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THE SET OF ALL ANALYTICALLY DEFINABLE SETS OF NATURAL NUMBERS CAN BE DEFINED ANALYTICALLY

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ABSTRACT. The author proves consistency with ZFC of the following assertion: the set of all analytically definable sets $x \subseteq \omega$ is analytically definable. A subset x of ω is said to be analytically definable if x belongs to one of the classes Σ^1_n of the analytic hierarchy. The same holds for $X \subseteq \mathcal{P}(\omega)$. Thus Tarski's problem on definability in the theory of types is solved for the case $p = 1$. The proof uses the method of forcing, with the aid of almost disjoint sets.

Bibliography: 14 titles.

§1. Introduction

1.1. *Formulation of the Fundamental Theorem.* The language of second-order arithmetic with variables for sets (of natural numbers) is described in [1], p. 492. It contains two types of variables: variables of type 0 with range the set ω of natural numbers, and variables of type 1 with range the collection $R = \mathcal{P}(\omega)$ of all subsets of ω. In addition to predicates of elementary arithmetic for variables of type 0, this language also contains the binary membership predicate "$k \in x$"; k is assumed to be a variable of type 0, and x a variable of type 1.

The formulas of this language will be called *analytic formulas*.

Free variables of analytic formulas can be replaced by parameters of the corresponding type; that is, for type 0 from ω, and for type 1 from R.

If $\varphi(k)$ is an analytic formula with k (of type 0) as its only free variable, and not containing parameters from R, then the set $\{k \in \omega: \varphi(k)\}$ defined by this formula will be said to be *analtyically definable*. The collection of all analytically definable $x \in R$ will be denoted by An.

Similarly, if $\varphi(x)$ is an analytic formula with x (of type 1) as its only free variable, and not containing parameters from R, then the set $\{x \in R: \varphi(x)\}$ defined by this formula will be said to be *analytically definable*.

Sets $x \in R$ and $X \subseteq R$ are analytically definable if and only if they occur in one of the classes Σ^1_n, Π^1_n or Δ^1_n of the analytic hierarchy. (See [1], §16.2, Theorem XII, and §16.1, Corollary 1. For a definition of the classes Σ^1_n, Π^1_n and Δ^1_n, see, for example, [1], §16.1, convention on notation.)

In the survey [6], problems P3110 and P3112 posed the question of the consistency with ZFC [3] of the following assertions:

1) $\mathrm{An} = R \cap L$.

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2) *The set* An *is analytically definable.*

(L is the class of all sets constructible in the sense of Gödel ([3], Chapter III).)

We shall give a positive answer to these questions:

FUNDAMENTAL THEOREM (FT). *The assertion* An $= R \cap L$ *is consistent with the theory* ZFC.

COROLLARY. *The assertion "The set* An *is analytically definable" is consistent with* ZFC.

The derivation of the Corollary from FT is trivial: The set $R \cap L$, as is well known, belongs to Σ_2^1 (cf. [7] or [2], Problem 10g). Thus, this set is analytically definable.

The Corollary of FT also gives a (partial) answer to a problem of Tarski. In [13], for every $n \geqslant 1$ and every $p \in \omega$, there was introduced the collection D_{np} of all sets of type n definable by formulas of type not higher than p, and it was asked whether $D_{1p} \in D_{2p}$ for $p \geqslant 1$. (All the remaining possibilities among the assertions $D_{np} \in D_{n+1,p}$ were examined in [13].)

We note that D_{11} is precisely the collection of all sets of type 1, that is, all $x \in R$ definable by analytic formulas without parameters; thus, D_{11} coincides with the collection An introduced above. Similarly, D_{21} is precisely the collection of all analytically definable sets $X \subseteq R$. Thus the Corollary can be rewritten as follows:

The assertion $D_{11} \in D_{21}$ *is consistent with* ZFC.

This also gives a partial answer (for $p = 1$) to Tarski's problem.

By a somewhat more complicated application of the methods of the present paper, one can prove that the assertion $D_{1p} \in D_{2p}$ is consistent with ZFC in general, for any $p > 1$. It is also consistent for $p = \infty$, where $D_{n\infty} = \bigcup_{p \in \omega} D_{np}$.

We remark that the assertion $D_{11} \notin D_{21}$ also is consistent with ZFC, since it follows from the axiom of constructibility (see the the next section).

The present paper consists of the proof of FT. Before presenting the plan of the proof, we shall make several observations.

1.2. *Axioms contradicting the assertions* An $= R \cap L$ *and "The set* An *is analytically definable"*. Among such axioms there is, in particular, the axiom of constructibility $V = L$ ([3], Chapter III, §1), asserting that every set is constructible.

In fact, let us assume the contrary: $V = L$ and An is analytically definable. Then the set $X = R -$ An also is analytically definable and is obviously nonempty (since R is nondenumerable and An is denumerable). On the other hand, $V = L$ implies [7] the existence of an analytically definable (more precisely, in Δ_2^1) well-ordering of the set R. The least, in the sense of this well-ordering, element x of the set X is also analytically definable; that is, it belongs to An, yielding a contradiction.

Since the axiom of constructibility is consistent with ZFC [3], the assertions whose consistency with ZFC is stated in FT and its corollary are independent of ZFC; that is, their denials are consistent with the theory ZFC.

We remark that the proof of the denials of the assertions just mentioned can be carried out with the help of hypotheses weaker than $V = L$. Let us cite several such hypotheses.

The ordinal ω_1^L *(the first ordinal nondenumerable in* L*) is nondenumerable in the universe of all sets.*

$R \cap L \nsubseteq$ An.

There exists an analytically definable well-ordering of some nondenumereble set $X \subseteq R$.

There exist a measurable cardinal κ and a normal measure μ on κ such that $V = L[\mu]$; that is, all sets are constructible from μ.

1.3. *Definability at a lower level.* The Corollary of FT shows that the set An of all analytically definable $x \in R$ "can be" analytically definable. This result contrasts with the following propositions, provable in ZF:

(a) *The set of all recursive $x \in R$ is nonrecursive.*

(b) *The set of all arithmetically definable $x \in R$ is not arithmetically definable.*

(c) *The set of all Δ^1_1-sets $x \in R$ does not belong to Δ^1_1.*

(d) *The set of all Δ^1_2-sets $x \in R$ does not belong to Δ^1_2.*

TERMINOLOGY. The *recursive* sets $x \in R$ and $X \subseteq R$ are defined in [1], §15.1. A set $x \in R$ is called *arithmetically definable* if there exists an analytic formula $\varphi(k)$, not containing parameters from R or quantifiers with variables of type 1, such that $x = \{k \in \omega: \varphi(k)\}$. In like manner, one defines the concept of arithmetically definable sets $X \subseteq R$. In general, "arithmetic" is "analytic" without quantifiers with variables of type 1.

Proposition (a) is obvious; the collection of all recursive $x \in R$ is denumerable, but every recursive $X \subseteq R$ must be clopen (open and closed) in R and, therefore, either empty or of the power of the continuum.

(b) and (c) are mentioned in [1], §16.7, in a somewhat altered form appropriate for second-order arithmetic with variables of type 1 for functions from ω into ω.

Finally, (d) can be easily obtained from the following "Basis Theorem" ([1], §16.7, Corollary XLV(c)): *Every nonempty Σ^1_2-set (and, therefore, every nonempty Δ^1_2-set) X contains an element $x \in X$ in the class Δ^1_2.* For the proof of (d), one must assume the contrary and apply the Basis Theorem to the complement of the set $\{x \in R: x \text{ is a } \Delta^1_2\text{-set}\}$.

We note that, under the assumption $V = L$, the assertion "The set of all Δ^1_n-sets $x \in R$ does not belong to Δ^1_n" is also provable for any $n > 3$. It would be of interest to determine whether the denial of this assertion for some $n > 3$ is consistent with ZFC.

It would also be of interest to determine whether the assertion "$R \cap L$ is precisely the collection of all Δ^1_n-sets $x \in R$" is consistent with ZFC.

1.4. *A second formulation of FT.* We shall prove the Fundamental Theorem in the following more convenient form.

FUNDAMENTAL THEOREM (second formulation). *Assume that ω_3^L (the third nondenumerable cardinal of the constructible universe L) is denumerable in the universe of all sets. Then there exists a set G such that in the class $L[G]$ of all sets constructible from G the following two assertions are true:*

(i) *Every $a \in L[G] \cap R$ analytically definable in $L[G]$ is constructible.*

(ii) *Every constructible $r \in R$ is analytically definable in $L[G]$.*

It is known that the assertion "ω_3^L is denumerable" is consistent with ZFC (see, for example, [3], Chapter IV, §10). In view of this fact, the validity of the second formulation of FT automatically follows from the validity of the first.

In accordance with the second formulation of FT, we shall assume in the proof that the ordinal ω_3^L is denumerable in the universe of all sets. However, this assumption actually is only used for the proof of the existence of generic filters (cf. 2.4 and 4.1).

1.5. *The plan of the proof.* In §2 we introduce the important concept of a *system.* With every system U we associate the set of forcing conditions $P(U)$ for generic extensions of the constructible universe L. The structure of $P(U)$ is similar to the structure of the set of forcing conditions in [8], §5.

We consider filters which are L-generic on $P(U)$ in the sense of [5], which, for brevity, we shall call U-generic filters, or U-g.f.'s. Every U-g.f. G generates, in a natural way, 1) a function g^G from ω onto $R \cap L$, and 2) a family of sets $\{S^G_{ni}: n, i \in \omega\}$.

Some general properties of these sets in generic extensions of the indicated form are also studied in §2.

In §3 we construct in L a certain specific system $U^* = (U^*_{ni}: n, i \in \omega)$ such that every U^*-g.f. G will be what is required in the second formulation of FT. The construction of U^* is organized in such a way that the complexity of the "nth-level" component of (U^*_{ni}: $i \in \omega$) increases as n increases. Moreover, for any $m \in \omega$, the components U^*_{ni} with $n \geqslant m$ have no influence, roughly speaking, on the forcing of formulas of complexity $\leqslant m$. This effect is achieved by the special "generic" way in which U^* is constructed in L.

In §§4 and 5 we prove proposition (i) of the second formulation of FT for any U^*-g.f. G. The decisive element is the following important proposition, which follows from the construction of the system U^*:

(A) *If $p \in P(U^*)$ and if an analytic formula $\psi(k)$, with k (of type 0) as its only free variable, does not have parameters, then one can select $q \in P(U^*)$, $q \geqslant p$, such that $\forall k$ [either q forces $\psi(k)$ or q forces $\sim \psi(k)$].*

(Cf. the proof of Theorem 4.8.2.)

This proposition implies (i) of the second formulation of FT for any U^*-g.f. G. The verification of (A) uses a special relation forc, which is introduced and studied in §4. Actually, (A) (more precisely, Proposition 4.8(*), from which (A) is easily obtained) is proved first for the relation forc in §5; but the "generic" character of U^* enables us to establish the agreement between forc and forcing with respect to $P(U^*)$ (Corollary 4.8.1 and Theorem 4.7).

The apparatus of the forc relation is similar, in its meaning and its role in the proof, to the corresponding apparatus of forc in [12].

For the purpose of greater clarity, we shall prove (i) of the second formulation of FT for any U^*-g.f. G (Theorem 4.9), using 4.8(*), before the proof of Proposition 4.8(*) itself in §5.

Finally, in §6 we prove proposition (ii) of the second formulation of FT for any U^*-g.f. G. We introduce a collection of formulas $\Phi_n(S, i)$ satisfying the following conditions:

1) *If $i \in g^G(n)$ (where g^G is the function from ω onto $R \cap L$ mentioned above), then $\Phi_n(S^G_{ni}, i)$ is true in $L[G]$.*

2) *If $i \notin g^G(n)$, then there is no $S \in L[G]$ such that $\Phi_n(S, i)$ is true in $L[G]$.*

3) *The set $\{i \in \omega: \exists S\ \Phi_n(S, i)$ is true in $L[G]\}$ is analytically definable in $L[G]$.*

(The U^*-g.f. G and $n, i \in \omega$ are arbitrary.)

Conditions 1) and 2) mean that every set $g^G(n)$ is definable in $L[G]$ by a formula $\exists S\ \Phi_n(S, i)$. Proposition 3) implies, in addition, the analytic definability of $g^G(n)$. But, since g^G is a function from ω onto $R \cap L$, every $r \in R \cap L$ has the form $g^G(n)$ for suitable n and is, therefore, analytically definable in the class $L[G]$.

That is the structure of the proof.

1.6. *Notation.* All of the basic set-theoretic notation is taken from [4], with the following change: the power of a set x is denoted by card(x).

We often use the abbreviation (. . . : . . .) for "indexed sets", that is, for functions. For example, the notation $(U^\alpha_{ni}: \alpha \in \omega_2)$ designates the function f defined on the set ω_2 by the condition $f(\alpha) = U^\alpha_{ni}\ \forall \alpha \in \omega_2$. Similarly, the notation $(U^\alpha_{ni}: n, i \in \omega)$ designates the function f defined on the set $\omega \times \omega$ by the condition $f(n, i) = U^\alpha_{ni}\ \forall n, i \in \omega$.

§2. Generic extensions, used for the proof of the Fundamental Theorem

2.1. *Preliminary definitions.* ω_1^L is the first ordinal which is nondenumerable in L.

Seq_α is the collection of all constructible functions from α into $2 = \{0, 1\}$.

$\mathrm{Seq} = \bigcup_{\alpha \in \omega_1^L} \mathrm{Seq}_\alpha$.

Fun is the collection of all constructible functions from ω_1^L into 2.

Assume $S \subseteq \mathrm{Seq}$, $f \in \mathrm{Fun}$, and $\gamma \in \omega_1^L$. If there is no $\alpha \in \omega_1^L$, $\alpha > \gamma$, such that $f|\alpha \in S$, we say that *S does not cover f above γ*. If $(\exists \gamma \in \omega_1^L)$ [S does not cover f above γ], we say that *S does not cover f*. In the contrary case, we say that *S covers f*. This holds if and only if

$$(\forall \gamma \in \omega_1^L)(\exists \alpha \in \omega_1^L)[\alpha > \gamma \text{ and } f|\alpha \in S].$$

Let us introduce the important definition of a system. By a *system* we mean a constructible function U defined on $\omega \times \omega$ and satisfying the condition $U(n, i) \subseteq \mathrm{Fun}$ for all n and i. To simplify the notation, we shall write U_{ni} instead of $U(n, i)$, and, similarly, U'_{ni} and U^*_{ni} instead of $U'(n, i)$ and $U^*(n, i)$, etc. (for systems U, U' and U^*). The set U_{ni} will be called the (n, i)th component of the system U.

2.2. *Forcing conditions.* Let U be a system. We wish to construct a set of forcing conditions which will produce a generic function g from ω onto $R \cap L$ and a generic family $(S_{ni}: n, i \in \omega)$ of subsets of the set Seq such that, for all $n \in \omega$, $i \in g(n)$ and $f \in \mathrm{Fun}$, the following equivalence holds: *S_{ni} covers f if and only if $f \notin U_{ni}$.*

A suitable set of forcing conditions is the collection $P(U)$ of all *constructible* functions p defined on $\{0\} \cup (\omega \times \omega)$ and satisfying the following seven conditions:

(i) *$e = p(0)$ is a function from some finite $x \subseteq \omega$ into $R \cap L$; we shall denote $x =$* dom(p(0)) *by* $|p|$.

(ii) *Every $p(n, i)$ is a pair (s_{ni}, X_{ni}).*

(iii) $s_{ni} \subseteq \mathrm{Seq}$ *and s_{ni} is at most countable in L.*

(iv) *X_{ni} is a collection, at most countable in L, of pairs of the form (γ, f), where $\gamma \in \omega_1^L$ and $f \in U_{ni}$.*

(v) *If $(\gamma, f) \in X_{ni}$, then s_{ni} does not cover f above γ.*

(vi) *If $n \notin |p|$ (where, by* (i), $|p| = \mathrm{dom}(p(0))$) *and $i \in \omega$, then $p(n, i) = (0, 0)$; that is, $s_{ni} = 0$ and $X_{ni} = 0$.*

(vii) *If $n \in |p|$ and $i \in \omega$, $i \notin e(n)$ (where, by* (i), *$e(n)$ belongs to $R \cap L$), then also* $p(n, i) = (0, 0)$.

This is the definition of $P(U)$. Before defining an order $\leqslant$ on $P(U)$, we agree to write $(s, X) \leqslant (s', X')$ if $s \subseteq s'$ and $X \subseteq X'$ (s, s', X and X' are arbitrary sets). Now we order $P(U)$ componentwise: $p \leqslant q$ if and only if $p(0) \subseteq q(0)$ and $p(n, i) \leqslant q(n, i)$ for all n and i.

By P_0 we denote the set $P(V)$, where $V = (\omega \times \omega) \times \{\text{Fun}\}$ (that is, $V_{ni} = \text{Fun}$ for all n and i). It is clear that every $P(U)$ is a constructible subset of P_0.

2.3. *Some properties of the ordering.* Before describing the use of sets of the form $P(U)$ for forcing and the construction of generic extensions, we shall consider in more detail the ordering $\leqslant$ on P_0.

Arbitrary $p, p' \in P_0$ are said to be *compatible* if there exists $q \in P_0$ such that $q \geqslant p$ and $q \geqslant p'$. In the contrary case, p and p' are said to be *incompatible*. We say that $Q \subseteq P_0$ is an *antichain* if any distinct $p, p' \in Q$ are incompatible.

LEMMA 1. *If $Q \subseteq P_0$ is a constructible antichain, then Q has power $\leqslant \omega_1^L$ in L.*

We carry out the proof in L. For every $p \in P_0$ we define a function s_p on $\{0\} \cup (\omega \times \omega)$ in the following way: $s_p(0) = p(0)$, and, if $p(n, i) = (s, X)$, then $s_p(n, i) = s$. Then, on the one hand, if $p, q \in P_0$ are incompatible, it is easy to check that $s_p \neq s_q$. But, on the other hand, the set $\{s_p : p \in P_0\}$ obviously has power $\leqslant \omega_1$, whence the desired result follows.

For $p_1, p_2 \in P_0$ we introduce $p_1 \vee p_2$ as the (unique) function q defined on $\{0\} \cup (\omega \times \omega)$ by the conditions $q(0) = p_1(0) \cup p_2(0)$ and $q(n, i) = (s, X)$, where $s = s_1 \cup s_2$, $X = X_1 \cup X_2$, and $(s_1, X_1) = p_1(n, i)$, $(s_2, X_2) = p_2(n, i)$. The following two assertions are valid, and their simple verification is left to the reader.

LEMMA 2. *Let $p_1, p_2 \in P_0$. Then p_1 and p_2 are compatible if and only if $p_1 \vee p_2 \in P_0$; and in that case $p_1 \leqslant p_1 \vee p_2$ and $p_2 \leqslant p_1 \vee p_2$.*

LEMMA 3. *If U is a system and $p_1, p_2 \in P(U)$ are compatible, then $p_1 \vee p_2 \in P(U)$, and thus p_1 and p_2 are also compatible in $P(U)$ (that is, there exists a $q \in P(U)$, namely $q = p_1 \vee p_2$, such that $q \geqslant p_1$ and $q \geqslant p_2$).*

The simple proof of the following lemma, giving a necessary and sufficient condition for compatibility, is also left to the reader.

LEMMA 4 (compatibility criterion). *Assume that $p, q \in P_0$, $p(0) = e$, $q(0) = e'$, and $p(n, i) = (s_{ni}, X_{ni})$ and $q(n, i) = (s'_{ni}, X'_{ni})$ for all $n, i \in \omega$. Then p and q are compatible if and only if the following three assertions are (simultaneously) satisfied*:

(1) *The functions e and e' are compatible*; *that is, there exists no $j \in |p| \cap |p'|$ such that $e(j) \neq e'(j)$.*

(2) *If $n, i \in \omega$ and $(\nu, f) \in X_{ni} - X'_{ni}$, then the set $s'_{ni} - s_{ni}$ does not cover f above ν.*

(3) *If $n, i \in \omega$ and $(\nu, f) \in X'_{ni} - X_{ni}$, then the set $s_{ni} - s'_{ni}$ does not cover f above ν.*

Before proving an important corollary of Lemma 4, we introduce the following definition.

Let $p \in P_0$ and $m \in \omega$. By $p\lfloor m$ we denote the function q defined on the set $\{0\} \cup (\omega \times \omega)$ by the following conditions:

1) $q(0) = p(0)|m$; that is, if $e = p(0)$ and $x = \text{dom}(e)$ $(= |p|)$, then $q(0)$ is the function e' defined on $x \cap m$ by the condition $e'(j) = e(j)$ for any $j \in x \cap m$;

2) $q(n, i) = p(n, i)$ for $n < m$;

3) $q(n, i) = (0, 0)$ for $n \geqslant m$.

It is not difficult to verify that $p\lfloor m \in P_0$, $p\lfloor m \leqslant p$ and $|p\lfloor m| \subseteq m$, and, in addition, if $|p| \subseteq m$, then $p\lfloor m = p$ (from 2.2(vi)), and, if p belongs to $P(U)$, so does $p\lfloor m$.

COROLLARY 5. *Assume that* $p, p' \in P_0$, $m \in \omega$, $|p'| \cap |p| \subseteq m$, *and* $p{\restriction}m \leqslant p'{\restriction}m$. *Then* $p \vee p' \in P_0$; *that is, according to Lemma* 2, p *and* p' *are compatible.*

PROOF. Let us put $p'' = p'{\restriction}m$, $e = p(0)$, $e' = p'(0)$, $e'' = p''(0)$, $p(n, i) = (s_{ni}, X_{ni})$ and $p'(n, i) = (s'_{ni}, X'_{ni})$ for all $n, i \in \omega$. Let us check conditions (1)–(3) of Lemma 4.

If $j \in |p| \cap |p'|$, then by hypothesis $j < m$; that is, $j \in |p''|$. Hence $e'(j) = e''(j)$. On the other hand, from $p{\restriction}m \leqslant p'{\restriction}m$ and $j < m$ it follows that $e(j) = e''(j)$. Now it is obvious that (1) holds.

Further, if $n < m$, then from $p{\restriction}m \leqslant p'{\restriction}m$, we have $s_{ni} \subseteq s'_{ni}$ and $X_{ni} \subseteq X'_{ni}$, whence (2) and (3) follow.

If $n \geqslant m$, then, by hypothesis, $n \notin |p'| \cap |p|$. Hence either $s_{ni} = X_{ni} = 0$ or $s'_{ni} = X'_{ni} = 0$. In both cases the requirements (2) and (3) of Lemma 4 are satisfied, and the corollary is proved.

In conclusion, we note that all the definitions, and, in general, all the arguments of 2.1–2.3 are, by their very meaning, relativizible to L.

2.4. *Generic filters and forcing.* Let us fix a system U and consider the set $P(U)$ with the ordering $\leqslant$ as the set of forcing conditions for generic extensions of the constructible universe.

The approach in which the constructible universe L is used as the initial model for generic extensions is apparently due to Jensen [9].

We assume that the reader is familiar with the theory of generic extensions in [5], pp. 5–7 (the definition of *dense subset of* $P(U)$, L-*generic filter on* $P(U)$, the corresponding *forcing*, and their properties). The forcing corresponding to $P(U)$ will be denoted by $\Vdash_U$ or simply by $\Vdash$ when it is clear which system U we are talking about. We make a few observations.

1. As in [5], $p \leqslant q$ signifies that q is "more informative" than p; that is, every formula forced by p is also forced by q. We note that in [2] the reverse convention is assumed.

2. Below we shall consider only L-generic filters, and so for brevity we write "U-g.f." instead of "L-generic filter on $P(U)$".

3. The requirement in [5], 1.1.8, that the set $C = \{Y \subseteq P(U)\colon Y \text{ is constructible}\}$ is at most countable is fulfilled in our case. In fact, $P(U)$ obviously has power $\leqslant \omega_2^L$ in L. Hence C has power $\leqslant \omega_3^L$ in L. But the ordinal ω_3^L is countable (in the universe of all sets) by the assumption in 1.4.

The requirement that C is countable implies, for every $p \in P(U)$, the existence of a U-g.f. G satisfying $p \in G$. It is also necessary for the proof of the basic properties of forcing (if the latter is defined by truth in generic extensions, as in [5]).

4. The language whose formulas are forced contains in [5] constants for a generic filter and for every x of the *initial model*; that is, for every $x \in L$ in our case. We shall denote these constants by **G** and **x**, respectively.

5. Below in §3 we shall construct a system U^* possessing the property that every U^*-g.f. G is what is required in the sense of the second formulation of FT. But now we consider certain properties of generic extensions not depending on the specific choice of U.

2.5. *Elementary properties of generic extensions.* To avoid repetition, we fix a system U and some U-g.f. G. We introduce the following sets belonging to the generic extension $L[G]$:

$G[0] = \{p(0): p \in G\}$;
$G[n, i] = \{p(n, i): p \in G\}$;
$g^G = \bigcup G[0]$;
$S_{ni}^G = \bigcup \{s$: there exists X such that $(s, X) \in G[n, i]\}$.

The following assertions hold:

1. *g^G is a function from ω onto $R \cap L$. Hence $R \cap L$ and ω_1^L are countable in $L[G]$.*

2. *Every S_{ni}^G is a subset of the set* Seq.

3. *If $n, i \in \omega$ and $i \notin g^G(n)$, then every $p \in G$ satisfies the condition $p(n, i) = (0, 0)$. Thus, in this case, $S_{ni}^G = 0$.*

PROOF. 1. The collection $\{p(0): p \in P(U)\}$ forms the usual set of forcing conditions for obtaining generic functions from ω onto $R \cap L$ (cf. [3], Chapter IV, §7, and [2], §9.8).

2 is obvious.

3. Assume, for the sake of contradiction, that $p \in G$ and $p(n, i) \neq (0, 0)$. Since $i \notin g^G(n)$ and G is generic, we may assume, without restricting generality, that $p \Vdash$ "$\mathbf{i} \notin \mathbf{g}^{\mathbf{G}}(\mathbf{n})$" (the subscript U is omitted). But the latter assertion, as can be easily verified, signifies that $n \in \operatorname{dom}(e)$ and $i \notin e(n)$, where $e = p(0)$. But this, together with $p(n, i) \neq (0, 0)$, contradicts 2.2(vii).

2.6. COVERING THEOREM. *Assume that U is a system, G is a U-g.f., $n \in \omega$, $i \in g^G(n)$, and $f \in$* Fun. *Then S_{ni}^G covers f if and only if $f \notin U_{ni}$.*

This important theorem is based upon requirements (iv) and (v) of the definition of 2.2 and reveals their specific character.

PROOF. *Necessity*. Assume, for the sake of contradiction, that $f \in U_{ni}$ and that S_{ni}^G covers f. This assumption quickly reduces to the existence of some $p \in P(U)$ (and even some $p \in G$, but this is not needed) such that

(1) $n \in \operatorname{dom}(e)$ *and* $i \in e(n)$, *where* $e = p(0)$, *and*

(2) $p \Vdash$ "$\mathbf{S}_{\mathbf{ni}}^{\mathbf{G}}$ *covers* $\mathbf{f}$" (we omit the subscript U from $\Vdash$).

Assume $p(n, i) = (s, X)$. Since $s \subseteq$ Seq is at most countable in L by 2.2(iii), there exists $\gamma \in \omega_1^L$ such that $\operatorname{dom}(h) < \gamma$ for all $h \in s$. We set $X' = X \cup \{(\gamma, f)\}$ and define $p' \in P(U)$ by the conditions $p'(0) = p(0)$, $p'(n, i) = (s, X')$ and $p'(m, j) = p(m, j)$ for $m \neq n \vee j \neq i$. Using $p \in P(U)$, $f \in U_{ni}$, and (1), we can easily check that, in fact, $p' \in P(U)$. It is also obvious that $p' > p$.

On the other hand, $p'(n, i) = (s, X')$ and $(\gamma, f) \in X'$. From this and part 2.2 (v) of the definition of $P(U)$ it follows that, if $p'' \in P(U)$, $p'' > p'$ and $p''(n, i) = (s'', X'')$, then s'' does not cover f above γ. Thus, by the definition of S_{ni}^G, we have $p' \Vdash$ "$\mathbf{S}_{\mathbf{ni}}^{\mathbf{G}}$ does not cover $\mathbf{f}$ above γ", which contradicts (2) and $p' > p$. The necessity is proved.

Sufficiency. Again we assume the contrary: $f \notin U_{ni}$, but S_{ni}^G does not cover f. This reduces to the existence of $p \in P(U)$ and $\gamma \in \omega_1^L$ for which the following two assertions hold:

(3) $n \in \operatorname{dom}(e)$ *and* $i \in e(n)$, *where* $e = p(0)$, *and*

(4) $p \Vdash$ "$\mathbf{S}_{\mathbf{ni}}^{\mathbf{G}}$ *does not cover* $\mathbf{f}$ *above* γ".

Again, assume $p(n, i) = (s, X)$. The collection $F = \{f' \in \text{Fun}: \exists \nu\, [(\nu, f') \in X]\}$ is constructible and at most countable in L, and $F \subseteq U_{ni}$ by definition 2.2(iv). Hence $f \notin F$. Therefore there exists $\delta \in \omega_1^L$ such that $\delta > \gamma$ and $f|\delta \neq f'|\delta$ for all $f' \in F$. We set $s' = s \cup \{f|\delta\}$ and we define $p' \in P(U)$ by the conditions $p'(0) = p(0)$, $p'(n, i) = (s', X)$ and $p'(m, j) = p(m, j)$ for $m \neq n \vee j \neq i$.

As above, $p' \in P(U)$ and $p' > p$. In addition, $p'(n, i) = (s', X)$ and $f|\delta \in s'$, $\delta > \gamma$. By definition of S^G_{ni}, this implies $p' \Vdash$ "$\sim[\mathbf{S}^{\mathbf{G}}_{\mathbf{ni}}$ does not cover $\mathbf{f}$ above $\gamma]$". This contradicts (4), and the theorem is proved.

2.7. *Codes and a representation theorem.* Below we shall have to carry out a careful investigation of sets $a \subseteq \omega$ and $S \subseteq \mathrm{Seq}$ belonging to generic extensions. For this study we shall introduce two constructible collections of codes for the indicated sets.

By cod we shall denote the collection of all constructible families of the form $c = (Q_k: k \in \omega)$ such that each Q_k is a subset of P_0 of power $< \omega_1^L$ in L. If c is as indicated, and $G \subseteq P_0$ (for example, if G is a U-g.f. for some system U), we introduce the "completion" $c^G = \{k \in \omega: G \cap Q_k \neq 0\}$. It is clear that $c^G \subseteq \omega$ and $c^G \in L[G]$.

Similarly, by Cod we denote the collection of all constructible families $c = (Q_h: h \in \mathrm{Seq})$ such that each Q_h is a subset of P_0 of power $< \omega_1^L$ in L. For such c and arbitrary $G \subseteq P_0$ we set $c^G = \{h \in \mathrm{Seq}: G \cap Q_h \neq 0\}$.

In addition, for every system U, we introduce $\mathrm{Cod}(U)$ as the collection of all $c = (Q_h: h \in \mathrm{Seq}) \in \mathrm{Cod}$ such that $Q_h \subseteq P(U)$ for arbitrary $h \in \mathrm{Seq}$.

The sets cod, Cod, and $\mathrm{Cod}(U)$ are obviously constructible.

The following theorem shows that a representation of $a \in L[G]$, $a \subseteq \omega$, in the form $a = c^G$, $c \in \mathrm{cod}$, is possible for all such a.

THEOREM 1. *Assume that U is a system, G is a U-g.f., $a \in L[G]$, and $a \subseteq \omega$. Then there exists $c \in$ cod such that $a = c^G$.*

PROOF. Being an element of $L[G]$, the set a is definable in $L[G]$ by some formula $\varphi(k)$ with constructible parameters and the parameter G: $a = \{k \in \omega: \varphi(k) \text{ is true in } L[G]\}$. Let the formula $\boldsymbol{\varphi}(k)$ be obtained from $\varphi(k)$ by replacing the parameter G by the constant $\mathbf{G}$ and by replacing every parameter $x \in L$ by a corresponding constant $\mathbf{x}$.

We reason within L. For every $k \in \omega$ we define $B_k = \{p \in P(U): p \Vdash \boldsymbol{\varphi}(\mathbf{k})\}$ (the subscript U of $\Vdash$ is omitted). The definition of B_k can be carried out within L since forcing is expressible within the "initial model" ([5], 1.1.9). Further, for every $k \in \omega$ we choose in B_k a maximal antichain $Q_k \subseteq B_k$ (that is, there is no $p \in B_k$ which is incompatible with arbitrary $q \in Q_k$). Then $\mathrm{card}(Q_k) < \omega_1$ by 2.3.1, and therefore $c = (Q_k: k \in \omega)$ belongs to cod. This ends the reasoning within L.

Let us prove that the c that was constructed is what is required; that is, $a = c^G$. By the genericity of G, the choice of φ, and the definition of B_k, we have $k \in a \equiv G \cap B_k \neq 0$. On the other hand, by definition of c^G, we have $k \in c^G \equiv G \cap Q_k \neq 0$. Thus it suffices to prove the equivalence $G \cap B_k = 0 \equiv G \cap Q_k = 0$ for every $k \in \omega$.

The implication from left to right in this equivalence is obvious, since, by definition, $Q_k \subseteq B_k$. The implication from right to left can be derived without difficulity from the choice of Q_k as a maximal antichain in B_k, the genericity of G, and Proposition 2.3.3. The details are left to the reader. This proves the theorem.

REMARK. By construction, $Q_k \subseteq P(U)$ for all k.

By using this remark and after inessential changes in the proof of Theorem 1, it is not difficult to obtain a proof of the following theorem.

THEOREM 2. *Assume that U is a system, G is a U-g.f., $S \in L[G]$, and $S \subseteq \mathrm{Seq}$. Then there exists $c \in \mathrm{Cod}$ such that $S = c^G$.*

§3. Construction of the system U^*

After the general arguments of §2, we shall construct in this section a system U^* possessing the property that every U^*-g.f. satisfies the second formulation of the fundamental Theorem 1.4.

All the reasoning in §3 is carried out within the constructible universe L.

3.1. *Preliminary definitions.* We say that the system V *extends* the system U if $U_{ni} \subseteq V_{ni}$ for all n and i. A system U is said to be *small* if every set U_{ni} has power $< \omega_1$. By PS we denote the collection of all pairs (U, V) such that U and V are small systems and $U_{ni} \cap V_{ni} = 0$ for all n and i.

If $m \in \omega$, then we set

$$PS_{<m} = \{(U, V) \in PS\colon U_{ni} = V_{ni} = 0 \text{ for all } n > m \text{ and } i \in \omega\},$$

and

$$PS_{>m} = \{(U, V) \in PS\colon U_{ni} = V_{ni} = 0 \text{ for all } n < m \text{ and } i \in \omega\}.$$

Let U be a system and $m \in \omega$. By $U[<m]$ we denote the system U' defined by the conditions $U'_{ni} = U_{ni}$ for $n < m$, and $U'_{ni} = 0$ for $n > m$. Similarly, by $U[>m]$ we denote the system U'' defined by the conditions $U''_{ni} = U_{ni}$ for $n > m$, and $U''_{ni} = 0$ for $n < m$.

If α is an ordinal and $(U^\gamma\colon \gamma \in \alpha)$ is a sequence of systems, then by $\lim_{\gamma\in\alpha} U^\gamma$ we denote the system U defined by the condition $U_{ni} = \bigcup_{\gamma\in\alpha} U^\gamma_{ni}$ for all n and i. A sequence of systems $(U^\gamma\colon \gamma \in \alpha)$ is said to be a *continuously increasing sequence* (c.i.s.) if: (1) every system U^γ is small, (2) U^β extends U^γ for $\gamma \in \beta \in \alpha$, and (3) $U^\beta = \lim_{\gamma\in\beta} U^\gamma$ for all limit ordinals $\beta \in \alpha$.

Two c.i.s.'s $(U^\gamma\colon \gamma \in \alpha)$ and $(V^\gamma\colon \gamma \in \alpha)$ are said to be *contrary* if $(U^\gamma, V^\gamma) \in PS$ for all $\gamma \in \alpha$.

Finally, if (U, V) and (U', V') belong to PS, and, in addition, U' extends U and V' extends V, then we write that (U', V') *extends* (U, V).

The system U^* that will be constructed below will have the form $U^* = \lim_{\alpha\in\omega_2} U^\alpha$, where $(U^\alpha\colon \alpha \in \omega_2)$ is a previously constructed c.i.s. At the same time, we shall construct another c.i.s. $(V^\alpha\colon \alpha \in \omega_2)$ contrary to the first c.i.s. The construction of both c.i.s.'s is carried out in 3.6 after some auxiliary arguments in 3.2–3.5.

3.2. *Definability in $H\omega_2$.* By $H\omega_2$ we denote the set $\{x\colon$ the power of the transitive closure of x is less than $\omega_2\}$. We use the standard notation Σ_n and Π_n for classes of $\in$-formulas [10].

For convenience, we denote the set $\Sigma_n^{H\omega_2} = \{X \subseteq H\omega_2\colon X$ is definable in $H\omega_2$ by some Σ_n-formula without parameters$\}$ by $\Sigma_n^{(2)}$. Similarly for $\Pi_n^{(2)}$ and for $\Delta_n^{(2)} = \Sigma_n^{(2)} \cap \Pi_n^{(2)}$.

Further, we define $\mathbf{\Sigma}_n^{(2)} = \{X \subseteq H\omega_2\colon X$ is definable in $H\omega_2$ by some Σ_n-formula in which parameters from $H\omega_2$ are permitted$\}$. Similarly for $\mathbf{\Pi}_n^{(2)}$, and for $\mathbf{\Delta}_n^{(2)} = \mathbf{\Pi}_n^{(2)} \cap \mathbf{\Sigma}_n^{(2)}$.

Let us state the following proposition about the definability in $H\omega_2$ of some sets which were defined earlier.

$R \cap L \in \Sigma_0^{(2)}$, $\omega_1 \in \Sigma_1^{(2)}$, and the following sets belong to $\Delta_2^{(2)}$: $\{\omega_1\}$, Seq, Fun, P_0, cod, Cod, *$\{U\colon U$ is a small system$\}$, PS, $PS_{<m}$ and $PS_{>m}$ (uniformly with respect to m)*(1) *$\{(U, p)\colon U$ is a small system and $p \in P(U)\}$, $\{(p, q)\colon p, q \in P_0$ and $p < q\}$ and $\{(p, q)\colon p, q \in P_0$ are compatible$\}$.*

(1)This means that the sets $\{(m, U)\colon m \in \omega$ and $U \in PS_{<m}$ (or $PS_{>m}$)$\}$ belong to Δ_2.

OUTLINE OF THE PROOF. Since all the arguments of §3 are carried out within L, $R \cap L = R \in \Sigma_0^{(2)}$. Further, let $\mathrm{cnt}(x)$ be the following Σ_1-formula expressing "at most countable": $\exists f\,[f$ is a function from ω onto $x \cup \{0\}]$. We have

$$\omega_1 = \{\gamma:\ \gamma \text{ is an ordinal and } \mathrm{cnt}(\gamma)\} \in \Sigma_1^{(2)}.$$

In precisely the same way,

$$\{\omega_1\} = \{\lambda:\ \lambda \text{ is an ordinal, } \sim\mathrm{cnt}(\lambda) \text{ and } (\forall\gamma\in\lambda)\,\mathrm{cnt}(\gamma)\};$$

that is, $\{\omega_1\}$ is definable in $H\omega_2$ by a formula which is a conjunction of the Σ_0-formula "λ is an ordinal", the Π_1-formula "$\sim \mathrm{cnt}(\lambda)$", and the Σ_1-formula "$(\forall\gamma \in \lambda)\mathrm{cnt}(\gamma)$" (the quantifier $(\forall\gamma \in \lambda)$ is bounded and does not affect the level of definability). From this it follows that $\{\omega_1\} \in \Delta_2^{(2)}$.

The rest of the sets listed above are definable in $H\omega_2$ by formulas which are built up out of formulas of the form $\mathrm{cnt}(x)$, $x \in R \cap L$, $x = \omega_1$, with the help of propositional connectives and bounded quantifiers. These formulas can be written down immediately from the definitions, and this is left to the reader. The last of the indicated sets requires special care; one must use 2.3.2 and write it in the form $\{(p, q)$: the sets p, q, and $p \vee q$ belong to $P_0\}$.

3.3. *Canonical well-ordering of $H\omega_2$.* Remember that all arguments of §3 are carried out within L. This means that there exists a canonical well-ordering of the (constructible) universe. By $\prec$ we denote the restriction of this well-ordering to the set $H\omega_2$. The relation $\prec$ on $H\omega_2$ is a well-ordering of $H\omega_2$ of type ω_2 and has definability properties expressed in the following proposition (cf. [10], p. 83):

1. *The sets $\{(x,y)$: $x, y \in H\omega_2$ and $x \prec y\}$ and $IS = \{\{y\colon y \prec x\}\colon x \in H\omega_2\}$ (the collection of all initial segments of $H\omega_2$ in the sense of $\prec$) both belong to $\Delta_1^{(2)}$.*

With the help of this proposition, we prove the "uniformization principle":

LEMMA 2. *Assume $X \subseteq H\omega_2 \times H\omega_2$ and $X \in \Delta_n^{(2)}$, $n \geq 1$. Then the set $F = \{(x,y) \in X$: y is the $\prec$-least of all y' such that $(x,y') \in X\}$ also belongs to $\Delta_n^{(2)}$.*

PROOF. The following two equations hold for F:

$$F = \{(x, y) : (x, y) \in X \text{ and } \forall y'\,[y' \prec y \to (x, y') \notin X]\},$$

$$F = \{(x, y) : (x, y) \in X \text{ and } \exists Z\,[Z \in IS \,\&\, y \in Z \,\&\, (\forall y' \in Z)\,[y' \neq y \to (x, y') \notin X]]\}.$$

The first of these yields a Π_n-definition of F in $H\omega_2$, and the second yields a Σ_n-definition. Reduction of the formulas written above to forms in Π_n and Σ_n can be carried out with the help of Proposition 1 and $X \in \Delta_n^{(2)}$. We note that the quantifier "$(\forall y' \in Z)$" is bounded and does not increase the complexity.

For a more detailed proof of a similar proposition, cf. [10], the theorem on pp. 86–87.

3.4. *The sequences m_α, t_α, $U^{(\alpha)}$, and $V^{(\alpha)}$.* These sets play an important role in the construction of U^α and V^α below in 3.6.

For every $\alpha \in \omega_2$, we construct sets m_α, t_α, $U^{(\alpha)}$, and $V^{(\alpha)}$ such that the following three conditions are fulfilled:

1. $m_\alpha \in \omega$, $t_\alpha \in H\omega_2$, $(U^{(\alpha)}, V^{(\alpha)}) \in PS$ *for all* α.

2. *The sequences* $(m_\alpha\colon \alpha \in \omega_2)$, $(t_\alpha\colon \alpha \in \omega_2)$, $(U^{(\alpha)}\colon \alpha \in \omega_2)$, *and* $(V^{(\alpha)}\colon \alpha \in \omega_2)$ *belong to the collection* $\Delta_2^{(2)}$.

3. *Assume that* $m \in \omega$, $t \in H\omega_2$, *and* $\kappa \subseteq \omega_2$ *is closed and bounded in* ω_2 *(that is,* $\bigcup \kappa = \omega_2$ *and* $\bigcup(\kappa \cap \alpha) \in \kappa$ *for all* $\alpha \in \omega_2$*). Assume also that* $(U^\alpha\colon \alpha \in \omega_2)$ *and* $(V^\alpha\colon \alpha \in \omega_2)$ *are contrary c.i.s.'s. Then there exists* $\alpha \in \kappa$ *such that* $m = m_\alpha$, $t = t_\alpha$, $U^\alpha = U^{(\alpha)}$, *and* $V^\alpha = V^{(\alpha)}$.

The construction of these sets will be based upon a sequence $(A_\alpha\colon \alpha \in \omega_2)$ of elements of the set $H\omega_2$ with the following properties:

(i) $(A_\alpha\colon \alpha \in \omega_2) \in \Delta_1^{(2)}$, *and*

(ii) *if* $\kappa' \subseteq \omega_2$ *is closed and bounded in* ω_2 *and* $(B_\alpha\colon \alpha \in \omega_2)$ *is a sequence of elements of* $H\omega_2$, *then there exists* $\alpha \in \kappa'$ *such that* $A_\alpha = (B_\gamma\colon \gamma \in \alpha)$.

The existence of such a sequence is, in turn, based upon Jensen's principle $\diamondsuit_{\omega_2}$, which is true upon assumption of $V = L$ (and $V = L$ is assumed in this section); cf. [10] and [14]. This principle asserts the existence of a sequence $(S_\alpha\colon \alpha \in \omega_2)$ of sets $S_\alpha \subseteq \alpha$ satisfying the following condition:

(1) If a set $\kappa \subseteq \omega_2$ is closed and bounded in ω_2, and if $X \subseteq \omega_2$, then there exists an $\alpha \in \kappa$ such that $X \cap \alpha = S_\alpha$.

Analyzing any standard construction of such a sequence (see [10] and [14]), we can verify without difficulty, with the help of the uniformization principle 3.3.2, that the sequence satisfies

(2) $(S_\alpha\colon \alpha \in \omega_2) \in \Delta_1^{(2)}$.

Further, for every $\alpha \in \omega_2$, let $C(\alpha)$ be the αth element of the set $H\omega_2$ in the sense of the ordering $\prec$ of 3.3. Then the function C from ω_2 onto $H\omega_2$ is constructible and belongs to $\Delta_1^{(2)}$ by virtue of the choice of the ordering $\prec$.

Starting from S_α and C, we construct A_α as follows: $A_\alpha = C''S_\alpha = \{C(\beta)\colon \beta \in S_\alpha\}$. From (2) and the membership of C in $\Delta_1^{(2)}$, we immediately obtain property (i) for the sequence $(A_\alpha\colon \alpha \in \omega_2)$. Let us verify (ii). Assume κ' and $(B_\alpha\colon \alpha \in \omega_2)$ are as indicated in (ii). Let us set

$$X = \{\beta \in \omega_2 : C(\beta) = \{(\gamma, B_\gamma)\} \text{ for some } \gamma \in \omega_2\}$$

and

$$\kappa'' = \{\alpha \in \omega_2 : \{C(\beta) : \beta \in X \cap \alpha\} = \{(\gamma, B_\gamma) : \gamma \in \alpha\}\}$$

and put $\kappa = \kappa' \cap \kappa''$. It is clear that $X \subseteq \omega_2$. In addition, it is not hard to verify that κ'' is closed and bounded in ω_2. Hence the intersection κ of two closed and bounded sets is itself closed and bounded. Now, by virtue of (1), there exists $\alpha \in \kappa$ such that $X \cap \alpha = S_\alpha$. We have

$$\begin{aligned}(B_\gamma : \gamma \in \alpha)\,(= \{(\gamma, B_\gamma) : \gamma \in \alpha\}) &= \{C(\beta) : \beta \in X \cap \alpha\} \quad (\text{since } \alpha \in \kappa'')\\ &= \{C(\beta) : \beta \in S_\alpha\} \quad (\text{since } X \cap \alpha = S_\alpha) = A_\alpha.\end{aligned}$$

But, by construction, $\alpha \in \kappa \subseteq \kappa'$, and the proof of (ii) is finished.

Now, having a sequence $(A_\alpha\colon \alpha \in \omega_2)$ with properties (i) and (ii), we define the sets m_α, t_α, $U^{(\alpha)}$, and $V^{(\alpha)}$.

Assume $\alpha \in \omega_2$. If A_α has the form $(a_\gamma\colon \gamma \in \alpha)$, every a_γ is a quadruple $(m, t, U^\gamma, V^\gamma)$ (where m and t are one and the same for all γ), and $(U^\gamma\colon \gamma \in \alpha)$ and $(V^\gamma\colon \gamma \in \alpha)$ are contrary c.i.s.'s, then we define $m_\alpha = m$, $t_\alpha = t$, $U^{(\alpha)} = \lim_{\gamma \in \alpha} U^\gamma$, and $V^{(\alpha)} = \lim_{\gamma \in \alpha} V^\gamma$.

If the indicated group of conditions does not hold, then we set $m_\alpha = t_\alpha = 0$ and $U^{(\alpha)} = V^{(\alpha)} = (\omega \times \omega) \times \{0\}$ (that is, $U^{(\alpha)}_{ni} = V^{(\alpha)}_{ni} = 0$ for all n and i).

This concludes the definition of m_α, t_α, $U^{(\alpha)}$, and $V^{(\alpha)}$. Let us verify the conditions stated above.

That 1 holds is obvious from the construction and the definition of lim.

Further, the group of conditions determining the two alternatives of the construction expresses a $\Delta^{(2)}_2$-relation (this is easily obtained from (i) and 3.2). This implies that 2 holds.

Finally, the proof of 3 is obtained by applying (ii) to the set $\kappa' = \{\alpha \in \kappa$: α is a limit ordinal$\}$ and the sequence $(B_\alpha$: $\alpha \in \omega_2)$, determined by the condition that $B_\alpha = (m, t, U^\alpha, V^\alpha)$ for every $\alpha \in \omega_2$. In this connection, it is necessary to consider the continuity of the sequences U^α and V^α, which implies the equations $U^\alpha = \lim_{\gamma \in \alpha} U^\gamma$ and $V^\alpha = \lim_{\gamma \in \alpha} V^\gamma$ for limit ordinals $\alpha \in \omega_2$.

The details are left to the reader.

3.5. *Blocking pairs.* Assume that $(U', V') \in PS$, $m \in \omega$, and $D \subseteq H\omega_2$. We write that the pair (U', V') *m-blocks D* if one of the following two conditions holds:

(i) $(U'[\geqslant m], V'[\geqslant m]) \in D$;

(ii) *there is no pair* $(U'', V'') \in D \cap PS_{\geqslant m}$ *extending* $(U'[\geqslant m], V'[\geqslant m])$.

The following proposition is obvious.

1. *If* $m \in \omega$, $\alpha \in \omega_2$, *and* $D \subseteq H\omega_2$, *then there exists a pair* $(U', V') \in PS$, *extending the pair* $(U^{(\alpha)}, V^{(\alpha)})$ *(constructed in* 3.4) *and m-blocking the set D, such that* $U'[< m] = U^{(\alpha)}[< m]$ *and* $V'[< m] = V^{(\alpha)}[< m]$.

(This proposition is true, it is clear, not only for the pair $(U^{(\alpha)}, V^{(\alpha)})$, but also for any pair $(U, V) \in PS$.)

By $F^m_D(\alpha)$ we denote the least such pair (U', V'), in the sense of the well-ordering $\prec$ of (3.3).

Further, for every $n \geqslant 1$, we fix a Σ_n-formula $\mathrm{un}_n(x, t)$ which is *universal* in the sense of the following assertion: If $X \subseteq H\omega_2$ and $X \in \Sigma^{(2)}_n$, then there exists $t \in H\omega_2$ such that $X = \{x \in H\omega_2$: $\mathrm{un}_n(x, t)$ is true in $H\omega_2\}$. We define $M^n_t = \{x \in H\omega_2$: $\mathrm{un}_n(x, t)$ is true in $H\omega_2\}$. By the choice of the formula un_n we have the following proposition:

2. $\{M^n_t$: $t \in H\omega_2\}$ *is exactly the collection of all* $\Sigma^{(2)}_n$*-subsets of* $H\omega_2$ *(assuming* $n \geqslant 1$).

Finally, for $\alpha \in \omega_2$ and $m \in \omega$, we define $U^{(m\alpha)}$ and $V^{(m\alpha)}$ so that $(U^{(m\alpha)}, V^{(m\alpha)}) = F^m_D(\alpha)$, where $D = M^{m+2}_t$ and $t = t_\alpha$. From the definitions we immediately obtain the following proposition:

3. *Let* $\alpha \in \omega_2$ *and* $m \in \omega$. *Then the pair* $(U^{(m\alpha)}, V^{(m\alpha)})$ *belongs to PS, extends the pair* $(U^{(\alpha)}, V^{(\alpha)})$, *m-blocks the set* $M^{m-2}_{t_\alpha}$, *and satisfies the equations* $U^{(m\alpha)}[< m] = U^{(\alpha)}[< m]$ *and* $V^{(m\alpha)}[< m] = V^{(\alpha)}[< m]$.

As usual, by $U^{(m\alpha)}_{ni}$ and $V^{(m\alpha)}_{ni}$ we denote the (n, i)th component of the systems $U^{(m\alpha)}$ and $V^{(m\alpha)}$ (cf. 2.1). We note that the last two equations of proposition 3 yield $U^{(m\alpha)}_{ni} = U^{(\alpha)}_{ni}$ and $V^{(m\alpha)}_{ni} = V^{(\alpha)}_{ni}$ whenever $n < m$.

Now we prove a lemma on definability of components.

LEMMA 4. *Assume* $m \in \omega$. *Then the sets* $(U^{(m\alpha)}_{ni}$: $\alpha \in \omega_2$ *and* $n, i \in \omega)$ *and* $(V^{(m\alpha)}_{ni}$: $\alpha \in \omega_2$ *and* $n, i \in \omega)$ *belong to* $\Delta^{(2)}_{m+3}$.

We carry out the proof only for the first set; the proof for the second is similar. It suffices to verify that the sequence $(U^{(m\alpha)}$: $\alpha \in \omega_2)$ belongs to $\Delta^{(2)}_{m+3}$. From 3.2, 3.4.2, and membership of the formula un_{m+2} in Σ_{m+2}, it is not hard to obtain the following fact: *the set* $\{(\alpha, U', V')$: *the pair* (U', V') *belongs to PS, extends the pair* $(U^{(\alpha)}, V^{(\alpha)})$, *m-blocks the set* $M_{t_\alpha}^{m+2}$, *and satisfies* $U'[<m] = U^{(\alpha)}[<m]$ *and* $V'[<m] = V^{(\alpha)}[<m]\}$ *belongs to* $\Delta^{(2)}_{m+3}$. More precisely, this set is definable in $H\omega_2$ by a conjunction of a Σ_{m+2}-formula describing condition (i) for (U', V') and the connection with $(U^{(\alpha)}, V^{(\alpha)})$ and t_α, and of a Π_{m+2}-formula transcribing (ii) for (U', V').

Now the membership of $(U^{(m\alpha)}: \alpha \in \omega_2)$ in $\Delta^{(2)}_{m+3}$ follows from the uniformization principle 3.3.2 and the definitions. This proves the lemma.

After the preliminary constructions in 3.2–3.5, we begin the construction of the sequences U^α and V^α discussed in 3.1.

3.6. THEOREM ON THE SEQUENCES U^α AND V^α. *There exist sequences* $(U^\alpha: \alpha \in \omega_2)$ *and* $(V^\alpha: \alpha \in \omega_2)$ *satisfying the following three conditions*:

(i) *These sequences are contrary c.i.s.'s.*

(ii) *If* $m \in \omega$, $\beta \in \omega_2$, *and the set* $D \subseteq H\omega_2$ *belongs to* $\Sigma^{(2)}_{m+2}$, *then there exists* $\alpha \in \omega_2$, $\alpha > \beta$, *such that the pair* (U^α, V^α) *m-blocks the set D.*

(iii) *If* $n \in \omega$, *then the sets* $(U^\alpha_{ni}: \alpha \in \omega_2$ *and* $i \in \omega)$ *and* $(V^\alpha_{ni}: \alpha \in \omega_2$ *and* $i \in \omega)$ *belong to* $\Sigma^{(2)}_{n+3}$.

PROOF. We shall describe the construction of U^α and V^α by induction on $\alpha \in \omega_2$.

(a) We set $U^0 = V^0 = (\omega \times \omega) \times \{0\}$; that is, $U^0_{ni} = V^0_{ni} = 0$ for all n and i.

(b) If $\beta \in \omega_2$ is a limit ordinal, then $U^\beta = \lim_{\gamma\in\beta} U^\gamma$ and $V^\beta = \lim_{\gamma\in\beta} V^\gamma$.

(c) Assume that $\alpha \in \omega_2$ and that U^α and V^α have been constructed with $(U^\alpha, V^\alpha) \in PS$. We shall construct $(U^{\alpha+1}, V^{\alpha+1}) \in PS$. For this, we fix $n, i \in \omega$ and describe the construction of the components $U^{\alpha+1}_{ni}$ and $V^{\alpha+1}_{ni}$. We point out that this construction will be based only upon the components U^α_{ni} and V^α_{ni} of the systems U^α and V^α (already constructed), but not upon the "entire" systems U^α and V^α.

The construction depends upon satisfaction of the following conditions:

$$m_\alpha \leqslant n, \quad U^\alpha_{ni} = U^{(\alpha)}_{ni}, \quad V^\alpha_{ni} = V^{(\alpha)}_{ni}. \tag{$*$}$$

If $(*)$ is satisfied, then we define $m = m_\alpha$, $U^{\alpha+1}_{ni} = U^{(m\alpha)}_{ni}$, and $V^{\alpha+1}_{ni} = V^{(m\alpha)}_{ni}$. ($U^{(m\alpha)}_{ni}$ and $V^{(m\alpha)}_{ni}$ are the (n, i)th components of the systems $U^{(m\alpha)}$ and $V^{(m\alpha)}$ that were constructed in 3.5.)

If $(*)$ is not satisfied, then we define $U^{\alpha+1}_{ni} = U^\alpha_{ni}$ and $V^{\alpha+1}_{ni} = V^\alpha_{ni}$. This concludes the definition of $U^{\alpha+1}_{ni}$ and $V^{\alpha+1}_{ni}$.

Carrying out the definition of $U^{\alpha+1}_{ni}$ and $V^{\alpha+1}_{ni}$ in this way for all $n, i \in \omega$, we obtain systems $U^{\alpha+1}$ and $V^{\alpha+1}$, the components of which are $U^{\alpha+1}_{ni}$ and $V^{\alpha+1}_{ni}$.

This concludes the inductive construction of U^α and V^α for $\alpha \in \omega_2$.

Let us prove that the constructed sequences satisfy requirements (i)–(iii) in the statement of the theorem. We begin with the following two assertions.

1. *If* $\alpha \in \omega_2$, *then* $(U^\alpha, V^\alpha) \in PS$ *and the pair* $(U^{\alpha+1}, V^{\alpha+1})$ *extends the pair* (U^α, V^α). *Thus, according to part* (b) *of the inductive construction, the constructed sequences are contrary c.i.s.'s; that is, requirement* (i) *is satisfied.*

2. *Assume that* $\alpha \in \omega_2$ *is such that* $U^\alpha = U^{(\alpha)}$ *and* $V^\alpha = V^{(\alpha)}$. *Then* $U^{\alpha+1} = U^{(m\alpha)}$, $V^{\alpha+1} = V^{(m\alpha)}$, *and the pair* $(U^{\alpha+1}, V^{\alpha+1})$ *m-blocks the set* M^{m+2}_t, *where* $m = m_\alpha$ *and* $t = t_\alpha$.

The simple proof of both assertions follows immediately from the definitions and Proposition 3.5.3. (In particular, one uses the equations $U^{(m\alpha)}[< m] = U^{(\alpha)}[< m]$ and $V^{(m\alpha)}[< m] = V^{(\alpha)}[< m]$ in 3.5.3.)

The details of the proof of 1 and 2 are left to the reader.

Now that we have proved requirement (i) of the statement of the theorem (assertion 1), in 3.7 and 3.8 we shall prove (ii) and (iii) for the constructed sequences, and so finish the proof of the theorem.

3.7. *Verification of* (ii). So, let us verify requirement (ii) of the statement of Theorem 3.6 for the sequences constructed in 3.6.

Let $m \in \omega$, $\beta \in \omega_2$, $D \subseteq H\omega_2$, and $D \in \Sigma^{(2)}_{m+2}$. Let us find $\alpha > \beta$ such that the pair (U^α, V^α) will m-block D. According to 3.5.2, the set D has the form $D = M_t^{m+2}$ for suitable $t \in H\omega_2$. We fix this t and apply 3.4.3 for the given m, t, the set $\kappa = \{\alpha \in \omega_2: \alpha > \beta\}$, and the sequences constructed in 3.6 (which are contrary c.i.s.'s by 3.6.1). We find $\alpha \in \omega_2$, $\alpha \geqslant \beta$, such that $m_\alpha = m$, $t_\alpha = t$, $U^{(\alpha)} = U^\alpha$ and $V^{(\alpha)} = V^\alpha$.

Now, taking into account the choice of t, we apply 3.6.2 to complete the proof.

3.8. *Verification of* (iii). We fix $n \in \omega$ and prove the membership in $\Sigma^{(2)}_{n+3}$ of the set $E = \{(\alpha, i, U^\alpha_{ni}): \alpha \in \omega_2 \text{ and } i \in \omega\}$ (this is a more explicit description of the set $(U^\alpha_{ni}: \alpha \in \omega_2 \text{ and } i \in \omega)$).

We write down the following formula:

$$\varphi_n(\alpha, i, u, v, u', v') \Leftrightarrow \alpha \in \omega_2 \& i \in \omega \& \theta,$$

where θ is the disjunction $\theta_1 \vee \theta_2$ of the following formulas θ_1 and θ_2:

$$\theta_1 \Leftrightarrow \bigvee_{m \leqslant n} [m = m_\alpha \& u = U^{(\alpha)}_{ni} \& v = V^{(\alpha)}_{ni} \& u' = U^{(m\alpha)}_{ni} \& v' = V^{(m\alpha)}_{ni}],$$
$$\theta_2 \Leftrightarrow [u' = u \& v' = v \& [m_\alpha > n \vee u \neq U^{(\alpha)}_{ni} \vee v \neq V^{\alpha}_{ni}]].$$

On the one hand, from the definition in 3.6 we obtain the following proposition.

1. *Take any $\alpha \in \omega_2$, $i \in \omega$, and $u', v' \in H\omega_2$. Then the formula $\varphi_n(\alpha, i, U^\alpha_{ni}, V^\alpha_{ni}, u', v')$ is true in $H\omega_2$ if and only if $u' = U^{\alpha+1}_{ni}$ and $v' = V^{\alpha+1}_{ni}$.*

On the other hand, from 3.4.2 and 3.5.4 we obtain

2. *The set $\{(\alpha, i, u, v, u', v'): \varphi_n(\alpha, i, u, v, u', v')$ is true in $H\omega_2\}$ belongs to the collection $\Delta^{(2)}_{n+3}$.*

Now we immediately begin the study of the complexity of the set E. We introduce another auxiliary formula:

> $\psi_n(\alpha, i, f, g) \Leftrightarrow \alpha \in \omega_2 \;\&\; i \in \omega \;\&\; [f$ and g are functions defined on the set $\alpha + 1] \;\&\; f(0) = g(0) = 0 \;\&\;$ [for every limit ordinal $\beta < \alpha$, $f(\beta) = \bigcup_{\gamma \in \beta} f(\gamma)$ and $g(\beta) = \bigcup_{\gamma\in\beta} g(\gamma)] \;\&\; (\forall \beta \in \alpha)$ $\varphi_n(\alpha, i, f(\beta), g(\beta), f(\beta + 1), g(\beta + 1))$.

We note at once that proposition 2 implies

3. *The set $\{(\alpha, i, f, g): \psi_n(\alpha, i, f, g)$ is true in $H\omega_2\}$ belongs to $\Delta^{(2)}_{n+3}$.*

In addition, from 1 and the definition of 3.6, one concludes that *$\psi_n(\alpha, i, f, g)$ is true in $H\omega_2$ if and only if $f(\beta) = U^\beta_{ni}$ and $g(\beta) = V^\beta_{ni}$ for all $\beta < \alpha$.* Hence

$$E = \{(\alpha, i, u): \exists f \exists g\ [\psi_n(\alpha, i, f, g) \& f(\alpha) = u] \text{ is true in } H\omega_2\}.$$

From this equation and 3, we obtain $E \in \Sigma^{(2)}_{m+3}$.

The verification of requirement 3.6(iii) for the second set (V^α_{ni}: $\alpha \in \omega_2$ and $i \in \omega$) is similar.

This concludes the verification of requirements (i)–(iii) of the statement of Theorem 3.6 for the sequences constructed above, and Theorem 3.6 is proved.

3.9. *The system U^*.* So, we have constructed sequences (U^α: $\alpha \in \omega_2$) and (V^α: $\alpha \in \omega_2$) satisfying the requirements 3.6(i)–(iii). These sequences are to remain fixed in the ensuing exposition. Remember that their construction, like all of the reasoning in §3, is carried out in the constructible universe L.

We define $U^* = \lim_{\alpha \in \omega_2} U^\alpha$ and $V^* = \lim_{\alpha \in \omega_2} V^\alpha$, and we state several propositions relating to U^* and V^*.

1. *If $\beta < \alpha \in \omega_2$, then the pair (U^α, V^α) belongs to PS, extends the pair (U^β, V^β), and satifies $P(U^\beta) \subseteq P(U^\alpha) \subseteq P(U^*)$.*

2. $U^*_{ni} = \bigcup_{\alpha \in \omega_2} U^\alpha_{ni}$, $V^*_{ni} = \bigcup_{\alpha \in \omega_2} V^\alpha_{ni}$ *and* $U^*_{ni} \cap V^*_{ni} = 0$.

3. $P(U^*) = \bigcup_{\alpha \in \omega_2} P(U^\alpha)$.

Propositions 1 and 2 are easily obtained from 3.6(i) and the definition of lim. For the proof of 3 we must note that, in every $p \in P_0$, at most countably many $f \in \mathrm{Fun}$ are "involved".

Somewhat less trivial is the following proposition.

LEMMA 4. *Assume $n, i \in \omega$. Then the set U^*_{ni} has power ω_2.*

PROOF. Assume the contrary: $\mathrm{card}(U^*_{ni}) < \omega_1$. Then $U^*_{ni} \in H\omega_2$, and therefore, by 3.2, the set $D = \{(U, V) \in PS\colon U_{ni} \not\subseteq U^*_{ni}\}$ belongs to $\Sigma^{(2)}_2$ (with parameter U^*_{ni}). Hence, according to 3.6(ii) (for $m = 0$), there exists $\alpha \in \omega_2$ such that either (a) or (b) holds:

(a) $(U^\alpha, V^\alpha) \in D$.

(b) *There is no pair $(U, V) \in D$ extending (U^α, V^α).*

If (a) holds, then, by definition of the set D, $U^\alpha_{ni} \not\subseteq U^*_{ni}$, which contradicts 2.

Assume that (b) holds. We show that in this case one also obtains a contradiction. Since $(U^\alpha, V^\alpha) \in PS$ by 3.6(i), the set V^α_{ni} has power $< \omega_1$. Thus, according to the assumption at the beginning of the proof, $V^\alpha_{ni} \cup U^*_{ni}$ also has power $< \omega_1$. Hence, there exists $f \in \mathrm{Fun}$ such that $f \notin V^\alpha_{ni} \cup U^*_{ni}$. Now we define a system U' as follows: $U'_{ni} = U^\alpha_{ni} \cup \{f\}$, and $U'_{mj} = U_{mj}$ for $m \neq n \vee j \neq i$. By the choice of f, it is clear that the pair (U', V^α) belongs to PS, extends the pair (U^α, V^α), and satisfies $f \in U'_{ni}$. The latter means that (U', V^α) belongs to D. But this contradicts (b).

Thus, both possibilities (a) and (b) lead to contradiction, and the lemma is proved.

LEMMA 5. *Assume $n \in \omega$. Then the sets $C_n = \{(i, f)\colon i \in \omega$ and $f \in U^*_{ni}\}$ and $C'_n = \{(i, f)\colon i \in \omega$ and $f \in V^*_{ni}\}$ belong to $\Sigma^{(2)}_{n+3}$.*

We shall carry out the proof for C_n; the proof for C'_n is similar. From proposition 2, we have $C_n = \{(i, f)\colon i \in \omega \ \&\ (\exists \alpha \in \omega_2)[f \in U^\alpha_{ni}]\}$. Now $C_n \in \Sigma^{(2)}_{n+3}$ follows from 3.6(iii).

§4. Constructibility of all sets of natural numbers analytically definable in $L[G]$

4.1. *Third formulation of the fundamental theorem.* The system U^* constructed in 3.9 will play a central role in the proof of the fundamental theorem. In fact, we now will prove FT in the following form.

FUNDAMENTAL THEOREM (third formulation). *If G is a U^*-g.f., then the following assertions hold*:

(i) *Every $a \in L[G] \cap R$, analytically definable in $L[G]$, is constructible.*

(ii) *Every constructible $r \subseteq \omega$ is analytically definable in $L[G]$.*

The derivation of the second formulation of FT (1.4) from the third is trivial by virtue of the existence of a U^*-g.f., 2.4.3.

§4 is devoted to the proof of assertion (i). First, we introduce in 4.2–4.6 the special apparatus of the forc relation in order to study analytic truth in $L[G]$, based upon the representation Theorem 2.7.1. Then assertion (i) is proved in 4.7–4.9. The proof uses a proposition, the restriction principle 4.8(*), the verification of which is done separately in §5.

All the reasoning in 4.2–4.6 and 4.8, like that in §3, is carried out within the constructible universe L.

4.2. $\sim \&\exists$-*formulas*. We shall introduce a special language for the study of analytic truth in generic extensions. Here and in the rest of §4, the letters m, n and k denote natural numbers and variables of type 0 (with range ω), the letter x is a variable of type 1 (with range $\mathcal{P}(\omega)$), and the letter c is a "code", an element of the set cod.

By an *elementary* $\sim \&\exists$-*formula* we mean a formula of one of the following forms: $m + n = k$, $m \cdot n = k$, $m = n$, $k \in x$, $k \in c$. Further, by a $\sim \&\exists$-*formula* we mean a formula obtained from elementary ones with the help of $\sim$, &, and quantifiers $\exists k$ and $\exists x$. The form of the logical symbols is restricted for technical reasons.

For each $\sim \& \exists$-formula φ, we define its complexity com(φ):

$$\begin{aligned}
&\mathrm{com}(\varphi) = 0 \text{ if } \varphi \text{ is an elementary formula;}\\
&\mathrm{com}(\varphi \,\&\, \psi) = \max(\mathrm{com}(\varphi), \mathrm{com}(\psi));\\
&\mathrm{com}(\exists k\, \varphi(k)) = \mathrm{com}(\exists x\, \varphi(x)) = \mathrm{com}(\varphi);\\
&\mathrm{com}(\sim \varphi) = \mathrm{com}(\varphi) + 1.
\end{aligned}$$

The interpretation of $\sim \&\exists$-formulas will be introduced in 4.7.

4.3. *The* forc *relation*. Before the definition of forc, we introduce the following definition:

$$T_m = \{(U, V) \in PS\colon \text{ there exists } \beta \in \omega_2, \text{ such that } U[<m] = U^\beta[<m] \text{ and } V[<m] = V^\beta[<m]\}.$$

It is clear that $T_{m+1} \subseteq T_m \subseteq T_0 = PS$.

Now we define the relation p forc_{UV} φ. As part of this notation, we assume that $(U, V) \in PS$, $p \in P(U)$, and φ is a closed $\sim \&\exists$-formula. The definition proceeds by induction on the length of φ.

(i) *Assume that the (closed) formula φ has one of the forms $m + n = k$, $m \cdot n = k$ or $m = n$. Then p forc_{UV} φ if and only if φ is true under the usual interpretation of addition, multiplication, or equality of natural numbers.*

(ii) *Assume φ is a formula $k \in c$, where $k \in \omega$ and $c = (Q_t\colon t \in \omega) \in$* cod. *Then p forc_{UV} φ if and only if there exists $q \in Q_k$ such that $p > q$.*

(iii) *p forc_{UV} $\varphi \,\&\, \psi$ if and only if p forc_{UV} φ and p forc_{UV} ψ.*

(iv) *p forc_{UV} $\exists k\, \varphi(k)$ if and only if there exists $k \in \omega$ such that p forc_{UV} $\varphi(k)$.*

(v) *p forc_{UV} $\exists x\, \varphi(x)$ if and only if there exists $c \in$* cod *such that p forc_{UV} $\varphi(c)$.*

(vi) *p forc_{UV} $\sim \varphi$ if and only if there do not exist a pair $(U', V') \in T_{\mathrm{com}(\varphi)}$ and some $p' \in P(U')$ such that (U', V') extends (U, V) and $p' > p$ and p' $\mathrm{forc}_{U'V'}$ φ.*

This concludes the definition of forc. We recommend that this definition be compared with the original definition of forcing by P. J. Cohen [3], Chapter IV, §3, in the light of the representation Theorem 2.7.1. In our definition, the role of "parameter space" is played by ω, the collection of constants for natural numbers, and by cod, the collection of constants for sets of natural numbers.

We note the following properties of forc:

1. *If* p $\text{forc}_{UV}\ \varphi$, *then* $(U, V) \in PS$, $p \in P(U)$, *and* φ *is a closed* $\sim \&\exists$*-formula* (*by definition*).

2. *If* p $\text{forc}_{UV}\ \varphi$, *the pair* $(U', V') \in PS$ *extends* (U, V), $p' \in P(U')$, *and* $p' \geqslant p$, *then* p' $\text{forc}_{U'V'}\ \varphi$.

3. *If* $(U, V) \in T_{\text{com}(\varphi)}$, *then* p $\text{forc}_{UV}\ \varphi$ *and* p $\text{forc}_{UV} \sim \varphi$ *cannot hold simultaneously*.

Assertion 2 can be verified without difficulty by induction, using the definition of the forc relation, and 3 follows from part (vi) of the definition.

4.4. *Definability of the relation* forc. In 4.7 below we shall prove that the relation forc is connected with truth in generic extensions in roughly the same way as ordinary forcing. In the proof of this fact it will be essential to use the specific characteristics of the construction of the system U^*, as well as the particular property 3.6(ii). In order to apply this property, we shall study the complexity of the forc relation.

Let $\varphi(k_1, \ldots, k_m, x_1, \ldots, x_n)$ be a $\sim \&\exists$-formula, with all its free variables explicitly indicated. We define the set

$$\text{Forc}_\varphi = \{(p, U, V, k_1, \ldots, k_m, c_1, \ldots, c_l) : (U, V) \in PS,\ p \in P(U),\ k_i \in \omega,\ c_i \in \text{cod and } p\ \text{forc}_{UV}\ \varphi(k_1, \ldots, k_m, c_1, \ldots, c_l)\}.$$

THEOREM. $\text{Forc}_\varphi \in \Sigma^{(2)}_{\text{com}(\varphi)+2}$. (*The definition of* $\Sigma^{(2)}_n$ *is in* 3.2.)

PROOF. For elementary formulas φ, the theorem follows immediately from various assertions in 3.2; in this connection, the quantifier "$\forall q \in Q_k$" in the definition 4.3(ii) is bounded and does not raise the level of definability. Further, we carry out the proof by induction on the length of the formula φ. The induction steps 4.3(iii)–(v) are no trouble and can be handled with the help of 3.2.

The only nontrivial part is the induction step 4.3(vi), which we examine now. For simplicity, we assume that the $\sim \&\exists$-formula φ is closed; that is, has no free variables. Then, by definition,

$$\text{Forc}_{\sim\varphi} = \{(p, U, V) : (U, V) \in PS,\ p \in P(U) \text{ and } \forall p', \forall U', \forall V' [(U', V') \in T_{\text{com}\,\varphi} \,\&\, (U', V') \text{ extends } (U, V) \,\&\, p' \in P(U') \,\&\, p' \geqslant p \rightarrow (p', U', V') \notin \text{Forc}_\varphi]\}.$$

If we now assume that $\text{Forc}_\varphi \in \Sigma^{(2)}_{\text{com}(\varphi)+2}$ (inductive hypothesis), from 3.2 and $T_m \in \Sigma^{(2)}_{m+2}$ we obtain

$$\text{Forc}_{\sim\varphi} \in \Pi^{(2)}_{\text{com}\,\varphi+2};$$

that is, $\text{Forc}_{\sim\varphi} \in \Sigma^{(2)}_{\text{com}(\sim\varphi)+2}$, since $\text{com}(\sim\varphi) = \text{com}(\varphi) + 1$.

Finally, the assertion $T_m \in \Sigma^{(2)}_{m+2}$, which was used in this argument, is obvious from property 3.6(iii) of the sequences U^α and V^α (cf. 3.9), as well as from 3.2 and the definition of T_m. The inductive step is finished, and the theorem is proved.

4.5. *The relation* forc *with pairs* (U^α, V^α). For brevity, we agree to write p $\text{forc}_\alpha\ \varphi$ instead of p $\text{forc}_{U^\alpha V^\alpha}\ \varphi$, where U^α and V^α are terms of the sequences fixed in 3.9. We shall prove two assertions about the relation forc_α.

1. *If* $p, q \in P(U^*)$, $p \leqslant q$, $\alpha \in \omega_2$, *and* p $\text{forc}_\alpha\ \varphi$, *then there exists a* $\lambda \in \omega_2$ *such that* q $\text{forc}_\lambda\ \varphi$.

2. *If* $p, p_1 \in P(U^*)$ *are compatible, and* α *and* α_1 *belong to* ω_2, *then* p $\text{forc}_\alpha \sim \varphi$ *and* p_1 $\text{forc}_{\alpha_1}\ \varphi$ *cannot hold simultaneously*.

Proof of 1. 3.9.1 and 3.9.3 imply the existence of $\lambda \in \omega_2$, $\lambda > \alpha$, such that $q \in P(U^\lambda)$. Now we apply 4.3.2, with the help of 3.9.1.

Proof of 2. Assume the contrary: p_1 $\text{forc}_{\alpha_1}\ \varphi$ and p $\text{forc}_\alpha \sim \varphi$. By 2.3.3, there exists $q \in P(U^*)$ such that $q \geqslant p$ and $q \geqslant p_1$. Arguing as in the proof of 1, we find $\beta \in \omega_2$ such that q $\text{forc}_\beta\ \varphi$ and q $\text{forc}_\beta \sim \varphi$. This contradicts 4.3.3, since the pair (U^β, V^β) obviously belongs to every set T_m.

4.6. Density Theorem. *Let* φ *be a closed* $\sim\&\exists$*-formula. Then the set*

$$Q = \{p \in P(U^*)\colon \text{there exists } \alpha \in \omega_2 \text{ such that } p \text{ forc}_\alpha\ \varphi \text{ or } p \text{ forc}_\alpha \sim \varphi\}$$

is dense in $P(U^*)$.

Proof. We fix $p \in P(U^*)$ and find $p' \in Q$ such that $p' \geqslant p$. We also fix $\delta \in \omega_2$ such that $p \in P(U^\delta)$. (Such a δ exists by 3.9.3.) We begin the proof with the following definitions: $m = \text{com}(\varphi)$ and $D = \{(U, V) \in PS_{\geqslant m}$: there exist a pair $(U', V') \in T_m$ and some $p' \in P(U')$ such that $U = U'[\geqslant m]$, $V = V'[\geqslant m]$, $p' \geqslant p$, and p' $\text{forc}_{U'V'}\ \varphi\}$. ($PS_{\geqslant m}$ and $U[\geqslant m]$ are defined in 3.1.) From 3.2, Theorem 4.4, and the proposition $T_m \in \Sigma^{(2)}_{m+2}$ in the proof of 4.4, we obtain $D \in \Sigma^{(2)}_{m+2}$. Hence, by 3.6(ii), there exists $\gamma \in \omega_2$, $\gamma \geqslant \delta$, such that the pair (U^γ, V^γ) m-blocks the set D. This means that one of the following two conditions is satisfied:

(a) *The pair* $(U^\gamma[\geqslant m], V^\gamma[\geqslant m])$ *belongs to the set* D.

(b) *There is no pair* $(U, V) \in D$ *extending* $(U^\gamma[\geqslant m], V^\gamma[\geqslant m])$.

Assume (a) holds. By definition of D, there exist $(U', V') \in T_m$ and $p' \in P(U')$ such that the following two conditions are fulfilled:

(1) $U'[\geqslant m] = U^\gamma[\geqslant m]$, $V'[\geqslant m] = V^\gamma[\geqslant m]$, *and* $p' \geqslant p$.

(2) p' $\text{forc}_{U'V'}\ \varphi$.

We note, further, that $(U', V') \in T_m$ implies, by the definition of T_m, the existence of $\beta \in \omega_2$ such that the following holds:

(3) $U'[< m] = U^\beta[< m]$ *and* $V'[< m] = V^\beta[< m]$.

We define $\alpha = \max(\beta, \gamma)$, and from (1), (3), and 3.9.1 we have that the pair (U^α, V^α) belongs to PS and extends the pair (U', V'). From this, (2), and 4.3.2, it follows that p' $\text{forc}_\alpha\ \varphi$; that is, $p' \in Q$. Finally, $p' \geqslant p$ holds by the choice of p' and (1). Thus, case (a) has been dealt with.

Now assume that (b) holds. Let us show that we then have p $\text{forc}_\gamma \sim \varphi$ (that is, p already belongs to Q). Assume the contrary. We note that from the choice of δ, 3.9.1, and $\gamma \geqslant \delta$ it follows that $p \in P(U^\gamma)$. Hence the assumption of the contrary implies, by the definition 4.3(vi), the existence of a pair $(U', V') \in T_m$, extending (U^γ, V^γ), and some $p' \in P(U')$ such that $p' \geqslant p$ and p' $\text{forc}_{U'V'}\ \varphi$. Now, denoting $U = U'[\geqslant m]$ and $V = V'[\geqslant m]$, we obviously have $(U, V) \in D$. In addition, the pair (U, V) will extend $(U^\gamma[\geqslant m], V^\gamma[\geqslant m])$, since (U', V') extends the pair (U^γ, V^γ). But this contradicts the hypothesis (b).

The contradiction completes the proof that p $\text{forc}_\gamma \sim \varphi$. Thus, p itself already belongs to the set D in case (b).

Both cases have been considered, and the theorem is proved.

4.7. *The agreement of truth with* forc. Up to this point, the reasoning in §4 has been carried out within the constructible universe (cf. 4.1). In this subsection, the reasoning will be carried out within the universe of all sets.

Before proving a theorem on the connection between truth and forc, which will play a central role in the whole forc apparatus, we shall introduce an interpretation of $\sim\&\exists$-formulas. If φ is a $\sim\&\exists$-formula and $G \subseteq P_0$ (for example, if G is a U^*-g.f.), we define φ^G to be the result of replacing in φ every constant $c \in \text{cod}$ by c^G. (The definition of c^G is in 2.7.) The formula φ^G is an analytic formula with parameters from $L[G]$. We note that φ^G coincides with φ if φ does not have constants from cod.

THEOREM. *Assume that φ is a closed $\sim\&\exists$-formula and that G is a U^*-g.f. Then φ^G is true in $L[G]$ if and only if there exist $\alpha \in \omega_2^L$ and $p \in G$ such that p $\text{forc}_\alpha\, \varphi$.*

The proof proceeds by induction on the length of φ. The consideration of formulas of 4.3(i) is trivial, and the induction steps 4.3(iii), (iv) are left to the reader. Let us consider a formula φ of the form 4.3(ii); that is, φ is a formula $k \in c$, where $k \in \omega$ and $c = (Q_t\colon t \in \omega) \in \text{cod}$.

Assume that φ^G is true in $L[G]$. This means that $k \in c^G$; that is, $G \cap Q_k \neq 0$ by definition of c^G (cf. 2.7).

We introduce the sets $Q^+ = \{p \in P(U^*)\colon \text{there exists } q \in Q_k \text{ such that } p > q\}$ and $Q^- = \{p \in P(U^*)\colon p \text{ is incompatible with every } q \in Q_k \cap P(U^*)\}$. It is not difficult to verify, using 2.3.3, that the set $Q = Q^+ \cup Q^-$ is dense in $P(U^*)$. In addition, Q is constructible, since c, and therefore Q_k, is constructible (cf. 2.7).

Thus, $G \cap Q \neq 0$. Assume $p \in G \cap Q$. We note that $p \in Q^-$ is impossible, since $G \cap Q_k \neq 0$ and $G \subseteq P(U^*)$ is a filter. Therefore, $p \in Q^+$ holds. But, by definition 4.3(ii), this means that p $\text{forc}_\alpha\, k \in c$, where $\alpha \in \omega_2^L$ is such that $p \in P(U^\alpha)$. (Such an α exists, by virtue of 3.9.3 and $p \in P(U^*)$.)

Conversely, assume $p \in G$, $\alpha \in \omega_2^L$, and p $\text{forc}_\alpha\, k \in c$. This means, by 4.3(ii), that $p > q$ for some $q \in Q_k$. We note that from $p > q$, $q \in P_0$, and $p \in P(U^*)$ it follows that $q \in P(U^*)$. Hence, since G is a filter and $p \in G$, we obtain that $q \in G$. Thus, $G \cap Q_k \neq 0$; that is, $k \in c^G$ by Definition 2.7.

Thus, the case of formulas φ of the form 4.3(ii) has been dealt with.

Now we shall do the inductive step 4.3(v); that is, we consider a $\sim\&\exists$-formula φ such that the theorem already has been proved for $\varphi(c)$ for arbitrary $c \in \text{cod}$, and we prove the theorem for the formula $\exists x\, \varphi(x)$.

Assume that $(\exists x\, \varphi(x))^G$ is true in $L[G]$. By Theorem 2.7.1, there exists $c \in \text{cod}$ such that $\varphi(c)^G$ is true in $L[G]$. The inductive hypothesis implies the existence of $p \in G$ and $\alpha \in \omega_2^L$ such that p $\text{forc}_\alpha\, \varphi(c)$. Thus, by 4.3(v), p $\text{forc}_\alpha\, \exists x\, \varphi(x)$.

Conversely, assume $p \in G$, $\alpha \in \omega_2^L$, and p $\text{forc}_\alpha\, \exists x\, \varphi(x)$. This means that p $\text{forc}_\alpha\, \varphi(c)$ for some $c \in \text{cod}$. By the inductive hypothesis, $\varphi(c)^G$ is true in $L[G]$, etc.

Finally, let us do the inductive step 4.3(vi). We assume that the theorem has been proved for a formula φ, and prove it for $\sim \varphi$.

Assume that $(\sim \varphi)^G$ is true in $L[G]$. This means that φ^G is false in $L[G]$, and thus, by the inductive hypothesis, there exist no $p \in G$ and $\alpha \in \omega_2^L$ such that p $\text{forc}_\alpha\, \varphi$. Hence, by Theorem 4.6 and the genericity of G, there exists $p \in G$ such that p $\mathrm{forc}_\alpha \sim \varphi$ for some $\alpha \in \omega_2^L$. Q.E.D. (Remark: the set Q of Theorem 4.6 is constructible, since all the reasoning in 4.2–4.6 was carried out within L.)

Conversely, assume $p \in G$, $\alpha \in \omega_2^L$, and p $\mathrm{forc}_\alpha \sim \varphi$. Assume the contrary: $(\sim \varphi)^G$ is false in $L[G]$; that is, φ^G is true in $L[G]$. By the inductive hypothesis, there exist $p_1 \in G$ and $\alpha_1 \in \omega_2^L$ such that p_1 forc_{α_1} φ. But G is a filter; that is, p and p_1 are compatible. This contradicts 4.5.2. This concludes the inductive step, and the theorem is proved.

4.8. *Restriction principle.* In this section we again reason within L. First we point out the following corollary of Theorem 4.7.

COROLLARY 1. *If $p \in P(U^*)$, $\alpha \in \omega_2$, and if a closed $\sim$ &∃-formula φ does not contain constants from* cod *and satisfies p* forc_α φ, *then $p \Vdash^*$ "φ is true (in L[G])".*

Some remarks about the formulation of this corollary. By $\Vdash^*$ we denote the relation $\Vdash_{U^*}$, where U^* is the system of 3.9. Further, as a direct consequence of Theorem 4.7, we should write $p \Vdash^*$ "$\varphi^{\mathbf{G}}$ is true in L[G]". But, since the corollary is restricted to formulas which do not have constants from cod, the superscript **G** can be omitted, since for every such formula φ and every $G \subseteq P$ the formula φ^G coincides with φ.

We also note that every closed $\sim$ &∃-formula not containing constants from cod is a (closed) analytic formula not containing parameters from R, and therefore one can speak about its truth or falsity (in $L[G]$).

Now let us state the following "restriction principle":

(*) *Let φ be a closed $\sim$ &∃-formula which does not contain constants from* cod. *Assume also that $m \in \omega$, $\mathrm{com}(\varphi) < m$, $(U, V) \in PS$, $p \in P(U)$, and p $\mathrm{forc}_{UV} \sim \varphi$. Then $p \restriction m$ $\mathrm{forc}_{UV} \sim \varphi$. (The definition of $p \restriction m$ is in* 2.3.)

This principle will be proved below. But here, let us prove with its help a theorem which leads directly to a proof of 4.1(i).

THEOREM 2. *If $p \in P(U^*)$ and $\psi(k)$ is a $\sim$ &∃-formula which has k as its only free variable and does not contain constants from* cod, *then $p \Vdash^*$ "the set $\{k \in \omega$: $\psi(k)$ is true (in L[G])$\}$ is constructible".*

PROOF. We define $Q_k = \{p \in P(U^*)$: there exists $\alpha \in \omega_2$ such that either $p \Vdash^* \psi(\mathbf{k})$ or $p \Vdash^* \sim \psi(\mathbf{k})\}$, and $Q = \bigcap_{k \in \omega} Q_k$. Thus, Q consists of all $p \in P(U^*)$ which "decide" every formula $\psi(\mathbf{k})$, $k \in \omega$.

The following proposition results from the definition of Q.

(1) *If $q \in Q$, then $q \Vdash^*$ "the set $\{k \in \omega$: $\psi(k)\}$ is constructible".*

In fact, let us define $r = \{k \in \omega$: $q \Vdash^* \psi(\mathbf{k})\}$. Then, on the one hand, r is constructible, since forcing is expressible in the "initial model" L ([5], 1.1.9). On the other hand, from $q \in Q$ it follows that $q \Vdash^*$ "$\{k \in \omega$: $\psi(k)\} = \mathbf{r}$".

Thanks to Proposition (1), for the proof of the theorem it suffices to prove the following lemma.

LEMMA. *If $p \in P(U^*)$, then there exists $q \in Q$ such that $q > p$.*

PROOF OF THE LEMMA. Since the set $|p|$ is finite by 2.2(i), there exists $m \in \omega$ such that $\mathrm{com}(\psi(k)) < m$ and $|p| \subseteq m$. Let us verify the following auxiliary proposition:

(2) *If $k \in \omega$, $p' \in P(U^*)$, and $|p'| \subseteq m$, then there exists $p'' \in Q_k$ such that $p'' > p'$ and $|p''| \subseteq m$.*

In fact, by Theorem 4.6 there exist $\alpha \in \omega_2$ and $t \in P(U^*)$ such that $t \geqslant p'$, and either t $\mathrm{forc}_\alpha \sim \psi(k)$ or t $\mathrm{forc}_\alpha \sim\sim \psi(k)$. Let us show that $p'' = t\lfloor m$ is what is required. Indeed, $p'' \in P(U^*)$ follows from $t \in P(U^*)$, $|p''| \subseteq m$ is obvious by construction, and $p'' \geqslant p'$ is obtained without difficulty from $t \geqslant p'$ and $|p'| \subseteq m$.

On the other hand, from the choice of m we obtain $\mathrm{com}(\psi(k)) < m$ and $\mathrm{com}(\sim \psi(k)) \leqslant m$. Hence, applying the restriction principle $(*)$, we obtain that either p'' $\mathrm{forc}_\alpha \sim \psi(k)$ or p'' $\mathrm{forc}_\alpha \sim\sim \psi(k)$. Corollary 1 allows us to change over to $\Vdash^*$: either $p'' \Vdash^* \sim \psi(\mathbf{k})$ or $p'' \Vdash^* \sim\sim \psi(\mathbf{k})$. But the latter assertion obviously implies $p'' \Vdash^* \psi(\mathbf{k})$. Thus, $p'' \in Q_k$. Q.E.D. This proves (2).

Let us continue the proof of the lemma. The assertion (2) that we have just proved enables us to construct, by induction on k, a sequence $(p_k\colon k \in \omega)$ satisfying the following three conditions:

(i) $p_0 = p$ *and* $p_{k+1} \in Q_k$ *for all* $k \in \omega$.

(ii) $|p_k| \subseteq m$ *for all* $k \in \omega$.

(iii) $p_k < p_{k+1}$ *for all* $k \in \omega$.

Let us show that this sequence is less than or equal to some $q \in P(U^*)$, which is what is required. We introduce the following notation:

$$e_k = p_k(\mathrm{U}), \quad (s^k_{ni}, X^k_{ni}) = p_k(n, i).$$

In addition, we define

$$e = \bigcup_{k \in \omega} e_k, \quad s_{ni} = \bigcup_{k \in \omega} s^k_{ni}, \quad X_{ni} = \bigcup_{k \in \omega} X^k_{ni}.$$

Finally, we introduce a function q, defined on the set $\{0\} \cup (\omega \times \omega)$ by the conditions $q(0) = e$ and $q(n, i) = (s_{ni}, X_{ni})$ for all $n, i \in \omega$.

First of all, let us prove that $q \in P(U^*)$. Since the whole construction of this section is carried out in L, q is automatically constructible. Now it is necessary to check conditions 2.2(i)–(vii) for q. We begin with 2.2(i).

From (iii) it follows that $(e_k\colon k \in \omega)$ is a nondecreasing sequence of finite functions, $e_k \subseteq \omega \times (R \cap L)$ for all k. But, according to (ii), $\mathrm{dom}(e_k) \subseteq m$ for all k. Hence $(e_k\colon k \in \omega)$ becomes constant and $e = e_k$ for some k. Thus requirement 2.2(i) holds for q.

After this, the verification of 2.2(ii)–(vii) for q can be carried out without any special difficulty, if we take into account (iii) and the fact that every p_k belongs to $P(U^*)$. The details are left to the reader.

Thus, $q \in P(U^*)$. Moreover, from the construction it is obvious that $q \geqslant p_k$ for all k, and therefore that $q \geqslant p = p_0$. Finally, from (i) it follows that $q \in Q_k$ for all k; that is, $q \in Q$. Thus, q is what is required, and the lemma is proved. This completes the proof of the theorem.

4.9. *Constructibility of analytically definable sets.* The reasoning of this subsection will be carried out within the universe of all sets. We already have sufficient information to prove Propostion 4.1(i).

THEOREM. *If G is a U^*-g.f. and $a \in L[G] \cap R$ is analytically definable in $L[G]$, then a is constructible.*

PROOF. Let $\varphi(k)$ be an analytic formula without parameters, defining a in $L[G]$: $a = \{k \in \omega\colon \varphi(k) \text{ is true in } L[G]\}$. Expressing in $\varphi(k)$ the symbols $\vee$, $\rightarrow$, $\equiv$, and $\forall$ by means of $\sim$, & and $\exists$, we obtain a $\sim\&\exists$-formula $\psi(k)$, without constants from cod, also satisfying $a = \{k \in \omega\colon \psi(k)$ is true in $L[G]\}$. Application of Theorem 4.8.2 in this situation, and using the well-known ([5], 1.1.9) relations between forcing and truth in generic extensions, completes the proof.

Thus, the desired result 4.1(i) has been proved under the assumption that the principle $(*)$ of 4.8 holds.

§5. Proof of principle 4.8(*)

The proof of this principle is based upon an idea used in the proof of a lemma in Chapter IV, §5 of [3]. First, in 5.1–5.3 we introduce and study some special transformations of sets occurring in the forc apparatus, and then, in 5.4–5.6, we prove principle 4.8(*).

All the reasoning of §5 proceeds within the constructible universe.

5.1. *Transformations.* Fix $m \in \omega$. By Γ_m we denote the collection of all bijections π of the set ω onto itself such that $\pi(j) = j$ for all $j < m$. With every pair $\pi \in \Gamma_m$, $b \subseteq \omega_1$, we associate a transformation $[\pi b]$, operating on sets of certain forms.

Thus, assume $\pi \in \Gamma_m$ and $b \subseteq \omega_1$; we shall define the operation of $[\pi b]$.

If $h \in$ Seq and $\alpha = \mathrm{dom}(h)$ $(\in \omega_1)$, then by bh we denote the function h' defined on α by the conditions $h'(\gamma) = h(\gamma)$ for $\gamma \in \alpha - b$ and $h'(\gamma) = 1 - h(\gamma)$ for $\gamma \in \alpha \cap b$. It is clear that $bh \in$ Seq and $\alpha = \mathrm{dom}(bh)$.

Similarly, if $f \in$ Fun, then by bf we denote the function $f' \in$ Fun defined as follows: $f'(\gamma) = f(\gamma)$ for $\gamma \in \omega_1 - b$ and $f'(\gamma) = 1 - f(\gamma)$ for $\gamma \in b$.

If $s \subseteq$ Seq and $u \subseteq$ Fun, then we set $bs = \{bh\colon h \in s\}$ and $bu = \{bf\colon f \in u\}$.

Assume now that U is a system. By $[\pi b]U$ we denote the system U' defined by the following conditions: $U'_{ni} = U_{ni}$ for $n < m$ and $U'_{\pi(n)i} = bU_{ni}$ for $n \geqslant m$. (Remember that $U_{ni} \subseteq$ Fun; cf. 2.1.) Since $\pi \in \Gamma_m$, these two conditions do not contradict each other and determine U'_{ki} for any $k \in \omega$.

If $X \subseteq \omega_1 \times$ Fun, we set $bX = \{(\gamma, bf)\colon (\gamma, f) \in X\}$.

Let $p \in P_0$. By $[\pi b]p$ we denote the unique $q \in P_0$ satisfying the following three conditions:

1) *If* $e = p(0)$, *then* $q(0)$ *is the function* e' *defined on the set* $\pi``|p|$ *by the condition* $e'(\pi(j)) = e(j)$ *for all* $j \in |p|$. (*Remember that* $|p| = \mathrm{dom}(p(0)) = \mathrm{dom}(e)$; *cf.* 2.2.)

2) *If* $n \geqslant m$ *and* $p(n, i) = (s, X)$, *then* $q(\pi(n), i) = (bs, bX)$.

3) *If* $n < m$, *then* $q(n, i) = p(n, i)$.

From $\pi \in \Gamma_m$ it follows that these three conditions determine a unique function q defined on $\{0\} \cup (\omega \times \omega)$. We suggest to the reader that he verify that this function q actually does belong to P_0.

In addition, if $c = (Q_k\colon k \in \omega) \in$ cod, then we set $[\pi b]c = (Q'_k\colon k \in \omega)$, where $Q'_k = \{[\pi b]p\colon p \in Q_k\}$ for all $k \in \omega$.

Finally, if φ is a $\sim\&\exists$-formula, then by $[\pi b]\varphi$ we denote the result of replacing in φ all constants $c \in$ cod by $[\pi b]c$.

This concludes the definition of the transformation $[\pi b]$. We note that this definition obviously depends, not only on π and b, but also on m. We do not explicitly indicate this dependence on m for the sake of convenience. Below, it will always be clear from the context that m is involved in the definition.

5.2. *Properties of the transformation* $[\pi b]$. The natural number m of 5.1 is to remain fixed. We also fix $\pi \in \Gamma_m$ and $b \subseteq \omega_1$. The simple, but sometimes quite tedious, verification of the following nine assertions is left to the reader.

1. *If* $p \in P_0$, *then* $[\pi b]p \in P_0$. (We already noted this in 5.1.)

2. *If* U *is a system, so is* $[\pi b]U$.

3. *If* $(U, V) \in PS$, *then* $([\pi b]U, [\pi b]V) \in PS$.

4. *If* $c \in$ cod, *then* $[\pi b]c \in$ cod.

5. *If* $p \in P(U)$, *then* $[\pi b]p \in P([\pi b]U)$.

6. *If* $p \in P_0$ *and* U *is a system, then* $([\pi b]p)\restriction m = p\restriction m$ *and* $U[< m] = ([\pi b]U)[< m] = [\pi b](U[< m])$.

7 (consequence of 6 and 3). *If* $(U, V) \in T_k$, $k \leqslant m$, *then* $([\pi b]U, [\pi b]V) \in T_k$.

8. *If* $p \leqslant q$, *then* $[\pi b]p \leqslant [\pi b]q$; *if a system* V *extends a system* U, *then* $[\pi b]V$ *will extend* $[\pi b]U$.

9. *The operation of* $[\pi b]$ *is invertible in the following sense: if, for example,* $p \in P_0$, *then* $[\pi^{-1}b]([\pi b]p) = [\pi b]([\pi^{-1}b]p) = p$. *We have a similar invertibility for operations on systems, codes (elements of the set* cod), *and* $\sim \&\exists$*-formulas.*

5.3. *Invariance of* forc. In this subsection, fix $m \in \omega$. In the statement and proof of the following theorem, the operation of all transformations $[\pi b]$ is carried out with respect to this m.

THEOREM. *If* $(U, V) \in PS$, $p \in P(U)$, $\pi \in \Gamma_m$, $b \subseteq \omega_1$, φ *is a closed* $\sim \&\exists$*-formula such that* $\mathrm{com}(\varphi) \leqslant m + 1$, *and finally,* p forc_{UV} φ, *then* $[\pi b]p$ $\mathrm{forc}_{[\pi b]U[\pi b]V}$ $[\pi b]\varphi$.

The proof proceeds by induction on the number of logical signs ($\sim$, &, $\exists$) in φ, for all sets U, V, p, π, b of the specified forms. It suffices to prove the following six assertions:

1. *The theorem is true for every formula* φ *of the form* 4.3(i).

2. *The theorem is true for every formula* φ *of the form* 4.3(ii).

3. *The theorem is true for a formula* φ_1 & φ_2 *if it is true for each of the formulas* φ_1 *and* φ_2.

4. *The theorem is true for a formula* $\exists k\ \varphi(k)$, *where* k *is a variable of type* 0, *if it is true for formulas* $\varphi(k)$ *for arbitrary* $k \in \omega$.

5. *The theorem is true for a formula* $\exists x\ \varphi(x)$, *where* x *is a variable of type* 1, *if it is true for formulas* $\varphi(c)$ *for arbitrary* $c \in$ cod.

6. *The theorem is true for a formula* $\sim \varphi$ *if it is true for* $[\pi b]\varphi$ *(and, naturally, under the assumption that* $\mathrm{com}(\sim \varphi) \leqslant m + 1$).

Assertions 1 and 2 are the initial step of the induction, and 3–6 form the inductive step. Before proving these assertions, we define $U' = [\pi b]U$, $V' = [\pi b]V$, and $p' = [\pi b]p$.

Assertion 1 is obvious from 4.3(i), since the conditions $(U', V') \in PS$ and $p' \in P(U')$ hold by virtue of 5.2.3 and 5.2.5.

PROOF OF 2. So, assume φ is $k \in c$, where $k \in \omega$ and $c = (Q_l\colon l \in \omega) \in$ cod. By definition 4.3(ii), p forc_{UV} φ means that there exists $q \in Q_k$ satisfying

(1) $p \geqslant q$.

From the assertions $p \in P(U)$ and (1), with the help of 5.2.5 and 5.2.8, we have

(2) *The set* $p' = [\pi b]p$ *belongs to* $P(U')$ *and satisfies* $p' \geqslant q'$, *where* $q' = [\pi b]q$.

But, by the definition of $[\pi b]c$ and 4.3(ii), assertion (2) implies p' $\mathrm{forc}_{U'V'}$ $k \in [\pi b]c$; that is, p' $\mathrm{forc}_{U'V'}[\pi b]\varphi$. Q.E.D. This completes the proof of 2.

The trivial proofs of 3 and 4 are left to the reader.

PROOF OF 5. By 4.3(v), p forc_{UV} $\exists x\ \varphi(x)$ implies p forc_{UV} $\varphi(c)$ for some $c \in$ cod. But the theorem is true for $\varphi(c)$ by hypothesis. Hence p' $\mathrm{forc}_{U'V'}$ $\varphi'(c')$, where $\varphi'(x)$ is the formula $[\pi b]\varphi(x)$ and $c' = [\pi b]c$ ($\in$ cod by 5.2.4). Therefore, by 4.3(v), we have p' $\text{forc}_{U'V'}$ $\exists x\, \varphi'(x)$; that is, p' $\text{forc}_{U'V'}[\pi b]\exists x\, \varphi(x)$. Q.E.D.

PROOF OF 6. By φ' we denote the formula $[\pi b]\varphi$. It is clear that $[\pi b] \sim \varphi$ is $\sim \varphi'$. Define $k = \text{com}(\varphi')$. Then $k \leqslant m$, since $\text{com}(\varphi') = \text{com}(\varphi)$, and $\text{com}(\sim \varphi) \leqslant m + 1$ by the statement of 6.

Let us assume the contrary: that the theorem is not true for $\sim \varphi$; that is, the following two assertions hold:

(3) p $\text{forc}_{UV} \sim \varphi$.

(4) *It is not true that* p' $\text{forc}_{U'V'} \sim \varphi'$.

Since $(U', V') \in PS$ and $p' \in P(U')$ follow from the hypothesis of the theorem and 5.2.3 and 5.2.6, by 4.3(vi) assertion (4) yields the existence of a pair $(\mathbf{U}', \mathbf{V}') \in T_k$, extending the pair (U', V'), and the existence of $\mathbf{p}' \in P(\mathbf{U}')$, $\mathbf{p}' > p'$, such that we have

(5) $\mathbf{p}'$ $\text{forc}_{\mathbf{U}'\mathbf{V}'}$ φ'.

But, by the hypothesis of assertion 6, the theorem is true for the formula $\varphi' = [\pi b]\varphi$. Hence one can carry out the transformation $[\pi^{-1}b]$ on (5) and obtain

(6) $\mathbf{p}$ $\text{forc}_{\mathbf{UV}}$ φ, *where* $\mathbf{p} = [\pi^{-1}b]\mathbf{p}'$, $\mathbf{U} = [\pi^{-1}b]\mathbf{U}'$, *and* $\mathbf{V} = [\pi^{-1}b]\mathbf{V}'$ (*the identity of* φ *and* $[\pi^{-1}b]\varphi$ *follows from* 5.2.9).

On the other hand, from the choice of $\mathbf{U}'$, $\mathbf{V}'$, $\mathbf{p}'$, the definitions of U', V', p' (at the beginning of the proof of the theorem), and various assertions of 5.2, applied to the transformation $[\pi^{-1}b]$, it is not difficult to obtain

(7) $(\mathbf{U}, \mathbf{V}) \in T_k$, $(\mathbf{U}, \mathbf{V})$ *extends* (U, V), $\mathbf{p} \in P(\mathbf{U})$, *and* $\mathbf{p} > p$.

Now, assertions (3), (6) and (7) lead to a contradiction, by virtue of 4.3(vi). This contradiction refutes the contrary hypothesis (the conjunction of (3) and (4)), and assertion 6 and the theorem are proved.

5.4. *Beginning of the proof of principle* 4.8(*). *Assumption of the contrary*. After the preliminary arguments in 5.1–5.3, we proceed to the proof of 4.8(*). Until the end of §5, we fix a closed $\sim \&\exists$-formula φ, not containing constants from the set cod. We also fix, in accordance with the statement of 4.8(*), m, k, p, U, and V such that the following assertion holds:

1. $m \in \omega$, $k = \text{com}(\varphi) \leqslant m$, $(U, V) \in PS$, $p \in P(U)$, *and* p $\text{forc}_{UV} \sim \varphi$.

Let us prove that $p \restriction m$ $\text{forc}_{UV} \sim \varphi$. Assume the contrary. Since $p\restriction m \in P(U)$ easily follows from $p \in P(U)$, by 4.3(vi) the assumption of the contrary implies the existence of sets $\tilde{p}$, $\tilde{U}$ and $\tilde{V}$ satisfying the following two conditions:

2. $(\tilde{U}, \tilde{V}) \in T_k$, $(\tilde{U}, \tilde{V})$ *extends* (U, V), *and* $\tilde{p} \in P(\tilde{U})$.

3. $\tilde{p} > p\restriction m$ *and* $\tilde{p}$ $\text{forc}_{\tilde{U}\tilde{V}}$ φ.

Below in 5.6 we intend to obtain a contradiction, which will refute the assumption of the contrary that we have made.

5.5. *Construction of* U', V' *and* p'. These sets, which will play an important role in obtaining a contradiction, will be constructed by carrying out on $\tilde{U}$, $\tilde{V}$, and $\tilde{p}$ a suitable transformation $[\pi b]$. We indicate the choice of π and b.

Choice of π. By the definition 2.2(i), the sets $|p|$ and $|\tilde{p}|$ will be finite sets of natural numbers. Hence there exists an $N \in \omega$ such that $|p| \cup |\tilde{p}| \subseteq N$. Now it is not hard to select $\pi \in \Gamma_m$ satisfying the following requirement:

$$\forall j[m \leqslant j < N \rightarrow \pi(j) \geqslant N].$$

Let us fix this $\pi \in \Gamma_m$.

Choice of b. We set $F = \bigcup_{n,i\in\omega}(\tilde{U}_{ni} \cup \tilde{V}_{ni})$. Since $(\tilde{U}, \tilde{V}) \in PS$, $F \subseteq \mathrm{Fun}$ will be a set of power $\leqslant \omega_1$. Let $((f_\alpha, g_\alpha): \alpha \in \omega_1)$ be some enumeration of the set $F \times F$. We define $b = \{\alpha \in \omega_1 : f_\alpha(\alpha) = g_\alpha(\alpha)\}$.

From the choice of b and the definition 5.1, we immediately obtain:

(a) *If $f \in F$, then $bf \notin F$.* (For the definition of bf, see 5.1.)

Thus, $\pi \in \Gamma_m$ and $b \subseteq \omega_1$ have been chosen. Now we define $p' = [\pi b]\tilde{p}$, $U' = [\pi b]\tilde{U}$, and $V' = [\pi b]\tilde{V}$. (The operation of $[\pi b]$ is carried out with respect to the natural number m specified at the beginning of 5.4.) Let us prove the following properties of the sets p', U' and V':

1. $(U', V') \in T_k$, $p' \in P(U')$ *and* $p' \downharpoonright m = \tilde{p} \downharpoonright m$.
2. $U'[< m] = \tilde{U}[< m]$ *and* $V'[< m] = \tilde{V}[< m]$.
3. $|p| \cap |p'| \subseteq m$.
4. p' $\mathrm{forc}_{U'V'}$ φ.
5. *If $n \geqslant m$ and $i \in \omega$, then* $U_{ni} \cap V'_{ni} = U'_{ni} \cap V_{ni} = 0$.

Assertion 1 follows from 5.4.2 and 5.2.7, 5.2.5 and 5.2.6.

Assertion 2 follows from 5.2.6.

For the proof of 3, we observe that $|p| \subseteq N$ by the choice of N. On the other hand, it is not hard to verify that $|p| = \pi"|\tilde{p}|$, and thus, by the choice of N and π, if $j \in |p'|$, then either $j < m$ or $j \geqslant N$. Now 3 is obvious.

Further, assertion 4 follows from 5.4.3 and Theorem 5.3; the identity of the formulas φ and $[\pi b]\varphi$ holds by virtue of the absence in φ of constants from cod.

It remains to verify 5. Assume $n \geqslant m$ and $i \in \omega$. Let us prove, for example, that $U_{ni} \cap V'_{ni} = 0$. (The second equality is proved similarly.) Assume, for the sake of contradiction, that $g \in U_{ni} \cap V'_{ni}$. Since the system $\tilde{U}$ extends U by 5.4.2, we obtain $g \in \tilde{U}_{ni}$, and thus $g \in F$.

On the other hand, from $\pi \in \Gamma_m$, $n \geqslant m$, $g \in V'_{ni}$, and the definition $V' = [\pi b]\tilde{V}$, one infers the existence of $n' \geqslant m$ (namely, $n' = \pi^{-1}(n)$) and $f \in \tilde{V}_{n'i}$ such that $g = bf$. Again, $f \in F$ by definition of F.

Thus, $f, g \in F$ and $g = bf$. But this contradicts (a), and assertion 5 is proved.

5.6. *Completion of the proof of* 4.8(∗). We have constructed sets U', V', and p' satisfying requirements 1–5 of 5.5. We shall use these sets to obtain a contradiction, which will refute the assumption of the contrary of 5.4 and thus will complete the proof of principle 4.8(∗).

We set $p'' = p \vee p'$. (For the definition of the operation $\vee$, see 2.3.) Let us introduce systems U'' and V'' by the conditions $U''_{ni} = U_{ni} \cup U'_{ni}$ and $V''_{ni} = V_{ni} \cup V'_{ni}$.

Let us prove the following auxiliary assertions.

1. *If $n < m$ and $i \in \omega$, then* $U_{ni} \subseteq U'_{ni}$ *and* $U_{ni} \subseteq V'_{ni}$.
2. *If $n, i \in \omega$, then* $U_{ni} \cap V'_{ni} = U'_{ni} \cap V_{ni} = 0$.
3. *The pair (U'', V'') belongs to PS and extends the pairs (U, V) and (U', V').*
4. $U''[< k] = U'[< k]$ *and* $V''[< k] = V'[< k]$ *(k was introduced in* 5.4).
5. *The pair (U'', V'') belongs to T_k.*
6. $p'' \in P_0$, $p'' \geqslant p$ *and* $p'' \geqslant p'$.
7. $p'' \in P(U'')$.

Assertion 1 follows from 5.4.2 and 5.5.2.

Assertion 2 for $n \geqslant m$ is obtained directly from 5.5.5. If $n < m$, then 2 follows immediately from 1 and the equality $U'_{ni} \cap V'_{ni} = 0$, which is a consequence of $(U', V') \in PS$ (and the latter is valid because of 5.5.1, since $T_k \subseteq PS$).

Further, $(U'', V'') \in PS$ follows from 2 and the fact that the pairs (U, V) and (U', V') belong to PS (for the pair (U, V) this was mentioned in 5.4.1). The remaining part of assertion 3 is obvious.

For the proof of 4, it suffices to apply 1, making use of $k < m$ and the definitions of U'' and V''. Now from 4, the definition of T_k, and the assertion $(U', V') \in T_k$ (cf. 5.5.1), we obtain a proof of 5.

For a proof of 6 we note that from 5.5.1 and 5.4.3 it follows that $p' \lfloor m > p \lfloor m$. Now 6 follows from 2.3.5, 2.3.2, and 5.5.3.

Finally, remember that $p \in P(U)$ and $p' \in P(U')$ (cf. 5.4.1 and 5.5.1). From this and the definition of the system U'' and the set $p'' = p \vee p'$, and by virtue of $p'' \in P_0$, which already has been proved, we have $p'' \in P(U'')$. Assertions 1–7 are proved.

Now we obtain a contradiction. From assertions 3, 6, 7, 5.4.1 and 4.3.2 it follows that p'' $\text{forc}_{U''V''} \sim \varphi$. Similarly, using 5.5.4 instead of 5.4.1, we have p'' $\text{forc}_{U''V''}\ \varphi$. But these last two assertions contradict each other according to 4.3.3, since $(U'', V'') \in T_k$ has been noted in 5, and $k = \text{com}(\varphi)$ holds by 5.4.1.

The contradiction that we have obtained refutes the assumption of the contrary in 5.4 and completes the proof of principle 4.8(*). Thus we have completely proved Theorem 4.9 and proposition (i) of the third formulation of FT in 4.1.

§6. The analytic definability in $L[G]$ of all constructible sets of natural numbers

In this section we intend to prove proposition (ii) of the third formulation of FT. Thus the fundamental theorem will be completely proved. We begin with the definition of formulas which will ensure the definability in $L[G]$ of every constructible $r \subseteq \omega$.

6.1. *The formulas* Φ_n. For every $n \in \omega$, we introduce the following formula $\Phi_n(S, i)$ with variables S and i and constructible parameters U^*, V^*, and n:

$$\Phi_n(S, i) \Leftrightarrow S \subseteq \text{Seq} \,\&\, i \in \omega \,\&\, (\forall f \in U^*_{ni})[S \text{ does not cover } f] \,\&\, (\forall f \in V^*_{ni})\,[S \text{ covers } f].$$

We shall explain the use of the formulas Φ_n for definability. Assume G is a U^*-g.f. Theorem 2.6 and the proposition $U^*_{ni} \cap V^*_{ni} = 0$ of 3.9.2 imply the following result.

COROLLARY. *If $n \in \omega$ and $i \in g^G(n)$, then the formula $\Phi_n(S^G_{ni}, i)$ is true in $L[G]$. Thus, since $S^G_{ni} \in L[G]$, the formula $\exists S\ \Phi_n(S, i)$ is true in $L[G]$.*

(The special properties of the construction of U^* do not enter into the validity of the corollary.)

We shall prove, using the special properties of U^*, that if $i \notin g^G(n)$, then $\exists S\ \Phi_n(S, i)$ is false in $L[G]$ (Theorem 6.3). Thus, the set $r = g^G(n)$ turns out to be definable in $L[G]$ by the formula $\exists S\ \Phi_n(S, i)$. Moreover, Proposition 2.5.1 enables us to define every constructible $r \subseteq \omega$ by a formula of that form. Finally, with the help of 3.9.5, we shall prove that every formula $\exists S\ \Phi_n(S, i)$ is equivalent in $L[G]$ to a suitable analytic formula (even occuring in Σ^1_{n+5}). This will complete the proof of (ii) in 4.1.

6.2 *A sharpening of the representation theorem.* We need a somewhat stronger form of Theorem 2.7.2, based on 2.5.3.

We introduce the following definition: the set $c = (Q_h\colon h \in \text{Seq}) \in \text{Cod}$ is said to be *(n, i)-evasive* if for all $h \in \text{Seq}$ and $p \in Q_h$ we have $p(n, i) = (0, 0)$.

COROLLARY (of Theorem 2.7.2). *Assume that G is a U^*-g.f., $S \in L[G]$, and $S \subseteq$ Seq. Assume also that $n, i \in \omega$ and $i \notin g^G(n)$. Then there exists an (n, i)-evasive $c \in \mathrm{Cod}(U^*)$ such that $S = c^G$.*

PROOF. By Theorem 2.7.2, there exists a (not necessarily (n, i)-evasive) $c' \in \mathrm{Cod}(U^*)$ satisfying $c'^G = S$. Let $c' = (Q'_h\colon h \in \mathrm{Seq})$. Define $Q_h = \{p \in Q'_h\colon p(n, i) = (0, 0)\}$ and $c = (Q_h\colon h \in \mathrm{Seq})$, and let us show that c is what is required.

By construction, $c \in \mathrm{Cod}(U^*)$ and c is (n, i)-evasive. Hence, by choice of c', it suffices to prove the equality $c'^G = c^G$; that is, the equivalence $G \cap Q'_h = 0 \equiv G \cap Q_h = 0$ for arbitrary $h \in$ Seq. But this equivalence is obtained directly from 2.5.3.

6.3. *Formulation and beginning of the proof of the "falsity theorem".* Our next goal is the proof of the following theorem.

THEOREM. *Assume that G is a U^*-g.f., $n, i \in \omega$, and $i \notin g^G(n)$. Then the formula $\exists S\ \Phi_n(S, i)$ is false in $L[G]$.*

The proof of this theorem will be concluded in 6.5. It begins by assuming the contrary: the formula $\Phi_n(S, i)$ is true in $L[G]$ for some $S \in L[G]$. By Corollary 6.2, there exists an (n, i)-evasive $c = (Q_h\colon h \in \mathrm{Seq}) \in \mathrm{Cod}(U^*)$ such that $S = c^G$. Thus, the formula $\Phi_n(c^G, i)$ is true in $L[G]$. This implies the existence of $p_0 \in G$ such that the following holds:

1. $p_0 \Vdash^* \Phi_n(\mathbf{c}^{\mathbf{G}}, \mathbf{i})$.

Here and below we shall write $\Vdash^*$ instead of $\Vdash_{U^*}$. Remember that $\mathbf{G}$ and $\mathbf{x}$, for every $x \in L$, are constants of the forcing language (see 2.4).

Now we shall obtain a contradiction. The sets n, i, G, p_0 and $c = (Q_h\colon h \in \mathrm{Seq})$ that we have introduced will be fixed in the reasoning in 6.4–6.5.

6.4. *The set D.* All the reasoning of this subsection is carried out in L. First of all, because of 3.9.3 we can fix $\beta \in \omega_2$ such that $p_0 \in P(U^\beta)$. Now we introduce the following definition.

If $\gamma \in \omega_1$, $f \in$ Fun, and $p \in P_0$ are such that $(\forall \mu \in \omega_1)[\mu > \gamma \to p$ is incompatible with every $q \in Q_{f|\mu}]$, then we shall write $\mathfrak{A}(\gamma, f, p)$.

REMARK. If $f \in$ Fun and $\mu \in \omega_1$, then $h = f|\mu$ belongs to Seq, and $Q_{f|\mu} = Q_h$ has been fixed in 6.3.

We shall prove two lemmas about the formula $\mathfrak{A}$.

LEMMA 1. *If $f \in V^*_{ni}$, $\gamma \in \omega_1$, $p \in P(U^*)$, and $p \geqslant p_0$, then $\sim \mathfrak{A}(\gamma, f, p)$.*

PROOF. From 6.3.1, $p \geqslant p_0$, $f \in V^*_{ni}$, and the definition of Φ_n it follows that $p \Vdash^*$ "$\mathbf{c}^{\mathbf{G}}$ covers $\mathbf{f}$". This means that there exists $p_1 \in P(U^*)$, $p_1 \geqslant p$, and $\mu \in \omega_1$, $\mu > \gamma$, such that $p_1 \Vdash^*$ "$\mathbf{h} \in \mathbf{c}^{\mathbf{G}}$", where $h = f|\mu \in$ Seq. By definition of c^G (cf. 2.7), this implies the existence of $q \in Q_h$ such that p_1 is compatible with q. The p also is compatible with q, since $p_1 \geqslant p$. This yields $\sim \mathfrak{A}(\gamma, f, p)$.

LEMMA 2. *If $f \in U^*_{ni}$, then there exist $t \in P(U^*)$ and $\gamma \in \omega_1$ such that $t \geqslant p_0$ and $\mathfrak{A}(\gamma, f, t)$.*

PROOF. From 6.3.1, $f \in U^*_{ni}$, and the definition of Φ_n one infers the existence of $\gamma \in \omega_1$ and $t \in P(U^*)$ such that $t \geqslant p_0$ and the following holds:

(1) $t \Vdash^*$ "$\mathbf{c}^{\mathbf{G}}$ *does not cover* $\mathbf{f}$ *above* γ".

Let us show that t is what is required; that is, $\mathfrak{A}(\gamma, f, t)$. Assume the contrary. Then there exist $\mu \in \omega_1$, $\mu > \gamma$, and $q \in Q_{f|\mu}$ such that t is compatible with q. Set $h = f|\mu$.

Note that $t \in P(U^*)$ holds by our choice of t, and $q \in P(U^*)$ follows from $c \in \mathrm{Cod}(U^*)$. Hence, applying 2.3.3, we deduce that the set $t' = t \vee q$ belongs to $P(U^*)$ and satisfies $t' \geqslant t$ and $t' \geqslant q$.

The latter assertion implies, in particular, that $t' \Vdash^*$ "$\mathbf{q} \in \mathbf{G}$", and, in addition, $t' \Vdash^*$ "$\mathbf{h} \in \mathbf{c}^{\mathbf{G}}$" by the definition of c^G and the fact that $q \in Q_h$. But this obviously contradicts (1) and the fact that $t' \geqslant t$ and $\mu > \gamma$. The lemma is proved.

Now we define D as the collection of all pairs $(U, V) \in PS$ such that there exist $\gamma \in \omega_1$, $f \in V_{ni}$, and $p \in P(U)$ satisfying $p \geqslant p_0$ and $\mathfrak{A}(\gamma, f, p)$. Using various parts of Proposition 3.2 on definability, we can easily verify that $D \in \Sigma_2^{(2)}$ (with parameters f, p_0, $c \in H\omega_2$). Hence, applying property 3.6(ii) of the sequences U^α and V^α that were fixed in 3.9, we can find $\alpha \in \omega_2$, $\alpha \geqslant \beta$, such that one of the following two assertions is satisfied:

3. $(U^\alpha, V^\alpha) \in D$.

4. *There is no pair* $(U, V) \in D$ *extending the pair* (U^α, V^α).

We shall show that each of these assertions leads to a contradiction. We begin with assertion 3.

So, assume $(U^\alpha, V^\alpha) \in D$. By the definition of D and by 3.9.1 and 3.9.2, this means that there exist $p \in P(U^*)$ and $f \in V^*_{ni}$ such that $p \geqslant p_0$ and $\mathfrak{A}(\gamma, f, p)$. But this contradicts Lemma 1. Thus assertion 3 has quickly led to a contradiction.

6.5. *Completion of the proof of Theorem* 6.3. Now we intend to reduce assertion 4 of 6.4 to a contradiction. The reasoning of this subsection, like that of 6.4, proceeds within L.

So, assume that there is no pair $(U, V) \in D$, extending (U^α, V^α).

For $p \in P_0$ and $m, j \in \omega$, we shall denote by $F_p(m, j)$ the set $\{g \in \mathrm{Fun}: \exists \nu[(\nu, g) \in X]\}$, where X is the second component of the pair $p(m, j)$; that is, $p(m, j) = (s, X)$ for some $s \subseteq \mathrm{Seq}$.

Definition 2.2 implies that $F_p(m, j) \subseteq \mathrm{Fun}$ is at most countable.

Further, the set U^α_{ni} has power $\leqslant \omega_1$, since $(U^\alpha, V^\alpha) \in PS$ (cf. 3.9.1). On the other hand, U^*_{ni} has power ω_2 by 3.9.4. Hence there exists $f \in U^*_{ni} - U^\alpha_{ni}$. Fix such an $f \in \mathrm{Fun}$. We shall prove the following result.

LEMMA. *There exist* $p \in P(U^*)$ *and* $\gamma \in \omega_1$ *such that* $p \geqslant p_0$, $f \notin F_p(n, i)$, *and* $\mathfrak{A}(\gamma, f, p)$ *holds.*

(Remember that n, i and p_0 were fixed, beginning with 6.3.)

PROOF. By 6.4.2, there exist $t \in P(U^*)$, $t \geqslant p_0$, and $\gamma \in \omega_1$ such that the following holds:

(1) $\mathfrak{A}(\gamma, f, t)$.

We shall change t slightly in order to obtain the desired p. Let $t(n, i) = (s, X)$. Set $X_1 = \{(\nu, g) \in X: g \neq f\}$; that is, X_1 is obtained from X by eliminating from X all pairs of the form (ν, f), where $f \in \mathrm{Fun}$ was fixed above.

Now we define $p \in P(U^*)$ by the conditions $p(0) = t(0)$, $p(n, i) = (s, X_1)$, and $p(m, j) = t(m, j)$ for $m \neq n \vee j \neq 1$. Let us show that the p just constructed, together with the γ chosen above, are what is required.

From $t \in P(U^*)$ and the construction of p, it is clear that p does indeed belong to $P(U^*)$. Moreover, the change to p from t has to do only with the function f in the (n, i)th component. On the other hand, $p_0 \in P(U^\alpha)$ (this follows from $\alpha > \beta$, the choice of β in 6.4, and 3.9.1), and $f \notin U^\alpha_{ni}$. From this, together with $t > p_0$, one easily obtains $p > p_0$.

The assertion $f \notin F_p(n, i)$ holds by definition of p.

It remains to verify $\mathfrak{A}(\gamma, f, p)$. Assume the contrary. This means that there exist $\mu \in \omega_1$, $\mu > \gamma$, and $q \in Q_{f|\mu}$ such that the following is satisfied.

(2) *p is compatible with q.*

Let us show that t also is compatible with q (this would contradict (1)). It suffices to verify that the compatibility criterion 2.3.4 is satisfied for t and q, using the fact that this criterion holds for p and q by (2).

By construction of p, the equations $p(0) = t(0)$ and $p(m, j) = t(m, j)$ for $m \neq n \vee j \neq i$ are valid. Hence, by (2) and criterion 2.3.4 for p and q, we conclude that for the proof of the compatibility of t and q it suffices to verify only conditions (2) and (3) of 2.3.4 for t and q, and, moreover, only for those n and i which were fixed, beginning with 6.3. But for these n and i the equation $q(n, i) = (0, 0)$ holds, since $q \in Q_{f|\mu}$ and the set $c = (Q_h: h \in \mathrm{Seq})$ is (n, i)-evasive (cf. 6.3). In this situation, 2.3.4 (2,3) obviously holds for t and q (and for the given n and i).

Thus, the compatibility of t and q has been established. But this fact contradicts assertion (1) be virtue of the choice of $\mu > \gamma$ and $q \in Q_{f|\mu}$.

This contradiction completes the proof of $\mathfrak{A}(\gamma, f, p)$ and the lemma.

Let us return to the proof of Theorem 6.3 (analysis of case 6.4.4). According to the lemma just proved, there exist $p \in P(U^*)$ and $\gamma \in \omega_1$ such that $f \notin F_p(n, i)$, and we have

(3) *$p > p_0$ and $\mathfrak{A}(\gamma, f, p)$.*

We define a system U by the conditions $U_{mj} = U^\alpha_{mj} \cup F_p(m, j)$ for all $m, j \in \omega$, and a system V by the conditions $V_{mj} = V^\alpha_{mj}$ for $m \neq n \vee j \neq i$, and $V_{ni} = V^\alpha_{ni} \cup \{f\}$. Taking into account that $f \notin U^\alpha_{ni}$ and $p \in P(U^*)$, we can easily verify that the pair (U, V) belongs to PS and extends the pair (U^α, V^α). It is also obvious that $p \in P(U)$ and $f \in V_{ni}$. Now $(U, V) \in D$ follows from (3).

Thus we have found a pair $(U, V) \in D$, extending the pair (U^α, V^α). But this contradicts the assumption 6.4.4.

Thus, each of the assumptions 6.4.3 and 6.4.4 leads to a contradiction. Therefore the assumption of the contrary in 6.3 also leads to a contradiction, and Theorem 6.3 is proved.

6.6 *Analytic definability of all constructible sets.* Now we can prove proposition (ii) of the third formulation of the fundamental theorem (cf. 4.1). We fix an arbitrary U^*-g.f. G, and prove that every constructible $r \subseteq \omega$ is analytically definable in $L[G]$. First, we obtain

COROLLARY 1. *If $r \subseteq \omega$ is constructible, then there exists an $n \in \omega$ such that*

$$r = \{ i \in \omega : \exists S\ \Phi_n(S, i) \text{ is true in } L[G] \}.$$

PROOF. By 2.5.1, there is an $n \in \omega$ such that $r = g^G(n)$. This n is what is required, by virtue of Corollary 6.1 and Theorem 6.3.

Now we show that every formula $\exists S\ \Phi_n(S, i)$ determines in $L[G]$ an analytically definable set.

LEMMA 2. *Assume $n \in \omega$ and $r = \{i \in \omega : \exists S\ \Phi_n(S, i)$ is true in $L[G]\}$. Then $r \in \Sigma^1_{n+5}$ in $L[G]$.*

PROOF. We define $H = \{x \in L[G]$: the transitive closure of x is at most countable in $L[G]\}$, and $Z = \{x \in L$: the power of the transitive closure of x is not greater than ω_1^L in $L\}$.

We note that ω_1^L is denumerable in $L[G]$ by 2.5.1, and ω_2^L is nondenumerable in $L[G]$ (this is derived in the usual way from 2.3.1 and 2.3.3; cf., for example, [4], Lemma 56). Hence $\omega_1^{L[G]} = \omega_2^L$. From this and the definition of H and Z it follows that $Z = H \cap L = \{z \in H$: "z is constructible" is true in $H\}$. But the formula "is a constructible set" is a Σ_1-formula (cf. [10], p. 38 or p. 82). Therefore we have

(1) $Z \in \Sigma_1^H$; *that is, Z is definable in H by some Σ_1-formula.*

On the other hand, it is clear that Z is the set $(H\omega_2)^L$. Hence the sets $C_n = \{(i, f)$: $i \in \omega$ and $f \in U_{ni}^*\}$ and $C_n' = \{(i, f)$: $i \in \omega$ and $f \in V_{ni}^*\}$ belong to Σ_{n+3}^Z by Lemma 3.9.5. From this and (1), we obtain

(2) *The sets C_n and C_n' belong to Σ_{n+3}^H.*

In addition, by the definition of Φ_n, C_n and C_n' we have the equation

$$r = \{i \in \omega: \exists S\, \forall f[[(i, f) \in C_n \to S \text{ does not cover } f] \\ \&[(i, f) \in C_n' \to S \text{ covers } f]] \text{ is true in } H\}.$$

This equation and (2) imply

(3) *The set r belongs to Σ_{n+4}^H.*

Finally, the definability in the collection H of all sets which are hereditarily countable in $L[G]$ is related to analytic definability in the following way (cf. [11], the lemma on p. 281):

Assume that $m \geqslant 1$ and that $a \subseteq \omega$ belongs to $L[G]$. Then $a \in \Sigma_m^H$ if and only if $a \in \Sigma_{m+1}^1$ in $L[G]$.

From this and (3) it follows that $a \in \Sigma_{n+5}^1$ in $L[G]$, and the lemma is proved.

Corollary 1 and Lemma 2 imply the basic result of this section:

THEOREM 3. *If $r \subseteq \omega$ is constructible, then r is analytically definable in $L[G]$ for any U^*-g.f. G.*

Combining this theorem with Theorem 4.9, we complete the proof of the fundamental theorem FT in its third formulation (cf. 4.1).

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