4. *If $m \in \omega$, $a \in R^m$ is a P-generic m-tuple, $Q \in L$, and Q is predense in P^m, then $a \in \bigcup_{q \in Q} \mathrm{Cl}_q$.* (Obvious)

(A set $Q \subseteq P^m$ is called *predense in P^m* if $\{p \in P^m : (\exists q \in Q)\, [p \geqslant q]\}$ is dense in P^m [7].)

5. *Assume $p \in P^\infty$. Then there exists a P-generic (over L) sequence $a \in {}^\omega R$ satisfying $p \in G_a$.*

In fact, the countability of $(\omega_2)^L$ (cf. 2.5) easily yields the countability (in the universe) of the collection of all constructible sets that are dense in P^∞; then we apply the argument of [6], I.1.8.

3.3. *Forcing.* We fix some $P \in L \cap$ Spl. A set $A \subseteq R$ is called a *P-generic set* if $A = \mathrm{rng}(a) = \{a_i : i \in \omega\}$ for some P-generic (over L) sequence $a = (a_i, i \in \omega)$. For the study of generic extensions of the form $L(A)$, where A is a P-generic set, we introduce a suitable forcing relation.

First we define the language $\mathcal{L}$ to be the extension of the usual ϵ-language obtained by adding the constants $\underline{A}$, $\underline{a}_k$ $(k \in \omega)$, and $\underline{x}$ $(x \in L)$. If φ is a formula of $\mathcal{L}$, and $a = (a_i, i \in \omega) \in {}^\omega R$, then we define the interpretation φ^a of the formula φ by changing $\underline{A}$, $\underline{a}_k$ and $\underline{x}$ to $\{a_i : i \in \omega\}$, a_k and x, respectively. Following [6], I.1.9, we introduce a forcing relation: *$p \Vdash_P \varphi$ if and only if φ^a is true in $L(\mathrm{rng}(a))$ for any P-generic (over L) sequence a satisfying $p \in G_a$.*

In this definition it is presupposed that $p \in P^\infty$ and that φ is a closed formula of the language L. If P is clearly determined by the context (as in this section), then we shall write $\Vdash$ instead of $\Vdash_P$.

The basic properties of the relation $\Vdash_P$ are as follows:

1. *The relation* $\Vdash$ *(i.e.,* $\Vdash_P$ *for a given* $P \in \mathrm{Spl}$*) is expressible in L in the following sense: if* $\varphi(x_1, \ldots, x_m)$ *is an* $\in$*-formula without parameters, then* $\{(p, c_1, \ldots, c_m)$: $p \in P$, $c_1, \ldots, c_m$ *are constants of the language* L *and* $p \Vdash \varphi(c_1, \ldots, c_m)\}$ *is a class in* L.

2. *Assume a is a P-generic sequence,* $A = \mathrm{rng}(a)$, φ *is a formula of the language* L *and* φ^a *is true in* $L(A)$. *Then there exists* $p \in G_a$ *such that* $p \Vdash \varphi$.

We refer the reader to [6], I. 1.9.

3.4. *Symmetry properties, and securing* $\mathfrak{A}1(A)$ *and* $\mathfrak{A}2(A)$ *in* $L(A)$. The set P of 3.3 remains fixed. We formulate some properties of generic extensions of type 3.3, studied in detail in [3], Chapter 4, §9.

1. *Assume φ is a closed formula of the language* L. *Then the set* $\{p \in P^\infty: p \Vdash \varphi$ *or* $p \Vdash \sim \varphi\}$ *is dense in* P^∞ *and belongs to* L. (This follows easily from 3.3.1 and 3.3.2.)

2. *Assume* $p \in P^\infty$, φ *is a formula of language* L, $m \in \omega$, *and every constant* $\underline{a}_i$ *occurring in* φ *satisfies* $i < m$. *Then* $p \Vdash \varphi$ *and* $p \restriction m \Vdash \varphi$ *are equivalent.*

3. *If φ and m are as in 2, then the set* $\{p \in P^m: p \Vdash \varphi$ *or* $p \Vdash \sim \varphi\}$ *is dense in* P^m *and belongs to* L.

4. *If* $A \subseteq R$ *is a P-generic set and* $z \in L(A)$, $z \subseteq \omega$, *then there exist* $m \in \omega$ *and* $a_0, \ldots, a_{m-1} \in A$ *such that* $z \in L(a_0, \ldots, a_{m-1})$.

Assertions 2 and 4 are common to all "permutation" extensions ([3], proof of the lemma on Russian p. 261), and 3 follows from 1 and 2.

THEOREM. *Assume* $A \subseteq R$ *is a P-generic set. Then* $\mathfrak{A}1(A)$ *and* $\mathfrak{A}2(A)$ *are satisfied in* $L(A)$.

We carry out the proof in $L(A)$. The validity of $\mathfrak{A}1(A)$ is obvious from 3.2.2. Let us prove that A is infinite and D-finite. That A is infinite follows from 3.2.2 again. Let us suppose that f is a bijection from ω into A ($f \in L(A)$). Applying assertion 4, we easily obtain that $f \in L(a_0, \ldots, a_{m-1})$ for some $m \in \omega$ and $a_0, \ldots, a_{m-1} \in A$. Then $\mathrm{rng}(f) \subseteq L(a_0, \ldots, a_{m-1})$. Again by 3.2.2, since $\mathrm{rng}(f) \subseteq A$, we conclude that $\mathrm{rng}(f) \subseteq \{a_0, \ldots, a_{m-1}\}$, which contradicts the fact that f is bijective. This contradiction completes the proof of the theorem.

§4. Application of Jensen's method for securing properties $\mathfrak{A}3(A)$ and $\mathfrak{A}i'_n(A)$, $i = 4, 5, 6$, in $L(A)$

Thus, we are able to construct a set $A \subseteq R$ such that $\mathfrak{A}1(A)$ and $\mathfrak{A}2(A)$ are true in $L(A)$. In fact, one must take any constructible $P \in \mathrm{Spl}$ (for example, $P = \mathrm{Perf} \cap L$) and any P-generic (over L) set A.

In this section we shall consider a method for constructing constructible $P \in \mathrm{Spl}$ such that $\mathfrak{A}3(A)$ and $\mathfrak{A}i'_n(A)$, $i = 4, 5, 6$, also hold in $L(A)$. This method consists in constructing P (in L) in the form $P = \bigcup_\alpha P_\alpha$, where $(P_\alpha, \alpha \in (\omega_1)^L)$ is a sequence constructed by the method of [7] (such sequences will be called *Jensen sequences*—see Definition 4.2). Imposing on the Jensen sequence $(P_\alpha, \alpha \in (\omega_1)^L)$ the additional requirement of definability in L, we obtain the fulfillment in $L(A)$ of properties $\mathfrak{A}3(A)$ and $\mathfrak{A}i'_n(A)$, $i = 4, 5, 6$ (for suitable $n \geqslant 2$) (Theorems 4.5 and 4.6).

Below in §6 we shall indicate another requirement on Jensen sequences leading to the fulfillment in $L(A)$ of properties $\mathfrak{A}i''_n(A)$. In this connection, in §4 we shall deal with not a particular Jensen sequence (as was done in [7]), but rather an "arbitrary" one. A particular choice of a Jensen sequence for the proof of FT is carried out in §7.

The ideas of [7] form the basis for the arguments of this section.

4.1. *Splitting.* Let us introduce some preliminary definitions. If $p, q \in \mathrm{Perf}$ and $p = q \cap (\langle e_1\rangle \cup \cdots \cup \langle e_m\rangle)$ for certain $e_1, \ldots, e_m \in q$, then we say that p is *closed-open* (clopen) in q; in that case, $\mathrm{Cl}_p \cap \mathrm{Cl}_q$ will be clopen in Cl_q in the topological sense. We say that $p = (p_0, \ldots, p_{m-1}) \in \mathrm{Perf}^m$ is *componentwise disjoint* if $p_i \cap p_j$ is finite (that is, $\mathrm{Cl}_{p_i} \cap \mathrm{Cl}_{p_j}$ is empty) for $i \neq j$. If $p \in \mathrm{Perf}^m$, $U \subseteq \mathrm{Perf}^m$, and $p \subseteq \bigcup Q$ for some finite $Q \subseteq U$, then we write $p \geqslant \bigvee U$.

Assume $P, Q \in \mathrm{Spl}$, and let Ξ be any set. We say that Q is a *splitting of P over Ξ* if the following assertions (1)–(4) hold:

(1) for every $p \in P$, there exists $q \in Q$ such that $q \subseteq p$;

(2) for every $q \in Q$, there exists $p \in P$ such that $q \subseteq p$;

(3) if $p \in P$ and $q \in Q$, then p is clopen in q;

(4) if $m \in \omega$, $U \in \Xi$ is predense in P^m, and $q \in Q^m$ is componentwise disjoint, then $q \geqslant \bigvee U$.

Let us show that, if (1), (2) and (3) hold, then (4) is equivalent to:

(5) if $m \in \omega$, $U \in \Xi$ is predense in P^m, and $a_0, \ldots, a_{m-1} \in \bigcup_{q\in Q}\mathrm{Cl}_q$ are pairwise distinct, then $(a_0, \ldots, a_{m-1}) \in \bigcup_{u\in U}\mathrm{Cl}_u$.

First let use prove (5) $\longrightarrow$ (4). Assume $q \in Q^m$ is componentwise disjoint. Then every $a = (a_0, \ldots, a_{m-1}) \in \mathrm{Cl}_q$ obviously satisfies $a_i \neq a_j$ for $i \neq j$. According to (5), this yields $a \in \bigcup_{u\in U}\mathrm{Cl}_u$. Thus, $\mathrm{Cl}_q \subseteq \bigcup_{u\in U}\mathrm{Cl}_u$. But, by (3), the set $\mathrm{Cl}_u \cap \mathrm{Cl}_q$ is clopen in Cl_q for any $u \in U$. From this, with the help of the compactness of R^m, we obtain $q \geqslant \bigvee U$, i.e. (4).

Let us prove the converse implication. Since $Q \in \mathrm{Spl}$, it is not difficult to choose a componentwise disjoint $q \in Q^m$ such that $a = (a_0, \ldots, a_{m-1}) \in \mathrm{Cl}_q$. But, according to (4), $q \geqslant \bigvee U$, and this yields $a \in \bigcup_{u\in U}\mathrm{Cl}_u$, i.e., (5) has been proved. This completes the proof of the equivalence.

4.2. *Jensen sequences.* Assume $\pi = (P_\alpha, \alpha \in \nu) \in L$ is a sequence of elements of Spl that are countable in L. We call π a Spl-*sequence* (of length ν) if, for every $\beta \in \nu$, the set P_β is a splitting of $P_{<\beta} = \bigcup_{\alpha\in\beta}P_\alpha$ over $L_{\gamma^*(\pi/\beta)}$, where by $\gamma^*(\pi/\beta)$ we mean the least $\gamma \in \mathrm{On}$ for which $\pi|\beta \in L_\gamma$ and β is countable in L_γ. Spl-sequences of length $(\omega_1)^L$ are called *Jensen sequences.* We note that every Spl-sequence (and Jensen sequence) is constructible by definition.

Jensen sequences were constructed for the first time in [7]. We shall formulate some trivial properties of Spl-sequences.

LEMMA 1. *Assume $\pi = (P_\alpha, \alpha \in \nu)$ is a* Spl-*sequence and $\alpha \in \nu$. Then P_α is predense in $P = \bigcup_{\alpha\in\nu} P_\alpha$* (cf. [7], Corollary 5).

The proof is easily obtained by induction on α and ν, taking into account that $\{P_\alpha : \alpha \in \beta\} \subseteq L_{\gamma^*(\pi|\beta)}$ for any $\beta \in \nu$.

From the lemma and the definitions, we obtain

COROLLARY 2. *Let π be as in Lemma 1, and assume $\alpha \in \beta \in \nu$.*

(1) *If $q \in P_\beta$, then $q \geqslant \bigvee P_\alpha$.*

(2) *If $p \in P_\alpha$, then there exists $q \in P_\beta$ such that $q \geqslant p$.*

4.3. *A property of antichains.* If $Q \subseteq U \subseteq \mathrm{Perf}^\infty$ and, for arbitrary $p, q \in Q$ and $u \in U$, $u \geqslant p \;\&\; u \geqslant q$ implies $p = q$, then we call Q an *antichain* in U. A maximal antichain in U obviously is a set which is predense in U.

THEOREM. *Assume $\pi = (P_\alpha, \alpha \in (\omega_1)^L)$ is a Jensen sequence, $P = \bigcup_{\alpha\in(\omega_1)^L} P_\alpha$, $m \in \omega$, and $Q \in L$ is a maximal antichain in P^m. Then Q is countable in L and there exists $\lambda \in (\omega_1)^L$ such that the following conditions are satisfied*:

(i) *$Q \subseteq P^m_{<\lambda}$ (where $P_{<\lambda} = \bigcup_{\alpha\in\lambda} P_\alpha$).*

(ii) *If $a_0, \ldots, a_{m-1} \in \bigcup_{p\in P_\lambda} \mathrm{Cl}_p$ are pairwise distinct, then $(a_0, \ldots, a_{m-1}) \in \bigcup_{q\in Q} \mathrm{Cl}_q$.*

(iii) *If $q \in P^m_\lambda$ is componentwise disjoint, then $q \geqslant \bigvee Q$.*

PROOF. The beginning of the argument is in L. We introduce the notation

$$Q_{<\lambda} = (P^m_{<\lambda}) \cap Q$$

(where $P_{<\lambda} = \bigcup_{\alpha\in\lambda} P_\alpha$) for every $\lambda \in \omega_1$. It is clear that ω_1, π, P and Q are elements of L_{ω_2}. This permits us to choose a countable set $M \in L_{\omega_2}$ containing the indicated sets and such that

(1) *M is an elementary submodel of L_{ω_2}.*

Obviously, M is an extensional set. So, there exist a transitive N and a $\in$-isomorphism φ of M onto N. We define $\lambda = \varphi(\omega_1)$, and we shall show that λ is what is required. Taking into account (1) and the choice of φ and N, we obtain the following sequence of propositions:

(2) *If $x \in M$ is countable, then $x \subseteq M$.* (One uses (1) and the formula "There exists a function from ω onto x".)

(3) *If $x \in M$ is hereditarily countable, then $\varphi(x) = x$.* (This follows from (2) by induction on the rank of x.)

(4) $\lambda = M \cap \omega_1$ (from (2) and (3)).

(5) $\varphi(\pi) = \pi|\lambda = (P_\alpha, \alpha \in \lambda)$ (from (1), (3), (4) and the choice of φ).

(6) $\varphi(P) = P_{<\lambda}$, $\varphi(Q) = Q_{<\lambda}$ (from (3) and (5)).

(7) *$Q_{<\lambda}$ is a maximal antichain in $P^m_{<\lambda}$.* (This follows from (1), (6), and the hypotheses.) *Hence, $Q_{<\lambda}$ is predense in $P^m_{<\lambda}$.*

(8) *$N = L_\delta$ for some $\delta \in \omega_1$.*

(9) *λ is nondenumerable in $N = L_\delta$* (since ω_1 is nondenumerable in L_{ω_2}); *thus, $\delta < \gamma^*(\pi|\lambda)$ and $N \subseteq L_{\gamma^*(\pi|\lambda)}$.*

(10) $Q_{<\lambda} \in L_{\gamma^*(\pi|\lambda)}$ (from (7) and (9)).

(11) $Q = Q_{<\lambda}$, *i.e.*, $Q \subseteq P^m_{<\lambda}$. In fact, suppose the contrary; that is, assume $v \in Q - Q_{<\lambda}$. By 4.2.2(2), one can select $\beta \in \omega_1$, $\beta \geqslant \lambda$, and a componentwise disjoint $q \in P^m_\beta$ such that $q \geqslant v$. (One also uses $P_\beta \in \mathrm{Spl}$.) Applying (7) and (10) and taking Definition 4.2, into account, we obtain $q \geqslant \bigvee Q_{<\lambda}$. From this, since u and $P_\beta \in \mathrm{Spl}$, we obtain the existence of $u \in Q_{<\lambda}$ and $w \in P^m_\beta$ such that $w \geqslant q$ and $w \geqslant u$. Thus, $w \geqslant u$, $w \geqslant v$, $w \in P^m$. But u and v automatically are distinct elements of Q ($u \in Q_{<\lambda}$; $v \in Q - Q_{<\lambda}$), contrary to the choice of Q as an antichain. This proves (11).

This ends the reasoning in L. Let us finish the proof of the theorem. From (11) we deduce (i) and the countability of Q in L, and (ii) and (iii) follow from (7), (10), Definition 4.2, and parts 4.1(4) and 4.1(5) of Definition 4.1.

So, λ is the required ordinal, and the theorem is proved.

4.4. *Definability of A in $L(A)$.* Assume $\pi = (P_\alpha, \alpha \in (\omega_1)^L)$ is a Jensen sequence, and $P = \bigcup_{\alpha\in(\omega_1)^L} P_\alpha$. In [7], with the help of an analogue of Theorem 4.3, it was established that, if $a \subseteq \omega$ is P-generic over L, then

$$\{a\} = \bigcap_{\alpha\in(\omega_1)^L} \bigcup_{p\in P_\alpha} \mathrm{Cl}_p$$

is true in $L(a)$. We shall prove here an analogue of this proposition for extensions of type 3.3.

THEOREM. *Let π and P be as above, and assume that $A \subseteq R$ is a P-generic set. Then $(\omega_1)^L = (\omega_1)^{L(A)}$ and*

$$L(A) \models A = \bigcap_{\alpha\in(\omega_1)^L} \bigcup_{p\in P_\alpha} \mathrm{Cl}_p.$$

PROOF. Let $A = \{a_k : k \in \omega\}$, where $a = (a_k, k \in \omega)$ is a P-generic sequence over L. We note that every constructible antichain in P^∞ is at most countable in L, by virtue of 4.3. From this, in the usual way ([4], Lemma 56), we obtain $(\omega_1)^L = (\omega_1)^{L(A)}$. In addition, every P_α, $\alpha \in (\omega_1)^L$, is predense in P by 4.2.1, and every a_k is P-generic. Applying 3.2.4, we obtain the inclusion from left to right in the statement of the theorem. For a proof of the reverse inclusion, we prove the following lemma:

LEMMA 1. *Assume $m \in \omega$, $b \subseteq \omega$, $b \notin \{a_0, \ldots, a_{m-1}\}$, and $b \in \bigcup_{p\in P_\alpha} \mathrm{Cl}_p$ for every $\alpha \in (\omega_1)^L$. Then $b \notin L(a_0, \ldots, a_{m-1})$.*

PROOF. By 3.2.1 it suffices to verify that $\mathbf{b} = (a_0, \ldots, a_{m-1}, b)$ is a P-generic $(m+1)$-tuple. Assume $U \in L$ and U dense in P^{m+1}. We select $Q \subseteq U$, $Q \in L$, such that Q is a maximal antichain in U. Then Q is a maximal antichain in P^{m+1} (since U is dense in P^{m+1}). Let $\lambda \in (\omega_1)^L$ be such that the requirement 4.3(ii) is satisfied. (It exists by Theorem 4.3.)

We note that $a_0, \ldots, a_{m+1}$ and b are pairwise distinct ($a_i \neq a_j$ by (3.2.2)) and are elements of the set $\bigcup_{p\in P_\lambda} \mathrm{Cl}_p$. (For b this follows from the hypothesis of the lemma, and for the a_i from the already proved inclusion from left to right in the theorem.) By the choice of λ, this gives $\mathbf{b} \in \bigcup_{q\in Q} \mathrm{Cl}_q$. A fortiori, $\mathbf{b} \in \bigcup_{u\in U} \mathrm{Cl}_u$ ($Q \subseteq U$). This means that $G_{\mathbf{b}} \cap U \neq 0$, which proves the lemma.

Let us return to the proof of the theorem. Assume $b \subseteq \omega$ and $b \in \bigcup_{p \in P_\alpha} \mathrm{Cl}_p$ for every $\alpha \in (\omega_1)^L$. Let us show that $b \in A$. In fact, from 3.4.4 it follows that $b \in L(a_0, \ldots, a_{m-1})$ for some $m \in \omega$. Now, by Lemma 1, we obtain $b \in \{a_0, \ldots, a_{m-1}\}$, i.e., $b \in A$. This proves the inclusion from right to left and the theorem.

4.5. *Securing* $\mathfrak{A}i'_n(A)$, $i = 4, 5, 6$. Let π and P be as in 4.4, and assume that $A \subseteq R$ is a P-generic set. The preceding theorem shows that A is definable in $L(A)$ by some $\in$-formula (with parameter π). So, imposing on π a certain condition of definability in L, we can expect the satisfaction in $L(A)$ of the properties $\mathfrak{A}i'_n(A)$, $i = 4, 5, 6$, for suitable n. Let us take a short pause to introduce certain concepts connected with *$\in$-definability in the collection T of all hereditarily at most countable sets.* (We remark that $T = L_{\omega_1}$ if the axiom of constructibility $V = L$ holds.)

Σ_n and Π_n are standard designations for classes of $\in$-formulas [9]. By Σ_n^T we denote the collection of all $X \subseteq T$ defined in T by some Σ_n-formula without parameters. Analogously, one defines Π_n^T; then $\Delta_n^T = \Sigma_n^T \cap \Pi_n^T$. There is a remarkable connection between definability in T and analytic definability:

PROPOSITION 1 ([11], the lemma on p. 281). *Assume $n \geqslant 1$ and $X \subseteq R$. Then $X \in \Sigma_{n+1}^1$ if and only if $X \in \Sigma_n^T$, and similarly for Π and Δ.*

We shall use this proposition to prove the following theorem.

THEOREM. *Let π, P and $A = \{a_k : k \in \omega\}$ be such that, as above, $n \geqslant 2$ and $\pi \in \Sigma_{n-1}^T$ is true in L. Then $\mathfrak{A}i'_n(A)$, $i = 4, 5, 6$, hold in $L(A)$.*

We carry out the proof in $L(A)$. From Theorem 4.4 it follows that $(\omega_1)^L = \omega_1$ $(= (\omega_1)^{L(A)})$. From this, by the hypothesis and the equation $T = L_{\omega_1}$ in L, we obtain that $\pi \in \Sigma_{n-1}^{L_{\omega_1}}$ (in $L(A)$).

On the other hand, $L_{\omega_1} \in \Sigma_1^T$ (cf., for example, [7], the proof of Corollary 9). This yields $\pi \in \Sigma_{n-1}^T$. This means, by Theorem 4.4, that $A \in \Pi_{n-1}^T$. Finally, using Proposition 1, we obtain $A \in \Pi_n^1$; that is, $\mathfrak{A}4'_n(A)$ (true in $L(A)$).

Let us turn to $\mathfrak{A}5'_n(A)$. Assume $a = a_k \in A$. From 3.2.2 it follows that $A \cap L(a) = \{a\}$, whence, just as in the proof of $\mathfrak{A}4'_n(A)$, we obtain $L(A) \models \{a\} \in \Pi_n^1$; that is, $\mathfrak{A}5'_n(A)$.

Finally, let us consider $\mathfrak{A}6'_n(A)$. Assume $k, l \in \omega$, $k \neq l$. Then $L(a_k) \cap \{a_l\} = 0$ by 3.2.2, and $L(a_l) \cap \{a_l\} \neq 0$ (obvious). It suffices to prove, therefore, that $\{a_l\} \in \Pi_n^1$ in $L(a_k, a_l)$. But $\{a_k, a_l\} = A \cap L(a_k, a_l)$ (by 3.2.2); from this, as above, we obtain $\{a_k, a_l\} \in \Pi_n^1$ in $L(a_k, a_l)$. Now it is obvious that $L(a_k, a_l) \models \{a_l\} \in \Pi_n^1$, which proves the theorem.

4.6. *Securing the minimality of elements of A.* The objects π, P and $A = \{a_k : k \in \omega\}$ of 4.5 remain fixed. We shall prove that every $a \in A$ is L-minimal. We note that our proof differs from the proof of minimality in [7].

THEOREM . $L(A) \models \mathfrak{A}3(A)$; *that is, every a_k is L-minimal.*

PROOF. By 3.2.3, it suffices to prove the L-minimality of a_0. $a_0 \notin L$ follows from 3.2.2. Assume now that $x \in L(a_0) - L$ and $x \subseteq \omega$. Let us prove that $a_0 \in L(x)$. From the choice of x it is clear that there exists a formula $\chi(k)$ of the language L (3.3), containing only the constant $\underline{a}_0$ and constants of the form $\underline{c}$, $c \in L$, and satisfying $x = \{k \in \omega: L(a_0) \models \chi^a(k)\}$. We write $\models$ instead of $\Vdash_P$ (cf. 3.3). We may assume that, if $p \in P^\infty$, then

$$p \Vdash [\{k \in \omega : L(\underline{a}_0) \models \chi(k)\} \notin L$$

(if this is not so, then we consider the formula

$$\chi'(k) \rightleftharpoons [\chi(k) \text{ if } \{k \in \omega : L(\underline{a}_0) \models \chi(k)\} \notin L; \text{ and } k \in \underline{a}_0 \text{ otherwise}]).$$

Then the set $U = \{(p, q) \in P^2 : \text{for some } k \in \omega, \text{ either } p \Vdash \chi(\underline{k})\ \&\ q \Vdash \sim \chi(\underline{k}) \text{ or } p \Vdash \sim \chi(\underline{k})\ \&\ q \Vdash \chi(\underline{k})\}$ is dense in P^2 (by 3.4.3 and the choice of χ), and $U \in L$ (by 3.3.1).

Now, choosing a maximal antichain and using Theorem 4.3, in a way similar to the proof of 4.4, we select $\lambda \in (\omega_1)^L$ such that

(1) *if $u, v \in P_\lambda$ and $u \cap v$ is finite (that is, (u, v) is componentwise disjoint), then $(u, v) \geqslant \bigvee U$.*

Assume Φ is the collection of all formulas obtained from formulas of the form $\chi(\underline{k})$, $k \in \omega$, with the help of the symbols $\sim$, & and $\vee$. From (1) and the definition of U, one easily obtains

(2) *If $u, v \in P_\lambda$, and $u \cap v$ is finite, then there exists a formula $\varphi \in \Phi$ such that $u \Vdash \varphi$, but $v \Vdash \sim \varphi$.*

In addition, for $\varphi \in \Phi$ we define its interpretation $x \models \varphi$ interpreting $\chi(\underline{k})$ as $k \in x$. We now observe that from 4.2.1 and 3.2.4 it follows that

$$a_0 \in \bigcup_{p \in P_\lambda} \mathrm{Cl}_p.$$

From this, since $P_\lambda \in \mathrm{Spl}$, on taking account of 3.4.2 and 3.3.2, we obtain that a_0 is the unique element in the set $\bigcap\{\mathrm{Cl}_u : u \in P_\lambda$, and, if $\varphi \in \Phi$ and $x \models \varphi$, then $\sim u \Vdash \sim \varphi\}$. Now $a_0 \in L(x)$ is obvious, and the theorem is proved.

§5. The language λ and ramified forcing

5.1. *Arithmetic functions.* Now we want to introduce a language for describing real numbers in extensions of the form $L(A)$, where $A \subseteq R$ is a P-generic set, and $P = \bigcup_{\alpha \in (\omega_1)^L} P_\alpha$ for some Jensen sequence $(P_\alpha, \alpha \in (\omega_1)^L)$. We desire, in particular, that this language contain constants for all subsets of ω in $L(A)$.

The well-known methods of introducing a "ramified" language and forcing (the parametric space of [3] or the Boolean-valued universe of [4]) cannot be used because of their great complexity (in the sense of definability) and their "nondescriptive" character. We offer a considerably simpler construction, connected with the use of arithmetic functions as constants for subsets of ω in extensions of the form $L(A)$.

A function $F : R \longrightarrow R$ is said to be an *arithmetic function* (a.f.) if $\{(x, m): x \subseteq \omega\ \&\ m \in F(x)\}$ is a set in the class Σ^0_∞ $(= \bigcup_{n \in \omega} \Sigma^0_n)$. It is clear that the collection $\mathcal{F}$ of all a.f.'s is a continuum, $\mathcal{F} = \{F_f : f \subseteq \omega\}$, where the enumeration F_f can be chosen so that

$$\{(x, f, m) : x, f \subseteq \omega\ \&\ m \in F_f(x)\} \in \Delta^1_1. \tag{1}$$

In fact, to every $f \subseteq \omega$ one can, in a canonical fashion, associate a construction of a Σ^0_∞-set $X_f \subseteq R \times \omega$, starting from elementary intervals; further, F_f is set equal to $\{(x, X_f''\{x\}) : x \subseteq \omega\}$. (1) is satisfied by virtue of the canonical character of the choice of X_f from f.

Let us extend the operation of F_f onto R^m, $m \in \omega$. If $m \geqslant 1$, we define a homeomorphism Φ_m by the condition $\Phi_m(x_0, \ldots, x_{m-1}) = \{mk + i : i < m \;\&\; k \in x_i\}$, and we set $F_f(x_0, \ldots, x_{m-1}) = F_f(\Phi_m(x_0, \ldots, x_{m-1}))$. Separately, for $m = 0$ we set $F_f(0) \leftrightharpoons f$.

5.2. REPRESENTATION THEOREM. *Assume* $\pi = (P_\alpha, \alpha \in (\omega_1)^L)$ *is a Jensen sequence* $P = \bigcup_{\alpha \in (\omega_1)^L} P_\alpha$, $a = (a_k, k \in \omega)$ *is a P-generic sequence, and* $A = \{a_k : k \in \omega\}$. *In this situation, every* $x \in R \cap L(A)$ *can be represented in the form of an a.f. with constructible coding, in the sense of the following theorem.*

THEOREM. *Assume* $m \in \omega$ *and* $x \in L(a \restriction m)$, $x \subseteq \omega$. *Then there exists* $f \in L$, $f \subseteq \omega$, *such that* $x = F_f(a \restriction m)$.

PROOF. Assume $\varphi(k)$ is a formula of the language L, containing as constants only constants of the form $\underline{a}_k$, $k < m$, and $\underline{c}$, $c \in \mathsf{L}$, and satisfying $x = \{k \in \omega : L(A) \models \varphi^a(k)\}$.

We reason within L. We define $U_k = \{p \in P^m : p \Vdash \varphi(\underline{k}) \text{ or } p \Vdash \sim \varphi(\underline{k})\}$. (The subscript P on $\Vdash$ is omitted; cf. 3.3.) From 3.4.3 and the choice of φ, we obtain that every U_k is dense in P^m. This permits us, using Theorem 4.3, to select a sequence $(Q_k, k \in \omega)$ of countable antichains maximal in P^m such that $Q_k \subseteq U_k$ for all k. Let us set

$$Q_k^0 = \{p \in Q_k : p \Vdash \sim \varphi\}, \quad Q_k^1 = \{p \in Q_k : p \Vdash \varphi\} \quad (Q_k^0 \cup Q_k^1 = Q_k),$$
$$Z_k^i = \bigcup_{p \in Q_k^i} \mathrm{Cl}_p.$$

It is clear that every Z_k^i is a $\underline{\Sigma}^0_2$-subset of the set R^m.

Hence there exists an $f \subseteq \omega$ such that:

(1) *if* $z \in Z_k^0 - Z_k^1$, *then* $k \notin F_f(z)$, *and*

(2) *if* $z \in Z_k^1 - Z_k^0$, *then* $k \in F_f(z)$.

This ends the reasoning in L. Let us prove that f is what is required; that is, $x = F_f(a \restriction m)$. Assume $k \in x$; we shall show that $k \in F_f(a \restriction m)$. First we establish that:

(3) $G_a \cap Q_k^1 \neq 0$ *and* $G_a \cap Q_k^0 = 0$. (For the definition of G_a, cf. 3.2.)

In fact, from 3.2.4 and the choice of Q_k as a maximal antichain in P^m, we obtain $Q_k \cap G_a \neq 0$. Assume $q \in Q_k \cap G_a$. From $k \in x$, the choice of φ, and 3.3.2, it follows that $q \in Q_k^1 - Q_k^0$, which is what was required. This proves (3). Now, defining

$$Y_k^i = \bigcup_{p \in Q_k^i} \mathrm{Cl}_p$$

(the analogue of Z_k^i in the universe), we obtain from (3)

(4) $a \restriction m \in Y_k^1 - Y_k^0$.

We note finally that (2) is equivalent to some Π^1_1-formula with parameters from L (the same in L and in the universe). Together with the Absoluteness Principle (cf. 2.1), this yields

(5) *if* $z \in Y_k^1 - Y_k^0$, *then* $k \in F_f(z)$ *(in the universe).*

From (4) and (5) we also obtain $k \in F_f(z)$.

The derivation of $k \notin x \longrightarrow k \notin F_f(z)$ is similar. Thus, f is what is required, and the theorem is proved.

5.3. *λ-formulas.* The theorem just proved provides the basis for introducing a ramified language for describing elements of $R \cap L(A)$, using as constants of type 1 (i.e., for R) arithmetic functions with coding from L (more precisely, we use the codings themselves, attaching to them the meaning of the a.f.'s). The precise definition is as follows.

Let $n \geqslant 1$. By $\Sigma\lambda_n^1$ we denote an "extension" of the collection of all Σ_n^1-formulas (without constants) by means of:

(1) allowing symbols of the form $\underline{l}$, $l \in \omega$, to be taken as constants of type 0, and symbols of the form $m{*}f$, where $m \in \omega$ and $f \in R \cap L$, as constants of type 1;

(2) allowing us to provide certain constants of type 1 with subscripts from ω.

Moreover, we require that:

(3) quantifiers of type 1, occurring in the right-most block of such quantifiers, do not have subscripts.

$\Pi\lambda_n^1$ is defined similarly. Formulas occurring in $\Sigma\lambda_n^1$ or $\Pi\lambda_n^1$ for some $n \geqslant 1$ are called *λ-formulas.* We note that λ-formulas are constructible (i.e., their transcriptions are constructible finite sequences), since every constant $m{*}f$ satisfies $f \in L$.

Assume φ is a λ-formula. By $\|\varphi\|$ we denote the least $m_0 \in \omega$ such that: (i) if $m{*}f$ occurs in φ, then $m \leqslant m_0$, and (ii) if a quantifier with subscript m occurs in φ, then $m \leqslant m_0$. By $|\varphi|$ we denote the least $m_0 \in \omega$ such that (i) is satisfied (note that $|\varphi| \leqslant \|\varphi\|$). If $\varphi \in \Sigma\lambda_n^1$, then by φ^- we denote the result of the canonical transformation of $\sim\varphi$ to a $\Pi\lambda_n^1$ form; similarly for $\varphi \in \Pi\lambda_n^1$.

We proceed to the interpretation of λ-formulas. Assume φ is a λ-formula and $a \in {}^\omega R$. We define φ^a to be the result of replacing in φ all constants of the form $\underline{l}$ and $m{*}f$ by l and $F_f(a \restriction m)$, respectively (cf. the definition of $a \restriction m$ in 3.1), and of quantifiers of the form $\exists(\forall)_m x$ by $\exists(\forall) x \in R \cap L(a \restriction m)$. (Quantifiers without subscripts are not changed.) We note that, if, in addition, $\varphi \in \Sigma\lambda_1^1 \cup \Pi\lambda_1^1$, then φ^a is a formula of second-order arithmetic with parameters from $R \cup \omega$. (This follows from (3).)

5.4. *Forcing* (forc). The forcing relation forc for λ-formulas which we define below is not formally connected with truth in P-generic extensions of the form $L(A)$ for some P. However, one can show that, in fact, it corresponds to $(\mathrm{Perf} \cap L)$-generic extensions of the indicated form. (This fact will be useful to have in mind, although it will not be used anywhere in what follows.)

We shall define a relation p forc φ, where it is presupposed that φ is a closed λ-formula and $p \in \mathrm{Perf}^\infty \cap L$, by induction on the complexity of φ:

(1) *If* $\varphi \in \Sigma\lambda_1^1 \cup \Pi\lambda_1^1$ *and, for every* $a \in {}^\omega R$ *satisfying* $a \restriction \|p\| \in \mathrm{Cl}_p$, φ^a *is true, then* p forc φ. (As we mentioned in 5.3, φ^a is a formula of second-order arithmetic with parameters from $R \cup \omega$ for $\varphi \in \Sigma\lambda_1^1 \cup \Pi\lambda_1^1$, and, therefore, one can speak about the truth of φ^a.)

(2) *If* φ *is* $\exists_m x \psi(x)$ *and* $\psi \in \Sigma\lambda_{k+1}^1 \cup \Pi\lambda_k^1$, $k \geqslant 1$, *then* p forc $\varphi \Leftrightarrow$ (*there exists* $f \in R \cap L$ *such that* p forc $\psi(m{*}f)$).

(3) *If* φ *is* $\exists x\, \psi(x)$, $\psi \in \Sigma\lambda_{k+1}^1 \cup \Pi\lambda_k^1$, $k \geqslant 1$, *then* p forc $\varphi \Leftrightarrow$ (*there exist* m *and* $f \in R \cup L$ *such that* p forc $\psi(m{*}f)$).

(4) *If* $\varphi \in \Pi\lambda_k^1$, $k \geqslant 2$, *then* p forc $\varphi \leftrightharpoons$ (*for every* $q \in \mathrm{Perf}^\infty \cap L$ *such that* $q \geqslant p$, q forc φ^- *does not hold*).

In (2) and (3), x is a variable of type 1.

5.5. *Some properties of* forc.

1. *The relation* forc *is expressible in* L; *more precisely*, $[p \text{ forc } \varphi] \equiv [L \models p \text{ forc } \varphi]$.

In fact, in the transcription of p forc φ, by the definition in 5.3, the constructibility of p and (the transcription of) φ is assumed. In addition, the proposition $(\forall a \in {}^\omega R)$ $[a \restriction \|p\| \in \mathrm{Cl}_p \longrightarrow \varphi^a$ is true$]$, written in 5.4(1), is obviously a formula of the class Π_1^1 (for $\varphi \in \Pi\lambda_1^1$) or Π_2^1 (for $\varphi \in \Sigma\lambda_1^1$) with constructible parameters. Thus, this proposition is absolute under relativization to L (cf. the Absoluteness Principle of 2.1). From this, we obtain what is required for $\varphi \in \Sigma\lambda_1^1 \cup \Pi\lambda_1^1$. If the formula φ is more complex, we carry out the proof by induction on the complexity of φ, taking into account the obvious absoluteness of the definitions of 5.4 (2, 3, 4). The details are trivial and are left to the reader.

In the following propositions, we fix a closed λ-formula φ and some $p \in \mathrm{Perf}^\infty \cap L$.

2. p forc φ *and* p forc φ^- *cannot hold simultaneously.*

3. *If* $\varphi \in \Pi\lambda_k^1$, $k \geqslant 2$, *and* $\sim p$ forc φ, *then there exists* $q \in L \cap \mathrm{Perf}^\infty$ *such that* $q \geqslant p$ *and* q forc φ^-.

4. *If* $q \in \mathrm{Perf}^\infty \cap L$ *and* $q \geqslant p$ *and* p forc φ, *then* q forc φ.

5. *If* $\|p\| \leqslant k \in \omega$, *then* p forc $\varphi \equiv p \restriction k$ forc φ.

The proofs of 2–4 are trivial, and 5 is easily proved by induction on the complexity of φ, with the help of the definition of $p \restriction k$ in 3.1. (For $k \geqslant \|p\|$ and $a \in {}^\omega R$, we have $a \restriction k \in \mathrm{Cl}_{p \restriction k} \equiv a \restriction \|p\| \in \mathrm{Cl}_p$; thus, p and $p \restriction k$ carry "the same information in the sense of forc".)

5.6. *A restriction theorem.* We want to prove in this subsection a proposition similar to 3.4.2 (for forc). It will be proved that the relation p forc φ actually depends only on $p \restriction |\varphi|$. We remark in passing that, in the transcription of φ^a, a_i can occur with $i > |\varphi|$ (at the cost of subscripts for quantifiers of type 1); therefore, the following theorem expresses a deeper fact than 3.4.2. (A direct analogue of 3.4.2 would have to contain $\|\varphi\|$, rather than $|\varphi|$.)

THEOREM. *Assume* φ *is a closed* λ-*formula*, $p \in \mathrm{Perf}^\infty \cap L$ *and* $\|p\| \geqslant |\varphi| = m$. *Then* p forc φ *and* $p \restriction m$ forc φ *are equivalent.*

The proof is carried out in L. (This is permissible by virtue of 5.5.1.) It is clear that $p \restriction m \leqslant p$. From this and 5.5.4, under the hypothesis of the theorem, one easily obtains the implication from right to left.

For the proof of the reverse implication, we consider certain order automorphisms of the set Perf^∞ (which is equal to $\mathrm{Perf}^\infty \cap L$, since the proof is carried out in L), and their extensions to arithmetic functions (a.f.) and λ-formulas. Let us introduce the appropriate definitions.

If $l \in \omega$, and $p = (p_0, \ldots, p_{l-1})$ and $q = (q_0, \ldots, q_{l-1})$ are elements of Perf^l, and the sequence $\eta = (H_i, i \in \omega)$ is such that every H_i is a homeomorphism of Cl_{p_i} onto Cl_{q_i} for $i < l$ and a homeomorphism of R onto R for $i \geqslant l$, then we write $\eta \in \mathrm{Hom}_{pq}$. If, in addition, $l \geqslant m$ and $H_i = \{(x, x): x \in \mathrm{Cl}_{p_i}\}$ (that is, in particular, $p_i = q_i$) is satisfied for all $i < m$, then η is said to be *m-preserving*. The following assertion is obvious.

1. *If $p, q \in \mathrm{Perf}^l$, $m \leqslant l$, and $p \restriction m = q \restriction m$, then there exists an m-preserving $\eta \in \mathrm{Hom}_{pq}$.*

Assume now that $p, q \in \mathrm{Perf}^l$ and $\eta = (H_i, i \in \omega) \in \mathrm{Hom}_{pq}$ are fixed. For $k \in \omega$ one can define a homeomorphism $H_{<k}: \mathrm{Cl}_{p \restriction k} \xrightarrow{\text{onto}} \mathrm{Cl}_{q \restriction k}$ by the condition

$$H_{<k}(x_0, \ldots, x_{k-1}) = (H_0(x_0), \ldots, H_{k-1}(x_{k-1})).$$

If, in addition, $u \in \mathrm{Perf}^k$, $u \geqslant p$, then the set $H''_{<k}\mathrm{Cl}_u$ obviously has the form Cl_v for some (unique) $v \in \mathrm{Perf}^k$, $v \geqslant q$; we denote it by $v = [\eta]_1(u)$. We have

2. *$[\eta]_1$ is a bijection of $\{u \in \mathrm{Perf}^\infty: u \geqslant p\}$ onto $\{v \in \mathrm{Perf}^\infty: v \geqslant q\}$, preserving $\leqslant$ and $\|\cdots\|$*; $[\eta]_1(p) = q$ (this is obvious from the definition).

Let us extend the operation of η to a.f.'s. If $f \subseteq \omega$ and $F = F_f^k$ (cf. 5.1), then we can define a function F' from R^k into R by the condition $F'(H_{<k}(x)) = F(x)$ for $x \in \mathrm{Cl}_{p \restriction k}$ and $F'(y) = y$ for $y \notin \mathrm{Cl}_{q \restriction k}$ $(= H''_{<k}\mathrm{Cl}_{p \restriction k})$. It is clear that F' is an a.f. (from R^k into R); that is, $F' = F_{f'}^k$ for some (not unique) $f' \subseteq \omega$; we denote one of these f' by $[\eta]_2^k(f)$. We note the following fact:

3. *Assume $k \leqslant m \leqslant l$, η is m-preserving, and $f \subseteq \omega$. Then $F_f^k = F_{f'}^k$, where $f' = [\eta]_2^k(f)$.*

This allows us to require in addition:

4. *Under the hypotheses of* 3, $[\eta]_2^k(f) = f$.

Finally, let us extend the operation of η to λ-formulas. If φ is a λ-formula, then by $[\eta]_3\varphi$ we denote the result of replacing in φ every constant $m{*}f$ by $m{*}f'$, where $f' = [\eta]_2^k(f)$. It is not difficult to verify that

5. *if $u \in \mathrm{Perf}^\infty$, $u \geqslant p$, $k = \|u\|$, $a, a' \in {}^\omega R$, $a \restriction k \in \mathrm{Cl}_u$ and $a' \restriction k = H_{<k}(a \restriction k)$, then φ^a coincides with $(\varphi')^{a'}$, where $\varphi' = [\eta]_3\varphi$.*

PROPOSITION 6 (invariance of forc). *Assume $p, q \in \mathrm{Perf}^l$, $\eta \in \mathrm{Hom}_{pq}$, $u \in \mathrm{Perf}^\infty$, $u \geqslant p$, φ is a λ-formula, $u' = [\eta]_1(u)$, and $\varphi' = [\eta]_3\varphi$. Then u* forc *φ and u'* forc *φ' are equivalent.*

The proof proceeds by means of a trivial induction on the complexity of φ, using the definition of 5.4. Moreover, in the case $\varphi \in \Sigma\lambda_1^1 \cup \Pi\lambda_1^1$, the desired result follows from 5 above, the passage from $\Pi\lambda_n^1$ to $\Sigma\lambda_{n+1}^1$ is obvious (one must use the operators $[\eta]_2^k$), and the passage from $\Sigma\lambda_n^1$ to $\Pi\lambda_n^1$, $n \geqslant 2$, is realized with the help of 2 above. The details are left to the reader.

Let us return to the proof of the theorem. Assume that m, p and φ are the objects in the hypotheses of the theorem and that p forc φ; we shall prove that $p \restriction m$ forc φ. We set $q = (p \restriction m) \restriction l$, where $l = \|p\|$ $(\geqslant m)$. It is clear that $p \restriction m = q \restriction m$, $\|p\| = \|q\| = l$. Applying 1, we find an m-preserving $\eta \in \mathrm{Hom}_{pq}$. From p forc φ and 6 (with $u = p$), it follows that $[\eta]_1(p)$ forc $[\eta]_3\varphi$. But $[\eta]_1(p) = q$ by 2, and $[\eta]_3\varphi = \varphi$ by 4 and the fact that $|\varphi| = m$. Thus, q forc φ, whence, from 5.5.5 and the definition of q, we obtain $q \restriction m$ forc φ; that is, $p \restriction m$ forc φ. Q.E.D.

§6. Conserving sequences

6.1. *Definition and formulation of the fundamental property of conserving sequences.* First we give some intuitive motivation. Assume $P = \bigcup_\alpha P_\alpha$; $(P_\alpha, \alpha \in (\omega_1)^L)$ is a Jensen sequence, $a = (a_k, k \in \omega)$ is a P-generic (over L) sequence of elements of R, and $A = \{a_k: k \in \omega\}$. What properties of P can guarantee that $\mathfrak{A}i''_n(A)$, $i = 4, 5, 6$, hold in $L(A)$?

One can prove that the truth in $L(A)$ of the indicated propositions follows from the following general requirement:

(*) *Assume $\varphi(k) \in \Sigma\lambda^1_n$, k is the unique free variable of φ (of type 0), and $m = |\varphi|$. Then*

$$\{k \in \omega : L(A) \models \varphi^a(k)\} \in L(a \restriction m).$$

For example, that part of $\mathfrak{A}5''_n(A)$ which asserts the constructibility of every element of R in $\Sigma^{1,L}_n$ follows directly from (*) for $m = 0$.

But how can (*) be ensured? If we prove that forc and truth in $L(A)$ agree (as in 3.3.2) up to $\Sigma\lambda^1_n$-formulas, then, in the light of Theorem 5.6, one can be sure that (*) will be satisfied.

In turn, how can we now ensure the indicated agreement? For formulas of $\Sigma\lambda^1_1 \cup \Pi\lambda^1_1$, agreement follows from Definition 5.4(1) and the Absoluteness Principle (cf. 2.1). Theorem 5.2 guarantees that the agreement is preserved in passing from $\varphi(x)$ to $\exists_m x(\exists x)\varphi(x)$. But, in passing from φ^- to φ, $\varphi \in \Pi\lambda^1_k$, $k \geqslant 2$, it is difficult to be sure of the preservation of the agreement, since, in Definition 5.4(4), $q \in \mathrm{Perf}^\infty \cap L$ is written, not $q \in P^\infty$. Thus, to preserve the agreement under the indicated transformation, we must use a property that is like an "elementary equivalence" of P^∞ and $\mathrm{Perf}^\infty \cap L$. This is brought about in the following definition.

A Jensen sequence $(P_\alpha, \alpha \in (\omega_1)^L)$ is said to be an *n-conserving sequence* (n-c.s.) if, for every $p \in P^\infty$ (where $P = \bigcup_{\alpha \in (\omega_1)^L} P_\alpha$) and every closed formula $\varphi \in \bigcup_{1 \leqslant k < n} \Pi\lambda^1_k$, there exists $q \in P^\infty$ such that $q \geqslant p$, and either q forc φ or q forc φ^-.

By 5.5.1, the formula "x is an n-c.s." is absolute with respect to relativization to L.

THEOREM. *Assume $n \geqslant 2$, $(P_\alpha, \alpha \in (\omega_1)^L)$ is an n-c.s., and P, a and $A = \{a_k, k \in \omega\}$ are as above. Then $\mathfrak{A}i''_n(A)$, $i = 4, 5, 6$, are true in $L(A)$.*

The rest of §6 is devoted to the proof of this theorem. Let us briefly outline the proof. In 6.2, with the help of Theorem 5.2, we prove the agreement of forc and truth in $L(A)$ up to $\Sigma\lambda^1_n$-formulas. From this and 5.6, in 6.3 we easily show that proposition (*) is satisfied. Finally, in 6.4–6.6, we prove the truth of $\mathfrak{A}i''_n(A)$, $i = 4, 5, 6$, in $L(A)$, with the help of (*), which already has been established.

The objects P, a and $A = \{a_k: k \in \omega\}$ of the hypothesis of the theorem remain fixed in 6.2–6.6.

6.2. THEOREM (on the connection of forc and truth). *Assume $1 \leqslant k \leqslant n$ and that $\varphi \in \Sigma\lambda^1_k$ is closed. Then φ^a is true in $L(A)$ if and only if there exists $p \in G_a$ such that p* forc φ. (Compare 3.3.2.)

1. We begin the proof with an auxiliary lemma.

LEMMA 1. *There exists* $p \in G_a$ *such that* p forc φ *or* p forc φ^-.

PROOF. By the definition of n-c.s., the set $Q = \{p \in P^\infty\colon\ p$ forc φ or p forc $\varphi^-\}$ is dense in P^∞, and from 5.5.1 it follows that $Q \in L$. From this and the definition of P-generic sequence (3.2), we obtain the desired result.

Let us proceed to a proof of the theorem by induction on $k \geqslant 1$.

2. $k = 1$. Assume first that $p \in G_a$ and p forc φ; we shall prove that $L(A) \models \varphi^a$. By Definition 5.4(1), p forc φ means that φ^b is true for any $b \in {}^\omega R$ satisfying $b \downharpoonright \|p\| \in \mathrm{Cl}_p$. In particular, φ^a is true ($a \downharpoonright \|p\| \in \mathrm{Cl}_p$, since $p \in G_a$). But φ^a obviously is a Σ^1_1-formula with parameters from $L(A)$. Hence, by the Absoluteness Principle (cf. 2.1), φ^a is true in $L(A)$, Q.E.D. This proves the implication from right to left.

Conversely, assume $L(A) \models \varphi^a$, and that there is no $p \in G_a$ such that p forc φ. Then, by Lemma 1, there exists $p \in G_a$ satisfying p forc φ^-. As above, it follows from this that $L(A) \models (\varphi^-)^a$; that is, $L(A) \models \sim \varphi^a$; but this contradicts the assumption $L(A) \models \varphi^a$.

Thus, the case $k = 1$ has been dealt with.

3. Let us assume that the theorem has been proved for some k, $1 \leqslant k < n$, and prove it for formulas φ of $\Sigma\lambda^1_{k+1}$. For simplicity, we assume that $\varphi = \exists_m x \psi(x)$ (x a variable of type 1), $m \in \omega$ and $\psi \in \Pi\lambda^1_k$; that is, the leftmost block of quantifiers of φ consists entirely of one quantifier, $\exists_m\, x$. (The general case of various quantifiers $\exists$ of type 1 is perfectly analogous.)

Let us prove it from left to right. Assume $L(A) \models \varphi^a$. According to the definition of 5.3, this means that there exists $x \in R \cap L(a \downharpoonright m)$ such that $L(A) \models \psi^a(x)$. By Theorem 5.2, x has the form $F^m_f(a \downharpoonright m)$ for some $f \in R \cap L$. Thus, $L(A) \models \psi^a(F^m_f(a \downharpoonright m))$; that is, $L(A) \models \psi(m{*}f)^a$, and, finally, $\sim L(A) \models (\psi(m{*}f)^-)^a$. Applying the inductive hypothesis to the formula $\psi(m{*}f)^- \in \Sigma\lambda^1_k$ and using Lemma 1, we deduce that there exists $p \in G_a$ such that p forc $\psi(m{*}f)$; that is, p forc φ (by 5.4(2)). This proves the implication from left to right.

Conversely, assume $p \in G_a$ and p forc φ. By 5.4(2), this means that p forc $\psi(m{*}f)$ for some $f \in R \cap L$. Let us prove that

$$L(A) \models \psi(m{*}f)^a. \tag{1}$$

In fact, in the contrary case, $L(A) \models (\psi(m{*}f)^-)^a$. Again applying the inductive hypothesis to $\psi(m{*}f)^-$, we can find $q \in G_a$ such that q forc $\psi(m{*}f)^-$. We note that G_a is a filter (cf. 3.2); that is, there exists $r \in G_a$ such that $r \geqslant p$ and $r \geqslant q$. Together with 5.5.4, this yields r forc $\psi(m{*}f)$ and r forc $\psi(m{*}f)^-$, which contradicts 5.5.2. The contradiction proves (1).

But (1) means that $\psi^a(x)$ is true in $L(A)$, where $x = F^m_f(a \downharpoonright m)$. From $f \in L$ and 5.1(1), it easily follows that $x \in L(a \downharpoonright m)$, whence we obtain $L(A) \models (\exists_m x \psi(x))^a$; that is, $L(A) \models \varphi^a$. Q.E.D.

This completes the induction step, and the theorem is proved.

6.3. PROOF OF PROPOSITION (*) OF 6.1. We note first the following strengthening of Theorem 6.2.

LEMMA 1. *Assume* $\varphi \in \Sigma\lambda^1_n$ *is closed. Then* φ^a *is true in* $L(A)$ *if and only if there exists* $p \in G_a \cap P^{|\varphi|}$ *such that* p forc φ.

The proof follows easily from 5.6 and 6.2.

THEOREM 2 ((*) of 6.1). *Assume* $\varphi(k) \in \Sigma\lambda_n^1$ *is a formula with the unique free variable* k (*of type* 0). *Then the set* $z = \{k \in \omega: L(A) \models \varphi^a(k)\}$ *belongs to* $L(a \downharpoonright m)$.

PROOF. Lemma 1 implies that $z = \{k \in \omega$: there exists $p \in P^m$ such that $a \downharpoonright m \in \mathrm{Cl}_p$ and p forc $\varphi(k)\}$. From this and 5.5.1, we easily obtain the desired result.

Let us derive another corollary of Lemma 1. Before formulating it, we note that $p \downharpoonright 0 = 0$ and $a \downharpoonright 0 \in \mathrm{Cl}_{p \downharpoonright 0}$ for arbitrary $p \in \mathrm{Perf}^\infty$ (cf. 3.1).

COROLLARY 3 (reformulation of Lemma 1 for $|\varphi| = 0$). *Assume* $\varphi \in \Sigma\lambda_n^1$ *is closed and* $|\varphi| = 0$. *Then* φ^a *is true in* $L(A)$ *if and only if* 0 forc φ.

6.4. THEOREM. $\mathfrak{A}4_n''(A)$ *is true in* $L(A)$.

PROOF. Assume $X \subseteq R$, $X \in L(A)$, $X \in \underline{\Sigma}_n^1$ in $L(A)$, and X does not have any subsets in $L(A)$ which are equinumerous with ω. We shall prove that X is finite. First we choose a $\Sigma\lambda_n^1$-formula defining the set X in $L(A)$.

By the choice of X, there exist a Σ_n^1-formula $\kappa(x, y)$ (without parameters) and a $y \in L(A)$ such that $X = \{x \in R \cap L(A): L(A) \models \kappa(x, y)\}$. Using 3.4.4 and 5.2, we find $f \in R \cap L$ such that $y = F_f^m(a \downharpoonright m)$. Let us consider the $\Sigma\lambda_n^1$-formula $\theta(x) \leftrightharpoons \kappa(x, m*f)$. By the definitions of 5.3 and the choice of f, it is clear that $\theta^a(x)$ coincides with $\kappa(x, y)$. Thus, $X = \{x \in L(A) \cap R: L(A) \models \theta^a(x)\}$. We note that $|\theta| = m$, by the definition of θ. We shall prove two auxiliary propositions.

1. *If* $l \in \omega$, *then* $X \cap L(a \downharpoonright l)$ *is finite.*

In fact, the class $L(a \downharpoonright l)$ has in $L(A)$ a canonical well-ordering, and X does not contain subsets in $L(A)$ which are equinumerous with ω.

2. *If* $l \in \omega$, *then* $X \cap L(a \downharpoonright l) \subseteq L(a \downharpoonright m)$.

It suffices to prove this proposition for $l \geqslant m$. We let X' denote $X \cap L(a \downharpoonright l)$, and we assume the contrary; that is, we assume $x \in X' - L(a \downharpoonright m)$. By the finiteness of X' (according to 1), there exist $s_0, \ldots, s_j, s_0', \ldots, s_j' \in \omega$ such that x is the unique element of X' containing every s_i, but not containing any s_i' $(i \leqslant j)$. Assume $\varphi(k)$ is the result of the canonical reduction of the formula

$$\exists \iota x\, [\theta(x) \,\&\, \underline{s}_0, \ \ldots, \ \underline{s}_j \in x \,\&\, \underline{s}_0', \ \ldots, \ \underline{s}_j' \notin x \,\&\, k \in x]$$

to $\Sigma\lambda_n^1$-form. Then, from the definitions of 5.3, the choice of s_i and s_i', and the definition of X', it easily follows that $x = \{k \in \omega: L(A) \models \varphi^a(k)\}$. We note that, by the construction of φ, $|\varphi| = |\theta| = m$ (the subscript l does not occur in the definition of $|\varphi|$; cf. 5.3). Applying 6.3.2, we obtain $x \in L(a \downharpoonright m)$, which obviously contradicts the choice of x. This contradiction proves 2.

Now, from 2 and 3.4.4, we obtain $X \subseteq L(a \downharpoonright m)$, from which the finiteness of X follows from 1. This proves the theorem.

6.5. THEOREM. $\mathfrak{A}5_n''(A)$ *is true in* $L(A)$.

PROOF. Assume $a = a_i \in A$, $x \in R \cap L$, $y \in R \cap L(a)$, and $y \in \Sigma_n^{1,x}$ in $L(A)$. We shall prove that $y \in L$ and $y \in \Sigma_n^{1,x}$ in L. By the equivalence of all the a_i (3.2.3), we may assume $i = 0$; that is, $a = a_0$. Assume $\psi(k)$ is a Σ_n^1-formula with parameter x, defining y in $L(a_0)$; that is, $y = \{k \in \omega: L(a_0) \models \psi(k)\}$. We define in the following way a formula $\varphi(k) \in \Sigma\lambda_n^1$:

(1) *We change in ψ every occurrence of x to $0{*}x$.* (We note here that $F_x^0(a \restriction 0) = F_x^0(0) = x$, according to the definition of 5.1.)

(2) *Every quantifier of ψ of type* 1 *in the rightmost block of such quantifiers is left without subscripts.*

(3) *All the remaining quantifiers of ψ of type* 1 *are provided with the subscript* 1.

It is not difficult to verify that

$$[L(A) \models \varphi^a(k)] \equiv [L(a_0) \models \psi(k)]$$

for any $k \in \omega$. (In fact, a quantifier $\exists(\forall)_i z$ is the relativization to $L(a_0) = L(a \restriction 1)$ of a quantifier $\exists(\forall)z$, according to the definition of 5.3; the quantifiers mentioned in (2) need not be relativized, by virtue of the Absoluteness Principle of 2.1; and the constants $0{*}x$ occurring in φ are converted in the formula φ^a into the parameter x, as indicated in (1).) Thus,

$$y = \{k \in \omega : L(A) \models \varphi^a(k)\} = \{k : 0 \text{ forc } \varphi(\underline{k})\}.$$

(The second equation follows from 6.3.3; $|\varphi| = 0$ is obvious from the construction of φ.) From this and 5.5.1 it follows that $y \in L$.

For the proof of $y \in \Sigma_n^{1,x}$ in L, we use the following proposition, which it is convenient for us to prove in §7.

(*) *In L, $\{k \in \omega: 0 \text{ forc } \varphi(\underline{k})\} \in \Sigma_n^{1,x}$ is true* (cf. 7.5.4).

From (*), we easily obtain that $y \in \Sigma_n^{1,x}$ in L. This proves the theorem.

6.6. THEOREM. *$\mathfrak{A}6_n''(A)$ is true in $L(A)$.*

PROOF. It is necessary to prove that, if $i, j \in \omega$, $z \in L(a_i, a_j)$ is a set in the class $\Sigma_n^{1,L}$ in $L(a_i, a_j)$, $Z \subseteq R$, and $Z \cap L(a_i) \neq 0$, then $Z \cap L(a_j) \neq 0$. By 3.2.3, we can assume that $i = 0$ and $j = 1$.

Assume, therefore, that $Z \in L(a_0, a_1)$, $Z \subseteq R$, $Z \cap L(a_0) \neq 0$, and $Z \in \Sigma_n^{1,L}$ holds in $L(a_0, a_1)$; we shall prove that

$$Z \cap L(a_1) \neq 0. \tag{1}$$

As in the proof of Theorem 6.5, we choose a formula $\varphi(x) \in \Sigma\lambda_n^1$ such that $|\varphi| = 0$, every quantifier of φ of type 1 except those occurring in the rightmost block of these quantifiers has subscript 2 (this corresponds to the relativization to $L(a \restriction 2) = L(a_0, a_1)$), and, finally, $L(A) \models Z = \{x \subseteq \omega: \varphi^a(x)\}$. Then $Z \cap L(a_0) \neq 0$ means that $(\exists_1 x\varphi(x))^a$ is true in $L(A)$ (since $L(a_0) = L(a \restriction 1)$); that is, 0 forc $\exists_1 x\varphi(x)$ (by 6.3.3). Assume now that $b = (b_l, l \in \omega) \in {}^\omega R$ is such that $a_1 = b_0$, $a_0 = b_1$, and $b_l = a_l$ for $l \geqslant 2$. Then b is a P-generic sequence by 3.2.3, and $A = \{b_l: l \in \omega\}$ holds. From this, applying 6.3.3 (to b instead of a) in the reverse direction, we obtain $L(A) \models (\exists_1 x\varphi(x))^b$; that is,

$$(\exists\, x \in R \cap L(b_0))\,[L(A) \models \varphi^b(x)].$$

We observe now that $b_0 = a_1$ by definition of b, and φ^b coincides with φ^a by the choice of φ. Hence the assertion just obtained can be rewritten in the following form:

$$(\exists\, x \in R \cap L(a_1))\,[L(A) \models \varphi^a(x)];$$

that is, $L(a_1) \cap X \neq 0$ (by the choice of φ), Q.E.D. (1) and the theorem are proved.

Combining Theorems 6.4–6.6, we obtain a proof of Theorem 6.1.

§7. Proof of the fundamental theorem

By the results of 3.4, 4.5, 4.6, and 6.1, we see that, for the proof of FT for a fixed $n \geqslant 2$, it suffices to construct an n-conserving Jensen sequence π satisfying $\pi \in \Sigma^T_{n-1}$ in L, and also to prove the proposition (*) of 6.5. Having such a π and applying 3.2.5, it is not difficult to obtain a set $A \subseteq R$ with the properties required in the statement of FT.

In this section we present a method for constructing a π of the indicated form. The construction of π is carried out within L in the form of a transfinite procedure.

We begin with the following theorem, asserting the existence of a splitting (cf. 4.1).

7.1. *Theorem on splittings.* Before formulating it, we shall prove the following lemma.

LEMMA 1. *Assume $P \in$ Spl and Ξ countable. Then there exists a countable splitting of the set P over Ξ.*

PROOF. Without restricting its generality, our argument may assume that P and Ξ are elements of the collection T of all hereditarily countable sets. We define $\mathbf{F}$ as the collection of all finite (as sets) functions $f \subseteq (\omega \times {}^{<\omega}2) \times P$ satisfying the following property: if $(k, e) \in \mathrm{dom}(f)$, $e' \in {}^{<\omega}2$, and $e' \subseteq e$, then $(k, e') \in \mathrm{dom}(f)$ and $f(k, e) \subseteq f(k, e')$.

A set $\mathbf{G} \subseteq \mathbf{F}$ is said to be *dense* in $\mathbf{F}$ if $(\forall f \in \mathbf{F})\,(\exists g \in \mathbf{G})\,[f \subseteq g]$. In [7] it was shown that one can construct a "total" function S from $\omega \times {}^{<\omega}2$ into P such that the following conditions are fulfilled:

(1) *$S(k, e) \subseteq S(k, e')$ for $e' \subseteq e$.*

(2) *If $\mathbf{G} \subseteq \mathbf{F}$ is dense in $\mathbf{F}$ and is defined in T by some $\in$-formula with parameters from $P \cup \Xi$ (a countable number of parameters), then there exists $f \in G$ such that $f \subseteq S$ (that is, $f = S \mid \mathrm{dom}(f)$).*

Now let us set

$$q_k = \bigcap_{m \in \omega} \bigcup_{e \in {}^m 2} S(k, e).$$

From (1) and (2) it follows that $q_k \in$ Perf for arbitrary k (cf. [7]). Thus, $Q = \{q_k \cap \langle e \rangle : k \in \omega \;\&\; e \in q_k\} \in$ Spl and Q is countable. Let us show that Q is a splitting of P over Ξ. In fact, 4.1 (1, 2, 3) easily follow from (1) and (2). (For example, for the proof of 4.1 (1) it suffices to verify that $\mathbf{G} = \{f \in \mathbf{F} : f(k, 0) \subseteq p$ for some $k \in \omega\}$ is dense in $\mathbf{F}$ for arbitrary $p \in P$.)

Finally, property 4.1(5) (from which 4.1(4) follows, as we proved in 4.1) is taken by us from [7] (Corollary 2).

A detailed proof of 4.1(5) is rather involved from a technical standpoint, and we leave it to the reader.

Thus, Q is what was required, and the lemma is proved.

THEOREM. *Assume $P \in \mathrm{Spl}$ and Ξ countable, $p \in P^m$, and U dense in* Perf^m. *Then there exist Q, a countable splitting of P over Ξ, and $q \in Q^m$ such that $q \geqslant p$ and $q \in U$.*

PROOF. Assume Q_0 is a countable splitting of P over Ξ. (It exists by Lemma 1.) Using 4.1(1), we choose $q' \in Q_0^m$ such that $q' \geqslant p$ and further, using the density of U, we choose $q = (q_0, \ldots, q_{m-1}) \in U$ such that $q \geqslant q'$. Finally, we define $Q = Q_0 \cup \{q_i \cap \langle e\rangle: i < m \;\&\; e \in q_i\}$. Then Q is the desired splitting. (4.1(1, 2, 3) are obvious, 4.1(5) for Q follows from 4.1(5) for Q_0 and $\bigcup_{q\in Q}\mathrm{Cl}_q = \bigcup_{q \in Q_0}\mathrm{Cl}_q$; finally, $q \in Q^m$, $q \geqslant p$ and $q \in U$ by construction.) This proves the theorem.

7.2. *Theorem on extensions.* All the reasoning of this subsection is carried out within L. We shall apply the results of 7.1 to prove a theorem playing a key role in the construction of the required sequence π. First we shall prove two auxiliary propositions.

LEMMA 1. *If $p \in \mathrm{Perf}^m$, and $B \subseteq R^m$ is a Π_1^1-set, then there exists $q \in \mathrm{Perf}^m$, $q \geqslant p$, such that either $\mathrm{Cl}_q \cap B = 0$ or $\mathrm{Cl}_q \subseteq B$.*

PROOF. $\mathrm{Cl}_p \cap B$ possesses the Baire property in Cl_p (cf. [1a], p. 153). Hence, one can assume that either $\mathrm{Cl}_p - B$ or $\mathrm{Cl}_p \cap B$ is a set of the first category in Cl_p. Let us assume the first (the second is perfectly analogous). Then $\mathrm{Cl}_p \cap B$ contains a subset of the form $X = \bigcap_{k\in\omega} X_k$, where every X_k is a dense (in the topological sense) open subset of Cl_p. Setting $P = \{p \cap \langle e\rangle: e \in p\}$ and $U_k = \{u \in P^m: \mathrm{Cl}_u \subseteq X_k\}$ we obtain that $P \in \mathrm{Spl}$ is countable, and every U_k is dense in P^m.

Let us apply 7.1.1 to P and $\Xi = \{U_k: k \in \omega\}$. Assume Q is a splitting of P over Ξ. Using 4.1(1), we choose $q \in Q$ such that $q \geqslant p$. Then $q \geqslant \bigvee U_k$ (by 4.1(4)); that is, $\mathrm{Cl}_q \subseteq \bigcup_{n\in U_k}\mathrm{Cl}_u$; in addition, $\mathrm{Cl}_q \subseteq X_k$. Thus, $\mathrm{Cl}_q \subseteq X = \bigcap_{k\in\omega} X_k$. This means that q is what was required, and the lemma is proved.

COROLLARY 2. *Assume φ is a closed λ-formula, and $m \geqslant |\varphi|$. Then the set $\{p \in P^m: p$ forc φ or p forc $\varphi^-\}$ is dense in P^m.*

PROOF. If $\varphi \in \Pi\lambda_1^1 \cup \Sigma\lambda_1^1$, then what is required follows easily from Lemma 1, taking into account the definition of 5.4(1). If $\varphi \in \Pi\lambda_k^1 \cup \Sigma\lambda_k^1$, $k \geqslant 2$, we apply 5.5.3 and 5.6.

Now let us prove the theorem on extensions.

THEOREM. *Assume $\rho = (P_\alpha, \alpha \in \nu)$ is a* Spl*-sequence, $\nu \in \omega_1$ $(= (\omega_1)^L$; all the reasoning of this subsection is carried out within $L)$, φ is a closed λ-formula, $P_{<\nu} = \bigcup_{\alpha\in\nu} P_\alpha$, and $p \in P^\infty_{<\nu}$. Then there exist P_ν and $q \in P_\nu^\infty$ such that $(P_\alpha, \alpha \leqslant \nu)$ also is a* Spl*-sequence (of length $\nu + 1$), $q \geqslant p$, and either q* forc *φ or q* forc *φ^-.*

PROOF. One can assume that $m = \|p\| \geqslant \|\varphi\|$. We apply Theorem 7.1 to $P = P_{<\nu}$, $\Xi = L_{\gamma^*(\rho)}$ (cf. the definition of γ^* in 4.2), p, and $U = \{q \in \mathrm{Perf}^m: q$ forc φ or q forc $\varphi^-\}$ (which is dense in Perf^m by Corollary 2).

Assume P_ν is a countable splitting of $P_{<\nu}$ over Ξ, $q \in P_\nu^m$, $q \geqslant p$, and $q \in U$. P_ν and q are what is required, and the theorem is proved.

7.3. *Construction of a Σ_1^T-definable Jensen sequence.* Because of considerations of a technical nature, we shall prove FT separately for $n = 2$ and $n \geqslant 3$. In the case of $n = 2$, Propositions (A), (B) and (C) of §2 permit us not to be concerned about $\mathfrak{A}i_n''(A)$, $i = 4, 5, 6$. Thus, it suffices to construct (in L) a Σ_1^T-definable Jensen sequence.

THEOREM (formulated and proved in L). *There exists a Jensen sequence π such that $\pi \in \Sigma_1^T$.*

PROOF. Let $\prec$ be a canonical well-ordering of the set $T = L_{\omega_1}$(cf. [9]). It has the following property:

1. $\prec \in \Delta_1^T$ *and* $\mathrm{Seg} = \{\{y : y \prec x\}: x \in T\} \in \Delta_1^T$ (cf. [9], Lemma 21, p. 83).

Now we shall estimate the complexity of certain sets.

PROPOSITION 2. *The following sets are in Δ_1^T:*

(1) $M_1 = \{(P, Q, \Xi) \in T$: *$Q$ is a splitting of P over Ξ*$\}$;

(2) $\{\pi \in T$: *π is a* Spl-*sequence*$\}$.

PROOF. (1) As is well known, $(L_\alpha, \alpha \in \omega_1) \in \Delta_1^T$ ([9], p. 38 or p. 82). On the other hand, if $\alpha \in \omega_1$ is a limit ordinal and $\xi = (P, Q, \Xi) \in L_\alpha$, then it is easily verified that $\xi \in M_1 \equiv L_\alpha \models \xi \in M_1$. This means that $M_1 = \{\xi \in T$: there exists a limit ordinal $\alpha \in \omega_1$ such that $\xi \in L_\alpha$ and $L_\alpha \models \xi \in M_1\} = \{\xi \in T$: if $\alpha \in \omega_1$ is a limit ordinal and $\xi \in L_\alpha$, then $L_\alpha \models \xi \in M_1\}$. From this, (1) is obvious; and (2) easily follows from (1) and the observation that $(L_\alpha, \alpha \in \omega_1) \in \Delta_1^T$. The details are left to the reader.

Let us return to the proof of the theorem. For every at most countable Spl-sequence ρ, by ρ^+ we shall denote the least (in the sense of $\prec$) countable Spl-sequence η satisfying $\rho \subsetneqq \eta$ (that is, $\nu = \mathrm{dom}(\rho) \in \mathrm{dom}(\eta)$ and $\rho = \eta|\nu$); such a sequence exists by Theorem 7.2.

Now we set $\rho_0 = 0$, $\rho_{\alpha+1} = \rho_\alpha^+$, $\rho_\beta = \bigcup_{\alpha\in\beta}\rho_\alpha$ for limit ordinals $\beta \in \omega_1$, and $\pi = \bigcup_{\alpha\in\omega_1} P_\alpha$. Let us show that π is the desired sequence. In fact, π is a Spl-sequence, since every ρ_α is a Spl-sequence and $\rho_\alpha \subseteq \rho_\beta$ for $\alpha \in \beta$. Moreover, $\mathrm{dom}(\pi) = \omega_1$, since $\mathrm{dom}(\rho_{\alpha+1})$ is strictly greater than $\mathrm{dom}(\rho_\alpha)$. Hence, π is a Jensen sequence. It remains to prove that $\pi \in \Sigma_1^T$.

LEMMA 3. *The set $M = \{(\rho, \eta) \in T$: ρ, η are* Spl-*sequences and $\eta = \rho^+\}$ belongs to Δ_1^T.*

PROOF. Let $M_0 = \{(\rho, \eta) \in T$: ρ, η are Spl-sequences and $\rho \subsetneqq \eta\}$. From 2(2) it follows that

$$M_0 \in \Delta_1^T. \tag{*}$$

Further, from the definitions of ρ^+ and Seg (cf. Proposition 1), we have

$$M = \{(\rho, \eta) \in M_0 : (\forall \zeta \in T)\,[\zeta \prec \eta \to (\rho, \zeta) \notin M_0]\}$$

$$= \{(\rho, \eta) \in M_0 : (\exists S \in \mathrm{Seg})\,[\eta \in S \,\&\, (\forall \zeta \in S)\,[(\rho, \zeta) \in M_0 \to \zeta = \eta]]\}.$$

From this and property 1, the lemma follows.

COROLLARY 4. *The set* $H = \{\rho_\alpha : \alpha \in \omega_1\}$ *belongs to* Σ_1^T.

PROOF. It is clear that $\rho \in H \equiv$ [there exists a function f such that $\nu = \mathrm{dom}(f)$ is a countable limit ordinal, $f(0) = 0, f(\alpha + 1) = f(\alpha)^+$ for all $\alpha \in \nu$, $f(\beta) = \bigcup f''\beta$ for limit ordinals $\beta \in \nu$, and $\rho \in \mathrm{rng}(f)$]. From this and 3, it follows that $H \in \Sigma_1^T$.

Let us return to the proof of the theorem. From the construction it follows that $\pi = \bigcup H$. From this, with the help of 4, we obtain that $\pi \in \Sigma_1^T$, and π is what was required. This proves the theorem.

7.4. *Complexity of the relation* forc. All the arguments of this subsection are carried out within L. (Nevertheless, the propositions proved are also true in the universe by virtue of 5.5.1.)

As usual, every λ-formula and every formula of second-order arithmetic will be identified with its transcription (a finite sequence of symbols), regarding, in this connection, all logical signs and other such symbols (encoded) to be sets of finite rank. Then every λ-formula will be a finite sequence of sets of finite rank and of constructible (by the definition of 5.3) $f \in R$, occurring in constants m^*f. Thus, every λ-formula is an element of the set $T = L_{\omega_1}$ (all our reasoning is carried out in L), and the collections $\Sigma\lambda_k^1$ and $\Pi\lambda_k^1$ will be Δ_1^T-subsets of the set T.

If $k \geqslant 1$, we define $\mathrm{Forc}_k^\Sigma = \{(p, \varphi) : p \in \mathrm{Perf}^\infty \mathbin{\&} \varphi \in \Sigma\lambda_k^1$ is closed $\&$ p forc $\varphi\}$; we define Forc_k^Π in a similar manner. Forc_k^Σ and Forc_k^Π are also subsets of T. Let us evaluate their complexity.

THEOREM (in L). $\mathrm{Forc}_1^\Pi \in \Delta_1^T$ *and* $\mathrm{Forc}_1^\Sigma \in \Pi_1^T$; *if* $k \geqslant 2$, *then* $\mathrm{Forc}_k^\Pi \in \Pi_{k-1}^T$ *and* $\mathrm{Forc}_k^\Sigma \in \Sigma_{k-1}^T$.

The proof consists of three lemmas.

LEMMA 1. $\mathrm{Forc}_1^\Pi \in \Delta_1^T$.

PROOF. Using the definitions of 5.3 and 5.4(1), to every $p \in \mathrm{Perf}^\infty$ and every closed $\Pi\lambda_1^1$-formula φ one can associate a closed Π_1^1-formula $\psi_{p\varphi}$ with parameters from R such that:

(1) p forc $\varphi \equiv \psi_{p\varphi}$ *is true, and*

(2) $p, \varphi \longmapsto \psi_{p\varphi}$ *is a transformation of the class* Δ_1^T.

(The stipulation 5.1(1) allows us to ensure (2).)

In addition, for every closed Π_1^1-formula ψ with parameters from R, in [9], §5, there is constructed a certain $r(\psi) \subseteq \omega \times \omega$ such that:

(3) ψ *is true* $\equiv r(\psi)$ *is a well-ordering, and*

(4) r *is a function of the class* Δ_1^T.

Finally, we note the following obvious fact:

(5) Word $= \{r \subseteq \omega \times \omega$: *$r$ is a well-ordering*$\} \in \Delta_1^T$.

From (1) and (3), we obtain

$$\mathrm{Forc}_1^\Pi = \{(p, \varphi) : p \in \mathrm{Perf}^\infty \mathbin{\&} \varphi \in \Pi\lambda_1^1 \textit{ is closed } \& \, r(\psi_{p\varphi}) \in \mathrm{Word}\}.$$

From this and (2), (4) and (5), the assertion of the lemma easily follows.

LEMMA 2. $\mathrm{Forc}_1^\Sigma \in \Pi_1^T$.

OUTLINE OF THE PROOF. For $p \in \mathrm{Perf}^\infty$ and closed $\varphi \in \Sigma\lambda_1^1$, one can construct a Σ_1^1-formula $\psi_{p\varphi}(x)$ with unique free variable x (of type 1) such that

$$p \text{ forc } \varphi \equiv \forall x \psi_{p\varphi}(x) \equiv (\forall x \subseteq \omega)[r(\psi_{p\varphi}(x)) \notin \mathrm{Word}].$$

The rest is as in Lemma 1.

Let us now prove the "main" part of the theorem by induction with respect to $k \geqslant 2$, formulating the induction step in the following form:

LEMMA 3. *Assume* $k \geqslant 1$ *and* $\mathrm{Forc}_k^\Pi \in \Delta_k^T$. *Then* $\mathrm{Forc}_{k+1}^\Sigma \in \Sigma_k^T$ *and* $\mathrm{Forc}_{k+1}^\Pi \in \Pi_k^T$.

The proof for $\mathrm{Forc}_{k+1}^\Sigma$ easily follows from the definitions of 5.4(2, 3) by virtue of the fact that transformations of the type $\exists_m x\psi(x) \longmapsto \psi(m^*f)$ are expressible by Δ_1^T-relations. Moreover, according to the definition of 5.4(4).

$$\mathrm{Forc}_{k+1}^\Pi = \{(p, \varphi) : p \in \mathrm{Perf}^\infty \;\&\; \varphi \in \Pi\lambda_{k+1}^1 \text{ is closed}$$

$$\& \;(\forall q \in \mathrm{Perf}^\infty)\,[q \geqq p \to (q, \varphi^-) \notin \mathrm{Forc}_{k+1}^\Sigma]\}.$$

From this expression, with the help of what already has been proved for $\mathrm{Forc}_{k+1}^\Sigma$, we obtain without difficulty that $\mathrm{Forc}_{k+1}^\Pi \in \Pi_k^T$. This proves the lemma.

Combining Lemmas 1, 2, and 3, we obtain a proof of the theorem.

COROLLARY 4 (Proposition (*) of 6.5). *Assume* $\varphi(k) \in \Sigma\lambda_n^1$, $n \geqslant 2$, $x \subseteq \omega$, *and* φ *contains only the constant* 0^*x *among constants of type* 1. *Then* $S = \{k \in \omega\colon 0 \text{ forc } \varphi(\underline{k})\} \in \Sigma_n^{1,x}$.

PROOF. From the choice of φ, it is clear that $k \longmapsto \varphi(\underline{k})$ is a $\Delta_1^{T,x}$-function. From this, with the help of the theorem just proved, we obtain $S \in \Sigma_{n-1}^{T,x}$. But, by 4.5.1, this means that $S \in \Sigma_n^{1,x}$. Q.E.D.

7.5. *Construction of a* Σ_{n-1}^T*-definable n-conserving Jensen sequence for* $n \geqslant 3$.

THEOREM (Formulated and proved in L). *Assume* $n \geqslant 3$. *Then there exists an n-conserving Jensen sequence* π *such that* $\pi \in \Sigma_{n-1}^T$.

PROOF. We choose any functions $\tilde{p}\colon \omega_1 \longrightarrow \mathrm{Perf}^\infty$ and $\tilde{\varphi}\colon \omega_1 \longrightarrow \bigcup_{1 \leqslant k < n} \{\varphi \in \Pi\lambda_k^1\colon \varphi \text{ is closed}\}$ satisfying the following conditions:

(1) *If* $p \in \mathrm{Perf}^\infty$, $1 \leqslant k < n$, *and* $\varphi \in \Pi\lambda_k^1$ *is closed, then there exists a nondenumerable set of* $\alpha \in \omega_1$ *such that* $p = \tilde{p}(\alpha)$ *and* $\varphi = \tilde{\varphi}(\alpha)$.

(2) $\tilde{p}$ *and* $\tilde{\varphi}$ *are functions of the class* Δ_1^T.

(Fulfillment of (2) can be secured with the help of the well-ordering $\prec$ of 7.3.)

Assume now that $\rho = (P_\alpha, \alpha \in \nu)$ is a Spl-sequence, $\nu \in \omega_1$, $P = \bigcup_{\alpha \in \nu} P_\alpha$, $p = \tilde{p}(\nu)$ and $\varphi = \tilde{\varphi}(\nu)$. From Theorem 7.2 we obtain that there exists a Spl-sequence $\eta \in T$

such that $\rho \subsetneqq \eta$, and if, in addition, $p \in P^\infty$, then for some $q \in (\bigcup \mathrm{rng}(\rho))^\infty$ we have $q \geqslant p$ and either q forc φ or q forc φ^-. By ρ^+ we denote the $\prec$-least of such η.

Moreover, as in 7.3, we define $\rho_0 = 0$, $\rho_{\alpha+1} = \rho_\alpha^+$, $\rho_\beta = \bigcup_{\alpha \in \beta} \rho_\alpha$ for limit ordinals β, and $\pi = \bigcup_{\alpha \in \omega_1} \rho_\alpha$. As in 7.3, π is a Jensen sequence. In addition, from (1) and the definition of ρ^+ it easily follows that π is n-conserving. ((1) takes care of all $p \in (\bigcup \mathrm{rng}(\pi))^\infty$ and of all $\varphi \in \bigcup_{1 \leqslant k < n} \Pi\lambda_k^1$.)

It remains to verify that $\pi \in \Sigma_1^T$.

LEMMA 1. $\{(\rho, \eta): \eta = \rho^+\} \in \Delta_{n-1}^T$.

The proof proceeds just as in 7.3.3, taking into account, in addition, (2), Theorem 7.4, and the condition $n \geqslant 3$. (If $n = 2$, then the proof of the lemma cannot be carried out by virtue of the fact that $\mathrm{Forc}_1^\Sigma \in \Pi_1^T$ but does not belong to Δ_1^T.) The details are left to the reader.

Now $\pi \in \Sigma_{n-1}^T$ follows from Lemma 1 in a way similar to the derivation of 7.3.4 from 7.3.3. Thus π is as required, and the theorem is proved.

We note that the proof of the theorem makes essential use of the condition $n \geqslant 3$. (For $n = 2$, it is impossible to prove Lemma 1.) Indeed, this was the reason for the necessity of a separate construction in 7.3.

7.6. *Completion of the proof of the Fundamental Theorem of* 2.4). Let $n \geqslant 2$ be fixed. We select a Jensen sequence $\pi = (P_\alpha, \alpha \in (\omega_1)^L)$ satisfying the condition $\pi \in \Sigma_{n-1}^T$ in L and which is an n-conserving sequence for $n \geqslant 3$. (The existence of a π with such properties is guaranteed by Theorem 7.3 for $n = 2$ and Theorem 7.5 for $n \geqslant 3$.) We set $P = \bigcup_{\alpha \in (\omega_1)^L} P_\alpha$ and, using 3.2.5, we choose a P-generic set $A \subseteq R$. (We note that the countability of $(\omega_2)^L$ is required in the hypothesis of FT.)

Using Theorems 3.4, 4.5, 4.6, and 6.1, as well as (for $n = 2$) Propositions (A), (B), and (C) of §2, we deduce that the proposition $\mathfrak{B}_n(A)$ is true in $L(A)$. This completes the proof of FT.

Received 6/OCT/1975
and 22/FEB/1977

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Translated by E. MENDELSON