Line comments to [H 1909a]: Die Graduierung nach dem Endverlauf

This paper accomplishes HAUSDORFF's cycle of publications devoted to the problem of graduation (*Graduierungsprobleme*) of real functions according to their behaviour at infinity. The first part of the paper (§§ 1 and 2) is an improved and significantly sharpened review of principal results of [H 1907a, Chapter V], while the second part has three remarkable additions: the existence of pantachies that are, in the same time, ordered non-archimedian fields in the sense of componentwise operations modulo finitely many positions (§ 3), the theorem of the existence of (ω_1, ω_1^*)-gaps (§ 4), and thoroughful study of the nature of "canonical" convergence/divergence gaps in arbitrary Pantachien (§ 5).

In the course of these Line comments, we'll sometimes refer to our Essay *Gaps in partially ordered sets and related problems* (Commentary to [H 1909a] and [H 1936b] in this Volume), which will be called simply *Gaps and partially ordered sets* throughout these Comments.

[1] S. 297 das Verhalten von Funktionen $f(x)$ ... zu *graduieren*
HAUSDORFF begins with an introduction into the *Graduierungsproblem*, that is, the problem of calibration or *graduation* of real functions defined on $[0, +\infty)$ according to their rate of growth, or, more generally, according to their behaviour at $+\infty$. Although the rate of growth comparison of real functions belongs to basics of analysis, it was only DU BOIS-REYMOND who considered the notion of growing faster or slower as a certain relation $\prec$ (see Comment [5]) on real functions in all their totality in *Sur la grandeur relative...*, Ann. di Mat. (2), **4** (1870), 338–353 and subsequent papers. DU BOIS-REYMOND's main technical achievement was his famous diagonal construction, which, given a countable family F of real positive increasing functions, allowed to define 1) a function increasing faster than all functions in F, 2) a function increasing slower than all functions in F, 3) given a gap in F, a function increasing with precisely such rate of growth as to fill in the gap.

Somewhat later DU BOIS-REYMOND interpreted functions growing faster as those representing stronger (larger) quantitative or ordinal infinities of some sort in *Ueber asymptotische Werke...*, Math. Ann., **8** (1875), 363–414, in fact following somewhat earlier ideas of J. THOMAE, *Abriss einer Theorie der complexen Functionen...*, Halle, Nebert, 1870.

HAUSDORFF cites several papers and books related to this problem in [H 1907a, Chapter V], footnote 1 on page 105. Some references can be added to his list, most notably J. HADAMARD, *Sur les caractères de convergence des séries...*, Acta Math., **18** (1894), 319–336, G. HARDY, *A theorem concerning the infinite cardinal numbers*, Quarterly Journ., **35** (1903), 87–94, and especially G. HARDY, *Orders of Infinity*, Cambridge University Press, 1910, with a good account of earlier studies in this direction and additional references.

As of modern historical surveys in matters of studies of DU BOIS-REYMOND and his close successors and their influence on HAUSDORFF, see a substantial paper of G. FISCHER, *The infinite and infinitesimal quantities of du*

Bois-Reymond and their reception, Arch. Hist. Exact Sci., **24** (1981), 101–163, very interesting historical comments to early HAUSDORFF's papers by J. M. PLOTKIN in *Hausdorff on ordered sets*, translated, edited and commented by J. M. Plotkin, History of Mathematics, 25, AMS and LMS, 2005, and the essays *Die Hausdorffsche Theorie der η_α-Mengen and ihre Wirkungsgeschichte* and *Zum Begriff des topologischen Raumes* in Band II.

[**2**] S. 298 die *Anodnung* der Funktionen nach gleichem, stärkerem, schwächerem Unendlich in Betracht zu ziegen
HAUSDORFF suggests to abandon attempts to project every function considered onto a pre-defined scale of infinities, but rather to compare the degrees of infinity by means of direct comparison of functions. See our Essay *Gaps and partially ordered sets*, Section 14, on this idea.

[**3**] S. 298 Eine wirkliche Schwierigkeit
HAUSDORFF observes that different attempts to compare functions in accordance with their rate of growth lead to the existence of incomparable functions. See our comment to Page 299 for more on that.

[**4**] S. 298 die Forderung stellen, daß...
HAUSDORFF suggests to consider only those graduation orderings $<$ which satisfy $f < g \Longrightarrow f <_{\mathtt{fro}} g$, where $<_{\mathtt{fro}}$ is the strict *final Rangordnung* (see Comment [5]). But this turns out to be a too restrictive condition since the orderings $\lhd$ and $<^*$ (see below) fail to satisfy it.

[**5**] S. 298 bisher vorgeschlagenen Graduierungen
HAUSDORFF obviously prefers to view a partial ordering as a combination of a strict partial order $<$ (and hence $>$ as well) and an associated equivalence relation, which he tends to denote with the equality symbol $=$. Nowadays it is more in custom to consider *non-strict partial quasi-orders (*PQO*s)* (formally, transitive reflexive relations) $\leq$, each of which has an associated *equivalence relation* $x \equiv y$ if both $x \leq y$ and $y \leq x$, and an associated *strict order* $x < y$ if $x \leq y$ but $y \not\leq x$. (And $x \equiv y$ does not necessarily imply the exact mathematical equality $x = y$.)

The graduation ordering based on the eventual size of the difference $f(x) - g(x)$ is the *final Rangordnung*. We denote it by $\leqslant_{\mathtt{fro}}$ (non-strict form) and $<_{\mathtt{fro}}$ (strict form) in these Comments, with $\equiv_{\mathtt{fro}}$ being the associated equivalence relation. Thus $f <_{\mathtt{fro}} g$ iff there is x_0 such that $f(x) < g(x)$ for all $x \geq x_0$, $f \equiv_{\mathtt{fro}} g$ iff there is x_0 such that $f(x) = g(x)$ for all $x \geq x_0$, and $f \leqslant_{\mathtt{fro}} g$ iff $f <_{\mathtt{fro}} g$ or $f \equiv_{\mathtt{fro}} g$. The index $_{\mathtt{fro}}$ stands for "final Rangordnung".

The graduation based on the limit $\mathtt{lim}_{x\to\infty} \frac{f(x)}{g(x)}$ is the original rate of growth ordering by DU BOIS-REYMOND, see, *e. g.*, his *Sur la grandeur relative...*, Ann. di Mat. (2), **4** (1870), 338–353, or *Ueber asymptotische Werke...*, Math. Ann., **8** (1875), 363–414. We denote it by $\preccurlyeq$ (non-strict form) and $\prec$ (strict form) in these Comments, with $\sim$ being the associated equivalence relation.

The graduation on the base of $\mathtt{lim\,sup}_{x\to\infty}\frac{f(x)}{g(x)}$ was discussed in [H 1907a], p. 106. We denote it by $\trianglelefteq$ (non-strict form) and $\triangleleft$ (strict form) in our *Gaps and partially ordered sets*, with $\bowtie$ being the associated equivalence relation. [1]

Elsewhere in [H 1936b] HAUSDORFF studies the *eventual domination* ordering (for infinite sequences): $f \leqslant^* g$ iff there is x_0 such that $f(x) \le g(x)$ for all $x \ge x_0$, with the associated equivalence relation $\equiv^*$ and the non-strict ordering $<^*$ defined by general rules.

[6] S. 299 Es existiert weder die „Infinitäre Pantachie" von P. du Bois-Reymond, in der jedes functionale Unendlich seinen bestimmten Platz einnimmt. . .

DU BOIS-REYMOND was not quite clear in his description of the idea of "infinitary pantachy" [2] in his book *Die allgemeine Funktionentheorie*, Tübingen, 1882, pp. 282–284. That is, on the one hand, he puts all $\sim$-equivalence classes into his "infinitary pantachy", but on the other hand he seems to view the result as a linearly ordered collection, in the ignorance of the (obvious for us but not for him?) existence of incomparable functions. Nevertheless the meaning of HAUSDORFF's negative claim seems to be that there is no natural method of graduation of functions defined on $[0, +\infty)$ (or infinite sequences of reals) in accordance with their rate of growth, so that any two functions (or sequences) are comparable, unless the method is restricted to very particular domains, like *e. g.* the domain of all polynomial-exponential-logarithmic functions, as in a paper by O. STOLZ, *Zur Geometrie der Alten, insbesondere über ein Axiom des Archimedes*, Mathematische Annalen, **22** (1883), 504–519.

The mathematical content of this negative claim deserves some comments. First of all, adopting HAUSDORFF's restrictive condition (that for $f < g$ it is necessary that $f(x) > g(x)$ for all sufficiently large x, see Comment [4]), one proves the existence of incomparable elements by elementary counterexamples outlined by HAUSDORFF on p. 298. But the problem becomes less trivial in the absense of the restrictive condition. See more on this in *Gaps and partially ordered sets*, Section 14.

[7] S. 300 eine *Pantachie* nennen

Thus, in modern terminology, HAUSDORFF's *pantachy* is a maximal **chain** (a subset linearly ordered in the sense of the strict order) in a given partially quasi-ordered set. The logic of study leads HAUSDORFF to linearly ordered subdomains of the partially ordered general domain (of functions or sequences), including maximal ones, *i. e.*, the pantachies.

And here we encounter interesting questions that go back to the universal graduation problem. Suppose that $L \subseteq \mathbb{R}^{\mathbb{N}}$ is a pantachy in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathtt{fro}} \rangle$, that is, a maximal $<_{\mathtt{fro}}$-linearly ordered set. Say that L is a *potential graduation*

[1] Note that in our Essay *Gaps and partially ordered sets* relations $\preccurlyeq$, $\prec$, $\sim$, as well as those based on $\mathtt{lim\,sup}$, are considered in their *differential* rather than *fractional* forms. But this does not matter much at least as long as functions with positive values are considered, because the logarithm transforms the fractional form to the differential one.

[2] Die Bezeichnung soll an das griechische Wort πανταχῆ (an allen Stellen, allerorten) erinnern. – *Copied from Band II, p. 650.*"

scale, if there is a map $\pi : \mathbb{R}^{\mathbb{N}} \to L$ such that

$$a \equiv_{\texttt{fro}} b \Longrightarrow \pi(a) = \pi(b)\,, \quad a <_{\texttt{fro}} b \Longrightarrow \pi(a) <_{\texttt{fro}} \pi(b)\,,$$

for $a, b \in \mathbb{R}^{\mathbb{N}}$, and π is the identity on L. In this case, the induced relation $a \leq b$ iff $\pi(a) \leqslant_{\texttt{fro}} \pi(b)$ is obviously an LQO on $\mathbb{R}^{\mathbb{N}}$ that can be considered as a universal graduation method based on L. Then, is any/every pantachy in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\texttt{fro}} \rangle$ a potential graduation scale? We don't know an answer to such a form of the universal graduation problem. It can be expected that **CH** still solves it in the positive.

[8] S. 300 Der Beweis, daß solche Pantachien existieren
The existence proof that follows, implicitly introduces what is now called the *maximal principle*.

[9] S. 304 Die Rangordnung der Zahlenfolgen
Obviously any ordering of functions $f : [0, +\infty) \to \mathbb{R}$ of real variable has a discrete analog for infinite sequences $\{a_n\}_{n \in \mathbb{N}}$ of reals a_n. The natural question, whether the properties of the "continual" (that is, for functions, not necessarily continuous) and the discrete structures are the same, is easily answered in the positive in some cases, but is still open, and probably difficult, in other cases, see our *Gaps and partially ordered sets* in this Volume, Section 3

[10] S. 304 $A < B$, $A = B$, $A > B$, $A \parallel B$
HAUSDORFF's idea to use $=$ for the final equality is in clear contradiction with the modern understanding of equality in mathematics (equal things is one and the same thing). We use $\equiv_{\texttt{fro}}$ and $<_{\texttt{fro}}$ (see Comment [5]) to denote the final equality and the strict final ordering.

[11] S. 304 Bereich
HAUSDORFF defines a *Bereich* to be any $<_{\texttt{fro}}$-chain $\mathfrak{P} \subseteq \mathbb{R}^{\mathbb{N}}$, that is, any two $a \neq b \in \mathfrak{P}$ are $<_{\texttt{fro}}$-comparable: $a <_{\texttt{fro}} b$ or $b <_{\texttt{fro}} a$. It follows that any $\equiv_{\texttt{fro}}$-class of elements of $\mathbb{R}^{\mathbb{N}}$ contains at most one element of $\mathfrak{P}$. Recall that a *pantachy* is just a maximal $<_{\texttt{fro}}$-chain.

[12] S. 304 Fundamentalsatz I
Theorem I (= Theorems (C) and (D) in [H 1907a,V]) goes back to the original DU BOIS-REYMOND's theorem that any countable set of real functions has a strict upper bound in the sense of the DU BOIS-REYMOND ordering $\prec$, see, *e. g.*, pp. 363–365 in *Ueber asymptotische Werke. . .*, Math. Ann., **8** (1875), 363–414. [3] See more on the history of related notions and results in the surveys by G. FISCHER and G. HARDY, mentioned in Anmerkung [1], and U. FELGNER's essay in Band II. The proofs for typical graduation orderings like $\prec$, $\lhd$, $<_{\texttt{fro}}$,

[3] Es ist also zu beweisen, dass, wenn eine unbegrenzte Schaar von fort und fort langsamer zunehmenden Functionen: $\lambda_1(x)$, $\lambda_2(x)$, $\lambda_1(x), \ldots$ vorliegt, die für jedes r die Bedingung $\texttt{lim}\,\frac{\lambda_r(x)}{\lambda_{r+1}(x)} = \infty$ erfüllen, man immer eine Function $\psi(x)$ angeben kann, die mit x unendlich wird, aber langsamer als irgend eine Function jenen Schaar. — *S. 365 of the paper cited.*

$<^*$ are slightly different in details, yet follow one and the same DU BOIS-REYMOND's idea, historically the first exposition of the *diagonal argument* usually attributed to CANTOR (see, *e.g.*, FELGNER in Band II, S. 650). [4]

[13] S. 305 Satz II: it claims, in modern terms, that any pantachy is an η_1-order. Generally, an η_α-*order* is a linearly ordered set L such that for any pair of subsets $A, B \subseteq L$ of cardinality $< \aleph_\alpha$, if $A < B$ (*i.e.*, $a < b$ for all $a \in A$, $b \in B$) then there is an element c with $A < c < B$. See U. FELGNER's essay *Die Hausdorffsche Theorie der η_α-Mengen and ihre Wirkungsgeschichte* in Band II regarding HAUSDORFF's theory of order types η_ν and their modern applications.

[14] S. 306 Wenn die wohlgeordneten Bereiche höchstens von der zweiten Mächtigkeit sind, so ist jede Pantachie mit Ω konfinal und mit Ω koinitial...

In context of the paper, the assumption means that any set of countable sequences, which is well-ordered by $<_{\mathtt{fro}}$, has length $< \omega_2$. This is a consequence of the continuum-hypothesis **CH** ($\mathfrak{c} = \aleph_1$), or course. Ω is ω_1, as usual. HAUSDORFF cites some results of his earlier papers.

[15] S. 306 Wenn sie dann keine $\Omega\Omega^*$-Lücken enthält

HAUSDORFF refers to a result in his earlier paper [H 1907a], Satz III on S. 128, accordingly to which if a linear order S of type η_1 (that is, H), does not contain (ω_1, ω_1^*)-gaps then S has the cardinality $\geq 2^{\aleph_1}$. Following SOLOVAY (see Footnote 5), we render the proof in modern terms is as follows. Let $W = 2^{\omega_1}$, the set of all maps $w : \omega_1 \to 2$ (or all dyadic sequences of length ω_1). Order W lexicographically. Let W_0 consist of all $w \in W$ such that the set $|w| = \{\xi < \omega_1 : w(\xi) = 1\}$ is at most countable. One easily defines an order preserving map $\Phi : W_0 \to S$, using the fact that S, as a set of order type η_1, is uncountably cofinal, uncountably coinitial, and has no (ω, ω^*)-gaps.

Now consider the set W_1 of all $w \in W$ such that both the set $|w|$ and its complement $\omega_1 \smallsetminus |w|$ are uncountable. Clearly any $w \in W_1$ defines a (ω_1, ω_1^*)-gap in W_0, and between two different elements of W_1 there is an element of W_0. Since S has no (ω_1, ω_1^*)-gaps, there is an order preserving extension $\Psi : W_0 \cup W_1 \to S$ of Φ. This yields the result since obviously $\mathtt{card}\, W_1 = 2^{\aleph_1}$.

In particular, if S is a pantachy (maximal chain) in the partial ordering $\langle \mathbb{R}^{\mathbb{N}} ; \leqslant_{\mathtt{fro}} \rangle$, and S contains no (ω_1, ω_1^*)-gaps [5], then S has an η_1-type, and

[4] Recall that CANTOR introduced his method, in the proof that no set is equinumerous with its power set, essentially later than DU BOIS-REYMOND did in his studies of the rate of growth. In fact the common part of both arguments (with Cantor's one in the form that no countable sequence of sets $X_n \subseteq \mathbb{N}$ contains all subsets of $\mathbb{N}$) can be described as follows: given an infinite sequence $\{X_n\}_{n \in \mathbb{N}}$ we define a function f on $\mathbb{N}$ so that any term $f(n)$ depends only on $X_0, \ldots, X_n$. In the DU BOIS-REYMOND case, $X_n = f_n : \mathbb{N} \to \mathbb{R}$ (we consider, for the sake of brevity, infinite sequences instead of functions), say, $f(k) = k\ \mathtt{sup}_{n \leq k} f_n(k)$ for all k, in CANTOR's case $f(k) = 1$ iff $k \notin X_k$.

[5] In this particular, but fully sufficient case, the proof of Satz III of [H 1907a] is given in R. SOLOVAY, *Introductory note to Gödel *1970a, *1970b, *1970c*, in: K. GÖDEL, *Collected works, Vol. III*, Clarendon, NY, 1995, 405–420, where there is no any related reference to HA-

hence $\mathtt{card}\, S \geq 2^{\aleph_1}$. But on the other hand $S \subseteq \mathbb{R}^{\mathbb{N}}$, so that $\mathtt{card}\, P \leq 2^{\aleph_0}$. It follows that the existence of such a pantachy S implies the equality $2^{\aleph_0} = 2^{\aleph_1}$.

HAUSDORFF proved in [H 1909a] the existence of pantachies containing (ω_1, ω_1^*)-gaps. In fact this is an easy consequence of his famous theorem on the existence of such gaps in the whole structure $\langle \mathbb{R}^{\mathbb{N}}; <_{\mathtt{fro}} \rangle$ established in [H 1909a]. As for those containing no (ω_1, ω_1^*)-gaps, the problem of their existence[6] remains open, and in fact it seems to be the oldest open problem in set theory explicitly stated in a suitable mathematical publication!

Let us call *gapless* any pantachy that does not have a (ω_1, ω_1^*)-gap. If a gapless pantachy exists then $2^{\aleph_0} = 2^{\aleph_1}$ (see above), therefore, the continuum-hypothesis fails! Remarkably, GÖDEL[7] used this fact in his attempt to prove $2^{\aleph_0} = \aleph_2$ from a plausible list of axioms. In fact GÖDEL's interest to HAUSDORFF's results on pantachies and their applications to the continuum problem goes back to the the first part of 1960s. See more details related to this story, including GÖDEL's exchange with COHEN and ULAM in 1964, in A. KANAMORI, *Gödel and set theory*, Bull. Symb. Log., **13** (2007), 153–188.

It can hardly be expected that the theory **ZFC** plus $2^{\aleph_0} = 2^{\aleph_1}$ outright proves or refutes the existence of gapless pantachies. In such a case, the practice of the forcing era in set theory leads to consistency questions. One can ask:

(A) is the existence of gapless pantachies consistent with **ZFC** + $2^{\aleph_0} = 2^{\aleph_1}$?

(B) is the absence of gapless pantachies consistent with **ZFC** + $2^{\aleph_0} = 2^{\aleph_1}$?

Both questions seem to remain unanswered. A somewhat stronger form of (A) (with the additional requirement that the pantachy does not contain strictly increasing or decreasing sequences of cardinality $\leq \aleph_2$) is in the list of (two) open questions in SOLOVAY's note cited in Footnote 5. SOLOVAY observes that the major problem in the construction of a model satisfying (A) is to avoid HAUSDORFF's (ω_1, ω_1^*)-gaps.

[16] S. 306 Typus H:
The type H is η_1 in the modern notation, see Comment [13]. HAUSDORFF notes that any two η_1-orders of cardinality $\aleph_1$ are order isomorphic. The proof, by an argument now called *the back-and-forth method*, was given in [H 1907a], p. 127, Satz II. In particular, under the assumption of **CH**, there is a unique (modulo order isomorphism) H-set, thus $H = \eta_1$ is the order type of any pantachy in this case.

USDORFF, or any other relevant reference. Section 4 in SOLOVAY's "Note" is entitled *Axioms 3 and 4 entail Luzin's hypothesis*, where Axioms 3 and 4 should be understood together as the existence of a pantachy in $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathtt{fro}} \rangle$ containing neither (ω_1, ω_1^*)-gaps (Axiom 4) nor ascending or descending sequences of cardinality $\geq \aleph_2$ (Axiom 3), while Luzin's hypothesis is $2^{\aleph_0} = 2^{\aleph_1}$. In fact Axiom 3 is not really involved in the argument. However if a pantachy S satisfies both Axioms 3 and 4 then all gaps in S (except for limits) are (ω_1, ω^*)-gaps and (ω, ω_1^*)-gaps, a remarkable uniformity similar to the Dedekind completeness of the reals.

[6] Explicitly posed and discussed by HAUSDORFF in an earlier paper [H 1907a, S. 151].

[7] K. GÖDEL, *Some considerations leading to the probable conclusion that the true power of the continuum is* $\aleph_2$, in: K. GÖDEL, *Collected works, Vol. III*, Clarendon, NY, 1995, 420–422.

[17] S. 307 so folgt aus $a \gtreqqless b$ auch $A \gtreqqless B$
In modern terms, this means that the map, say φ, which sends any function $f : [0, +\infty) \to \mathbb{R}$ to the sequence $\{f(n)\}_{n\in\mathbb{N}}$, is a *homomorphism* of the $\{<_{\mathtt{fro}}, \equiv_{\mathtt{fro}}\}$-structure of functions into $\langle \mathbb{R}^{\mathbb{N}}; \leqslant_{\mathtt{fro}}\rangle$, that is,

$$f <_{\mathtt{fro}} g \Longrightarrow \varphi(f) <_{\mathtt{fro}} \varphi(g) \quad \text{and} \quad f \equiv_{\mathtt{fro}} g \Longrightarrow \varphi(f) \equiv_{\mathtt{fro}} \varphi(g)\,.$$

The inverse implications obviously fail, *e.g.*, for $f(x) = \sin \pi x$ and $g(x) \equiv 0$. This means that φ is not a *reduction*.

It is an interesting *problem* to find a true reduction in this case, *i.e.*, a map φ from functions to sequences such that

$$f \leqslant_{\mathtt{fro}} g \Longleftrightarrow \varphi(f) \leqslant_{\mathtt{fro}} \varphi(g)$$

holds for all functions considered, say for all continuous $f, g : [0, +\infty) \to \mathbb{R}$, to begin with. Such a reduction from continuous functions to $\mathbb{R}^{\mathbb{N}}$ does exist in the weaker sence of the associated equivalence relation, that is, $f \equiv_{\mathtt{fro}} g \Longleftrightarrow \varphi(f) \equiv_{\mathtt{fro}} \varphi(g)$ holds for all $f, g \in \mathbf{C}[0, +\infty)$, see Section 13 in *Gaps and partially ordered sets*. But as far as the rate of growth order $\preccurlyeq$ is concerned, we don't know of any reduction even in the weaker sense of the associated equivalence relation $\sim$.

On the other hand, a reduction from sequences to continuous functions can be defined by just a simple interpolation.

[18] S. 307 $A \prec B$, $A \sim B$, $A \succ B$
These relations are just the restrictions of the Du Bois-Reymond rate of growth ordering (see Comment [5]) to the domain $\mathbb{R}^{\mathbb{N}}$ of all infinite real sequences.

[19] S. 308 eingeschränkter Gebiete von Zahlenfolgen
Hausdorff notes that Fundamentalsatz I (see Comment [12]) remains true within several important subdomains of the whole domain $\mathbb{R}^{\mathbb{N}}$ of countable real sequences, including *e.g.* the domain of all sequences that converge to a specific limit a (including $a = +\infty$ and $a = +\infty$), but fails for the domain of convergent all sequences since this domain is countably cofinal.

[20] S. 309 Satz III: the converse is not true!

[21] S. 311 rationaler Bereich, rationale Pantachie
Hausdorff's definitions on S. 311 amount to the following. Suppose that $\mathfrak{R} \subseteq \mathbb{R}^{\mathbb{N}}$ is a *Bereich* (for instance, a pantachy, see Comment [11]). Then for any $a, b \in \mathfrak{R}$ there is at most one $c \in \mathfrak{R}$ such that $c(n) = a(n) + b(n)$ for all but finite n. (Because a *Bereich* has at most one element in common with each $\equiv_{\mathtt{fro}}$-class). Such an element c — if it exists in $\mathfrak{R}$ — is denoted by $a + b$. (Note that $a + b$ depends on $\mathfrak{R}$, too, not only on a, b.) Similarly, Hausdorff defines $a - b$ and ab, and also $\frac{a}{b}$ provided $b(n) \neq 0$ for all but finite n. All of them are elements of $\mathfrak{R}$ if exist.

If each constant sequence $\bar{x}$ ($x \in \mathbb{R}$) represents its $\equiv_{\mathtt{fro}}$-class in $\mathfrak{R}$ then every $b \in \mathfrak{R}$ except for the constant 0 is eventually $\neq 0$, which is compatible with a field structure. Thus, a *Bereich* or a pantachy is *rational* if it is a field in the sense of the abovedefined operations.

One may ask whether there exist pantachies $\mathfrak{P} \subseteq \mathbb{R}^{\mathbb{N}}$ that are fields in the sense of straightforward componentwise operations, that is, for instance, $a+b=c$ iff $c(n)=a(n)+b(n)$ for $\underline{\text{all}}$ n. This easily answers in the negative. Indeed, otherwise by $a-a$ the constant 0 belongs to $\mathfrak{P}$ and is the null element of $\mathfrak{P}$ as a field. It follows that any other $c \in \mathfrak{P}$ satisfies $c(k) \neq 0$ for all k. Then we have $a(k) \neq b(k)$ for all k and $a \neq b \in \mathfrak{P}$, and hence for any $r \in \mathbb{R}$ and k there is at most one $a = a_{rk} \in \mathfrak{P}$ with $a_{rk}(k) = r$. Now define $b \in \mathbb{R}^{\mathbb{N}}$ so that $b(n) \neq a_{rk}(n)$ for all $k < n$ — then $a \equiv_{\mathtt{fro}} b$ is impossible for any $a \in \mathfrak{P}$, contrary to the maximality of $\mathfrak{P}$.

[22] S. 312 Unser Ziel ist, eine rationale Pantachie herzustellen
HAUSDORFF defines a pantachy that is a real closed field (an algebraic pantachy in his terminology). The principal tool is the same as in the construction of a pantachy with no algebraic properties, but the step of adding an element to a non-maximal rational or algebraic *Bereich* is now much more difficult (Satz VIII on S. 319).

[23] S. 319 direkte und inverse Ähnlichkeit aller Pantachienstrecken
By simple rational transformations, every rational pantachy is order isomorphic to any its open interval (including semiinfinite intervals), also in the sense of the inverse order. HAUSDORFF studied this type of *homogeneous* pantachies elsewhere in [H 1907a] and some other earlier papers. The problem of the existence of **non**-homogeneous pantachies was discussed in [H 1907a, S. 152].

CH easily implies that all pantachies, and all open intervals of pantachies, are order-isomorphic, therefore all pantachies are homogeneous.

On the other hand, the existence of a non-homogeneneous pantachy is consistent with **ZFC** (by necessity, with the negation of **CH**). Indeed, it is consistent with **ZFC** that there exist towers of different cofinalities (even of *many* different cofinalities), see P. DORDAL, *Towers in* $[\omega]^{\omega}$ *and* ω^{ω}, Annals of Pure and Appl. Logic, **45** (1989), 247-276. Extending a pair of towers of different cofinalities to pantachies, we get a pair of pantachies of different cofinalities, hence, non-isomorphic. "Gluing" together such a pair of pantachies, we obtain a non-homogeneous one.

It is perhaps still an open problem whether the non-existence of non-homogeneous pantachies is consistent with the negation of **CH**, for instance, with $2^{\aleph_0} = 2^{\aleph_1}$. Clearly, **MA** (the Martin axiom) plus not-**CH** implies that at least all pantachies have the same cofinality. But we don't know whether **MA** implies that moreover all pantachies are order isomorphic. The straightforward back-and-forth construction fails because we may encounter an empty (ω_1, ω_1^*)-gap in one of the pantachies while the corresponding gap in the other pantachy will be non-empty.

[24] S. 320 die Existenz einer $\Omega\Omega^*$-Lücke beweisen
This is HAUSDORFF's famous gap existence theorem in its first established

form: the structure $\langle \mathbb{R}^{\mathbb{N}}; <_{\mathtt{fro}} \rangle$ contains (ω_1, ω_1^*)-gaps. The proof follows on pp. 320–323.

[25] S. 321 die jeden Abschnitt von $\mathfrak{A}$ ungleichmäsig übertrifft
Here Hausdorff means any *proper* cut, that is, different from $\mathfrak{A}$ itself.

[26] S. 324 folgende Fragen kleiden
Hausdorff shows that all three questions are equivalent to each other. While the equivalence of 1) and 2) is quite obvious, the equivalence of 1) and 3) is non-obvious and requires rather elaborate arguments in the remainder of § 4. This equivalence was re-established much later by F. Rothberger, *On some problems of Hausdorff and of Sierpiński*, Fund. Math., **35** (1948), 29–46 for the suborder $\langle \mathbb{N}^{\mathbb{N}}; <_{\mathtt{fro}} \rangle$ of $\langle \mathbb{R}^{\mathbb{N}}; <_{\mathtt{fro}} \rangle$. It has to be noted that the strength of Hausdorff's methods related to this equivalence and the gap existence theorem is such that they easily accomodate to other similar structures.

[27] S. 327 …Konvergente und divergente Reihe
Hausdorff calls a real sequence $a \in \mathbb{R}^{\mathbb{N}}$ *convergent* in § 5 iff so is the series $\sum_n a(n)$, and *divergent* otherwise. Any sequence a convergent in this sense satisfies $\lim_{n\to\infty} a(n) = 0$, but not the other way around.

[28] S. 328 I
Hausdorff demonstrates that Fundamentalsatz I remains true in the domain $\mathbb{R}_+^{\mathbb{N}}$ of all positive real sequences, with certain specifications that reflect the phenomena of convergece and divergence.

[29] S. 330 üblichen „Skalen" von Konvergenz- und Divergenzkriterien
Hausdorff means different countable scales of calibrating sequences with positive terms (or positive real functions), in which every next sequence (function) converges (or diverges) slower (or, on the contrary, faster) than the previous one. In particular several such scales were given by Du Bois-Reymond, see, e. g., *Ueber asymptotische Werke…*, Math. Ann., **8** (1875), S. 364. It follows from Hausdorff's Satz I (S. 328) that no such countable scale can universally calibrate convergent and divergent series.

[30] S. 330 das Gebiet positiver Funktionen
The functions $f(x)$ considered have to be summable on every finite interval as otherwise any analogy with sequences fails.

[31] S. 330 "ideale" Elemente (Reichen oder Funktionen)
Fischer in his survey mentioned in Comment [1] gives an account of studies of the rate of growth, including the controversy related to "ideal functions", that is, here, an imaginary function separating convergent series from divergent (which Hausdorff obviously dismisses).

[32] S. 331 II
Working in the domain $\mathbb{R}_+^{\mathbb{N}}$ of sequences with strictly positive (non-0) real terms, Hausdorff considers the "canonical convergence-divergence gap"

$\mathfrak{P} = \mathfrak{P}_c \cup \mathfrak{P}_d$ of an arbitrary pantachy $\mathfrak{P} \subseteq \mathbb{R}_+^{\mathbb{N}}$, which consists of the initial segment $\mathfrak{P}_c$ of all sequences convergent in the sense of Comment [27] and the final segment $\mathfrak{P}_d$ of all sequences divergent in the sense of Comment [27]. It follows from Satz II that there are pantachies $\mathfrak{P} \subseteq \mathbb{R}_+^{\mathbb{N}}$ such that the convergent part $\mathfrak{P}_c$ contains a $<_{\mathtt{fro}}$-largest element, that can be interpreted as "the largest convergent sequence" in the sense of $\mathfrak{P}$, and similarly there are pantachies $\mathfrak{P}$ such that the divergent part $\mathfrak{P}_d$ contains a $<_{\mathtt{fro}}$-least element, viewed accordingly as "the least divergent sequence". This clearly undermines any plan to use arbitrary pantachies in the role of graduation methods that respect such an important analytic phenomenon as the partition of series into convergent and divergent.

[33] S. 334 Ob auch der Fall $\mathfrak{P}_c = 0$ hier realisierbar ist
Here HAUSDORFF most likely assumes that $\mathfrak{P}$ is a pantachy in the domain of *monotone* positive real sequences.

Indeed, it had been already established in [H 1907a], p. 150, that there exists a pantachy $\mathfrak{P}$ in the structure $\langle \mathbb{R}_+^{\mathbb{N}}; <_{\mathtt{fro}} \rangle$ (of all positive real sequences, not necessarily monotone), such that $\mathfrak{P}_c = \emptyset$. (That is, $\mathfrak{P}$ contains no sequences convergent in the sense of Comment [27].) The construction of [H 1907a] involves a transformation of infinite sequences called *mixing*. HAUSDORFF begins in [H 1907a], p. 148, with a pantachy, say $\mathfrak{D} \subseteq \mathbb{R}_+^{\mathbb{N}}$, which contains the constant-1 sequence $\mathbb{1}$ and is homogeneous enough for the sub-pantachies $\mathfrak{D}_< = \{a \in \mathfrak{D} : a <_{\mathtt{fro}} \mathbb{1}\}$ and $\mathfrak{D}_> = \{b \in \mathfrak{D} : \mathbb{1} <_{\mathtt{fro}} b\}$ to be order-isomorphic. Let $f : \mathfrak{D}_< \xrightarrow{\text{onto}} \mathfrak{D}_>$ be any order isomorphism. Then (p. 150) HAUSDORFF introduces a pantachy, say $\mathfrak{P}$, by mixing corresponding elements from $\mathfrak{D}_<$ and $\mathfrak{D}_>$. More exactly, if $a \in \mathfrak{D}_<$ and $b = f(a) \in \mathfrak{D}_>$ then the sequence c, defined so that $c(2n) = a(n)$ and $c(2n+1) = b(n)$ for all n, belongs to $\mathfrak{P}$, and $\mathfrak{P}$ consists of only such sequences c.

That $\mathfrak{P}$ is a pantachy follows by an agrument on p. 147. For instance, suppose towards the contrary that $c \in \mathbb{R}_+^{\mathbb{N}}$ satisfies $c <_{\mathtt{fro}} \mathfrak{P}$. Then the subsequence $a(n) = c(2n)$ of all even terms of c still belongs to $\mathbb{R}_+^{\mathbb{N}}$ and obviously satisfies $a <_{\mathtt{fro}} \mathfrak{D}_<$, which contradicts to the assumption that $\mathfrak{D}$ is a pantachy.

Moreover $\mathfrak{P}$ obviously consists of sequences divergent in the sense of Comment [27], so that $\mathfrak{P}_c = \emptyset$, as required.

But this argument does not seem to work in the subdomain $(\mathbb{R}_+^{\mathbb{N}})_{\mathtt{mon}}$ of monotone sequences in $\mathbb{R}_+^{\mathbb{N}}$ because the mixture, in the abovedefined sense, of two monotone sequences is not necessarily monotone. Generally speaking, it is perhaps still an open problem whether there is a pantachy $\mathfrak{P}$ in the structure $\langle (\mathbb{R}_+^{\mathbb{N}})_{\mathtt{mon}}; <_{\mathtt{fro}} \rangle$ such that $\mathfrak{P}_c = \emptyset$.

In fact continuum-hypothesis **CH** easily yields such a pantachy, and the key observation in the construction is that

1) if $g \in ((\mathbb{R}_+^{\mathbb{N}})_{\mathtt{mon}})_d$ and $f \in ((\mathbb{R}_+^{\mathbb{N}})_{\mathtt{mon}})_c$ then there is a sequence $h \in ((\mathbb{R}_+^{\mathbb{N}})_{\mathtt{mon}})_d$ such that $h <_{\mathtt{fro}} g$ but $f \not<_{\mathtt{fro}} h$, and

2) if $X \subseteq ((\mathbb{R}_+^{\mathbb{N}})_{\mathtt{mon}})_d$ is at most countable then, by the Fundamentalsatz, there is a sequence $g \in ((\mathbb{R}_+^{\mathbb{N}})_{\mathtt{mon}})_d$ such that $g <_{\mathtt{fro}} g'$ for all $g' \in X$.

Now, if $X \subseteq ((\mathbb{R}_+^{\mathbb{N}})_{\mathtt{mon}})_d$ is a countable pre-pantachy (that is, X is linearly ordered by $<_{\mathtt{fro}}$), then choose g by 2) and h by 1), thus $X \cup \{h\}$ is still a pre-pantachy, but no pantachy $\mathfrak{P}$ satisfying $X \cup \{h\} \subseteq \mathfrak{P}$ can contain f.

What happens under the negation of **CH** is not known.