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UNDECIDABLE AND DECIDABLE PROPERTIES OF CONSTITUENTS

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ABSTRACT. Some assertions concerning decidability and undecidability of Luzin's problems on constituents are proved.
Bibliography: 34 titles.

Introduction. Origin of problems about constituents

Mathematicians working in the domain of set theory have always considered one of the most important problems to be the relationship between the two simplest uncountable cardinalities—the cardinality $\aleph_1$ of the set of all at most countable ordinals and the cardinality of the continuum $\mathfrak{c}$. The attention of the experts was principally drawn to the following fundamental problems:

Can one construct (with the use of the axiom of choice or without that axiom) a one-to-one correspondence between the countable ordinals and all real numbers; that is, can one prove Cantor's continuum hypothesis $\mathfrak{c} = \aleph_1$ (Cantor, Hilbert, Luzin)?

Can one effectively construct (that is, without the axiom of choice—with the axiom of choice the construction in the given case can be easily carried out) a set of cardinality $\aleph_1$ consisting of real numbers; that is, can one effectively prove the inequality $\aleph_1 \leqslant \mathfrak{c}$ (Lebesgue, Luzin)?

These problems, which belong not only to set theory but also to the foundations of mathematics in a broad sense, were considered by many leading mathematicians of the beginning and the first half of the twentieth century. They aroused great interest, in particular, in N. N. Luzin, who attempted to find an approach to their solution with the help of methods of descriptive set theory. A penetrating analysis of this circle of problems led Luzin to the remarkable idea of an analogy between points of the real line and Borel sets of bounded order, in the light of which it was very natural to pose the following questions (in [1], [2], Chapter III, and [3]–[7]):

The restricted (or narrow) continuum problem: *Can one effectively partition the continuum of real numbers (or Baire space) into $\aleph_1$ nonempty Borel sets of bounded order?*

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The restricted (or narrow) Lebesgue problem: *Can one effectively construct a sequence of $\aleph_1$ pairwise distinct Borel sets of the real line (or Baire space) of uniformly bounded order?*

The order (Luzin used the term "class") of a Borel set characterizes the complexity of that set with respect to the Borel construction, that is, the least possible (finite or denumerable) length of a construction of the set from open sets of the given space by means of the operations of forming countable unions and taking complements. More precisely, by the order of a Borel set X we mean the least ordinal number ξ such that X belongs to the class Δ^0_ξ of the Borel hierarchy. Let us remember that the classes Δ^0_ξ (where $1 \leqslant \xi < \omega_1$ and ω_1 denotes the first uncountable ordinal) strictly increase as the index ξ increases, and every Borel set occurs in one of these classes (and, therefore, in all classes with larger indices). Everything that has been said refers to Borel sets of any *perfect Polish* space (which is taken to mean a separable complete metric space without isolated points), and, in particular, to sets of the real line and of Baire space.

A collection of Borel sets is said to be *bounded with respect to order* (or a collection of bounded order) if all the sets of the collection have order less than some fixed ordinal $\xi_0 < \omega_1$.

By the end of the 1920's, experts in descriptive set theory already knew ways of effectively constructing sequences of $\aleph_1$ Borel sets by means of the sieve operation that had been developed by Luzin. Applying these constructions to the problems formulated above, Luzin posed a series of more specific problems on the nature of constituents, the complete solutions of which were found only many years later. The present paper is devoted to these problems.

§1. Statement of the problems about constituents

Let us begin with several definitions connected with sieves. First of all, we agree to consider as the basic space the *Baire space* $\mathscr{N} = \omega^\omega$, consisting of all functions defined on the set of natural numbers $\omega = \{0, 1, 2, \dots\}$ with values in ω (that is, the topological product of ω copies of the set of natural numbers). This approach was introduced in Luzin's "Lectures" [2] and is accepted in modern work in descriptive set theory (see, for example, [17], [18] and [19]). In some publications of Luzin among those cited above (for example, in [6]), the real line $\mathbf{R}$ is taken to be the basic space. However, it is not difficult to show (which we shall not do here) that the spaces $\mathscr{N}$ and $\mathbf{R}$ (and, in general, all perfect Polish spaces) are completely equivalent with respect to the problems that will be discussed below.

The sieve operation, which has played an exceptional role in the development of descriptive set theory, was introduced by Luzin in 1927, and then was perfected by him (especially in [2]) and by other specialists. We shall present this operation in its most convenient variant, occurring in the monographs [17] and [18]. (For more details about sieves, see [20].) We denote by $\mathbf{Q}$ the set of all rational points of the line $\mathbf{R}$. By a *sieve* (for sifting sets of the Baire space $\mathscr{N}$) we mean any family $C = \langle C_q \colon q \in \mathbf{Q} \rangle$ of sets $C_q \subseteq \mathscr{N}$ of any kind. The sets C_q are said to be the *elements* of the sieve C. We shall call the sieve C *open* (*Borel*) when all of its elements C_q are open (respectively, Borel) sets in $\mathscr{N}$. In general, if we are given any class K of sets of the space $\mathscr{N}$, then we shall say that a sieve is of *class* K if every element of the sieve belongs to the class K.

Any sieve $C = \langle C_q : q \in \mathbf{Q} \rangle$ determines a partition of the space $\mathcal{N}$ into two sets, called *internal* (or *sifted*) and *external*. To the *external* set $[C]$ belong all points $\alpha \in \mathcal{N}$ for which the "vertical cross-section" $C/\alpha = \{q : \alpha \in C_q\}$ is well-ordered in the sense of the natural ordering of rational points. The remaining points are contained in the *internal* set $[C]_*$.

With every ordinal number $\nu < \omega_1$, we associate the sets

$$[C]_\nu = \{\alpha \in [C] : \text{the order type of } C/\alpha \text{ is equal to } \nu\};$$

$$[C]_{*\nu} = \{\alpha \in [C]_* : \text{the order type of the largest well-ordered initial segment of } C/\alpha \text{ is equal to } \nu\},$$

which are called the νth *external* and the νth *internal constituents*, respectively. The external constituents are pairwise disjoint and yield the external set $[C]$ as their union. Similarly, the internal constituents are pairwise disjoint and yield $[C]_*$ as their union.

If a sieve C is Borel (in particular, this holds for open sieves), then all the constituents $[C]_\nu$ and $[C]_{*\nu}$ are Borel sets (see [2], Chapter III, or [5]). Thus, with the help of a sieve one can construct a sequence of $\aleph_1$ pairwise distinct Borel sets: it is only necessary to make sure that, among the internal or external constituents determined by the Borel sieve, uncountably many are nonempty (and, therefore, also pairwise distinct)—and the desired sequence is at hand. The external constituents are of particular interest in this connection, since Luzin ([2], Chapter III) found a convenient criterion for having an uncountable number of nonempty external constituents of a Borel sieve, consisting in the requirement that the external set *not be Borel*. (For internal constituents, such a criterion is not fulfilled.)

Considering the sequences of external constituents in connection with the problems mentioned in the introduction, Luzin posed in his papers ([6], §1, and [7], §8) the following series of four problems about external constituents.

PROBLEM I. *Does there exist an open sieve C such that every constituent $[C]_\nu$ contains exactly one point?*

PROBLEM II. *Does there exist an open sieve C such that uncountably many of the constituents $[C]_\nu$ are nonempty and every $[C]_\nu$ contains not more than a countable set of points?*

PROBLEM III. *Does there exist an open sieve C such that uncountably many of the constituents $[C]_\nu$ are nonempty and all these constituents form a collection of Borel sets that is of uniformly bounded order?*

PROBLEM IV. *Does there exist an open sieve C such that uncountably many of the constituents $[C]_\nu$ are nonempty and all the constituents $[C]_\nu$ can be included in pairwise disjoint Borel sets that are of bounded order?*

The numbering of these four problems has been taken by us from Luzin's papers [6] and [7], where each of the problems is printed in a separate paragraph, with the word "problem" and the indicated number. With this numbering (identical in both of the cited papers), Luzin essentially confers proper names on the formulated problems.

An increase in the number of the problem corresponds to a decrease in the strength of the requirement imposed on the sieve C by the problem. In particular, the requirements imposed by Problem III are weaker (at least in a nonstrict sense) than the requirements of Problem II in view of the fact that every at most countable set (in general, every F_σ) belongs to the Borel class Δ^0_3 and, consequently, has order not higher than 3.

Of the four problems formulated above, Problem III especially attracted the attention of Luzin, and he turned to an analysis of that problem and some of its aspects in a number of other publications in the 1930's, in addition to those already indicated (see, in particular, [2], Chapter III, [3] and [4]). There Luzin posed several other questions about the existence of sieves with certain requirements on the orders of the constituents. Two of these questions are considered in the present paper. For the sake of unity of terminology and reference, we shall formulate these questions under the names Problem IIIa and Problem IIIb, since, in a sense, they are variants of Problem III, although it is certain that Luzin accorded less significance to these questions than to Problem III itself, and strictly speaking, there is no basis for assigning to these questions the status of the above Luzin problems.

PROBLEM IIIa (see [3], §1). *Does there exist an open sieve C with uncountably many nonempty constituents $[C]_\nu$ such that the orders of the nonempty sets $[C]_\nu$ do not converge to ω_1 (this means that there exists an ordinal $\xi < \omega_1$ for which the number of nonempty constituents $[C]_\nu$ having order less than or equal to ξ is uncountable)?*

PROBLEM IIIb (see [4], §5). *Does there exist an open sieve C such that all the constituents $[C]_\nu$ are nonempty and the orders of these constituents do not converge to the ordinal ω_1?*

All six of the problems formulated above in this section are connected with the Lebesgue problem and the restricted Lebesgue problem. On the other hand, the following problem is related to the restricted continuum problem. This problem was posed by Luzin in [5], §5, where he called it the "main problem of the theory of analytic sets" (without providing it with a number). For lack of a more suitable alternative numbering, we shall confer on this question the number 0.

PROBLEM 0. *Does there exist an open sieve C such that, among the constituents $[C]_\nu$ and $[C]_{*\nu}$ determined by it, uncountably many are nonempty (of at least one type) and all the constituents $[C]_\nu$ and $[C]_{*\nu}$ form a collection of bounded order?*

Luzin's problems constantly appeared as important unsolved problems in Soviet surveys of classical descriptive set theory (see, for example, [12], [13] and [15]). In connection with the construction of uncountable sequences of Borel sets of bounded order, constituents are discussed in Kuratowski's monograph [17] (§39, III). P. S. Novikov [10], [11], L. V. Keldysh [14], A. A. Lyapunov [16], and E. A. Selivanovskiĭ [34] turned to the study of Luzin's problems, as did, later on in the 1960's and 1970's, foreign specialists, in particular, Solovay [27], [28] and Stern [30], [31]. The paper [20] is especially devoted to aspects of these problems connected with the philosophy and history of mathematics, and therefore we shall not go into such matters specifically. We shall only present information about the history of investigations of problems concerning constituents when that information is essential for the succeeding exposition.

Luzin ([2], Chapter III) established that Problem II is equivalent to the problem of the existence of an uncountable analytic complement (that is, a Π^1_1-set in modern terminology) not having perfect subsets; that is, if there exists a sieve satisfying the requirements of Problem II, then there exists (in the space $\mathscr{N}$) an analytic complement of the indicated form, and conversely.

In P. S. Novikov's paper [10] it was proved that Problem II is equivalent to a weak variant of Problem I, where one requires that every constituent $[C]_\nu$ contain not more than one point, and, in addition, that the number of nonempty constituents $[C]_\nu$ be uncountable. In another paper [11] by the same author, the following progress was achieved. It was discovered that, with the methods of the Zermelo-Fraenkel axiomatic set theory ZFC with the axiom of choice (concerning this theory, see [19]; it is customary to assume that the theory ZFC formalizes all known methods of mathematical reasoning), it is impossible to prove the *nonexistence* of uncountable analytic complements which do not contain perfect subsets, and, therefore, by what was said above, it is impossible to solve Problem II in the negative (that is, to prove the nonexistence of sieves of the required form).

This result of Novikov was supplemented by the American mathematician Solovay, who found [28] that it is also impossible within ZFC to prove the *existence* of uncountable analytic complements without a perfect kernel, and, thus, that it is also impossible to solve Problem II in the affirmative. Thus, Problem II turns out to be undecidable: it is impossible with ordinary mathematical arguments either to prove or to refute the existence of a sieve satisfying the requirements of the problem. (Parenthetically, let us note that many other problems of classical descriptive set theory turn out to be undecidable. However, not all are undecidable; there are important exceptions, among problems about constituents in particular.)

In [22], this result was extended to Problems III and IV. Namely, it was proved in [22] that each of these two problems is equivalent to Problem II and is, therefore, undecidable. (Two problems are equivalent if, from the assumption that one of them has a positive solution, it follows that the other also has a positive solution.) However, it was established in [22] that Problem 0 is decidable, and, indeed, in the negative; that is, one can prove the nonexistence of sieves of the required form. In the present paper, we shall also determine the status of Problems I, IIIa and IIIb: the first of these, like Problem 0, is decidable in the negative, but the two others, like Problems III and IV, are equivalent to Problem II and undecidable.

All these results on undecidability and decidability of Luzin's problems will be obtained at the end of the next section as corollaries of a general theorem (the main theorem of this paper).

At the same time, it is clear that Luzin's problems that we are considering in no way exhaust the list of possible problems of the same type about the existence of sieves with given properties of their constituents. For example, analogues of Problems I–IV, formulated for internal constituents, are essentially just as interesting as Problems I–IV themselves. In the formulation of the problems, one can introduce certain changes and obtain new variants. (Two variants of Problem III, Problems IIIa and IIIb, were proposed by Luzin himself.) Finally, one can consider sieves of a more general form than open sieves. In this way, one obtains a rather long list of problems about the existence of sieves, including Luzin's problems. The author undertook an attempt at systematization and investigation of such problems, a result of which is the present paper.

§2. Classification of problems, formulation of the main theorem, and the derivation therefrom of corollaries about the undecidability and decidability of Luzin's problems

Each of the problems presented in the preceding section is a problem about the existence of a sieve satisfying a certain requirement (more precisely, a complex of requirements), connected with the class of the sieve (it is required that it be open) and with the constituents that it determines. As notation for these and similar requirements, we shall agree to use five-termed symbols of the form $[nij, p]_\tau$ (Here, the subscript τ can actually be omitted, as will be seen below; the absence of the subscript τ will be assumed to be a special value of that subscript.) We shall explain the meaning which the symbols n, i, j, p and τ bear.

The letter n refers to the requirement placed on the class of the sieve. As in Luzin's problems of §1, we shall consider open sieves, assigning the value $n = 0$ of the letter n to the requirement of openness; the value $n = 1$ is assigned to Borel sieves, and the value $n = 2$ to sieves of the projective class Δ^1_2. More complicated projective sieves will not be considered here.

The subscript τ denotes the type of constituents to which the requirement $[nij, p]_\tau$ refers. The absence of any symbol in place of τ (that is, an expression of the form $[nij, p]$) will signify that the requirement refers only to external constituents (like, for example, the requirements included in the statements of Luzin's Problems I, II, III, IIIa, IIIb, and IV of §1). The symbol $*$ in place of τ refers to internal constituents, while the symbol $+$ in place of τ indicates that we are considering constituents of both types (as in the statement of Problem 0).

The indicator i assumes the values 0 and 1. By $i = 1$ we mean the requirement that all the constituents of type τ are nonempty (Problems I and IIIb), and by $i = 0$ the weaker requirement of nonemptiness of uncountably many constituents of type τ (the remaining problems).

Finally, the meaning of the letter j ($j = 0$ or 1) and the letter p ($p =$ I, II, III, IV). One requires that the family of all nonempty constituents of type τ (for $j = 1$), or some uncountable subfamily of that family (for $j = 0$), possesses, depending on the value of p, the following property:

For $p =$ I: every constituent of the indicated family (or of a subfamily—in accordance with the value of j) contains exactly one point.

For $p =$ II: every constituent of the indicated family (or subfamily) is at most countable.

For $p =$ III: the indicated family (or subfamily) is a collection of Borel sets of bounded order.

For $p =$ IV: the property of order-bounded pairwise separation: there exists an ordinal number $\xi < \omega_1$ such that any two distinct constituents of the given family (or subfamily) can be separated from each other by a Borel set of order at most ξ. (The set Z *separates* X from Y if $X \subseteq Z$ and $Y \cap Z = \varnothing$.)

Thus, $j = 1$ for Problems I, II, III, IV, and 0, and $j = 0$ for Problems IIIa and IIIb. Problem I corresponds to the value $p =$ I, Problem II to the value $p =$ II, Problems III, IIIa, IIIb, and 0 to the value $p =$ III, and the requirement corresponding to Problem IV is, so to say, enclosed between $p =$ III and $p =$ IV.

An assertion of the existence of a sieve satisfying the complex of requirements $[nij, p]_\tau$ will be abbreviated by $\exists[nij, p]_\tau$. All together, we have $3 \cdot 2 \cdot 2 \cdot 4 \cdot 3 = 144$ different symbols $[nij, p]_\tau$ (for the indicated values of the letters occurring in these symbols), and, therefore, 144 assertions of the form $\exists[nij, p]_\tau$. From the point of view of their interrelations with the system ZFC, these assertions can be divided into three large groups, denoted here by the letters A, B and C. The division into groups is given by the following tables:

TABLE 1

n	ij			
	11	01	10	00
0	C	A	A	A
1	C	A	A	A
2	A	A	A	A

No subscript τ (external constituents), p arbitrary

TABLE 2

n	ij			
	11	01	10	00
0	C	C	A	A
1	C	C	A	A
2	A	A	A	A

$\tau = *$ (internal constituents), p arbitrary

TABLE 3

n	ij			
	11	01	10	00
0	C	C	A	A
1	C	C	A	A
2			A	A

$\tau = +$ (both types of constituents), p arbitrary

TABLE 4

p	
I or II	B
III or IV	A

$\tau = +$, $n = 2$, $j = 1$, i arbitrary

The principal task of this paper is the proof of the following theorem, which determines the status of assertions of groups A, B, and C in the axiomatic set theory ZFC.

MAIN THEOREM. (a) *All* 100 *assertions of group* A *are equivalent to each other in ZFC and are undecidable in ZFC.*

(b) *All four assertions of group* B *are also pairwise equivalent and undecidable in ZFC. Moreover, the assumption of the truth of any of these assertions implies the truth of all the assertions of group* A, *but the converse is false.*

(c) *All* 40 *assertions of group* C *are false in ZFC; that is, from the axioms of ZFC one can deduce the nonexistence of sieves of the required form.*

Let us show how, from the Main Theorem, one can obtain the results presented in §1 about the undecidability and decidability of the various Luzin problems. It suffices to sort out which symbols correspond to these problems in our system. As a typical example, let us consider Problem I. It is obvious that the assertion contained in the statement of this problem (that is, the assertion of the existence of an open sieve C such that every constituent $[C]_\nu$ contains exactly one point) receives in our notation the symbolic designation $\exists[011, \mathrm{I}]$ and, thus, belongs to group C of Table 1. Therefore, this assertion is false in ZFC by clause (c) of the Main Theorem, and Problem I has, as a consequence, a negative solution (that is, a sieve of the desired form does not exist), as was claimed at the end of §1.

In exactly the same way, the assertions contained in the statements of Problems II, III, IIIa, and IIIb receive the designations ∃[001, II], ∃[001, III], ∃[000, III], and ∃[010, III], respectively, belong to group A of Table I, and are undecidable and pairwise equivalent by clause (a) of the Main Theorem.

The assertion about the existence of a sieve, contained in the statement of Problem IV, is of a strength situated between assertions ∃[001, III] and ∃[001, IV], belonging to group A of Table 1. Hence, by clause (a) of the Main Theorem, the assertion of Problem IV is equivalent to all assertions of group A, and, in particular, to the assertions extracted from Problems II, III, IIIa, and IIIb, and so it is undecidable.

Thus, Problems II, III, IIIa, IIIb, and IV are equivalent to each other (in the sense indicated at the end of §1) and are undecidable.

Finally, the assertion of the existence of a sieve satisfying the requirements of Problem 0 receives the symbolic designation $\exists[001, \mathrm{III}]_+$, belongs to group C of Table 3, and, therefore, is false by clause (c) of the Main Theorem, whence Problem 0 itself receives a negative solution: a sieve of the required form does not exist.

§3. Statement of five theorems, from which the Main Theorem follows

Of course, in proving the Main Theorem, we will not deal separately with each of the assertions in groups A, B and C. It is entirely sufficient to limit ourselves to the minimal and maximal (with respect to the requirements imposed on the sieve) assertions in groups A and B and to the minimal assertions in group C. In particular, since the strength of the requirement $[nij, p]_\tau$ decreases as (independently) n increases from 0 to 2, i decreases from 1 to 0, j decreases from 1 to 0, and p increases from I to IV, one may have left in group A only the minimal assertions ∃[200, IV], $\exists[200, \mathrm{IV}]_*$, and $\exists[200, \mathrm{IV}]_+$, and the maximal assertions ∃[001, I], ∃[010, I], ∃[211, I], $\exists[010, \mathrm{I}]_*$, $\exists[211, \mathrm{I}]_*$, $\exists[010, \mathrm{I}]_+$, and $\exists[211, \mathrm{III}]_+$, in group B only the minimal assertion $\exists[201, \mathrm{II}]_+$ and the maximal $\exists[211, \mathrm{I}]_+$, and, finally, in group C only the minimal assertions ∃[111, IV], $\exists[101, \mathrm{IV}]_*$, and $\exists[101, \mathrm{IV}]_+$.

In order to more conveniently handle these remaining assertions, we shall use the following two propositions (not connected with sieves and constituents, but well established within axiomatic set theory (see, for example, [19c], §3)):

$$\exists \pi \in \mathcal{N}\left(\omega_1^{L[\pi]} = \omega_1\right), \qquad \exists \pi \in \mathcal{N}\left(\mathcal{N} \subseteq L[\pi]\right).$$

In these propositions, as usual, we denote by $\omega_1^{L[\pi]}$ the first uncountable ordinal in the class $L[\pi]$ of all sets that are constructible relative to π.

Both of these propositions do not contradict the axioms of ZFC, since they follow from the (consistent, see [19a], Chapter 5) axiom of constructibility. Their negations also do not contradict ZFC: this was proved by Lévy [26] with the help of a model which was used later in [28]. Thus, both propositions are undecidable in ZFC. Moreover, the first of them is a trivial consequence of the second, but the second, on the contrary, is not derivable from the first, since, in the simplest model of Cohen, refuting the continuum hypothesis (model II in [24]), the first proposition is true, but the second is false.

In light of everything that has been said, for the proof of the Main Theorem it completely suffices to prove the following five theorems:

THEOREM A1. *If the proposition* $\exists \pi \in \mathcal{N}(\omega_1^{L[\pi]} = \omega_1)$ *is true (that is, if there exists a point* $\pi \in \mathcal{N}$ *satisfying the equality* $\omega_1^{L[\pi]} = \omega_1$*), then the assertions* ∃[001, I], ∃[010, I],

$\exists[211, \mathrm{I}]$, $\exists[010, \mathrm{I}]_*$, $\exists[211, \mathrm{I}]_*$, $\exists[010, \mathrm{I}]_+$ *and* $\exists[211, \mathrm{III}]_+$ *are also true*; *that is, the required sieves exist.*

THEOREM A2. *If at least one of the assertions* $\exists[200, \mathrm{IV}]$, $\exists[200, \mathrm{IV}]_*$ *or* $\exists[200, \mathrm{IV}]_+$ *is true, then the proposition* $\exists\,\pi \in \mathcal{N}(\omega_1^{L[\pi]} = \omega_1)$ *is also true.*

THEOREM B1. *If the proposition* $\exists\,\pi \in \mathcal{N}(\mathcal{N} \subseteq L[\pi])$ *is true, then the assertion* $\exists[211, \mathrm{I}]_+$ *also holds.*

THEOREM B2. *If the assertion* $\exists[201, \mathrm{II}]_+$ *holds, then the proposition* $\exists\,\pi \in \mathcal{N}(\mathcal{N} \subseteq L[\pi])$ *is true.*

THEOREM C. *The propositions* $\exists[111, \mathrm{IV}]$, $\exists[101, \mathrm{IV}]_*$, *and* $\exists[101, \mathrm{IV}]_+$ *are false*; *that is, no sieve satisfying the requirements of any of these assertions exists.*

The entire remaining part of the paper consists of the proof of these theorems. In the following fourth section, after some remarks having to do with different hierarchies, we shall formulate and prove two propositions about sieves, one of which shows the connection between the class of a sieve and the class of its cross-section function, while the meaning of the second is that Borel sieves do not yield anything new with respect to the properties of constituents under consideration, in comparison with open sieves. After this, we shall prove Theorems A1 and B1 in §5, using the apparatus of constructible sets together with some rather subtle constructions (in particular, we shall use a method introduced by Hausdorff and Luzin for partitioning the continuum into $\aleph_1$ nonempty sets of the class Π_3^0). The following section, §6, includes definitions and some simple facts concerning the encoding of Borel sets, as well as a proof, based upon a result of Louveau, of a theorem on Π_1^1-expression for separability. In the first appoximation, this theorem then opens up the possibility of expressing the fact of $\Pi_{1+\rho}^0$-separation (where $\rho < \omega_1$ is given) of two Σ_1^1-sets by means of a Π_1^1-formula, containing as variables parameters from $\mathcal{N}$ occurring in the definitions of the given sets. In §7 we shall prove two key lemmas on separation, which will provide a procedure for proving Theorems A2, B2, and C. The proofs themselves of these theorems are located in the last two sections.

§4. Reference information and some auxiliary propositions about the projective hierarchy, analytic formulas, and sieves

All the main set-theoretic definitions and notation will be taken from [19]. Below, we shall denote points of Baire space $\mathcal{N}$ by the letters α, β, γ, δ, ε, π only, while ordinals (that is, natural numbers and transfinite ordinals) will be denoted by the letters ξ, η, μ, ν, ρ, λ, κ. As usual, ω_ξ will denote the ξth infinite cardinal ($\omega_0 = \omega$).

By a *transitive model* of ZFC we shall mean any transitive set or proper class which satisfies all the axioms of ZFC. If M is such a model and $\xi \in M$ is an ordinal, then by ω_ξ^M we shall denote the ξth infinite cardinal in M.

We shall use the standard notation Σ_ξ^0, Π_ξ^0 and Δ_ξ^0 (where $1 \leqslant \xi < \omega_1$) for Borel classes and the standard notation Σ_n^1, Π_n^1 and Δ_n^1 (where $n \in \omega$) for projective classes. We shall also use the notation $\Sigma_n^{1,\pi}$, $\Pi_n^{1,\pi}$ and $\Delta_n^{1,\pi}$ (where $\pi \in \mathcal{N}$) for the effective subclasses of the corresponding projective classes Σ_n^1, Π_n^1 and Δ_n^1. Definitions and elementary information about these classes may be found in [19b] and [23], Chapter 7.

For definitions of projective sets lying in spaces of the form $\mathcal{N}^m \times \omega^k$ ($m, k \in \omega$), it is convenient to use formulas of the language of second-order arithmetic. This language contains two types of variables: variables ranging over ω (denoted by the letters i, j, k, l, m) and variables ranging over $\mathcal{N}$ (the letters α, β, γ, δ, ε, π). Elementary formulas are those of the form $k + l = m$, $kl = m$, $k = l$, and $\alpha(k) = l$. Formulas of this language are said to be *analytic*, and, if an analytic formula does not contain bound variables over $\mathcal{N}$, then it is called *arithmetic*.

The collection of Σ^1_n*-formulas* (where $n \geqslant 1$) is to contain all analytic formulas of the form $\exists \alpha_1 \forall \alpha_2 \exists \alpha_3 \cdots \Box \alpha_n \psi$, where ψ is an arithmetic formula, and the symbol $\Box$ denotes the quantifier $\forall$ for even n and $\exists$ for odd n. The collection of analytic Π^1_n*-formulas* is introduced in exactly the same way, except that the left-most quantifier must be the quantifier $\forall$ and the value of $\Box$ must change accordingly.

In a somewhat wider sense, by a Σ^1_n-formula we shall mean any analytic formula for which one can construct an equivalent Σ^1_n-formula in the sense of the strict definition just given. The construction of equivalent formulas can be carried out with the help of the well-known devices for transforming analytic formulas (see [23], Chapter 7). For example, if $\varphi(\alpha)$, $\psi(\alpha)$, and $\Phi(\varepsilon, \alpha)$ are Σ^1_1-formulas and $\Psi(\varepsilon, \alpha)$ is a Π^1_1-formula, then the formula

$$\exists \varepsilon \forall \alpha((\varphi(\alpha) \to \Psi(\varepsilon, \alpha)) \wedge (\psi(\alpha) \to \neg \Phi(\varepsilon, \alpha)))$$

will be considered a Σ^1_2-formula. We shall interpret in a similarly wide sense the concept of a Π^1_n-formula.

Among projective classes, their effective subclasses, and the corresponding classes of analytic formulas, there is the following simple connection. If $n \geqslant 1$ and $\pi \in \mathcal{N}$, then the class $\Sigma^{1,\pi}_n$ consists precisely of those sets of the spaces of the form $\mathcal{N}^m \times \omega^k$ which can be defined by Σ^1_n-formulas not containing parameters in $\mathcal{N}$ other than the parameter π.

Analytic formulas of the second level obey the following remarkable proposition (for a proof, see [19c], p. 305).

THE SHOENFIELD ABSOLUTENESS PRINCIPLE. *Let M be a transitive model of ZFC containing all countable ordinals, and let φ be a closed Σ^1_2-formula or Π^1_2-formula with parameters from M. Then φ is absolute for M; that is, the truth of the formula in the universe of all sets implies its truth in M, and conversely.*

Analytic formulas are not entirely suitable for the description of sets arising in the course of transfinite constructions. Here it is more advantageous to use formulas of the set-theoretic $\in$-language. The definition of the classes of Σ_n-formulas and Π_n-formulas of this language may be found in [19], Chapter 5, §4; moreover, we shall sometimes interpret such formulas in an extended sense, analogous to that presented above for Σ^1_n-formulas.

If $P \subseteq X$ is an arbitrary set, then $\Sigma^X_n(P)$ will be understood to denote the collection of all sets $Y \subseteq X$ which can be defined in X by means of Σ_n-formulas with parameters from P. Similar meaning is to be given to the notation $\Pi^X_n(P)$ and $\Delta^X_n(P) = \Sigma^X_n(P) \cap \Pi^X_n(P)$. In the important special cases where $P = \varnothing$ or $P = X$, we shall write Σ^X_n and $\Sigma_n(X)$, respectively, instead of $\Sigma^X_n(P)$, and likewise for Π and Δ.

Analytic definability is connected with $\in$-definability in the set HC of all hereditarily countable sets. (By *hereditarily countable* sets we mean sets whose transitive closures are at most countable.) The connection is given by the following well-known lemma, which is proved, for example, in [19c], §9.

TRANSLATION LEMMA. *Assume $n \geqslant 1$, $\pi \in \mathcal{N}$, and $X \subseteq \mathcal{N}$. Then $X \in \Sigma^{\mathrm{HC}}_n(\{\pi\})$ if and only if $X \in \Sigma^{1,\pi}_{n+1}$, and similarly for Π and Δ classes.*

Now a few words about cross-section functions of sieves. It is clear that the elements C_q of any given sieve $C = \langle C_q \colon q \in \mathbf{Q} \rangle$ are uniquely determined if we are given the cross-section function of the sieve C; that is, the function which associates with each point

$\alpha \in \mathcal{N}$ the corresponding cross-section $C/\alpha = \{q \in \mathbf{Q}\colon \alpha \in C_q\}$ (in fact, $C_q = \{\alpha \in \mathcal{N}\colon q \in C/\alpha\}$ for any index q). Below, we shall construct and analyze sieves that are given, as a rule, by means of their cross-section functions, and not in the form of an indexed set of elements as in §1. Keeping this in mind, we now formulate a proposition that connects the class of a sieve (in the sense of §1) with the class of its cross-section function, mapping $\mathcal{N}$ into the space $\mathscr{P}(\mathbf{Q})$ of all subsets of $\mathbf{Q}$. Before stating it, let us agree on the following. First, the topology of $\mathscr{P}(\mathbf{Q})$ is the topology of the Cantor discontinuum, the subbase of which consists of all sets of the form $\{Q \subseteq \mathbf{Q}\colon q \in Q\}$ (where $q \in \mathbf{Q}$) and the complements of such sets. Second, a function F, mapping $\mathcal{N}$ into $\mathscr{P}(\mathbf{Q})$ is said to be *Borel* (or, in general, a function of some class K) if its graph (as a set of pairs lying in the product $\mathcal{N} \times \mathscr{P}(\mathbf{Q})$) is a Borel set (or, respectively, a set of the class K).

PROPOSITION 1. *To continuous cross-section functions correspond clopen (open-closed) sieves; that is, sieves with elements that are clopen in $\mathcal{N}$. More precisely, a sieve is clopen if and only if its cross-section function is continuous.*

To Borel cross-section functions correspond Borel sieves. Finally, sieves of the class $\mathbf{\Delta}^1_2$ correspond to cross-section functions of the class $\Delta^{\mathrm{HC}}_1(\mathcal{N})$.

The assertion is perfectly obvious for continuous functions and clopen seives. Let us consider Borel functions and sieves. If a sieve $C = \langle C_q\colon q \in \mathbf{Q}\rangle$ is Borel (this means that all the sets C_q are Borel), then the cross-section function $\alpha \mapsto C/\alpha$ is also Borel, since

$$Q = C/\alpha \leftrightarrow \forall q \in \mathbf{Q}(q \in Q \leftrightarrow \alpha \in C_q).$$

Conversely, if the cross-section function of a sieve C is Borel, then each of the elements C_q of this sieve will also be a Borel set, since

$$C_q = \{\alpha\colon \exists Q(Q = C/\alpha \wedge q \in Q)\} = \{\alpha\colon \forall Q(Q = C/\alpha \to q \in Q)\}.$$

Just such an argument (plus the Translation Lemma) enables us also to prove that sieves of the class $\mathbf{\Delta}^1_2$ correspond to cross-section functions of the class $\Delta^{\mathrm{HC}}_1(\mathcal{N})$.

To conclude this section, we state a proposition whose general meaning is that Borel sieves do not give us anything fundamentally new, in comparison with open sieves, as to the realization of the properties of constituents dealt with in §2.

PROPOSITION 2. *Let the letters i, j, p and τ assume any of the values indicated in* §2. *Then the existence of a sieve satisfying the requirements $[1ij, p]_\tau$ implies the existence of a sieve satisfying $[0ij, p]_\tau$.*

We shall present the proof only for the cases in which this proposition is used in the present paper (see §5.2); that is, for the cases where $p = \mathrm{I}$ and either $i = 1$ and $j = 0$ or, conversely, $i = 0$ and $j = 1$.

So, assume the sieve C satisfies the requirements $[1ij, p]_\tau$, and, in particular, is Borel. According to Proposition 1, the graph P of the cross-section function of the sieve is a Borel set in the space $\mathcal{N} \times \mathscr{P}(\mathbf{Q})$. From a classical theorem of descriptive set theory (see [2], Chapter II (= [9], p. 108) or [17], Chapter II, §31.II), the set P, after deletion of an at most countable subset, turns out to be a continuous one-to-one image of the Baire space. In other words, there exists a set $P' \subseteq P$ such that the difference $D = P - P'$ is at most countable and there is a continuous one-to-one mapping F of $\mathcal{N}$ onto P'.

If $\beta \in \mathcal{N}$ and $F(\beta) = \langle \alpha, Q\rangle \in P'$, then we define $G(\beta) = \alpha$ and $S/\beta = Q$. The function $\beta \mapsto S/\beta$ (from $\mathcal{N}$ into $\mathscr{P}(\mathbf{Q})$) is continuous, and the function $G\colon \mathcal{N} \to \mathcal{N}$ is one-to-one and also continuous, where the full image of G is obtained by removing from

$\mathcal{N}$ an at most countable set $X = \{\alpha\colon \langle \alpha, C/\alpha \rangle \in D\}$. Finally, it is obvious that $S/\beta = C/G(\beta)$ for any point β. From everything that has been said, it follows that the sieve S given by its cross-section function $\beta \mapsto S/\beta$ is open (and even clopen, by Proposition 1). Moreover, for any index $\nu < \omega_1$, the equations

$$[S]_\nu = G^{-1}\text{''}([C]_\nu - X), \qquad [S]_{*\nu} = G^{-1}\text{''}([S]_{*\nu} - X)$$

hold. Thus, since X is countable and G is one-to-one, if the sieve C satisfies the requirements $[101, \mathrm{I}]_\tau$ (for any fixed τ), then the sieve S satisfies the requirements $[001, \mathrm{I}]_\tau$, which is what we need.

In the case where $i = 1$ and $j = 0$, the indicated construction may not give the required result, since some of the constituents of type τ of the sieve S may turn out to be empty (when all points of the corresponding constituent of the sieve C belong to the deleted set X). Therefore, the construction becomes somewhat more complicated. We divide the space $\mathcal{N}$ into clopen Baire intervals $U_m = \{\beta\colon \beta(0) = m\}$ (where $m \in \omega$). The set U_0 is homeomorphic to the whole space $\mathcal{N}$, and, therefore, the construction of functions S and G can be performed so that they are defined only on U_0, and not on all of $\mathcal{N}$. In addition, since the set X is at most countable, let $X = \{\alpha_1, \alpha_2, \dots\}$. We introduce a new cross-section function S', setting

$$S'/\beta = \begin{cases} S/\beta & \text{when } \beta \in U_0, \\ C/\alpha_m & \text{when } \beta \in U_m \text{ and } m \geqslant 1. \end{cases}$$

This function is continuous together with S, and the corresponding sieve is open. In addition, the requirements contained in the values $i = 1$ and $j = 0$ of the letters i and j go over (for $p = \mathrm{I}$ and arbitrary τ) from the sieve C to the sieve S'.

§5. Proof of Theorems A1 and B1; construction of sieves with the help of the apparatus of constructible sets

The assertions about the existence of sieves that occur in the indicated theorems divide up, from the point of view of their methods of construction, into three groups (where the third entirely consists of one assertion). Let us consider these groups in turn, beginning with the easiest.

5.1. *Construction of sieves satisfying requirements* $[211, \mathrm{I}]_+$, $[211, \mathrm{I}]$, *and* $[211, \mathrm{I}]_*$. The proposed construction is based on the following fact of the theory of construcible sets: if $\pi \in \mathcal{N}$, then there exists a well-ordering $<_\pi$ of the set $\mathrm{HC}[\pi] = \mathrm{HC} \cap L[\pi]$ ($= L_{\omega_1}[\pi]$), satisfying the "bounded quantifier lemma": if a set $P \subseteq \mathrm{HC}[\pi]^3$ belongs to the class $\Delta_1^{\mathrm{HC}[\pi]}(\{\pi\})$, then the set

$$\{\langle y, z\rangle \in \mathrm{HC}[\pi]^2\colon \forall x <_\pi y(\langle x, y, z\rangle \in P)\}$$

also has class $\Delta_1^{\mathrm{HC}[\pi]}(\{\pi\})$; see [19c], §8.

We begin with the construction of a sieve of the form $[211, \mathrm{I}]_+$ (that is, a sieve C of class Δ^1_2 such that any of the constituents $[C]_\nu$ and $[C]_{*\nu}$, where $\nu < \omega_1$, contains exactly one point) under the assumption that there exists a point $\pi \in \mathcal{N}$ satisfying $\mathcal{N} \subseteq L[\pi]$. This will yield a proof of Theorem B1.

We partition the space $\mathcal{N}$ into two sets $X = \{\alpha \in \mathcal{N}\colon \alpha(0) = 0\}$ and $X_* = \mathcal{N} - X$. For every ordinal $\nu < \omega_1$, we denote by α_ν and $\alpha_{*\nu}$ the νth (with respect to the order $<_\pi$) points of the sets X and X_*, respectively. By Q_ν we denote the $<_\pi$-least of the well-ordered sets $Q \subseteq \mathbf{Q}$ of order type ν. Finally, by $Q_{*\nu}$ we denote the $<_\pi$-least of the sets $Q \subseteq \mathbf{Q}$ which are not well-ordered but have maximal well-ordered initial segments of type ν. The sequences of points α_ν, $\alpha_{*\nu}$ and sets Q_ν, $Q_{*\nu}$ have class $\Sigma_1^{\mathrm{HC}}(\{\pi\})$; for example,

$$\alpha = \alpha_\nu \leftrightarrow \exists f(f \text{ is a function} \wedge \operatorname{dom} f = \nu + 1 \wedge f(\nu) = \alpha \wedge \forall \mu \leqslant \nu(f(\mu) \in X - \{f(\xi) \colon \xi < \mu\} \wedge \forall \beta <_\pi f(\mu)(\beta \in X \to \beta \in \{f(\xi) \colon \xi < \mu\}))),$$

and the required result is yielded by the bounded quantifier lemma.

We give the required sieve C by means of its cross-section function, setting $C/\alpha_\nu = Q_\nu$ and $C/\alpha_{*\nu} = Q_{*\nu}$ for all $\nu < \omega_1$. By what was said above about the sequences of points α_ν and $\alpha_{*\nu}$ and of sets Q_ν and $Q_{*\nu}$, this cross-section function will belong to the class $\Sigma_1^{\mathrm{HC}}(\{\pi\})$, since

$$Q = C/\alpha \leftrightarrow \exists \nu((\alpha = \alpha_\nu \wedge Q = Q_\nu) \vee (\alpha = \alpha_{*\nu} \wedge Q = Q_{*\nu})),$$

and, therefore, it also belongs to the class $\Delta_1^{\mathrm{HC}}(\{\pi\})$, since

$$Q = C/\alpha \leftrightarrow \forall Q'(Q' = C/\alpha \to Q = Q').$$

Hence, the sieve C has class Δ_2^1 by Proposition 1 of §4. Finally, it is clear that $[C]_\nu = \{\alpha_\nu\}$ and $[C]_{*\nu} = \{\alpha_{*\nu}\}$ for any $\nu < \omega_1$. Thus, the sieve C that we have constructed satisfies the requirements $[211, \mathrm{I}]_+$.

Now let us turn to the construction of sieves of the forms [211, I] and $[211, \mathrm{I}]_*$. In accordance with the hypothesis of Theorem A1, assume that we have a point $\pi \in \mathcal{N}$ such that $\omega_1^{L[\pi]} = \omega_1$. It is known that the existence of such a point implies the existence of an uncountable set $X \subseteq \mathcal{N}$ of class $\Pi_1^{1,\pi}$ which does not have perfect subsets; see [25], [27], or [19c], §11. All points of this set must belong to $L[\pi]$ (see [19c], §16.7). Hence, by the absoluteness principle of §4, one can conclude that $X \in L[\pi]$ and $X \in \Pi_1^{1,\pi}$ in $L[\pi]$. We claim now that $X \in \Delta_1^{\mathrm{HC}}(\{\pi\})$ and $X \in \Delta_1^{\mathrm{HC}[\pi]}(\{\pi\})$. The first relation is obtained immediately by the Translation Lemma of §4, and for the proof of the second it suffices to apply the same lemma, because the hypothesis $\omega_1^{L[\pi]} = \omega_1$ implies the equality

$$\mathrm{HC}[\pi] = \{x \in L[\pi] \colon \text{it is true in } L[\pi] \text{ that } x \text{ is hereditarily countable}\}.$$

For every $\nu < \omega_1$, we denote by α_ν the νth point of the set X, in the sense of the ordering $<_\pi$. Arguing as in the preceding construction and taking into account the fact that $X \in \Delta_1^{\mathrm{HC}[\pi]}(\{\pi\})$, one can show that the sequence $\langle \alpha_\nu \colon \nu < \omega_1 \rangle$ has class $\Sigma_1^{\mathrm{HC}[\pi]}(\{\pi\})$. Let us verify that this sequence also belongs to the class $\Sigma_1^{\mathrm{HC}}(\{\pi\})$. Assume

$$\alpha = \alpha_\nu \leftrightarrow \exists f \in \mathrm{HC}[\pi] \varphi(\alpha, \nu, f),$$

where φ is a Δ_0-formula (that is, it does not have unbounded quantifiers; see [19a], p. 408) with the parameter π. We have

$$\alpha = \alpha_\nu \leftrightarrow \exists f \in \mathrm{HC}(f \in \mathrm{HC}[\pi] \wedge \varphi(\alpha, \nu, f)).$$

This yields the required fact about the class of the sequence of points α_ν, since $\mathrm{HC}[\pi] \in \Sigma_1^{\mathrm{HC}}(\{\pi\})$ (see [19c], p. 300).

By precisely the same kind of arguments, one can verify that the sequence, defined above, of sets Q_ν ($\nu < \omega_1$) has class $\Sigma_1^{\mathrm{HC}}(\{\pi\})$. Therefore (see the construction above of a sieve of the form $[211, \mathrm{I}]_+$) the sieve C, whose cross-section function is given by the conditions $C/\alpha_\nu = Q_\nu$ for $\nu < \omega_1$ and $C/\alpha = \mathbf{Q}$ for $\alpha \notin X$, will have class Δ_2^1. Moreover, it is obvious that $[C]_\nu = \{\alpha_\nu\}$ for any index ν; that is, the sieve C satisfies the requirements [211, I].

One constructs a sieve of the form [211, I]$_*$ in exactly the same way: it is only necessary to define $C/\alpha_\nu = Q_{*\nu}$ and $C/\alpha = \varnothing$ for $\alpha \notin X$.

5.2. *Construction of sieves of the forms* [001, I], [010, I], [010, I]$_*$ *and* [010, I]$_+$. By Proposition 2 of §4, one can limit oneself to the construction of *Borel* sieves with the necessary properties of constituents; that is, sieves of the forms [101, I], [110, I], [110, I]$_*$ and [110, I]$_+$. Keeping in mind the hypothesis of Theorem A1, we fix a point $\pi \in \mathcal{N}$ satisfying $\omega_1^{L[\pi]} = \omega_1$.

Let us denote by T the theory containing all axioms of ZFC except the power-set axiom, and also the axiom $V = L[\pi]$ and the axiom that every set is at most countable. By a *T-model* we shall mean any transitive set that satisfies all axioms of T (and, naturally, contains π). The set $\mathrm{HC}[\pi] = L_{\omega_1}[\pi]$ forms a natural T-model (here it is important that $\omega_1^{L[\pi]} = \omega_1$). In addition, there exist uncountably many ordinals $\nu < \omega_1$ such that the set $L_\nu[\pi]$ is a T-model. For $\xi < \omega_1$, we shall denote by ν_ξ the ξth such ordinal (in increasing order), and let us set $\Gamma = \{\nu_\xi: \xi < \omega_1\}$.

For every limit index ξ, we define $\mu_\xi = \sup_{m \in \omega} \nu_{\xi+m}$ and we denote by Ξ the set of all such ordinals μ_ξ. We shall construct a sieve C of the form [101, I] such that, for $\mu \in \Xi$, the constituent $[C]_\mu$ will contain exactly one point, and all the remaining external constituents will be empty.

Let $\alpha \in \mathcal{N}$, and define

$$M[\alpha] = \omega \cup \{\omega + k: \alpha(2k) = 1\};$$

$$\in_\alpha = \{\langle i, j\rangle: i \in j \in \omega\} \cup \{\langle i, \omega + k\rangle: \alpha(4(2^i \cdot 3^k) + 1) = 1\}$$
$$\cup \{\langle \omega + k, \omega + l\rangle: \alpha(4(2^i \cdot 3^k) + 3) = 1\}.$$

We wish to consider sets of the form $M[\alpha]$ (with the corresponding relations $\in_\alpha$, which are understood to be present everywhere below, but are not explicitly mentioned) as (nonstandard) models of the theory T. Here a certain inconvenience arises, connected with the fact that the sets $M[\alpha]$ clearly do not formally contain, say, the points of $\mathcal{N}$. However, if the ordinal $\eta = \omega + k \in M[\alpha]$ is such that it is true in $M[\alpha]$ that "η is a function from ω into ω", then one can assume that the point $\beta \in \mathcal{N}$ given by the condition "$\beta(k) = l$ when $\langle k, l\rangle \in \eta$ is true in $M[\alpha]$" actually belongs to $M[\alpha]$. In this case, we remove the ordinal η from $M[\alpha]$ and insert instead the indicated point β, changing the relation $\in_\alpha$ in the appropriate way. One can deal in similar fashion with the case where η is a rational number or a set consisting of rational numbers in $M[\alpha]$. In what follows, we understand by $M[\alpha]$ and $\in_\alpha$ (for a given point α) the result of such reconstructions, carried out for all η of $M[\alpha]$ that represent in $M[\alpha]$ points of $\mathcal{N}$, rational numbers, and sets of rational numbers.

We shall say that a point α is a *π-model point* if $\pi \in M[\alpha]$, all axioms of T hold in $M[\alpha]$, and the natural number sequence of $M[\alpha]$ coincides with the "genuine" set of natural numbers ω. By a *π-model* we mean a structure $M[\alpha]$ (with relation $\in_\alpha$) such that α is a π-model point. Every π-model $M[\alpha]$ has an "internal" well-ordering $(<_\pi)^{M[\alpha]}$, which we shall denote by $<_{\pi\alpha}$ and which may or may not be a well-ordering from the "external" point of view. If, however, $<_{\pi\alpha}$ actually well-orders $M[\alpha]$ (in such a case the π-model $M[\alpha]$ will be said to be *grounded*), then there exists a unique ordinal $\nu \in \Gamma$ such that the model $M[\alpha]$ is isomorphic to the T-model $L_\nu[\pi]$; moreover, the corresponding isomorphism (which is also unique) transforms $<_{\pi\alpha}$ into the relation $<_\pi \restriction L_\nu[\pi]$. Let us consider the set

$$W = \{\alpha \in \mathcal{N}: \forall m[(\alpha)_m \text{ is a } \pi\text{-model point} \wedge (\alpha)_m \in M[(\alpha)_{m+1}] \wedge$$
$$(\text{the following two propositions are true in } M[(\alpha)_{m+1}]:$$
$$1) \text{ there exists a largest transitive } T\text{-model and this model is isomorphic to } M[(\alpha)_m], \text{ and}$$
$$2) \ (\alpha)_m \text{ is constructible relative to } \pi, \text{ and there is no point } \beta\text{s} \in \mathcal{N} \text{ such that } \beta <_\pi (\alpha)_m \text{ and the model } M[\beta] \text{ is isomorphic to } M[(\alpha)_m])] \wedge$$
$$(\text{it is true in } M[(\alpha)_0] \text{ that there is no largest transitive } T\text{-model})\}.$$

(In this definition, we denote, as usual, by $(\alpha)_m$ the point of $\mathcal{N}$ satisfying for each k the equality $(\alpha)_m(k) = \alpha(2^m(2k+1)-1)$.)

Let us take a point $\alpha \in W$. For every natural number m, reasoning in $M[(\alpha)_{m+1}]$, we denote by Q_m the $<_\pi$-least well-ordered (in $M[(\alpha)_{m+1}]$) subset Q of the interval $[m, m+1)$ such that $Q \subseteq \mathbf{Q}$, $Q \in M[(\alpha)_{m+1}]$, and the π-model $M[(\alpha)_m]$ is isomorphic to the set $L_\mu[\pi]$ (where we denote by μ the order type of Q in the model $M[(\alpha)_{m+1}]$). Finally, we set $S/\alpha = \bigcup_{m \in \omega} Q_m$.

We specify the sieve S by defining its cross-section function $\alpha \mapsto S/\alpha$ as indicated above for $\alpha \in W$, while for $\alpha \notin W$ we set $S/\alpha = \mathbf{Q}$. The truth of any formulas in the model $M[\beta]$ can be expressed by arithmetic formulas with the variable β. Therefore, the set W and the mapping $\alpha \mapsto S/\alpha$ from $\mathcal{N}$ into $\mathcal{P}(\mathbf{Q})$ are Borel. By Proposition 1 of §4, this implies that our sieve S is Borel.

Now we come to the analysis of the constituents $[S]_\nu$. It is not difficult to verify the following. Assume a point $\alpha \in W$ is such that all the π-models $M[(\alpha)_m]$ are grounded. Then there exists a unique limit index $\xi < \omega_1$ such that, for any m, the model $M[(\alpha)_m]$ is isomorphic to $L_{\nu_{\xi+m}}[\pi]$. In that case, for any m, the point $(\alpha)_m$ coincides with the $<_\pi$-least point β satisfying the condition that $L_{\nu_{\xi+m}}[\pi]$ and $M[\beta]$ be isomorphic; that is, there is a one-to-one correspondence between points of W of the indicated form and the ordinals which occupy limit positions in Γ. Finally, in this case, the set $S/\alpha \subseteq \mathbf{Q}$ has order type μ_ξ.

If at least one of the π-models $M[(\alpha)_m]$ is not grounded, then it is evident that the set S/α cannot be well-ordered. If, in general, $\alpha \notin W$, then, by our construction, $S/\alpha = \mathcal{N}$ is again not a well-ordered set.

Taking into account what has been said, we see that, for any ordinal $\mu = \mu_\xi \in \Xi$, the constituent $[S]_\mu$ contains a unique point, namely, that point α which corresponds, in the sense indicated above, to the ordinal $\nu_\xi \in \Gamma$. But all the constituents $[S]_\nu$ with $\nu \notin \Xi$ will be empty sets. Thus, all the conditions of [101, I] hold for the sieve S that has been constructed, which is what was required.

For the construction of a sieve satisfying [110, I], we make a small change in the construction just carried out so that every constituent $[S]_\nu$ with index $\nu \notin \Xi$ will turn out to be nonempty. We note that the sets $Q \subseteq \mathbf{Q}$ can be indexed by subscripts $\gamma \in \mathcal{N}$ in such a way that every $Q \subseteq \mathbf{Q}$ receives at least one index and the mapping $\gamma \mapsto Q_\gamma$ is continuous (see the proof of Theorem 1 in §6). Let us consider the (Borel, like W) set

$$U = \{\alpha \notin W: \text{the point } (\alpha)_0 \text{ is a } \pi\text{-model point} \wedge (\alpha)_1 \in M[(\alpha)_0]$$
$$\wedge (\text{it is true in } M[(\alpha)_0] \text{ that } Q_{(\alpha)_1} \text{ is not order-isomorphic to any ordinal of the collection } \Xi)\}.$$

For $\alpha \in U$, we set $S/\alpha = Q_{(\alpha)_1}$. It is obvious that for $\alpha \in U$ the set S/α cannot be well-ordered with an order type from Ξ, while, at the same time, for any ordinal $\nu \in \Xi$ there exists a point $\alpha \in U$ such that the order type of S/α is equal to ν. Hence, keeping the definition of S/α for $\alpha \in W$ from the preceding construction and defining $S/\alpha = \mathscr{N}$ for all points $\alpha \notin W \cup U$, we obtain a Borel sieve S such that all constituents $[S]_\mu$ are nonempty and uncountably many of them are singletons (all those corresponding to indices $\mu \in \Xi$); that is, a sieve satisfying the requirements [110, I].

Making obvious minor changes in this construction, one can also construct (assuming, as before, that $\omega_1^{L[\pi]} = \omega_1$) sieves of the forms $[110, \text{I}]_*$ and $[110, \text{I}]_+$. We will not carry out these constructions here.

5.3. *Construction of a sieve satisfying* $[211, \text{III}]_+$. This construction is based on a construction, first carried out by Hausdorff in [33] and then perfected by Luzin in [8], that gives a (noneffective) decomposition of the continuum into $\aleph_1$ nonempty Borel sets of bounded order. This construction consists in the following. With the help of the axiom of choice, one can construct two transfinite sequences $\langle x_\nu \colon \nu < \omega_1\rangle$ and $\langle y_\nu \colon \nu < \omega_1\rangle$ of sets x_ν, $y_\nu \subseteq \omega$, possessing the following properties:

1) If $\nu < \mu$, then the differences $x_\nu - x_\mu$ and $y_\nu - y_\mu$ are finite, while the reverse differences $x_\mu - x_\nu$ and $y_\mu - y_\nu$ are infinite.

2) For any $\nu < \omega_1$, the intersection $x_\nu \cap y_\nu$ is finite.

3) There does not exist any set $z \subseteq \omega$ such that all the differences $x_\nu - z$ and all the intersections $y_\nu \cap z$ are finite.

The nature of the construction of these sequences is such that, under the assumption $\omega_1^{L[\pi]} = \omega_1$, by choosing at each step $\nu < \omega_1$ the pair $\langle x_\nu, y_\nu\rangle$ to be the $<_\pi$-least of the pairs $\langle x, y\rangle$ which are related in a certain way (see the cited papers) to the already constructed sequences $\langle x_\mu \colon \mu < \nu\rangle$ and $\langle y_\mu \colon \mu < \nu\rangle$, one can ensure the fulfillment of another condition:

4) The sequences $\langle x_\nu \colon \nu < \omega_1\rangle$ and $\langle y_\nu \colon \nu < \omega_1\rangle$ belong to the class $\Sigma_1^{\mathrm{HC}}(\{\pi\})$.

Having sequences with properties 1)–4), we define

$$Y_\nu = \{\alpha \in \mathscr{N} \colon \text{at least one of the sets } x_{\nu+1} - |\alpha| \text{ or } y_{\nu+1} \cap |\alpha| \text{ is infinite, where } |\alpha| = \{m \colon \alpha(m) = 0\}\}$$

for every $\nu < \omega_1$. By properties 1) and 2), the sets Y_ν increase with increasing ν; moreover, every Y_ν is strictly bigger than $\bigcup_{\mu<\nu} Y_\mu$ (for example, by using the characteristic function of the set x_ν). Thanks to property 3), the union of the sets Y_ν coincides with the whole space $\mathscr{N}$. In addition, it is not difficult to verify that every Y_ν has class Π_2^0 (that is, G_δ). Thus, we have obtained a partition of the space $\mathscr{N}$ into $\aleph_1$ nonempty Borel Π_3^0-sets—namely, the sets $Z_\nu = Y_\nu - \bigcup_{\mu<\nu} Y_\mu$. This is the content of Hausdorff's construction.

Let us observe that every set Z_ν contains both points α with $\alpha(0) = 0$ and points α with $\alpha(0) > 0$, since, in general, membership of α in a fixed Z_ν is unchanged by changing finitely many values of α. Therefore, for $\nu < \omega_1$, the sets

$$X_\nu = \{\alpha \in Z_\nu \colon \alpha(0) = 0\}, \qquad X_{*\nu} = Z_\nu - X_\nu$$

are nonempty, and all these sets belong to Π_3^0 by preceding results.

Choosing sets Q_ν, $Q_{*\nu} \subseteq \mathbf{Q}$ for each $\nu < \omega_1$ as indicated in §5.1 (so, both sequences of sets belong to the class $\Sigma_1^{\mathrm{HC}}(\{\pi\})$; see §5.1), we introduce a sieve C by specifying its cross-section function by the equalities $C/\alpha = Q_\nu$ for $\alpha \in X_\nu$ and $C/\alpha = Q_{*\nu}$ for

$\alpha \in X_{*\nu}$. By arguing as in §5.1, one can show that C is a sieve of class Δ^1_2. In addition, it is obvious that $[C]_\nu = X_\nu$ and $[C]_{*\nu} = X_{*\nu}$ for all $\nu < \omega_1$; that is, all external and internal constituents of C are nonempty sets of class Π^0_3 (and, therefore, of order at most 4, since $\Pi^0_3 \subseteq \Delta^0_4$). Thus, the sieve that has been constructed satisfies the requirements $[211, \mathrm{III}]_+$.

§6. Borel codes and a theorem on Π^1_1-expression for separation

This section begins the proof of Theorems A2, B2, and C of §3. The principal task to be carried out here consists in the effective separation (under certain conditions) of two sets of the Baire space $\mathscr{N}$, about which it is known that they can be separated by means of a Borel set of a given order. We shall turn directly to that task in the next section. This section contains only the definition and exposition of the elementary properties of encodings of Borel sets, as well as the proof of a theorem on Π^1_1-expression for separation that will be very important in what follows. This proof uses an effective separation theorem of Louveau [32]. We note that the theorem on Π^1_1-expression enables us here to avoid using the principle of Borel determinacy [29], which played an essential role in the author's preceding paper [22] devoted to the Luzin problems.

However, let us proceed to the subject of the present section. The idea of encoding the construction of a Borel set by means of grounded trees is well known, and it is only necessary to agree on the method of obtaining the initial sets and on the form of the intermediate operations. We shall use here almost without change the realization of this idea which was introduced in [22], in which one takes as a basis the operation of complementation of a countable union.

For every ordinal λ, we denote by Seq_λ the set of all finite sequences of ordinals less than λ (including the finite sequence Λ of length 0). We set $\mathrm{Seq} = \bigcup_{\lambda \in \mathrm{Ord}} \mathrm{Seq}_\lambda$ (as usual, Ord is the class of all ordinals). The expression $u \subset v$ means that the finite sequence v is a proper extension of the finite sequence u. If $u \in \mathrm{Seq}$ and $\xi \in \mathrm{Ord}$, then we agree to denote by $\xi^\wedge u$ and $u^\wedge \xi$ the finite sequences obtained by attaching the term ξ to the left or the right, respectively, of the terms of the finite sequence u. By a *tree* we mean any nonempty set $T \subseteq \mathrm{Seq}$ that possesses the property that, for $u \in \mathrm{Seq}$ and $v \in T$, $u \in T$ follows from $u \subset v$. We say that a tree T is *grounded* if there is no infinite path $u_0 \subset u_1 \subset u_2 \subset \cdots$ consisting of finite sequences $u_k \in T$. In such a case, to each $u \in T$ one can assign a unique ordinal $|u|_T$ so that

$$|u|_T = \sup_{v \in T,\, u \subset v} \big(|v|_T + 1\big).$$

Let us define $|T| = |\Lambda|_T$ (the *height* of the tree T), and let us denote by $\sup T$ the least ordinal λ such that $T \subseteq \mathrm{Seq}_\lambda$.

By a *Borel code* we mean any pair $\langle T, d\rangle$ consisting of a grounded tree $T \subseteq \mathrm{Seq}$ and any set $d \subseteq T \times \mathrm{Seq}_\omega$. If $\langle T, d\rangle$ is such a pair, then to every finite sequence $u \in T$ we can associate, by induction on $|u|_T$, a set $[T, d, u] \subseteq \mathscr{N}$ by means of the following system of equalities:

$$[T, d, u] = \neg \bigcup_{\langle u, w\rangle \in d} \mathscr{N}_w \quad \text{for } |u|_T = 0,$$

where $\mathcal{N}_w = \{\alpha \in \mathcal{N}: w \subset \alpha\}$ is a Baire interval for every finite sequence $w \in \mathrm{Seq}_\omega$, and the symbol $\neg$ denotes the complementation operation, that is, $\neg X = \mathcal{N} - X$;

$$[T, d, u] = \neg \bigcup_{u^\wedge\xi \in T} [T, d, u^\wedge\xi] \quad \text{for } |u|_T \geqslant 1.$$

We also define $[T, d] = [T, d, \Lambda]$ (the finite sequence Λ belongs to any tree T).

It is obvious that, if $\sup T < \omega_1$ (in general, if the tree T is at most countable), then $[T, d]$ is a Borel set of class $\Pi^0_{1+|T|}$. The converse is also true, where, for the construction of all $\Pi^0_{1+\rho}$-sets in the form $[T, d]$, one can (for a fixed $\rho < \omega_1$) restrict oneself entirely to one tree T_ρ, varying only the sets d. The trees T_ρ are given by a very simple construction by induction on ρ. Namely, $T_0 = \{\Lambda\}$, and, if $\rho \geqslant 1$ and all the trees T_ξ with indices $\xi < \rho$ have already been constructed, then

$$T_\rho = \{\Lambda\} \cup \left\{(\omega\xi + k)^\wedge u: \xi < \rho \wedge k \in \omega \wedge u \in T_\xi\right\}.$$

The reader will have no difficulty in proving the following lemma:

LEMMA 1. *Assume $\rho < \omega_1$. Then T_ρ is a grounded tree satisfying $|T_\rho| = \rho$ and $\sup T_\rho = \omega\rho$. In addition, for every $\Pi^0_{1+\rho}$-set $X \subseteq \mathcal{N}$, there is a set $d \subseteq T_\rho \times \mathrm{Seq}_\omega$ such that $X = [T_\rho, d]$.*

Before stating another lemma which will be needed in §7, we associate with every pair $\pi \in \mathcal{N}$, $\rho \in \mathrm{Ord}$ the collection $K_{\rho\pi}$ of all Borel codes $\langle T, d\rangle \in L[\pi]$ such that $\sup T \leqslant \omega_\rho^{L[\pi]}$ and $|T| \leqslant \rho$, and by $[K_{\rho\pi}]$ we denote the family of all sets of the form $[T, d]$, where $\langle T, d\rangle$ is a code in $K_{\rho\pi}$. The author found the idea of the following lemma in a paper by Stern [30].

LEMMA 2. *Assume that $\rho \in \mathrm{Ord}$, $\pi \in \mathcal{N}$, $\langle T, d\rangle \in L[\pi]$ is a Borel code, and $|T| \leqslant \rho$. Then $[T, d] \in [K_{\rho\pi}]$.*

PROOF. Reasoning in $L[\pi]$, we construct from every code $\langle T, d\rangle \in L[\pi]$ another code $\langle T', d'\rangle \in K_{|T|\pi}$ such that in the universe of all sets the equality $[T', d'] = [T, d]$ holds. The construction is carried out by induction on $|T|$. If $|T| = 0$, then $T = \{\Lambda\}$ and we simply take $\langle T', d'\rangle = \langle T, d\rangle$.

Now we construct $\langle T', d'\rangle$ for a code $\langle T, d\rangle$, assuming that $\rho = |T| > 0$ and that the required construction already has been carried out for all codes with trees of height $< \rho$. We set $U = \{\xi \in \mathrm{Ord}: \langle\xi\rangle \in T\}$ ($\langle\xi\rangle$ is the finite sequence with a single element ξ), and

$$T_\xi = \{u: \xi^\wedge u \in T\}, \qquad d_\xi = \left\{\langle u, w\rangle: \langle \xi^\wedge u, w\rangle \in d\right\}$$

for every $\xi \in U$. It is clear that $|T_\xi| < \rho$ for any $\xi \in U$, and so for every $\xi \in U$ the code $\langle T'_\xi, d'_\xi\rangle$ of the collection $K_{<\rho,\pi} = \bigcup_{\lambda<\rho} K_{\lambda\pi}$ such that $[T_\xi, d_\xi] = [T'_\xi, d'_\xi]$ is already constructed. However, the collection $K_{<\rho,\pi}$ has cardinality $\kappa = \omega_\rho^{L[\pi]}$ in $L[\pi]$; that is, there exists a function $f \in L[\pi]$, $f: \kappa \to K_{<\rho,\pi}$, such that the family of all codes of the form $\langle T'_{f(\eta)}, d'_{f(\eta)}\rangle$ (where $\eta < \kappa$) exactly coincides with the family of all codes $\langle T'_\xi, d'_\xi\rangle$ ($\xi \in U$). Now, defining

$$T' = \{\Lambda\} \cup \left\{\eta^\wedge u: \eta < \kappa \wedge u \in T'_{f(\eta)}\right\};$$

$$d' = \left\{\langle \eta^\wedge u, w\rangle: \eta < \kappa \wedge \langle u, w\rangle \in d'_{f(\eta)}\right\},$$

we obtain $\langle T', d'\rangle \in K_{\rho\pi}$ and, in addition,

$$[T', d'] = \neg \bigcup_{\eta<\kappa} \left[T'_{f(\eta)}, d'_{f(\eta)}\right] = \neg \bigcup_{\xi \in U} [T_\xi, d_\xi] = [T, d].$$

This concludes the proof of Lemma 2.

Borel codes will occur below in many calculations. In particular, they occur in the proof of the following theorem about a Π_1^1-expression for the concept of separation of Σ_1^1-sets. Before stating it, let us say that a set $X_1 \subseteq \mathcal{N}$ is *$\Pi_{1+\rho}^0$-separable from* $X_2 \subseteq \mathcal{N}$ (where $\rho < \omega_1$) if there exists a set $Z \subseteq \mathcal{N}$ of class $\Pi_{1+\rho}^0$ such that $X_1 \subseteq Z$ and $X_2 \cap Z = \varnothing$.

THEOREM 1. *Let $\varphi_1(\beta,\alpha)$ and $\varphi_2(\beta,\alpha)$ be a pair of Σ_1^1-formulas with parameters from some transitive model M of the theory ZFC, and let $\rho < \omega_1^M$. Then there exists a Π_1^1-formula $\theta(\beta)$ with parameters from M such that, in any transitive model of ZFC that is an extension of M (in particular, in M itself and in the universe of all sets), the following is true*:

$$\forall \beta \in \mathcal{N}\big(\theta(\beta) \leftrightarrow \text{the set } \{\alpha \in \mathcal{N}\colon \varphi_1(\beta,\alpha)\} \text{ is } \Pi_{1+\rho}^0\text{-separable from the set } \{\alpha\colon \varphi_2(\beta,\alpha)\}\big).$$

We begin the proof of this theorem with some definitions connected with the formulation of a result from [32]. In what follows, a fixed recursive enumeration $\mathbf{Q} = \{q_k\colon k \in \omega\}$ of the set $\mathbf{Q}$ of all rational numbers is presupposed. We introduce the collection WO of all points $\gamma \in \mathcal{N}$ such that the set $Q_\gamma = \{q_k\colon \gamma(k) = 0\}$ is well-ordered. For $\gamma \in WO$, we denote by $\mathrm{otp}(\gamma)$ the order type of the set Q_γ, and we set $WO_\nu = \{\gamma \in WO\colon \mathrm{otp}(\gamma) = \nu\}$ when $\nu < \omega_1$. We regard points of WO_ν as codes of ordinal ν.

Let us fix a point $\gamma \in WO_{\omega\rho}$. By means of γ, we associate with each natural number k of the set Q_γ the ordinal ξ_k that is equal to the order type of the set $\{q \in Q_\gamma\colon q < q_k\}$; here, the equality $\{\xi_k\colon k \in Q_\gamma\} = \{\xi\colon \xi < \omega\rho\}$ holds.

We also fix a recursive (and, in particular, a member of M) enumeration $\mathrm{Seq}_\omega = \{w_n\colon n \in \omega\}$ of the set Seq_ω such that $w_0 = \Lambda$. If the finite sequence w_n corresponding to a given index n is made up only of numbers in Q_γ, then we denote by u_n the finite sequence obtained from w_n by replacing each of its terms k by the corresponding ordinal ξ_k. For any other n, we define $w_n = \Lambda$. Then, $u_0 = \Lambda$ and $\{u_n\colon n \in \omega\} = \mathrm{Seq}_{\omega\rho}$.

In order to make the notation less cumbersome, let us agree in what follows to assume that the formulas φ_1 and φ_2 in the hypothesis of the theorem contain just one parameter $\pi \in M \cap \mathcal{N}$. Without restricting generality, one may assume that the point γ and the sets

$$S = \{m\colon u_m \in T_\rho\}, \quad S_0 = \{m \in S\colon |u_m|_{T_\rho} = 0\}, \quad S_1 = S - S_0,$$

$$S_2 = \{2^i \cdot 3^m \cdot 5^k\colon u_i = u_m{}^\wedge \xi_k\}, \qquad S_3 = \{2^n \cdot 3^i \cdot 5^{w_n(i)}\colon n \in \omega \wedge i < \mathrm{dom}\, w_n\}$$

are all recursive relative to π. Under this assumption, all of the following reasoning does not depend upon the specific choice of the point $\gamma \in WO_{\omega\rho}$, in view of which we shall write the index ρ instead of γ in the following definition. If $\varepsilon \in \mathcal{N}$, then we set

$$d_{\rho\pi\varepsilon} = \{\langle u_m, w_n\rangle\colon m \in S \wedge \varepsilon(2^m \cdot 3^n) = 0\}$$

(thus, $d_{\rho\pi\varepsilon} \subseteq T_\rho \times \mathrm{Seq}_\omega$) and $W_{\rho\pi\varepsilon} = [T_\rho, d_{\rho\pi\varepsilon}]$. By Lemma 1, the family of all sets of the form $W_{\rho\pi\varepsilon}$, where $\varepsilon \in \mathcal{N}$, exactly coincides with the class of all $\Pi_{1+\rho}^0$-sets of the space $\mathcal{N}$. The idea of [32] consists in taking not all, but only hyperarithmetic points ε. More precisely, for any $\beta \in \mathcal{N}$, we introduce the collection $\Pi_{1+\rho}^{*\pi\beta}$ of all sets of the form $W_{\rho\pi\varepsilon}$, where $\varepsilon \in \mathcal{N}$ is a point of class $\Delta_1^{1,\pi,\beta}$.

THEOREM 2 (effective separation theorem [32]). *Assume that $\beta \in \mathcal{N}$ and $X_1, X_2 \subseteq \mathcal{N}$ is a pair of disjoint sets of class $\Sigma_1^{1,\pi,\beta}$ such that the first is $\Pi_{1+\rho}^0$-separable from the second. Then X_1 can be separated from X_2 by a set of class $\Pi_{1+\rho}^{*\pi\beta}$.*

In order to use this theorem (which is accepted here without proof) in the proof of Theorem 1, we construct a Σ^1_1-formula $\Phi(\varepsilon, \alpha)$ and a Π^1_1-formula $\Psi(\varepsilon, \alpha)$, both with parameters from M (actually, with the single parameter π) such that, in any model of ZFC which is an extension of M,

$$\forall \varepsilon \forall \alpha\big(\alpha \in W_{\rho\pi\varepsilon} \leftrightarrow \Phi(\varepsilon, \alpha) \leftrightarrow \Psi(\varepsilon, \alpha)\big).$$

As $\Phi(\varepsilon, \alpha)$ one can take the formula

$$\exists g\colon \omega \to \{0,1\}\big(g(0) = 1 \wedge \big[\forall m \in S_0\big(g(m) = 1 \leftrightarrow \forall n\big(\varepsilon(2^m \cdot 3^n) = 0 \to \alpha \notin \mathcal{N}_{w_n}\big)\big)$$
$$\wedge \forall m \in S_1\big(g(m) = 1 \leftrightarrow \forall k \forall i \in S\big(u_i = u_m{}^\wedge\xi_k \to g(i) = 0\big)\big)\big]\big).$$

(The sets S, S_0, S_1, are defined above; the meaning of the formula in square brackets consists in the fact that

$$\forall m\big(g(m) = 1 \leftrightarrow \alpha \in \big[T_\rho, d_{\rho\pi\varepsilon}, u_m\big]\big)\big).$$

Of course, the proposed formula, as it is written, is not only not a Σ^1_1-formula but not even an analytic formula. However, it is not hard to transform it into an equivalent formula of the required form, using the characteristic functions of the sets S, S_0, S_1, S_2, and S_3 (the last set is used to express the relation $\alpha \notin \mathcal{N}_{w_n}$ by means of the formula $\exists i < \operatorname{dom} w_n(\alpha(i) \neq w_n(i))$).

As $\Psi(\varepsilon, \alpha)$ one can take the formula $\forall g\colon \omega \to \{0,1\}$ $([\cdots] \to g(0) = 1)$, transformed in a similar manner, where, in the square brackets, we put the same expression as in Φ.

To complete the proof of Theorem 1, we denote by $\theta(\beta)$ the formula

$$\exists \varepsilon \in \Delta^{1,\pi,\beta}_{1+\rho} \forall \alpha\big(\big(\varphi_1(\beta, \alpha) \to \Psi(\varepsilon, \alpha)\big) \wedge \big(\varphi_2(\beta, \alpha) \to \neg\Phi(\varepsilon, \alpha)\big)\big).$$

By Theorem 2, this formula does, in fact, express separation in the required way (see the hypothesis of Theorem 1). Furthermore, the expression inside the outer parentheses is a Π^1_1-formula with parameters from M, by virtue of the way the formulas φ_1, φ_2, Φ, and Ψ were chosen. The quantifier $\forall \alpha$ does not alter this class. Finally, it is known that a quantifier $\exists \varepsilon \in \Delta^1_1$, applied to a Π^1_1-formula, yields a formula of the same class and with the same set of parameters (see, for example, [19c], p. 288).

§7. Separation of sets of class $\Sigma_1(\mathrm{HC})$

Constituents determined by sieves of a class not higher than $\boldsymbol{\Delta}^1_2$ (and only such sieves are considered in the present paper) are sets of class $\boldsymbol{\Sigma}^1_2$ (see §8), and, therefore, also of class $\Sigma_1(\mathrm{HC})$ by the Translation Lemma of §4. In this connection, it is necessary to carry out an investigation of the phenomenon of separation of sets of the indicated class by means of Borel sets of a fixed order. This is the aim of the present section. The first lemma, however, is not directly connected with separation. Before stating it, for any triple $\pi \in \mathcal{N}$, $\rho \in \mathrm{Ord}$, and $X \subseteq \mathcal{N}$, we denote by $(X)_{\rho\pi}$ the intersection of all sets of the collection $[K_{\rho\pi}]$ (see §6) which include X. It is clear that $X \subseteq (X)_{\rho\pi}$.

LEMMA 3. *In the situation just described, if* $\omega_\rho^{L[\pi]} < \omega_1$ *and* $X \in \Sigma^{\mathrm{HC}}_1(\omega_1 \cup \{\pi\})$, *then* $(X)_{\rho\pi} \in [K_{\rho\pi}]$.

PROOF. Let us show that the family $K_{\rho\pi}(X)$ of all codes $\langle T, d\rangle \in K_{\rho\pi}$ such that $X \subseteq [T, d]$ belongs to the class $L[\pi]$. The lemma is easily obtained from this. In $L[\pi]$, let us index the indicated family: $K_{\rho\pi}(X) = \{\langle T_\xi, d_\xi\rangle\colon \xi < \kappa\}$, where κ is some ordinal. By definition of $K_{\rho\pi}$, every tree T_ξ satisfies the inequality $|T_\xi| \leqslant \rho$. In this situation, reasoning in $L[\pi]$, we can easily construct from the codes $\langle T_\xi, d_\xi \rangle$ a Borel code $\langle T, d \rangle \in L[\pi]$ such that $|T| \leqslant \rho$ and $[T, d] = \bigcap_{\xi<\kappa}[T_\xi, d_\xi]$; that is, $[T, d] = (X)_{\rho\pi}$. However, $[T, d] \in [K_{\rho\pi}]$ by Lemma 2 of §6, which is what we required.

Beginning the proof of the relation $K_{\rho\pi}(X) \in L[\pi]$, we assume, to make it less cumbersome, that the Σ_1-formula which defines our set X in HC contains only one ordinal $\nu < \omega_1$ as a parameter. In that case, there exists a set $P \subseteq \omega_1 \times \mathcal{N}$ of class $\Sigma_1^{\mathrm{HC}}(\{\pi\})$ such that $X = \{\alpha\colon \langle \nu, \alpha \rangle \in P\}$. The set

$$P' = \{\langle \gamma, \alpha \rangle\colon \gamma \in WO \wedge \langle \operatorname{otp}(\gamma), \alpha \rangle \in P\}$$

also belongs to the class $\Sigma_1^{\mathrm{HC}}(\{\pi\})$, and, therefore (by the Translation Lemma of §4), it also belongs to the class $\Sigma_2^{1,\pi}$. Using a construction from the standard proof of the theorem on the partition of a Σ_2^1-set into $\aleph_1$ sets of class Σ_1^1 (see, for example, [19b], p. 794), one can select a Σ_1^1-formula $\varphi(\gamma, \alpha, \delta)$ with parameter π, satisfying the equivalence

$$\forall \gamma \forall \alpha (\langle \gamma, \alpha \rangle \in P' \leftrightarrow \exists \delta \in WO \varphi(\gamma, \alpha, \delta)).$$

In particular, if $\gamma \in WO_\gamma$, then $X = \{\alpha\colon \exists \delta \in WO \varphi(\gamma, \alpha, \delta)\}$.

The next step of the proof of Lemma 3 involves the use of the method of forcing with the set Seq_λ, where $\lambda = \max\{\nu, \omega_\rho^{L[\pi]}\}$, as the set of forcing conditions. The order on Seq_λ is assumed to be reverse of inclusion: $p \leqslant q$ when $q \subseteq p$. The expression $p \leqslant q$ means that the forcing condition q is more informative (that is, forces more formulas) than p. The condition Λ (the empty sequence) is maximal and the least informative.

As the initial model (to be extended), we shall consider the universe V of all sets, in which Lemma 3 is proved, as well as its part $L[\pi] = L^V[\pi]$ consisting of all sets $x \in V$ that are constructible in V relative to π. Usually the technique of forcing "over" the universe V is realized in the form of Boolean-valued models. Here, for the purpose of greater clarity, we shall assume that the universe V is a countable model in some larger universe V^+. Then, for any condition $p \in \mathrm{Seq}_\lambda$, in V^+ there exist sets $G \subseteq \mathrm{Seq}_\lambda$, containing p and Seq_λ-generic over V. However, the translation of all our reasoning into a well-defined language of Boolean-valued models does not present any difficulties.

All the terminology used below, having to do with forcing and generic extensions, is taken from [19a].

Assume $\langle T, d \rangle \in K_{\rho\pi}$. The idea is to prove the equivalence of the inclusion $X \subseteq [T, d]$ in V and the relation $\Lambda \Vdash F^*(T, d)$, where $\Vdash$ denotes forcing corresponding to the initial model $L[\pi]$ and the set of forcing conditions Seq_λ, and $F^*(T, d)$ is the following formula:

$$\forall \gamma \in WO_\nu \forall \delta \in WO \forall \alpha (\varphi^*(\gamma, \alpha, \delta) \to \alpha \in [T^*, d^*]).$$

(In the system of [19a], Chapter 4, $x^* = \mathrm{Seq}_\lambda \times x$ is the term corresponding to the set x in the language of forcing with Seq_λ as the set of forcing conditions. By φ^* we denote the formula obtained from φ by replacing π by π^*.) We claim that

$$X \subseteq [T, d] \text{ in } V \leftrightarrow \Lambda \Vdash F^*(T, d) \tag{1}$$

for any code $\langle T, d \rangle \in K_{\rho\pi}$. From this equivalence it directly follows that $K_{\rho\pi}(X) \in L[\pi]$, since the forcing relation $\Vdash$ (over $L[\pi]$) is expressible in the initial model $L[\pi]$.

For the proof of the implication from left to right in (1), assume $X \subseteq [T, d]$, that is, it is true in V that

$$\forall \gamma \in WO_\nu \forall \delta \in WO \forall \alpha (\varphi(\gamma, \alpha, \delta) \to \alpha \in [T, d]),$$

which we shall denote by $F(T, d)$. Assume the contrary: $\Lambda \Vdash F^*(T, d)$ does not hold. Then there is a condition $p \in \mathrm{Seq}_\lambda$ which forces $\neg F^*(T, d)$. Let us consider any set $G \in V^+$ such that G is Seq_λ-generic over V (and, therefore, also over $L[\pi]$), $G \subseteq \mathrm{Seq}_\lambda$, and G contains p. By the choice of p, the formula $F(T, d)$ will be false in $L[\pi][G]$.

Let us consider a common extension $V[G]$ of the models V and $L[\pi][G]$. Let us verify that $F(T, d)$ is true in $V[G]$. By the absoluteness principle of §4 and the assumption of the truth of $F(T, d)$ in V, it suffices to construct a Π^1_2-formula with parameters from V that is equivalent to $F(T, d)$ in V and in $V[G]$. For the construction of such a formula, we observe that the tree T satisfies in V the inequality $\sup T < \omega_1$, since $\omega_\rho^{L[\pi]} < \omega_1$ by the hypothesis of Lemma 3, while $\langle T, d\rangle \in K_{\rho\pi}$. In this situation, there exists a Σ^1_1-formula $\Phi(\alpha)$ with parameters from V such that $\forall \alpha(\alpha \in [T, d] \leftrightarrow \Phi(\alpha))$ is true in V and in $V[G]$ (see the construction of Φ in §6). As the required Π^1_2-formula one can take the proposition

$$\forall \gamma \forall \delta \forall \alpha\big(\mathrm{eq}(\gamma, \gamma') \wedge \mathrm{wo}(\delta) \wedge \varphi(\gamma, \alpha, \delta) \to \Phi(\alpha)\big),$$

where $\gamma' \in V \cap WO_\nu$ is arbitrary, the Σ^1_1-formula $\mathrm{eq}(\gamma, \gamma')$ expresses in the canonical way the order-isomorphism of the sets Q_γ and $Q_{\gamma'}$ (that is, the membership of γ in WO_ν), and the Π^1_1-formula $\mathrm{wo}(\delta)$ expresses in the canonical way the membership of δ in WO.

Thus, $F(T, d)$ is true in $V[G]$. However, exactly the same kind of reasoning, but starting from the falseness of $F(T, d)$ in $L[\pi][G]$, can show that this formula is, at the same time, false in $V[G]$, which yields the desired contradiction. (The inequality $\sup T < \omega_1^{L[\pi][G]}$, which is necessary for the construction of the required formula Φ with parameters from $L[\pi][G]$, follows here from the choice of λ and the inequality $\lambda < \omega_1^{L[\pi][G]}$. The latter is obtained from the characteristic property of the "collapsing" set of forcing conditions Seq_λ: the function UG belongs to $L[\pi][G]$ and maps ω onto λ, see, for example, [24], §18).

This proves the implication from left to right in (1). The reverse implication is proved in an entirely analogous way.

The principal contribution to the proof of Theorems A2, B2, and C is made by the following lemma, which the author proved earlier in certain special cases (in [19c], §16, and in [22]). It should be noted that the idea of the argument, permitting us to analyze the phenomenon of separation of sets of the class $\Sigma_1^{\mathrm{HC}}(\omega_1 \cup \{\pi\})$ by means of Borel sets of a definite order, is due to Stern (see his note [30], where this idea is used to study transfinite sequences of Borel sets of bounded order in the Lévy-Solovay models).

LEMMA 4. *Assume that $\pi \in \mathcal{N}$, $\rho < \omega_1^{L[\pi]}$, and $X_1, X_2 \subseteq \mathcal{N}$ is a pair of disjoint sets of the class $\Sigma_1^{\mathrm{HC}}(\omega_1 \cup \{\pi\})$ such that the first is $\Pi^0_{1+\rho}$-separable from the second. Then X_1 can be separated from X_2 by a set of the family $[K_{\rho\pi}]$. In particular, by Lemma* 3, *the set $(X_1)_{\rho\pi}$ will separate X_1 from X_2.*

PROOF. As in the proof of the preceding lemma, one can select a pair of ordinals $\nu_i < \omega_1$ (here and below in the proof, the subscript i assumes the values 1 and 2) and a pair of Σ^1_1-formulas $\varphi_i(\gamma, \alpha, \delta)$ with unique parameter π, satisfying the equalities

$$X_i = \{\alpha \in \mathcal{N}: \exists \delta \in WO \varphi_i(\gamma_i, \alpha, \delta)\} \tag{2}$$

for any choice of $\gamma_i \in WO_{\nu_i}$. Let us denote by Ω the first uncountable ordinal $\omega_1 = \omega_1^V$ of the universe V in which Lemma 4 is proved. We set $\lambda = \max\{\nu_1, \nu_2, \Omega\}$ and we consider a set $G \subseteq \mathrm{Seq}_\lambda$ that is Seq_λ-generic over V (and belongs to the larger universe V^+; see the proof of Lemma 3).

By $\Phi_i(\gamma, \alpha, \delta)$ (where $i = 1, 2$) we denote the Σ_1^1-formula

$$\exists \gamma' \exists \delta' \big(\mathrm{eq}(\gamma', \gamma) \wedge \mathrm{Is}(\delta', \delta) \wedge \varphi_i(\gamma', \alpha, \delta')\big),$$

in which $\mathrm{eq}(\gamma', \gamma)$ and $\mathrm{Is}(\delta', \delta)$ are Σ_1^1-formulas which, in canonical fashion, respectively express the order-isomorphism of the sets $Q_{\gamma'}$ and Q_γ (that is, the equality $\mathrm{otp}(\gamma') = \mathrm{otp}(\gamma)$, when $\gamma', \gamma \in WO$) and the order-isomorphism of the set $Q_{\delta'}$ with some initial segment of the set Q_δ that is distinct from Q_δ itself (that is, the inequality $\mathrm{otp}(\delta') < \mathrm{otp}(\delta)$).

By Theorem 1 of §6, there exists a Π_1^1-formula $\theta(\gamma_1, \gamma_2, \delta)$ with parameters from V which is equivalent in V and in $V[G]$ to the assertion of the $\Pi_{1+\rho}^0$-separation of the set $\{\alpha: \Phi_1(\gamma_1, \alpha, \delta)\}$ from $\{\alpha: \Phi_2(\gamma_2, \alpha, \delta)\}$ (the theorem is applied in the universe $V[G]$ to the model $M = V$).

For what follows, we fix a pair of points $\gamma_i \in WO_{\nu_i} \cap V$. By what has just been said, the proposition $\forall \delta(\mathrm{wo}(\delta) \to \theta(\gamma_1, \gamma_2, \delta))$ is a Π_1^1-formula with parameters from V, and this proposition is true in V by virtue of (2). Hence, it is true also in $V[G]$ by Shoenfield's principle in §4.

SUBLEMMA. *There exists a set $d \in L[\pi][G]$, $d \subseteq T_\rho \times \mathrm{Seq}_\omega$ (the construction of the tree T_ρ may be found in §6), such that the following is true in $V[G]$: for any $\delta \in WO_\Omega$, the set $[T_\rho, d]$ separates the set $\{\alpha: \Phi_1(\gamma_1, \alpha, \delta)\}$ from $\{\alpha: \Phi_2(\gamma_2, \alpha, \delta)\}$.*

REMARK ON THE SUBLEMMA. The ordinal Ω is at most countable in the class $V[G]$ and even in $L[\pi][G]$, since $\Omega \leqslant \lambda$ by our choice of λ, and $\lambda < \omega_1^{L[\pi][G]}$ (see the proof of Lemma 3). Thus, the set WO_Ω is nonempty in $V[G]$ and in $L[\pi][G]$. By similar reasoning, the sets WO_{ν_i} are nonempty in $L[\pi][G]$. This fact is used in the proof of the sublemma.

PROOF OF THE SUBLEMMA. We choose in arbitrary fashion $\gamma_i' \in WO_{\nu_i} \cap L[\pi][G]$ and $\delta' \in WO_\Omega \cap L[\pi][G]$. As we saw above, $\theta(\gamma_1, \gamma_2, \delta')$ is true in $V[G]$. By the content of the formulas Φ_i, this implies that $\theta(\gamma_1', \gamma_2', \delta')$ is also true in $V[G]$; that is, it is true in $V[G]$ that the set $\{\alpha: \Phi_1(\gamma_1', \alpha, \delta')\}$ is $\Pi_{1+\rho}^0$-separable from $\{\alpha: \Phi_2(\gamma_2', \alpha, \delta')\}$. Here, Theorem 2 of §6 enables us to select a separating set in the form $[T_\rho, d]$, where $d = d_{\rho\pi\varepsilon}$ for a suitable point ε of class $\Delta_1^{1, \pi, \gamma_1', \gamma_2', \delta'}$. The point ε must belong to $L[\pi][G]$, since $\pi, \gamma_i', \delta' \in L[\pi][G]$. Hence, $d \in L[\pi][G]$. Finally, again by virtue of the content of the formulas Φ_i, one can conclude that the set $[T_\rho, d]$ separates $\{\alpha: \Phi_1(\gamma_1, \alpha, \delta)\}$ from $\{\alpha: \Phi_2(\gamma_2, \alpha, \delta)\}$, which is what was required.

Continuing the proof of Theorem 4, we consider the set d given by the sublemma. This set belongs to the generic extension $L[\pi][G]$ of the class $L[\pi]$. Hence, there exists a set $t \in L[\pi]$, $t \subseteq \mathrm{Seq}_\lambda \times (T_\rho \times \mathrm{Seq}_\omega)$, such that

$$d = I_G(t) = \{\langle u, w\rangle : \exists p \in G(\langle p, u, w\rangle \in t)\}$$

(see [19a], Chapter 4, Lemma 2.5). Denoting by $\Vdash$ the forcing that now corresponds to the initial model V and the set of forcing conditions Seq_λ, we find a condition $p_0 \in G$ such that

$$p_0 \Vdash \forall \delta \in WO_{\Omega^*} \big(\text{the set } [T_\rho^*, t] \text{ separates}$$
$$\{\alpha: \Phi_1^*(\gamma_1^*, \alpha, \delta)\} \text{ from } \{\alpha: \Phi_2^*(\gamma_2^*, \alpha, \delta)\}\big). \tag{3}$$

(Here, as in the proof of Lemma 3, $x^* = \mathrm{Seq}_\lambda \times x$ for every $x \in V$, and Φ_i^* is obtained from Φ_i by replacing the parameter π by π^*.) For every pair $u \in T_\rho$, $p \in \mathrm{Seq}_\lambda$, we define

$$Z_{up} = \{\alpha \in \mathcal{N} \cap V: p \Vdash \alpha^* \in [T_\rho^*, t, u^*]\}.$$

Using classical lemmas on the forcing of formulas of various logical structures ([19a], Chapter 4, §2), together with the definition of the sets $[T, d, u]$ in §6, it is not hard to obtain the following system of equalities, relating the sets Z_{up} in the universe V:

$$Z_{up} = \neg \bigcup_{w \in S_{up}} \mathcal{N}_w \quad \text{for } |u|_{T_\rho} = 0, \tag{4}$$

where $S_{up} = \{w \in \mathrm{Seq}_\omega \colon \exists q,\, r \in \mathrm{Seq}_\lambda (r \leqslant p,\, q \wedge \langle q, u, w\rangle \in t)\}$;

$$Z_{up} = \neg \bigcup_{q \leqslant p,\, u^\wedge \xi \in T_\rho} Z_{u^\wedge \xi, q} \quad \text{for } |u|_{T_\rho} \geqslant 1. \tag{5}$$

Since $t \in V$, these equalities incidentally show that the indexed family of sets Z_{up} belongs to V. We shall deduce one more important assertion:

The set $Z_{\Lambda p_0}$ separates X_1 from X_2 in the universe V. (6)

In fact, assume $\alpha \in X_1$. To verify that $\alpha \in Z_{\Lambda p_0}$, we fix any set H that contains p_0 and is Seq_λ-generic over V, and we shall prove that $\alpha \in [T_\rho, I_H(t)]$ in $V[H]$. By (2) and the choice of the points $\gamma_i \in WO_{\nu_i} \cap V$, there exists a point $\delta' \in WO \cap V$ such that the proposition $\varphi_1(\gamma_1, \alpha, \delta')$ is true in V. Then, by Shoenfield's principle in §4, this proposition turns out to be true in $V[H]$ also. We select in an arbitrary way a point $\delta \in V[H] \cap WO_\Omega$. We have $\mathrm{otp}(\delta) = \Omega$ and, at the same time, $\mathrm{otp}(\delta') < \Omega$, since $\delta' \in V$, and $\Omega = \omega_1{}^V$. Hence, $\Phi_1(\gamma_1, \alpha, \delta)$ is true in $V[H]$. However, $p_0 \in H$, and so by (3) we must obtain $\alpha \in [T_\rho, I_H(t)]$ in $V[H]$, which is what was required.

This argument proves the inclusion $X_1 \subseteq Z_{\Lambda p_0}$. In a similar way, one proves that $X_2 \cap Z_{\Lambda p_0} = \varnothing$.

The set $t \in L[\pi]$ and the family of sets Z_{up}, which belongs to V and satisfies in the universe V the requirements (4), (5) and (6)—this is all that we need to complete the proof of Lemma 4. From this point on, all the reasoning is carried out within the universe V of Lemma 4. By (6), it actually suffices to verify that $Z_{\Lambda p_0} \in [K_{\rho\pi}]$. To this end, we construct in $L[\pi]$, by induction on the ordinal $|u| = |u|_{T_\rho}$, a family of Borel codes $\langle T_{up}, d_{up}\rangle \in L[\pi]$ (where $u \in T_\rho$ and $p \in \mathrm{Seq}_\lambda$) with trees T_{up} such that $|T_{up}| = |u|$, and satisfying the equalities $Z_{up} = [T_{up}, d_{up}]$ in V. From this, for $u = \Lambda$ and $p = p_0$, Lemma 2 of §6 enables us to obtain the required result.

If $|u| = 0$, then we take $T_{up} = \{\Lambda\}$ and $d_{up} = \{\langle \Lambda, w\rangle \colon w \in S_{up}\}$; the equality $Z_{up} = [T_{up}, d_{up}]$ follows from (5).

If $|u| \geqslant 1$, then, fixing a bijection f of the set

$$E = \left\{\langle q, \xi\rangle \colon q \in \mathrm{Seq}_\lambda \wedge q \leqslant p \wedge \xi \in \mathrm{Ord} \wedge u^\wedge \xi \in T_\rho\right\}$$

on some ordinal κ ($f \in L[\pi]$), we define

$$T_{up} = \{\Lambda\} \cup \left\{f(q, \xi)^\wedge v \colon \langle q, \xi\rangle \in \mathrm{E} \wedge v \in T_{u^\wedge \xi, q}\right\};$$

$$d_{up} = \left\{\left\langle f(q, \xi)^\wedge v, w\right\rangle \colon \langle q, \xi\rangle \in \mathrm{E} \wedge \langle v, w\rangle \in d_{u^\wedge \xi, q}\right\}.$$

(Remark: if $u^\wedge \xi \in T_\rho$, then $|u^\wedge \xi| < |u|$.) The required equality $Z_{up} = [T_{up}, d_{up}]$ is guaranteed by the inductive hypothesis and (5). This concludes the proof of Lemma 4.

The proofs of Lemmas 3 and 4 presented here leave a somewhat strange impression through their use of the method of forcing, not for consistency proofs, but for the derivation of "positive" assertions about sets. Certain parallels between these arguments and constructions in the proof of Theorem 2 of §6 in Louveau's paper [32] indicate that a search for proofs of Lemmas 3 and 4 without use of the method of forcing would be promising.

Now, from the lemmas that were just proved, we shall obtain two corollaries that are directly connected with the proofs of Theorems A2, B2, and C.

COROLLARY 1. *Assume $\pi \in \mathcal{N}$, $\rho < \omega_1$, and $\omega_{\rho+1}^{L[\pi]} < \omega_1$. Then every family consisting of sets lying in $\mathcal{N}$ that are pairwise $\Pi^0_{1+\rho}$-separable and are in the class $\Sigma_1^{\mathrm{HC}}(\omega_1 \cup \{\pi\})$ is at most countable.*

PROOF. Let F be a family of the indicated form. By Lemma 4, all sets of the form $(X)_{\rho\pi}$, where $X \in F$, are pairwise distinct, and by Lemma 3 all such sets belong to the collection $[K_{\rho\pi}]$. However, as can easily be seen, the latter has cardinality $\omega_{\rho+1}^{L[\pi]}$ in $L[\pi]$; that is, by the hypothesis of the corollary, it is countable in the universe of all sets. From this the desired result is obvious.

COROLLARY 2. *Assume that $\pi \in \mathcal{N}$ and the set $X \subseteq \mathcal{N}$ has class $\Delta_1^{\mathrm{HC}}(\omega_1 \cup \{\pi\})$. Then $X \subseteq L[\pi]$.*

PROOF. Both of the sets X and $Y = \mathcal{N} - X$ belong to $\Sigma_1^{\mathrm{HC}}(\omega_1 \cup \{\pi\})$; moreover, the second is Π^0_2-separable from the first. Applying Lemma 4 with $\rho = 1$, we find a Borel code $\langle T, d\rangle \in K_{1\pi}$ such that the set $[T, d]$ separates Y from X; that is, it actually coincides with Y. By the definition of $K_{\rho\pi}$ in §6, the inequality $|T| \leqslant 1$ holds; moreover, one may assume that $|T| = 1$, since the case $|T| = 0$ corresponds to the case where the set X is open, and, therefore, by virtue of its countability, empty. Under this assumption, denoting by U the set of all ordinals ξ such that $\langle \xi\rangle \in T$, and defining

$$S_\xi = \{w: \langle\langle\xi\rangle, w\rangle \in d\}, \qquad X_\xi = \neg \bigcup_{w \in S_\xi} \mathcal{N}_w$$

for every $\xi \in U$, we obtain $X = \bigcup_{\xi \in U} X_\xi$. It remains to verify that $X_\xi \subseteq L[\pi]$ for any ξ. To this end we note that, by the definition of $K_{\rho\pi}$, $d \in L[\pi]$, and so $S_\xi \in L[\pi]$. This implies that every X_ξ is, by virtue of the countability of X, a countable set of class Π_1^{0,π_ξ} for a suitable $\pi_\xi \in \mathcal{N} \cap L[\pi]$. (One can take as π_ξ, say, the characteristic function of the set $\{n \in \omega: w_n \in S_\xi\}$, where w_n denotes the nth element of the set Seq_ω with respect to some fixed recursive enumeration of that set.) In this case, as is well known (see, for example, [19c], Corollary 4.12), the set X_ξ consists only of points of the class Δ_1^{1,π_ξ}, and thus the inclusion $X_\xi \subseteq L[\pi]$ holds.

§8. Proof of Theorems A2 and B2

In order to apply the results of the preceding section to the study of the constituents of sieves of class Δ^1_2, it is necessary to evaluate the definability of constituents. To this end, we shall prove the following lemma.

LEMMA 5. *Let C be a sieve of class Δ^1_2. Then there exists a point $\pi \in \mathcal{N}$ such that all the constituents $[C]_\nu$ and $[C]_{*\nu}$ of C have class $\Delta_1^{\mathrm{HC}}(\omega_1 \cup \{\pi\})$.*

PROOF. Let us recall that, by Proposition 2 of §4, the cross-section function $\alpha \mapsto C/\alpha$ of the sieve C is a function of class $\Delta_1^{\mathrm{HC}}(\mathcal{N})$. Hence, there exists a point $\pi \in \mathcal{N}$ such that this function has class $\Delta_1^{\mathrm{HC}}(\{\pi\})$.

Let us introduce the following three formulas:

$$\Psi_1(f,\mu,X) \leftrightarrow \mu \text{ is an ordinal} \wedge X \subseteq \mathbf{Q} \wedge f \text{ is an order-preserving bijection of } \mu \text{ onto } X;$$
$$\Psi_2(X,Y) \leftrightarrow X \subseteq Y \subseteq \mathbf{Q} \wedge X \text{ is an initial segment of } Y;$$
$$\Psi_3(X,Y) \leftrightarrow X \subseteq Y \subseteq \mathbf{Q} \wedge \text{the difference } Y - X \text{ is nonempty and has a least element.}$$

It is obvious that these formulas are bounded (or Δ_0-formulas (see [19a], p. 408); that is, these formulas can be rewritten so that all their quantifiers have the form $\exists x \in y$ or $\forall x \in y$). Now the lemma is yielded by the following easily verifiable equivalences:

$$\alpha \in [C]_\nu \leftrightarrow \exists f \Psi_1(f,\nu,C/\alpha) \leftrightarrow \forall f \forall X \forall \mu \leqslant \nu(\Psi_1(f,\mu,X) \wedge \Psi_2(X,C/\alpha) \rightarrow (\mu = \nu \wedge X = C/\alpha) \vee (\mu < \nu \wedge \Psi_3(X,C/\alpha)));$$

$$\alpha \in [C]_{*\nu} \leftrightarrow \exists f \exists X(\Psi_1(f,\nu,X) \wedge \Psi_2(X,C/\alpha) \wedge (C/\alpha - X \neq \varnothing) \wedge \neg\Psi_3(X,C/\alpha)) \leftrightarrow \forall f \forall X \forall \mu \leqslant \nu(\Psi_1(f,\mu,X) \wedge \Psi_2(X,C/\alpha) \rightarrow (\mu = \nu \wedge (C/\alpha - X \neq \varnothing) \wedge \neg\Psi_3(X,C/\alpha)) \vee (\mu < \nu \wedge \Psi_3(X,C/\alpha))).$$

The quantifiers $\exists f$, $\exists X$, $\forall f$ and $\forall X$ in these formulas can clearly be restricted to the set HC.

By means of Lemma 5 and the results of §7, the proof of Theorems A2 and B2 does not present any difficulties. In general, Theorem B2 follows automatically from Corollary 2 of §7, since the union of the external and internal constituents gives all points of $\mathcal{N}$.

Let us turn to Theorem A2. If we assume that there exists a sieve that ensures the truth of one of the assertions referred to in that theorem, then, by Corollary 1 of §7 and Lemma 5, there would exist a point $\pi \in \mathcal{N}$ and an ordinal $\rho < \omega_1$ such that $\omega_{\rho+1}^{L[\pi]} \geqslant \omega_1$. On the other hand, it is known [26] that if the inequality $\omega_1^{L[\pi]} < \omega_1$ holds for all $\pi \in \mathcal{N}$, then for all π and for all $\xi < \omega_1$ the inequality $\omega_\xi^{L[\pi]} < \omega_1$ also will hold, which would contradict $\omega_{\rho+1}^{L[\pi]} \geqslant \omega_1$.

It is clear that our argument can be applied not only to the study of sequences of constituents of sieves of class Δ^1_2, but also to the larger class of Σ_1(HC)-definable sequences. (Naturally, any sequence $\langle X_\nu\colon \nu < \omega_1\rangle$ of sets $X_\nu \subseteq \mathcal{N}$ is said to be Σ_1(HC)-*definable* if the set $\{\langle \nu,\alpha\rangle\colon \nu < \omega_1 \wedge \alpha \in X_\nu\}$ belongs to the class Σ_1(HC). Such are, by Lemma 5, all sequences of constituents of sieves of class Δ^1_2.) Here it is actually proved that the existence of a point $\pi \in \mathcal{N}$ satisfying the equality $\omega_1^{L[\pi]} = \omega_1$ follows from the existence of a Σ_1(HC)-definable sequence from whose terms one can select a subfamily of nonempty sets possessing the property of order-bounded pairwise separation (see §2).

§9. Proof of Theorem C

The initial form of the proof of this theorem, presented in [19c], §19, and in [22], was connected with the principle of Borel determinacy. Here we shall present another method of proof, based on the theorem about Π^1_1-expression in §6. Let us assume the contrary: there exists a sieve satisfying one of the requirements [111, IV], [111, IV]$_*$ or [111, IV]$_+$; we shall show that this yields a contradiction.

We shall begin with a derivation of a contradiction from the assumption of the existence of a sieve of the form [111, IV]. So, assume that $\rho < \omega_1$ and all constituents $[C]_\nu$ given by a Borel sieve $C = \langle C_q\colon q \in \mathbf{Q} \rangle$ are nonempty and pairwise $\Pi^0_{1+\rho}$-separable. Having in mind some recursive enumeration $\mathbf{Q} = \{q_k\colon k \in \omega\}$ of the set $\mathbf{Q}$ of all rational numbers, let us consider the set $\{\langle \alpha, k \rangle\colon \alpha \in C_{q_k}\}$. This set is Borel in the space $\mathcal{N} \times \omega$, since all C_q are Borel sets. Therefore, one can select a point $\pi \in \mathcal{N}$ as well as a Σ^1_1-formula $\varphi(\alpha, k)$ and a Π^1_1-formula $\psi(\alpha, k)$, both with π as sole parameter, so that

$$\forall \alpha \forall k \big(\alpha \in C_{q_k} \leftrightarrow \varphi(\alpha, k) \leftrightarrow \psi(\alpha, k)\big). \tag{7}$$

We set $\lambda = \omega^{L[\pi]}_{\rho+1}$ and fix a set $G \subseteq \mathrm{Seq}_\lambda$ (see the proof of Lemma 3 in §7) that is Seq_λ-generic over V (that is, over the universe in which Theorem C is proved). We introduce an analogue $C^{\#}$ of the sieve C in the model $V[G]$, setting

$$C^{\#}_{q_k} = \{\alpha\colon \varphi(\alpha, k)\} = \{\alpha\colon \psi(\alpha, k)\}$$

in $V[G]$ for every k (that this is well defined is guaranteed by the fact that (7) holds in the universe V and by Shoenfield's principle in §4).

Let us define $\omega^{\#}_1 = \omega_1^{V[G]}$. To obtain the desired contradiction, it suffices to verify that the sieve $C^{\#}$ has in $V[G]$ the same property that the sieve C has in V; that is, that all constituents $[C^{\#}]_\nu$, $\nu < \omega^{\#}_1$, are nonempty and pairwise $\Pi^0_{1+\rho}$-separable. In fact, if this property of C is proved, then, by Lemma 5 of §8, the constituents $[C^{\#}]_\nu$ (constructed in $V[G]$) form in $V[G]$ an uncountable family of nonempty pairwise $\Pi^0_{1+\rho}$-separable sets of class $\Sigma^{\mathrm{HC}}_1(\omega_1 \cup \{\pi\})$, in view of which $\omega^{L[\pi]}_{\rho+1} \geqslant \omega^{\#}_1$ by Corollary 1 of §7. However, by the definition of λ and the choice of G, we have exactly the reverse inequality $\omega^{L[\pi]}_{\rho+1} = \lambda < \omega^{\#}_1$.

The proof of the indicated property of the sieve $C^{\#}$ in $V[G]$ consists in expressing this property by a special Π^1_2-formula with parameters from V, whose truth in $V[G]$ will be obtained by means of Shoenfield's principle. First of all, there is a Σ^1_1-formula $\Phi(\gamma, \alpha)$ with parameter π such that the assertion

$$\forall \alpha \forall \gamma \in WO\big(\alpha \in [C]_{\mathrm{otp}(\gamma)} \leftrightarrow \Phi(\gamma, \alpha)\big),$$

is true in the universe V, and the analogous assertion for $C^{\#}$ is true in $V[G]$. (We can take, for example, a Σ^1_1-formula that canonically expresses the existence of an order-isomorphism of the sets C/α and Q_γ; the occurrence of C/α in this formula can be eliminated by means of the formulas φ and ψ that were introduced above. On the construction of such formulas, see [22], §2.) By Theorem 1 of §6, there exists a Π^1_1-formula $\theta(\gamma_1, \gamma_2)$ with parameters from V that expresses in V and in $V[G]$ the assertion of the $\Pi^0_{1+\rho}$-separability of the set $\{\alpha\colon \Phi(\gamma_1, \alpha)\}$ from the set $\{\alpha\colon \Phi(\gamma_2, \alpha)\}$. Now the Π^1_2-proposition

$$\begin{aligned}&\forall \gamma(\mathrm{wo}(\gamma) \to \exists \alpha \Phi(\gamma, \alpha))\\ &\quad \wedge \forall \gamma_1 \forall \gamma_2(\mathrm{wo}(\gamma_1) \wedge \mathrm{wo}(\gamma_2) \wedge \neg \mathrm{eq}(\gamma_1, \gamma_2) \to \theta(\gamma_1, \gamma_2))\end{aligned}$$

with parameters from V (concerning the formulas wo and eq, see the proof of Lemma 3 in §7) is equivalent in V to the assertion of the nonemptiness and pairwise $\Pi^0_{1+\rho}$-separability of all constituents $[C]_\nu$, $\nu < \omega_1$, and is equivalent in $V[G]$ to the analogous assertion about the constituents $[C^{\#}]_\nu$, where $\nu < \omega^{\#}_1$, of the sieve $C^{\#}$.

But by our choice of C this proposition is true in V. Hence it is also true in $V[G]$ by Shoenfield's absoluteness principle of §4, which is what was required.

The derivation of a contradiction from the assertion of the existence of a sieve of the form [101, IV]$_*$ differs from the preceding argument only in that it is necessary (when proving the truth in $V[G]$) to express by a Π^1_2-formula not an assertion of the nonemptiness of all $[C^{\#}]_\nu$, but rather an assertion of the uncountability of the set of all nonempty internal constituents $[C^{\#}]_{*\nu}$ (under the assumption that this assertion holds in V for the sieve C). In order to construct the desired Π^1_2-formula, we introduce a Π^1_1-formula $\Psi(\alpha)$ with parameter π that canonically expresses the fact that the set C/α is well-ordered in V, or that the set $C^{\#}/\alpha$ is well-ordered in $V[G]$. We also introduce a Σ^1_1-formula $\Phi^*(\gamma, \alpha)$ with parameter π that canonically expresses the existence of an order-isomorphism of the set Q_γ onto some initial segment of the set C/α (or the set $C^{\#}/\alpha$). Thus, the relations

$$\forall \alpha(\alpha \in [C]_* \leftrightarrow \neg\Psi(\alpha))$$

and

$$\forall \alpha \forall \gamma \in WO\Big(\alpha \in \bigcup_{\nu > \mathrm{otp}(\gamma)} [C]_{*\nu} \leftrightarrow \neg\Psi(\alpha) \wedge \Phi^*(\gamma, \alpha)\Big),$$

are true in the universe V, while analogous relations are true for $C^{\#}$ in $V[G]$. Now the desired Π^1_2-formula can be written as follows:

$$\forall \gamma\big(\mathrm{wo}(\gamma) \rightarrow \exists \alpha(\neg\Psi(\alpha) \wedge \Phi^*(\gamma, \alpha))\big).$$

Finally, to derive a contradiction from the assertion of the existence of a sieve satisfying the requirements [101, IV]$_+$, it suffices to show that such a sieve, if it existed, would also have to satisfy [101, IV]$_*$, which is impossible by the argument that has just been presented. In turn, to prove the implication [101, IV]$_+ \rightarrow$ [101, IV]$_*$, it is sufficient to verify that, if a Borel sieve C has uncountably many nonempty external constituents, then the number of nonempty internal constituents $[C]_{*\nu}$ also will be uncountable.

Let us prove this. If $[C]_\nu \neq \varnothing$ for uncountably many indices ν, then, by the criterion mentioned in §1, the set $[C]$ will not be Borel. Its complement $[C]_* = \mathscr{N} - [C]$ will also, naturally, not be Borel. However, $[C]_*$ coincides with the union of all internal constituents $[C]_{*\nu}$, and each of the latter is a Borel set (see the reference in §1). Obviously, this yields what is required.

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