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LARGE SYSTEMS

Geometric Relationship between Parallel Hyperplanes, Quadrics, and Vertices of a Hypercube

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Abstract—In a space of dimension 30 we find a pair of parallel hyperplanes, uniquely determined by vertices of a unit cube lying on them, such that strictly between the hyperplanes there are no vertices of the cube, though there are integer points. A similar two-sided example is constructed in dimension 37. We consider possible locations of empty quadrics with respect to vertices of the cube, which is a particular case of a discrete optimization problem for a quadratic polynomial on the set of vertices of the cube. We demonstrate existence of a large number of pairs of parallel hyperplanes such that each pair contains a large number of points of a prescribed set.

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1. PROBLEM SETTING

Our results are valid for any linearly ordered field, but for brevity we will speak about the real field $\mathbb{R}$. A *hypercube* (or simply a *cube*) is a polyhedron in a space of dimension n whose vertices have coordinates 0 or 1. The *weight of a vertex* of a cube is the number of its unit coordinates. A *quadric* in $\mathbb{R}^n$ is the set of zeros of a real (possibly, reducible) quadratic polynomial $f(x_1, \dots, x_n)$. A quadric $f = 0$ is said to be *empty* if values of the polynomial f at vertices of the cube are either all nonnegative or all nonpositive. In what follows, we assume the first case. We consider the structure of the set of minimum points of a quadratic polynomial f on the set of all vertices of the cube, i.e., location of an empty quadric with respect to the cube. A particular case of a quadric is a pair of parallel hyperplanes. In more detail, we consider the possibility of special arrangements of parallel hyperplanes with respect to vertices of the cube or to an arbitrary set of points in general position in an n-space.

Finding the minimum of a general quadratic polynomial on the set of vertices of the cube is an algorithmically hard problem. Efficient algorithms—for instance, the pseudo-Boolean programming method [1]—are applicable in particular cases only. In [2, 3], an overview of heuristic algorithms is presented. A particular case of optimization problems is the case of problems with separated variables. In [4] (see also references therein), a multicriteria minimax problem is considered where optimization of quadratic forms is performed over sets of vertices of two unit cubes of different dimensions. Results of [5] imply that vertices of the cube lying on an empty quadric lie on a empty cylinder (non-full-rank quadric) that does not contain other vertices. A degenerate case consists in describing vertices of the cube that lie on a pair of coinciding hyperplanes.

Many papers (see, e.g., [6–9]) give estimates for the number of vertices lying on a quadric. A hyperplane contains at most half of all vertices of the cube. However, if the corresponding linear function depends on each of the n variables nontrivially, the fraction of vertices of the cube lying on this hyperplane tends to zero with growing n [6]. A similar result was obtained for a quadratic polynomial with sufficiently many monomials: the fraction of vertices of the cube at which such a polynomial takes a fixed value tends to zero with growing dimension of the cube [7]. An estimate for the number of vertices of the cube lying in a half-space is given in [8].

An important discrete optimization problem is as follows: Is a given vertex of the cube nearest to a given hyperplane? In other words, we are given parallel hyperplanes, the original one and a parallel hyperplane passing through the given vertex, and it is required to determine whether there is at least one vertex of the cube lying strictly between them. Since a pair of hyperplanes can be represented as the zero set of a reducible quadratic polynomial, this problem is a particular case of the quadratic polynomial minimization problem. To find the minimum, it is important to know how minimum points can be located among vertices of the cube [10]. In [11], coverings of vertices of the n-cube by hyperplanes are considered. The following result is proved: For any m hyperplanes ($m \le n$) that do not cover all the vertices of the cube, at least 2^{n-m} vertices of the cube remain uncovered.

Search for (or proving nonexistence of) a vertex of the cube lying strictly between two hyperplanes given by equations with integer coefficients can be performed by means of a dynamic programming algorithm proposed in [12]. For an overview of such algorithms, see [13]. In this problem, the complexity of the dynamic programming algorithm depends polynomially on the dimension of the space but rapidly grows with the number of digits in coefficients of equations. Such algorithms are said to be pseudopolynomial. At the same time, mutually coprime integer coefficients of an equation specifying a hyperplane can be large, which reduces the efficiency of the dynamic programming algorithm.

It is much easier to determine whether there is an integer point strictly between two parallel hyperplanes given by equations with integer coefficients. Here the runtime is determined by the complexity of the Euclidean algorithm for finding the greatest common divisor; as is well known, this runtime is bounded by a polynomial of the total number of digits in the coefficients of the equation. In low dimensions, if one of the parallel hyperplanes is uniquely determined by vertices of the cube lying on it, nonexistence of vertices of the cube lying strictly between the hyperplanes is equivalent to nonexistence of integer points strictly between them. This is easily verified in dimensions two and three by exhaustive search over a small number of variants. In what other dimensions does this hold? Theorem 1 shows that in dimension 30 this is not the case (apparently, the result of the theorem can be strengthened). It is not clear whether checking the nonexistence of vertices of the cube between parallel hyperplanes can be reduced to checking the nonexistence of integer points between them in intermediate dimensions. Thus, Theorem 1 illustrates the computational complexity of the considered problem. An example in this theorem possesses an interesting arithmetic structure, which could be of use when searching for analogous examples.

The problem of reconstructing a matrix given a set of its entries or relations between them often arises in analysis of incomplete data and in factor analysis. This is an applied problem occurring in image recognition, NMR analysis of molecule structures, and data filtering design in computer networks. Efficiency of matrix reconstruction algorithms is usually based on the assumption of uniqueness of reconstruction. Problems of finding a symmetric matrix given relations between its entries are considered in [14,15]. A pair of parallel hyperplanes corresponds to a symmetric matrix of rank one, i.e., matrix of a quadratic form.

2. EXAMPLE OF THE GEOMETRIC RELATIONSHIP BETWEEN PARALLEL HYPERPLANES, VERTICES OF THE CUBE, AND INTEGER POINTS IN AN n-SPACE

Theorem 1. (a) *In the space $\mathbb{R}^{30}$ there exist a hyperplane H uniquely determined by vertices of the cube lying on it and a parallel hyperplane H' containing at least one vertex of the cube such that the open domain between H and H' contains no vertices of the cube but contains integer points.*

Table 1

	t	u_1	u_2	u_3	u_4	u_5	x_1	y_1	z_1
$\boldsymbol{v}$	1	1	0	0	0	0	1	1	1
$\boldsymbol{v}'$	1	0	1	0	0	0	1	1	1
$\boldsymbol{v}''$	1	0	0	1	0	0	1	1	1
$\boldsymbol{v}'''$	1	0	0	0	1	0	1	1	1
$\boldsymbol{v}''''$	1	0	0	0	0	1	1	1	1
$\boldsymbol{w}$	0	1	1	1	1	1	1	1	1

(b) *In the space $\mathbb{R}^{37}$ there exists a hyperplane H uniquely determined by vertices of the cube lying on it such that the distance from H to the nearest vertex of the cube that does not lie on H is strictly greater than the distance to the nearest integer point that does not lie on H.*

Proof. (a) Consider a linear form in 30 variables

$$f = 1260t + 315(u_1 + \ldots + u_5) + 252(x_1 + \ldots + x_6) + 180(y_1 + \ldots + y_8) + 140(z_1 + \ldots + z_{10}).$$

Its coefficients can be expanded into (pairwise coprime) factors 4, 5, 7, and 9: $1260 = 4 \cdot 5 \cdot 7 \cdot 9$, $315 = 5 \cdot 7 \cdot 9$, $252 = 4 \cdot 7 \cdot 9$, $180 = 4 \cdot 5 \cdot 9$, and $140 = 4 \cdot 5 \cdot 7$. Thus, one variable has coefficient $4 \cdot 5 \cdot 7 \cdot 9$, and the number of coefficients of the form $\dfrac{4 \cdot 5 \cdot 7 \cdot 9}{p}$ is $p + 1$.

Let a hyperplane H be given by $f = 2147$, where

$$2147 = 1260 + 315 + 252 + 180 + 140.$$

Let us show that vertices of the cube that lie on H generate the whole space $\mathbb{R}^{30}$. Consider vertices $\boldsymbol{v}$, $\boldsymbol{v}'$, $\boldsymbol{v}''$, $\boldsymbol{v}'''$, $\boldsymbol{v}''''$, and $\boldsymbol{w}$ lying on the hyperplane $f = 2147$. Their coordinates are given in Table 1, where rows correspond to the vertices, and columns to coordinates (unspecified coordinates of all these vertices are zero).

The linear combination $\boldsymbol{a} = \boldsymbol{v} + \boldsymbol{v}' + \boldsymbol{v}'' + \boldsymbol{v}''' + \boldsymbol{v}'''' - \boldsymbol{w}$ has only four nonzero coordinates: $a_t = 5$ and $a_{x_1} = a_{y_1} = a_{z_1} = 4$. Similarly, there exist other linear combinations of vertices with exactly four nonzero coordinates: $\boldsymbol{b}$ with $b_t = 6$ and $b_{u_1} = b_{y_1} = b_{z_1} = 5$, $\boldsymbol{c}$ with $c_t = 8$ and $c_{u_1} = c_{x_1} = c_{z_1} = 7$, and $\boldsymbol{d}$ with $d_t = 10$ and $d_{u_1} = d_{x_1} = d_{y_1} = 9$. The vertices $\boldsymbol{v}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are linearly independent. Indeed, one easily checks that

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 5 & 0 & 4 & 4 & 4 \\ 6 & 5 & 0 & 5 & 5 \\ 8 & 7 & 7 & 0 & 7 \\ 10 & 9 & 9 & 9 & 0 \end{pmatrix}$$

is a full-rank matrix. Hence, one can compose linear combinations of $\boldsymbol{v}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ such that only one coordinate among t, u_1, x_1, y_1, and z_1 is nonzero.

Other cases differ only by permutations of coordinate indices, which preserve the form f. Hence, for each of the coordinates there exists a linear combination of vertices of the cube lying on H for which only this coordinate is nonzero. Hence, vertices of the cube satisfying the equation $f = 2147$ lie on a unique hyperplane.

Since the greatest common divisor of all coefficients of the linear form f is 1, this form takes all possible integer values at integer points. Let us prove by contradiction that this form never takes the value 2146 at vertices of the cube. Let $\boldsymbol{v}$ be such that $f(\boldsymbol{v}) = 2146$. Since $2146 \equiv 2 \pmod 4$,

Table 2

	t	u_1	u_2	u_3	u_4	u_5	u_6	x_1	y_1	z_1
$\boldsymbol{v}$	1	1	0	0	0	0	0	1	1	1
$\boldsymbol{v}'$	1	0	1	0	0	0	0	1	1	1
$\boldsymbol{v}''$	1	0	0	1	0	0	0	1	1	1
$\boldsymbol{v}'''$	1	0	0	0	1	0	0	1	1	1
$\boldsymbol{v}''''$	1	0	0	0	0	1	0	1	1	1
$\boldsymbol{v}'''''$	1	0	0	0	0	0	1	1	1	1
$\boldsymbol{w}$	0	1	1	1	1	1	1	1	1	1

$315 \equiv 3 \pmod 4$, and the other coefficients are divisible by 4, equality of residues modulo 4 occurs only if $v_{u_1} + v_{u_2} + v_{u_3} + v_{u_4} + v_{u_5} = 2$. Similarly, $2146 \equiv 1 \pmod 5$ and $252 \equiv 2 \pmod 5$, whence $v_{x_1} + \ldots + v_{x_6} = 3$. Similarly, $2146 \equiv 4 \pmod 7$ and $180 \equiv 5 \pmod 7$, whence $v_{y_1} + \ldots + v_{y_8} = 5$. Similarly, $2146 \equiv 4 \pmod 9$ and $140 \equiv 5 \pmod 9$, whence $v_{z_1} + \ldots + v_{z_{10}} = 8$. But then the sum of the last four terms of the form f is $315 \cdot 2 + 252 \cdot 3 + 180 \cdot 5 + 140 \cdot 8 = 3406 > 2146$, a contradiction.

Since $f(\mathbf{0}) = 0 < 2147$, there exist hyperplanes parallel to H and passing through vertices of the cube in the corresponding half-space. Denote by H' the hyperplane nearest to H among such hyperplanes. Between H and H' there are no vertices of the cube but there are integer points.

Remark 1. On the other hand, the form attains the value 2148 at each of the vertices $\boldsymbol{v}$ satisfying $v_t = v_{u_1} = v_{u_2} = v_{u_3} = v_{u_4} = v_{u_5} = 0$, $v_{x_1} + \ldots + v_{x_6} = 4$, $v_{y_1} + \ldots + v_{y_8} = 4$, and $v_{z_1} + \ldots + v_{z_{10}} = 3$. Indeed, by substituting these equalities into f, we obtain

$$f(\boldsymbol{v}) = 252 \cdot 4 + 180 \cdot 4 + 140 \cdot 3 = 2148.$$

Hence, it is *not* true that the distance from H to the nearest vertex of the cube that does not lie on H is strictly greater than the distance to the nearest integer point that does not lie on H.

Remark 2. Statement (a) of Theorem 1 becomes obvious if we do not require that both hyperplanes H and H' contain vertices of the cube.

Remark 3. This statement does not hold in spaces of low dimensions. For instance, for $\mathbb{R}^3$ and $\mathbb{R}^4$ this can easily be shown by exhaustive search over all possible variants. It would be interesting to find the lowest dimension for which the statement is true.

(b) Consider a linear form in 37 variables

$$f = 3465t + 693(u_1 + \ldots + u_6) + 495(x_1 + \ldots + x_8) + 385(y_1 + \ldots + y_{10}) + 315(z_1 + \ldots + z_{12}).$$

Its coefficients can be expanded into (pairwise coprime) factors 5, 7, 9, and 11: $3465 = 5 \cdot 7 \cdot 9 \cdot 11$, $693 = 7 \cdot 9 \cdot 11$, $495 = 5 \cdot 9 \cdot 11$, $385 = 5 \cdot 7 \cdot 11$, and $315 = 5 \cdot 7 \cdot 9$. Thus, one variable has coefficient $5 \cdot 7 \cdot 9 \cdot 11$, and the number of variables with a coefficient of the form $\dfrac{5 \cdot 7 \cdot 9 \cdot 11}{p}$ is $p + 1$.

Let a hyperplane H be given by the equation $f = 5353$, where

$$5353 = 3465 + 693 + 495 + 385 + 315.$$

Let us show that vertices of the cube that lie on H generate the whole space $\mathbb{R}^{37}$. Consider vertices $\boldsymbol{v}$, $\boldsymbol{v}'$, $\boldsymbol{v}''$, $\boldsymbol{v}'''$, $\boldsymbol{v}''''$, $\boldsymbol{v}'''''$, and $\boldsymbol{w}$ lying on the hyperplane $f = 5353$. Their coordinates are given in Table 2, where rows correspond to the vertices, and columns to coordinates (unspecified coordinates of all these vertices are zero).

The linear combination $\boldsymbol{a} = \boldsymbol{v} + \boldsymbol{v}' + \boldsymbol{v}'' + \boldsymbol{v}''' + \boldsymbol{v}'''' + \boldsymbol{v}''''' - \boldsymbol{w}$ has only four nonzero coordinates: $a_t = 6$ and $a_{x_1} = a_{y_1} = a_{z_1} = 5$. Similarly, there exist other linear combinations of vertices with exactly four nonzero coordinates: $\boldsymbol{b}$ with $b_t = 8$ and $b_{u_1} = b_{y_1} = b_{z_1} = 7$, $\boldsymbol{c}$ with $c_t = 10$ and $c_{u_1} = c_{x_1} = c_{z_1} = 9$, and $\boldsymbol{d}$ with $d_t = 12$ and $d_{u_1} = d_{x_1} = d_{y_1} = 11$. The vertices $\boldsymbol{v}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, and $\boldsymbol{d}$ are linearly independent. Indeed, one can easily check that

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 6 & 0 & 5 & 5 & 5 \\ 8 & 7 & 0 & 7 & 7 \\ 10 & 9 & 9 & 0 & 9 \\ 12 & 11 & 11 & 11 & 0 \end{pmatrix}$$

is a full-rank matrix. Arguing in the same way as in the proof of statement (a), we conclude that vertices of the cube satisfying the condition $f = 5353$ lie on a unique hyperplane.

Since the greatest common divisor of all coefficients of the linear form f is 1, the form takes all possible integer values at integer points. Let us prove by contradiction that the form never takes the value 5352 at a vertex of the cube. Let a vertex $\boldsymbol{v}$ be such that $f(\boldsymbol{v}) = 5352$. Since $5352 \equiv 2 \pmod 5$, $693 \equiv 3 \pmod 5$, and the other coefficients are divisible by 5, equality of residues modulo 5 occurs only if $v_{u_1} + \ldots + v_{u_6} = 4$. Similarly, $5352 \equiv 4 \pmod 7$ and $495 \equiv 5 \pmod 7$, whence $v_{x_1} + \ldots + v_{x_8} = 5$. Similarly, $5352 \equiv 6 \pmod 9$ and $385 \equiv 7 \pmod 9$, whence $v_{y_1} + \ldots + v_{y_{10}} = 6$. Similarly, $5352 \equiv 6 \pmod{11}$ and $315 \equiv 7 \pmod{11}$, whence $v_{z_1} + \ldots + v_{z_{12}} = 4$. But then the sum of the last four terms of f is $693 \cdot 4 + 495 \cdot 5 + 385 \cdot 6 + 315 \cdot 4 > 5352$, a contradiction.

Now we prove in the same way that the form never takes the value 5354 at a vertex of the cube. Let $\boldsymbol{v}$ be such that $f(\boldsymbol{v}) = 5354$.

Since $5354 \equiv 4 \pmod 5$ and $693 \equiv 3 \pmod 5$, we have $v_{u_1} + \ldots + v_{u_6} = 3$. Similarly, $5354 \equiv 6 \pmod 7$ and $495 \equiv 5 \pmod 7$, whence $v_{x_1} + \ldots + v_{x_8} = 4$. Similarly, $5354 \equiv 8 \pmod 9$ and $385 \equiv 7 \pmod 9$, whence $v_{y_1} + \ldots + v_{y_{10}} = 5$. Similarly, $5354 \equiv 8 \pmod{11}$ and $315 \equiv 7 \pmod{11}$, whence $v_{z_1} + \ldots + v_{z_{12}} = 9$. But then the sum of the last four terms of f is $693 \cdot 3 + 495 \cdot 4 + 385 \cdot 5 + 315 \cdot 9 > 5354$, a contradiction.

Thus, both "neighboring" hyperplanes do not contain vertices of the cube but contain integer points. △

Remark 4. It is interesting to note that in the proof of both parts of Theorem 1 the determinants of the considered matrices are equal to the values of the form f on the hyperplane H, i.e., to 2147 in the first case and 5353 in the second.

Question. Can the condition that the hyperplane H is uniquely determined be replaced by the condition that it contains sufficiently many vertices of the cube with linearly independent tensor squares (more precisely, tensor squares of vectors composed of coordinates of the vertices)?

3. ADMISSIBLE ARRANGEMENTS OF VERTICES OF THE CUBE ON AN EMPTY QUADRIC

Theorem 2. *Let us be given an empty quadric $f = 0$ in an n-dimensional space $\mathbb{R}^n$ and an integer w, $2 \le w \le n-1$. If all vertices of the cube of weights 0 and w lie on this quadric, then all other vertices of the cube also lie on the quadric.*

Proof. The origin lies on the quadric. Hence, the polynomial f is of the form

$$f(x_1, \ldots, x_n) = \sum_{i,j} f_{ij} x_i x_j + \sum_i f_i x_i,$$

where $f_{ij} = f_{ji}$ form a symmetric matrix of the quadratic form. Denote $D = \sum_i (f_{ii} + f_i)$ and $B = \sum_{i \neq j} f_{ij}$. By summing values of the polynomial at vertices of weight w, we obtain the equality

$$\frac{(n-1)!}{(n-w)!\,(w-1)!} D + \frac{(n-2)!}{(n-w)!\,(w-2)!} B = 0;$$

i.e.,

$$\gamma D + B = 0, \tag{1}$$

where $\gamma = \dfrac{n-1}{w-1} > 1$. Since the quadric is empty, values of the polynomial f are nonnegative at all vertices of the cube. In particular,

$$f(1, \ldots, 1) = D + B \geq 0. \tag{2}$$

The nonnegativity of values of f at all vertices of weight 2 implies that for any pair of indices $i \neq j$ we have

$$(f_{ii} + f_i) + (f_{jj} + f_j) + (f_{ij} + f_{ji}) \geq 0. \tag{3}$$

The nonnegativity of values of f at vertices of weight 1 implies that for each index i we have $f_{ii} + f_i \geq 0$. Hence, $D \geq 0$. Subtracting (1) from (2), we obtain $D(1 - \gamma) \geq 0$, whence $D \leq 0$. Therefore, $D = 0$, and for each index i we have $f_{ii} + f_i = 0$. Also, from (1) we obtain $B = 0$. By (3), for each pair of indices $i \neq j$ we have $f_{ij} \geq 0$. Now from $B = 0$ we obtain $f_{ij} = 0$. The polynomial $f = \sum_i (f_{ii} x_i^2 + f_i x_i)$ with $f_{ii} + f_i = 0$ for each i is zero at each vertex of the cube. △

Example. The empty quadric $\sum_{i>j} x_i x_j = 0$ contains all vertices of the cube of weights 0 and 1 and only them. The empty quadric

$$n \sum_{i=1}^{n} (x_i)^2 - \left(\sum_{i=1}^{n} x_i \right)^2 = 0$$

contains all vertices of the cube of weights 0 and n and only them. Therefore, bounds for the weight w in Theorem 2 cannot be improved.

Remark 5. Theorem 2 remains valid for any affine image of the cube in any dimension under the corresponding redefinition of the weight. This allows one to find 2^k vertices of the original cube on a quadric if he knows $k+1$ such vertices arranged in a certain way on some k-dimensional parallelepiped embedded in the cube.

In [9], the following statement is proved, a particular case of which is given in [11].

Proposition 1. *Let $f(x_1, \ldots, x_n)$ be a polynomial in n variables of degree d over an arbitrary field $\Bbbk$. If at least one vertex of the cube does not lie on the variety $f = 0$, then the fraction of vertices lying on it is at most $1 - 2^{-d}$.*

In particular, if seven of eight vertices of a 3-dimensional cube lie on a quadric, the eighth also does. This is the only restriction on a possible location of vertices of the 3-dimensional cube lying on arbitrary quadrics. If we restrict ourselves to empty quadrics only, it is possible to obtain a constraint involving four vertices only. For instance, if four vertices of the 3-dimensional cube with coordinates $(0,0,0)$, $(0,1,1)$, $(1,1,0)$, and $(1,0,1)$ lie on an empty quadric, the same holds for all the other vertices.

4. NUMBER OF PAIRS OF PARALLEL HYPERPLANES CONTAINING A GIVEN SET OF POINTS

Let $\Bbbk$ be an arbitrary field. Consider the question of the number of parallel hyperplanes in the n-space $\Bbbk^n$ such that every pair contains a prescribed set of points of this space.

Proposition 2. *For any field* $\Bbbk$ *any set* S *consisting of* $n+1$ *points in* $\Bbbk^n$ *in general position there exist exactly* $2^n - 1$ *parallel hyperplanes such that every pair contains all points of* S.

Proof. Let us show that for any partition of S into two nonempty sets I and J there exists a unique pair of parallel hyperplanes such that one of them contains the vertices from I and the other from J. Without loss of generality we may assume that the origin belongs to J. Then coefficients of a linear form defining both hyperplanes must satisfy an inhomogeneous system of linear equations where free terms equal zero or one according to the partition into I and J. The system is nondegenerate because of independence of points of S. Hence, it has a unique solution for a fixed J. The number of proper subsets J is $2^{n+1} - 2$. Since interchanging I and J preserves the pair of hyperplanes, the total number of such hyperplanes is $2^n - 1$. △

Proposition 3. *For* $n \geq 2$ *there exists a set of* $n+2$ *points in* $\Bbbk^n$ *lying on* $3 \cdot 2^{n-2} - 1$ *pairs of parallel hyperplanes.*

Proof. Consider a set S of $n+1$ points in general position. Take any three points of S and complete a (plain) parallelogram. The sought-for set is the union of S and the fourth vertex of the parallelogram. Partition this set into subsets I and J as in the proof of Proposition 2. Linear dependence of vertices of the parallelogram imposes an extra constraint: J contains an even number of vertices of the parallelogram. If J contains exactly two such vertices, then these are vertices of one side of the parallelogram. Then $6/8 = 3/4$ of all choices of I and J are suitable, except for the cases $I = \varnothing$ or $J = \varnothing$. Thus, in total we have $\frac{3}{4}(2^{n-2}/2 - 1) = 3 \cdot 2^{n-1} - 1$ partitions. Each of these partitions defines a unique pair of parallel hyperplanes. △

Propositions 2 and 3 show nontriviality of the optimization problem: if we know in advance that vertices of the cube under consideration lie on a pair of parallel hyperplanes (or on a quadric of low rank), then there may be many choices for such hyperplanes, and the complexity of the corresponding integer programming problem varies depending on a particular choice.

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