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A Cofinal Family of Equivalence Relations and Borel Ideals Generating Them

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Abstract—An increasing ω_1-sequence of Borel equivalence relations on a Polish space that is cofinal (in the sense of Borel reducibility) in the family of all Borel equivalence relations is defined as a development of Rosendal's construction. It is proved that equivalence relations from this sequence are generated by explicitly defined Borel ideals.

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INTRODUCTION

In Moscow in August 2004, an international mathematical conference was organized that was dedicated to the 100th birthday of Lyudmila Vsevolodovna Keldysh. The authors of the present paper were among the participants of this conference. One of research directions of L.V. Keldysh in descriptive set theory was the construction of effective examples (i.e., those defined by explicit formulas) of sets of higher Borel classes. (These examples were also called arithmetic examples because they were defined by formulas of arithmetic character; see the final work by Keldysh [10] in this field.)

In modern descriptive set theory, effective examples of Borel sets have become of more complex character: usually, one deals with an example of a set with additional properties within a fixed structure. In this relation, of particular interest is the structure of Borel reducibility $\le_{\mathrm{B}}$ of equivalence relations on Polish (i.e., complete separable metric) spaces. Recall that the relation $\mathsf{E} \le_{\mathrm{B}} \mathsf{F}$ (where E and F are equivalences on Polish spaces X and Y, respectively) implies the existence of the injection of E-classes into F-classes that admits a Borel lifting $X \to Y$ (exact definitions are given below).

The family of Borel and then analytic equivalence relations, together with the Borel reducibility $\le_{\mathrm{B}}$, has been the object of intensive study in descriptive set theory since the early 1990s. The interest in this problem is motivated by its close relation to many classification problems in which the equivalence relation is defined as an isomorphism between two mathematical structures in a fixed class of structures (see, for example, Hjorth's monograph [5]).

The structure of Borel reducibility starts from *smooth* Borel equivalence relations defined by the condition $\mathsf{E} \le_{\mathrm{B}} \mathsf{D}_{\mathbb{R}}$, where D_X is the equality relation on X, which is considered as an equivalence relation. The classification of these equivalence relations (in the sense of $\le_{\mathrm{B}}$) is based directly on the number of equivalence classes: a finite number, a countable number $\aleph_0$, or a continuum $\mathfrak{c} = 2^{\aleph_0}$. The Vitali equivalence relation on the real axis $\mathbb{R}$ (i.e., two numbers are equivalent if their difference is a rational number) is $\le_{\mathrm{B}}$-minimal among all nonsmooth Borel equivalence relations according to the theorem by Harrington, Kechris, and Louveau [4] (see also the survey [7]). Further investigations (see, for example, the survey written by Kechris [12]) have shown that there exist mathematically meaningful Borel equivalence relations E_1, E_2, and E_3 that are $\le_{\mathrm{B}}$-minimal over

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the Vitali equivalence and $\leq_B$-incomparable with each other. Wide classes of Borel and analytic equivalence relations have been considered, for example, *countable* equivalence relations (i.e., such that all equivalence classes are countable [1]), c_0-equalities [2], and many others. The general theorem by Louveau and Velickovic [14] shows that the structure of Borel equivalence relations (as a partially ordered set with the relation $\leq_B$) is quite complicated.

There is a maximal element, called a *complete* (or universal) analytic equivalence relation, at the uppermost level in the structure of Borel reducibility of analytic equivalence relations; it can easily be determined starting from the universal analytic set (see Example 6 below). Complete analytic equivalence relations are also known in certain classes of special equivalence relations, especially those induced by the actions of topological groups on Polish spaces.

On the other hand, complete *Borel* equivalence relations (i.e., those that are $\leq_B$-maximal among all Borel equivalence relations) do not exist (see, for example, [3, 6]). Moreover, there exist strictly $\leq_B$-increasing $\leq_B$-cofinal ω_1-sequences of Borel equivalence relations: indeed, any complete analytic equivalence relation naturally generates such a sequence on the basis of its decreasing sequence of upper Borel approximations (see Theorem 16 in Section 6).

The goal of the present paper is to construct a cofinal sequence of this kind that has a number of additional properties according to the following theorem.

Theorem 1. *There exists an ω_1-sequence of Borel equivalence relations E_ν, $\nu < \omega_1$, in the space 2^ω with the following properties: this sequence is $\leq_B$-cofinal in the family of all Borel equivalence relations, the relations E_ν have mathematically meaningful definitions, and, moreover,*

(i) *the relation E_ν belongs to the Borel class $\mathbf{\Sigma}^0_{1+2\nu+1}$;*

(ii) *the relation E_ν is generated by an ideal $I_\nu \subseteq \mathcal{P}(\mathbb{N})$ of the class $\mathbf{\Sigma}^0_{\omega+2\nu+1}$ in the sense that, for any $x, y \in D$, the relation $x \mathrel{\mathsf{E}_\nu} y$ is equivalent to the fact that the set $\{n \in \mathbb{N} \colon x(n) \neq y(n)\}$ belongs to I_ν, and the ideals I_ν also have mathematically meaningful definitions.*

Several remarks concerning Theorem 1. The index ν enters the definition of the relations E_ν via the restriction on the ranks of certain trees. Namely, each E_ν is actually defined as the $(\omega\nu + 2)$th canonical upper Borel approximation $\mathsf{E}_{\rm NT}^{\omega\nu+2}$ of the same complete analytic equivalence relation $\mathsf{E}_{\rm NT}$ (the equivalence relation of normal trees), which was earlier defined in [13] and then investigated in [17].

A complete analytic equivalence relation can be defined by an elementary construction that employs a universal analytic set (Example 6). This construction is inapplicable to the Borel classes $\mathbf{\Sigma}^0_\xi$ and $\mathbf{\Pi}^0_\xi$ (although they also contain universal sets). Nevertheless, complete relations are known for certain initial Borel classes, for example, for the class of σ-compact sets (a part of the class $\mathbf{\Sigma}^0_2 = \mathbf{F}_\sigma$; see, for example, [17]). Relations that are "complete" for higher Borel classes in a certain weaker sense (namely, completeness with respect to the equivalence relations induced by continuous actions of the group of all permutations of $\mathbb{N}$) were constructed in [6]. Unfortunately, for the present, our methods do not allow the construction of complete equivalence relations for Borel classes.

The paper is organized as follows. The first three sections contain auxiliary material, which is mainly related to different types of trees on countable sets and their transformations, including the concept of normal tree in Section 1 and one key transformation in Section 3. Then, in Section 4, we introduce an equivalence relation $\mathsf{E}_{\rm NT}$ for normal trees from [13, 17] and, in Section 5, introduce an ω_1-sequence of its upper Borel approximations $\mathsf{E}^\xi_{\rm NT}$. We establish that all $\mathsf{E}^\xi_{\rm NT}$ are Borel equivalence relations and estimate their Borel class. It turns out that the sequence of equivalence relations $\mathsf{E}^\xi_{\rm NT}$ is $\subseteq$-decreasing, $\bigcap_{\xi<\omega_1} \mathsf{E}^\xi_{\rm NT}$ coincides with $\mathsf{E}_{\rm NT}$, and this intersection has an important canonical intersection property (Section 6), which is known from classical works on constituents. Then, in Section 7, we show that the approximating Borel equivalence relations $\mathsf{E}^\xi_{\rm NT}$ are related to certain Borel ideals in the same way as $\mathsf{E}_{\rm NT}$ itself is related to a certain analytic ideal according to a result of [17]. This actually completes the proof of Theorem 1, because it only remains to set $\mathsf{E}_\nu = \mathsf{E}_{\rm NT}^{\omega\nu+2}$.

Then, in Section 8, we consider a problem of the completeness of approximating equivalence relations in appropriate Borel classes.

In the final Section 9, we discuss certain open questions.

For the reader's convenience, the material is presented so that, in addition to the proof of Theorem 1, we give the proofs of a number of related results from [13, 17], because they are obtained in fact from the same technical lemmas. The approach of the present paper is a generalization of the corresponding points in [13, 17] in the sense that, instead of the property of ill-foundedness of certain trees, we consider the property "to have rank $\geq \xi$," where ξ is a given countable ordinal (which involves the ill-foundedness by definition).

1. NOTATIONS

We will use conventional set-theoretic notations, including notations related to the Borel projective classes, as well as other concepts of descriptive set theory (see [11] or [8]). In particular, "analytic" means "belonging to the class $\mathbf{\Sigma}^1_1$."

If P is an n-ary relation, then $P(x_1, \dots, x_n)$ denotes $\langle x_1, \dots, x_n \rangle \in P$.

If X is a countable set, then 2^X is considered as a topological space with the product topology; this is a Polish space (a separable space metrizable by a complete metric) that is homeomorphic to the Cantor discontinuum. The degree set $\mathcal{P}(X) = \{Y : Y \subseteq X\}$ is also considered as a Polish space that is isometric to 2^X by a map that assigns to each $Y \subseteq X$ its characteristic function $\chi_Y \in 2^X$.

The symmetric difference is denoted by $X \,\Delta\, Y = (X \setminus Y) \cup (Y \setminus X)$.

Equivalence relations. As regards the Borel reducibility of equivalence relations, the expression $\mathsf{E} \leq_{\rm B} \mathsf{F}$ implies that E and F are equivalence relations on certain Polish spaces X and Y and that there exists a Borel map $\vartheta\colon X \to Y$ such that $x \,\mathsf{E}\, x' \Leftrightarrow \vartheta(x) \,\mathsf{F}\, \vartheta(x')$ for any $x, x' \in X$. Such a map ϑ, called a (Borel) reduction of E to F, obviously induces an injection of E-classes into F-classes that is defined as $[x]_{\mathsf{E}} \mapsto [\vartheta(x)]_{\mathsf{F}}$.

The associated equivalence is denoted by $\sim_{\rm B}$, i.e., $\mathsf{E} \sim_{\rm B} \mathsf{F}$ if $\mathsf{E} \leq_{\rm B} \mathsf{F}$ and $\mathsf{F} \leq_{\rm B} \mathsf{E}$. Finally, $<_{\rm B}$ is the strict reducibility relation, i.e., $\mathsf{E} <_{\rm B} \mathsf{F}$ when $\mathsf{E} \leq_{\rm B} \mathsf{F}$ but not $\mathsf{F} \leq_{\rm B} \mathsf{E}$.

Ideals. Any ideal $\mathcal{I}$ on the set X ($\mathcal{I} \subseteq \mathcal{P}(X)$, $\mathcal{I}$ is closed with respect to finite unions and is $\subseteq$-closed downward) generates an equivalence relation $\mathsf{E}_{\mathcal{I}}$ on $\mathcal{P}(X)$ so that, for any $Y, Z \subseteq X$, the relations $Y \,\mathsf{E}_{\mathcal{I}}\, Z$ and $Y \,\Delta\, Z \in \mathcal{I}$ are equivalent. Using the above-mentioned identification, one can also define $\mathsf{E}_{\mathcal{I}}$ as an equivalence relation on 2^X such that, for any $a, b \in 2^X$, the relations $a \,\mathsf{E}_{\mathcal{I}}\, b$ and $\{x \in X : a(x) \neq b(x)\} \in \mathcal{I}$ are equivalent.

Trees. For any set X, denote by X^n the set of all sequences of length n that consist of elements of X and denote by $X^{<\omega} = \bigcup_{n \in \mathbb{N}} X^n$ the set of all finite sequences of elements of X.

Λ is an empty sequence; it belongs to $X^{<\omega}$ for any $X \neq \varnothing$.

$\mathtt{lh}\, s$ is the length of a finite sequence s.

$s \subseteq t$ implies that the sequence t continues s (possibly, $t = s$).

$s^\wedge x$ is the sequence obtained by adding x as the rightmost element to s.

$x^\wedge s$ is understood similarly: x becomes the leftmost element.

A *tree* on a set X is an arbitrary subset $T \subseteq X^{<\omega}$ that is closed with respect to restrictions; i.e., if $t \in T$, $s \in X^{<\omega}$, and $s \subseteq t$, then $s \in T$. Below, we will consider trees on products of sets. Note that any $s \in (X_1 \times \ldots \times X_n)^{<\omega}$ is formally a sequence of n-tuples $\langle x_1, \dots, x_n \rangle$, where $x_i \in X_i\ \forall\, i$; however, we will identify such s with the n-tuple $\langle s_1, \dots, s_n \rangle$, where all $s_i \in {X_i}^{<\omega}$ have the same length as s itself and $s(i) = \langle s_1(i), \dots, s_n(i) \rangle$ for any i.

Ranks. Let ∞ stand for a formal element that is greater than all the ordinals. It is well-known that any tree $R \subseteq X^{<\omega}$ admits a *rank function*, a unique map $\mathtt{rnk}_R\colon R \to \mathtt{Ord} \cup \{\infty\}$ that satisfies the following conditions:

(a) $\mathtt{rnk}_R(r) = -1$ for any $r \notin R$;

(b) $\mathtt{rnk}_R(r) = \sup_{r^\wedge n \in R} \mathtt{rnk}_R(r^\wedge n)$ for any $r \in R$.[2] In particular, $\mathtt{rnk}_R(r) = 0$ if and only if $r \in R$ is the $\subseteq$-maximal element of R;

(c) $\mathtt{rnk}_R(r) = \infty$ if and only if R has an infinite branch that contains r, i.e., there exists $\gamma \in X^\omega$ such that $\gamma \restriction n \in R$ for any n and $\gamma \restriction \mathtt{lh}\, r = r$.

In addition, set $\mathtt{rnk}(\varnothing) = -1$ for the empty tree $\varnothing$ and $\mathtt{rnk}(R) = \mathtt{rnk}_R(\Lambda)$ for any nonempty tree R. (Note that the empty sequence Λ belongs to any tree $\varnothing \neq R \subseteq X^{<\omega}$.) A tree R is well-founded (i.e., does not have infinite branches) when $\mathtt{rnk}(R) < \infty$.

Normal trees. If X is an ordered set (for example, $X = \mathbb{N}$) and $s, t \in X^{<\omega}$, then $s \leq_{\mathtt{cw}} t$ means that $\mathtt{lh}\, s = \mathtt{lh}\, t$ and $s(i) \leq t(i)$ for any $i < \mathtt{lh}\, s$. (This is a *componentwise* order, which is different from the lexicographic order.) If a binary operation of addition $+$ is defined on X (for example, $X = \mathbb{N}$) and $s, t \in X^{<\omega}$, $\mathtt{lh}\, s = \mathtt{lh}\, t$, then $s +_{\mathtt{cw}} t$ is a sequence of the same length defined in terms of the componentwise addition $(s +_{\mathtt{cw}} t)(i) = s(i) + t(i)$ for all i. The following definition introduces a class of trees that plays a key role in the present study.

Definition 2 (see [13, 17]). A tree T on $2 \times \mathbb{N}$ is called *normal* if, for any $u \in 2^{<\omega}$ and $s, t \in \mathbb{N}^{<\omega}$ such that $\mathtt{lh}\, u = \mathtt{lh}\, s = \mathtt{lh}\, t$ and $s \leq_{\mathtt{cw}} t$, we have $\langle u, s\rangle \in T \Rightarrow \langle u, t\rangle \in T$.

NT is the set of all nonempty normal trees $T \subseteq (2 \times \mathbb{N})^{<\omega}$. $\square$

For example, $(2 \times \mathbb{N})^{<\omega}$ itself is a normal tree.

2. TRANSFORMATIONS OF TREES

Here we consider certain transformations of trees on $\mathbb{N}$. The main goal is to estimate the rank of transformed trees compared with the rank of initial trees.

Finite union. Consider any pair of trees $S, T \subseteq \mathbb{N}^{<\omega}$. *We argue that* $\mathtt{rnk}(S \cup T) = \mathtt{max}\{\mathtt{rnk}(S), \mathtt{rnk}(T)\}$. Indeed, it suffices to prove the equality

$(*)$ $\mathtt{rnk}_{S\cup T}(r) = \mathtt{max}\{\mathtt{rnk}_S(r), \mathtt{rnk}_T(r)\}$

for any $r \in S \cup T$. If $\mathtt{rnk}_{S\cup T}(r) = \infty$, then the tree $S \cup T$ contains an infinite branch; hence, this is valid at least for one of the trees S and T, so we also have $\mathtt{max}\{\mathtt{rnk}_S(r), \mathtt{rnk}_T(r)\} = \infty$. It remains to verify $(*)$ by means of transfinite induction on $\mathtt{rnk}_{S\cup T}(r)$ in the case when $\mathtt{rnk}_{S\cup T}(r) < \omega_1$. If this ordinal is equal to 0, then r is $\subseteq$-maximal in $S \cup T$, which obviously implies that $\mathtt{rnk}_S(r) \leq 0$, $\mathtt{rnk}_T(r) \leq 0$, and at least one of these ranks is exactly 0. If $0 < \mathtt{rnk}_{S\cup T}(r) < \omega_1$, then, by the induction hypothesis,

$$\mathtt{rnk}_{S\cup T}(r) = \sup_n\bigl\{\mathtt{rnk}_{S\cup T}(r^\wedge n)\bigr\} = \sup_n \mathtt{max}\bigl\{\mathtt{rnk}_S(r^\wedge n), \mathtt{rnk}_T(r^\wedge n)\bigr\} = \mathtt{max}\bigl\{\mathtt{rnk}_S(r), \mathtt{rnk}_T(r)\bigr\},$$

which was to be proved.

Contraction. Let $S \subseteq 2^{<\omega}$ be a tree. Fix a bijection $b\colon \mathbb{N}^2 \xrightarrow{\text{onto}} \mathbb{N}$ once and for all. For any finite sequence $s = \langle k_0, k_1, \dots, k_n\rangle \in 2^{<\omega}$ of length $\mathtt{lh}\, s = n + 1 \geq 2$, we define a sequence $\widehat{s} = \langle b(k_0, k_1), k_2, \dots, k_n\rangle$ of length n. Denote by $\widehat{S} = \{\Lambda\} \cup \{\widehat{s}\colon s \in S \wedge \mathtt{lh}\, s \geq 2\}$ a *contracted tree*. Then, $\mathtt{rnk}(\widehat{S}) = \mathtt{rnk}(S) - 1$, where $\infty - 1 = \infty$, $\lambda - 1 = \lambda$ for any limit ordinal λ or $\lambda = 0$, and, finally, $(\xi + 1) - 1 = \xi$.

[2] We define $\sup\Omega$ for $\Omega \subseteq \mathtt{Ord}$ as the least ordinal that is strictly greater than all ordinals in Ω. We also define $\sup\Omega = \infty$ when Ω contains ∞.

Countable union. The equality $\mathtt{rnk}(S\cup T)=\mathtt{max}\{\mathtt{rnk}(S),\mathtt{rnk}(T)\}$ is obviously violated for countable unions. However, there is another useful operation. For any sequence of trees $T_n\subseteq\mathbb{N}^{<\omega}$, let $\sum_n^* S_n$ denote the tree $T=\{\Lambda\}\cup\{n^\wedge t\colon t\in T_n\}$. It is clear that $\mathtt{rnk}_T(n^\wedge t)=\mathtt{rnk}_{T_n}(t)$; therefore, $\mathtt{rnk}(T)=\mathtt{sup}_n\,\mathtt{rnk}(T_n)$.

Countable intersection. Given a sequence of trees $T_n\subseteq\mathbb{N}^{<\omega}$, how should one find a tree T that satisfies $\mathtt{rnk}(T)=\mathtt{inf}_n\,\mathtt{rnk}(T_n)$? In principle, one may use an equal-length subproduct of the usual Cartesian product $\prod_{n\in\mathbb{N}}T_n$. Indeed, let T consist of all $t\in\prod_{n\in\mathbb{N}}T_n$ such that $\mathtt{lh}\,t(n)=\mathtt{lh}\,t(m)$ for any $m,n\in\mathbb{N}$ with componentwise ordering ($s\preccurlyeq t$ when $s(n)\subseteq t(n)$ for any n); this tree T is the required one. However, this construction is not quite useful for our purposes because the tree T is generally uncountable.

To solve this problem, we define $\prod_n^* T_n$ as the set of all finite sequences of the form $t=\langle t_0,\dots,t_n\rangle$, where $t_n\in T_n$ and $\mathtt{lh}\,t_k=n$ for any $k\le n$. We assume that $\langle t_0,\dots,t_n\rangle\preccurlyeq\langle s_0,\dots,s_m\rangle$ when $n\le m$ and $t_k\subseteq s_k$ (in $\mathbb{N}^{<\omega}$) for any $k\le n$. In addition, we introduce Λ in T by setting $\Lambda\preccurlyeq t$ for any $t\in T$. It is obvious that $\langle T;\preccurlyeq\rangle$ is an at most countable tree that is order-isomorphic to a certain tree in $\mathbb{N}^{<\omega}$.

Lemma 3. *Suppose that T_n, $n\in\mathbb{N}$, are trees in $\mathbb{N}^{<\omega}$ and $T=\prod_n^* T_n$. Then, $\mathtt{rnk}(T)\le\mathtt{min}_{n\in\mathbb{N}}\,\mathtt{rnk}(T_n)+n$. Moreover, $\mathtt{rnk}(T)=\infty$ is equivalent to $\mathtt{rnk}(T_n)=\infty$ for any n.*

Proof. Simple arguments involving transfinite induction on $\mathtt{rnk}_T(t)$ show that $\mathtt{rnk}_T(t)\le\mathtt{rnk}_{T_n}(t_n)$ whenever $n\le m$ and $t=\langle t_0,\dots,t_m\rangle\in T$. Hence, $\mathtt{rnk}(T)\le n+\mathtt{min}_{t\in T_n,\,\mathtt{lh}\,t=n}\,\mathtt{rnk}_{T_n}(t_n)$; therefore, $\mathtt{rnk}(T)\le\mathtt{rnk}(T_n)+n$. This inequality also holds when $\mathtt{rnk}(T_n)<n$ (i.e., T_n does not contain sequences of length $\ge n$); then in fact $\mathtt{rnk}(T)\le n$. Thus, in any case, $\mathtt{rnk}(T)\le\mathtt{rnk}(T_n)+n$.

Note also that an infinite branch in T can easily be obtained once an infinite branch is contained in every tree T_n; conversely, the existence of an infinite branch in the tree T immediately implies the existence of infinite branches in all the trees T_n. □

The additional term $+n$ in the lemma is annoying; however, it is insignificant in the most important case of application of the lemma in this paper: $\mathtt{rnk}(T)\le\lambda$ whenever λ is a limit ordinal and $\mathtt{rnk}(T_n)<\lambda$ for any n.

Componentwise sum. Set $S+_{\mathtt{cw}}T=\{s+_{\mathtt{cw}}t\colon s\in S\wedge t\in T\wedge\mathtt{lh}\,s=\mathtt{lh}\,t\}$ for any trees $S,T\subseteq\mathbb{N}^{<\omega}$. The following technical lemma will be repeatedly used below. It shows that componentwise summation of trees behaves to a certain extent like an equal length Cartesian product, i.e., $S\times T=\{\langle s,t\rangle\colon s\in S\wedge t\in T\wedge\mathtt{lh}\,s=\mathtt{lh}\,t\}$.

Lemma 4. *Let $S,T\subseteq\mathbb{N}^{<\omega}$ be arbitrary trees and $W=S+_{\mathtt{cw}}T$.*

If $s\in S$, $t\in T$, and $\mathtt{lh}\,s=\mathtt{lh}\,t$, then $\mathtt{rnk}_W(s+_{\mathtt{cw}}t)\ge\mathtt{min}\{\mathtt{rnk}_S(s),\mathtt{rnk}_T(t)\}$.

Conversely, for any $w\in W$, there exist sequences $s\in S$ and $t\in T$ such that $\mathtt{lh}\,s=\mathtt{lh}\,t=\mathtt{lh}\,w$, $s+_{\mathtt{cw}}t=w$, and $\mathtt{rnk}_W(w)=\mathtt{min}\{\mathtt{rnk}_S(s),\mathtt{rnk}_T(t)\}$.

Therefore, $\mathtt{rnk}(W)=\mathtt{min}\{\mathtt{rnk}(S),\mathtt{rnk}(T)\}$.

Proof. The first inequality is proved by induction on the ordinal $\mu_{st}=\mathtt{min}\{\mathtt{rnk}_S(s),\mathtt{rnk}_T(t)\}$. If $\mu_{st}=0$, then there is nothing to prove. Suppose that $\mu_{st}=\mu+1$. By definition, there exist numbers $i,j\in\mathbb{N}$ such that $s'=s^\wedge i\in S$, $t'=t^\wedge j\in T$, and $\mu_{s't'}\ge\mu$. Then, $\mathtt{rnk}_W(s'+_{\mathtt{cw}}t')\ge\mu$ by the induction hypothesis and, hence, $\mathtt{rnk}_W(s+_{\mathtt{cw}}t)\ge\mu+1$ because $s'+_{\mathtt{cw}}t'=(s+_{\mathtt{cw}}t)^\wedge(i+j)$. The limit step, i.e., when μ_{st} is a limit ordinal $<\omega_1$, and the step in the case when one or both ranks $\mathtt{rnk}_S(s)$ and $\mathtt{rnk}_T(t)$ are equal to ∞ (the case of ill-founded trees) are considered similarly.

Let us prove the second assertion by induction on $\mu_w=\mathtt{rnk}_W(w)$.

If $\mu_w=\mu+1$ (a nonlimit induction step), then there exists $k\in\mathbb{N}$ such that $w'=w^\wedge k\in W$. Applying the induction hypothesis, we find $s'=s^\wedge i\in S$ and $t'=t^\wedge j\in T$ such that $w'=s'+_{\mathtt{cw}}t'$ and $\mathtt{rnk}_W(w')=\mathtt{min}\{\mathtt{rnk}_S(s'),\mathtt{rnk}_T(t')\}$. Now, it is obvious that $s+_{\mathtt{cw}}t=w$ and $\mathtt{rnk}_W(w)=\mathtt{min}\{\mathtt{rnk}_S(s),\mathtt{rnk}_T(t)\}$.

Consider the limit induction step: $\mu_w = \lambda < \omega_1$ is a limit ordinal. For any $\xi < \lambda$, there exists $k_\xi \in \mathbb{N}$ such that $w^\wedge k_\xi \in S$ and $\mathtt{rnk}_W(w^\wedge k_\xi) \geq \xi$; hence, by the induction hypothesis, there exist $s'_\xi = s_\xi{}^\wedge i_\xi \in S$ and $t'_\xi = t_\xi{}^\wedge j_\xi \in T$ such that $s'_\xi +_{\mathtt{cw}} t'_\xi = w^\wedge k_\xi$ and $\mathtt{min}\{\mathtt{rnk}_S(s'_\xi), \mathtt{rnk}_T(t'_\xi)\} \geq \xi$. Then, $s_\xi +_{\mathtt{cw}} t_\xi = w$ for any $\xi < \lambda$; in particular, s_ξ and t_ξ belong to the finite set $\{t \in \mathbb{N}^{\mathtt{lh}\, w}\colon t \leq_{\mathtt{cw}} w\}$. Hence, there exist $s \in S$ and $t \in T$ such that $\mathtt{lh}\, s = \mathtt{lh}\, t = \mathtt{lh}\, w$ and $s +_{\mathtt{cw}} t = w$, and we have $s_\xi = s$ and $t_\xi = t$ for a set of ordinals ξ that is cofinal in λ. It is clear that these s and t are such as required.

Finally, assume that $\mu_w = \infty$; thus, there exists a $\gamma \in \mathbb{N}^\omega$ such that $\gamma \restriction \mathtt{lh}\, w = w$ and $\gamma \restriction n \in W$ for any n. Then, for any n, there exist $s_n \in S$ and $t_n \in T$ of length n that satisfy $s_n +_{\mathtt{cw}} t_n = \gamma \restriction n$. Then, the sequences s_n and t_n belong to the set $\{t \in \mathbb{N}^{<\omega}\colon t \leq_{\mathtt{cw}} \gamma \restriction \mathtt{lh}\, t\}$, which is a tree with finite branchings. Therefore, according to König's lemma, there exist infinite sequences $\alpha, \beta \in \mathbb{N}^\omega$ such that $\forall\, m\, \exists\, n \geq m\, (\alpha \restriction m = s_n \restriction m \wedge \beta \restriction m = t_n \restriction m)$. Obviously, $\alpha +_{\mathtt{cw}} \beta = \gamma$ and, at the same time, $\alpha \restriction m \in S$ and $\beta \restriction m \in T$ hold for any m. Let us define $s = \alpha \restriction \mathtt{lh}\, w$ and $t = \beta \restriction \mathtt{lh}\, w$; then, $\mathtt{rnk}_S(s) = \mathtt{rnk}_T(t) = \infty$ and $w = s +_{\mathtt{cw}} t$, which was to be proved. □

3. THE LOUVEAU–ROSENDAL TRANSFORM

Consider an arbitrary $\boldsymbol{\Sigma}^1_1$-set A in the space $2^\omega \times 2^\omega$. It is well known from the elementary topology of Polish spaces that any $\boldsymbol{\Sigma}^1_1$-set in a Polish space S coincides with the projection of a certain closed subset of the space $S \times \mathbb{N}^\omega$ onto S. Thus, there exists a closed set $P \subseteq 2^\omega \times 2^\omega \times \mathbb{N}^\omega$ that satisfies $A = \mathtt{dom}\, P = \{\langle x, y\rangle\colon \exists\, z\, P(x, y, z)\}$. Next, there exists a tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ (a tree on $2 \times 2 \times \mathbb{N}$) such that $P = [R] = \{\langle x, y, \gamma\rangle\colon \forall\, n\, R(x \restriction n, y \restriction n, \gamma \restriction n)\}$; therefore,

$$\langle x, y\rangle \in A \;\Leftrightarrow\; \exists\, \gamma \in \mathbb{N}^\omega\, \forall\, n\, R(x \restriction n, y \restriction n, \gamma \restriction n) \;\Leftrightarrow\; R_{xy} \text{ is ill-founded,} \tag{1}$$

where, for any tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ and any $x, y \in 2^\omega$, $R_{xy} = \{s \in \mathbb{N}^{<\omega}\colon R(x \restriction \mathtt{lh}\, s, y \restriction \mathtt{lh}\, s, s)\}$. (It is clear that R_{xy} is a subtree of $\mathbb{N}^{<\omega}$.) If A is an arbitrary $\boldsymbol{\Sigma}^1_1$-set, then one can hardly establish anything concerning the structure of the tree R, which generates A in the sense of (1). However, assuming that $A = \mathsf{E}$ is an equivalence relation on 2^ω, we can hope that R behaves better, and it really does.

Theorem 5. *Suppose that $Q \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ is a tree and the set*

$$\begin{aligned} \mathsf{E} &= \bigl\{\langle x, y\rangle \in 2^\omega \times 2^\omega\colon\ \exists\, \gamma \in \mathbb{N}^\omega\, \forall\, n\, Q(x \restriction n, y \restriction n, \gamma \restriction n)\bigr\} \\ &= \bigl\{\langle x, y\rangle\colon\ Q_{xy} \text{ is ill-founded, i.e., } \mathtt{rnk}(Q_{xy}) = \infty\bigr\} \end{aligned} \tag{2}$$

is an equivalence relation on 2^ω. Then, there exists a tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ that satisfies the following conditions:

(i) *symmetry*: *$R(u, v, s) \Leftrightarrow R(v, u, s)$, and then $R_{xy} = R_{yx}$ for all $x, y \in 2^\omega$;*

(ii) *if $u \in 2^\omega$, $s \in \mathbb{N}^\omega$, and $\mathtt{lh}\, s = \mathtt{lh}\, u$, then $R(u, u, s)$;*

(iii) *normality, as in Section* 1: *if $R(u, v, s)$, $t \in \mathbb{N}^\omega$, and $s \leq_{\mathtt{cw}} t$, then $R(u, v, t)$;*

(iv) *transitivity*: *if $R(u, v, s)$ and $R(v, w, t)$, then $R(u, w, s +_{\mathtt{cw}} t)$;*

(v) *for any $x, y \in 2^\omega$, $\mathtt{rnk}(R_{xy}) = \infty$ if and only if $\mathtt{rnk}(Q_{xy}) = \infty$; therefore,* (2) *also holds for the tree R instead of Q.*

This theorem is equivalent to Theorem 4 in [13].

Proof. *Part* 1. Note that the tree

$$\widehat{Q} = Q \cup \bigl\{\langle u, u, s\rangle\colon\ u \in 2^\omega \wedge s \in \mathbb{N}^\omega \wedge \mathtt{lh}\, s = \mathtt{lh}\, u\bigr\} \cup \{\langle u, v, s\rangle\colon Q(v, u, s)\}$$

satisfies $\widehat{Q}_{xy} = Q_{xy} \cup Q_{yx} \cup D_{xy}$, where $D_{xy} = \mathbb{N}^{<\omega}$ for $x = y$ and $D_{xy} = \varnothing$ otherwise. However, if $x = y$, then $x \mathsf{E} y$ (since E is an equivalence relation); therefore, $\mathtt{rnk}(Q_{xy}) = \infty$. Hence, $\mathtt{rnk}(\widehat{Q}_{xy}) = \mathtt{max}\{\mathtt{rnk}(Q_{xy}), \mathtt{rnk}(Q_{yx})\}$ for any x and y (we refer to the result for unions in Section 2); hence, (2) still holds for $\widehat{Q}$. Moreover, it is obvious that $\widehat{Q}$ satisfies both (i) and (ii) and $\widehat{Q}$ is ξ-bounded if so is Q.

Thus, *we may assume from the very beginning that Q satisfies* (i) *and* (ii).

Part 2. Under this assumption, to satisfy (iii), we set

$$\widehat{Q} = \big\{\langle u, v, t\rangle \in (2 \times 2 \times \mathbb{N})^{<\omega} \colon \exists \langle u, v, s\rangle \in Q \ (s \leq_{\mathtt{cw}} t)\big\}.$$

This is still a tree on $2 \times 2 \times \mathbb{N}$ that includes Q and satisfies (i), (ii), and (iii). In addition, we have $\widehat{Q}_{xy} = Q_{xy} +_{\mathtt{cw}} 2^{<\omega}$ for any $x, y \in 2^\omega$; therefore, $\mathtt{rnk}(Q_{xy}) = \mathtt{rnk}(\widehat{Q}_{xy})$ according to Lemma 4; hence, (2) holds for $\widehat{Q}$ and $\widehat{Q}$ is ξ-bounded if so is Q.

Thus, *we may assume from the very beginning that Q satisfies conditions* (i), (ii), and (iii).

Part 3. A slightly more difficult problem is to satisfy (iv). Proceeding directly, we could determine a new tree R so that it would contain all triples of the form $\langle u_0, u_{n+1}, s_0 +_{\mathtt{cw}} \ldots +_{\mathtt{cw}} s_k\rangle$, where $\langle u_i, u_{i+1}, s_i\rangle \in Q$ for any $i = 0, 1, \ldots, k$. However, for this procedure to be correct, such a construction must be equipped with a counter for the number k of steps in the finite chain involved. Let us proceed to the implementation of this idea.

Reasoning under the assumption that Q satisfies (i), (ii), and (iii) (see Part 2), we determine a tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ as follows. Suppose that $n \in \mathbb{N}$, $u, v \in 2^n$, $s \in \mathbb{N}^n$, $k \in \mathbb{N}$, and $i, j \in 2 = \{0, 1\}$. We say that $\langle u^\wedge i, v^\wedge j, k^\wedge s\rangle \in R$ if

$$\exists\, u_0, u_1, \ldots, u_k \in 2^n \ \big(u_0 = u \wedge u_k = v \wedge \forall\, \ell < k\, Q(u_\ell, u_{\ell+1}, s)\big). \tag{3}$$

In addition, we naturally assume that $\langle \Lambda, \Lambda, \Lambda\rangle \in R$ (where Λ is an empty sequence). It is clear that R is a tree on $2 \times 2 \times \mathbb{N}$ because so is Q.

We argue that, under our assumptions, the tree R satisfies all the conditions (i)–(v).

(i) If $u_0, \ldots, u_k$ guarantee $R(u^\wedge i, v^\wedge j, k^\wedge s)$, then the reverse sequence $u_k, \ldots, u_0$ guarantees $R(v^\wedge j, u^\wedge i, k^\wedge s)$ in the sense of (3) because the tree Q satisfies (i).

(iii) Suppose that $\langle u^\wedge i, v^\wedge j, k^\wedge s\rangle \in R$, and let $u_0, \ldots, u_k$ guarantee (3). Denote $n = \mathtt{lh}\, u = \mathtt{lh}\, v = \mathtt{lh}\, s = \mathtt{lh}\, u_\ell \ \forall \ell$. Assume that $k \leq k'$ and $s \leq_{\mathtt{cw}} s'$ (hence, $\mathtt{lh}\, s' = n$). Set $u_\ell = v$ whenever $k < \ell \leq k'$. Then $Q(u_\ell, u_{\ell+1}, s)$ also holds for $k < \ell < k'$ according to (ii) for Q (indeed, $u_\ell = u_{\ell+1}$ in this case). Hence, $Q(u_\ell, u_{\ell+1}, s')$ holds for any $\ell < k'$ according to (iii) for Q. By definition, this guarantees that $\langle u^\wedge i, v^\wedge j, k'^\wedge s'\rangle \in R$, which was to be proved.

(ii) If $k = 0$ and $u = v$, then (3) holds in view of obvious arguments (with an empty list of intermediate sequences $u_1, \ldots, u_{k-1}$); therefore, $R(u^\wedge i, u^\wedge j, 0^\wedge s)$ holds for any $u \in 2^\omega$ and $s \in \mathbb{N}^\omega$ of equal length; in particular, $R(u, u, 0^n)$ for any n and $u \in \mathbb{N}^\omega$ with $\mathtt{lh}\, u = n$. It remains to apply property (iii), which we have just proved.

(iv) Suppose that the triples $\langle u^\wedge i, v^\wedge j, k^\wedge s\rangle$ and $\langle v^\wedge j, w^\wedge \rho, \kappa^\wedge \sigma\rangle$ belong to R and n is the length of each of the sequences u, v, s, w, and σ. Assume that $u_0, \ldots, u_k$ guarantee $R(u^\wedge i, v^\wedge j, k^\wedge s)$ in the sense of (3) and, accordingly, $v_0, \ldots, v_\kappa$ guarantee $R(v^\wedge j, w^\wedge \rho, \kappa^\wedge \sigma)$. (All u_ℓ and v_ℓ belong to 2^n.) Since Q satisfies (iii), the same sequences guarantee $R(u^\wedge i, v^\wedge j, k^\wedge t)$ and $R(v^\wedge j, w^\wedge \rho, \kappa^\wedge t)$, where $t = s +_{\mathtt{cw}} \sigma$ (componentwise). This implies that the combined complex $u_0, \ldots, u_{k-1}, u_k = v_0$, $v_1, \ldots, v_\kappa$ guarantees $R(u^\wedge i, w^\wedge \rho, (k + \kappa)^\wedge t)$, which was to be proved.

(v) Notice that, by definition, $Q(u, v, s) \Rightarrow R(u^\wedge i, v^\wedge j, 1^\wedge s)$ for any $i, j = 0, 1$. Hence, for any $x, y \in 2^\omega$, $s \in Q_{xy} \Rightarrow 1^\wedge s \in R_{xy}$; therefore, R_{xy} is ill-founded if so is Q_{xy}.

The converse implication in (v) requires more effort. This reasoning belongs to Louveau and Rosendal [13]. Suppose that $\mathtt{rnk}(R_{xy}) = \infty$, i.e., there exists an infinite sequence $\delta \in \mathbb{N}^\omega$ such that $\forall\, n\; R(x \restriction n, y \restriction n, \delta \restriction n)$. Set $k = \delta(0)$ and $\gamma(m) = \delta(m+1)$ for any m so that $\delta = k^\wedge\gamma$. By definition, for any n, there exist sequences $u_0^n, \dots, u_k^n \in 2^n$ that satisfy $u_0^n = x \restriction n$, $u_k^n = y \restriction n$, and $Q(u_\ell^n, u_{\ell+1}^n, \gamma \restriction n)$ for any $\ell < k$. Each $(k+1)$-tuple $\langle u_0^n, \dots, u_k^n\rangle \in (2^n)^{k+1}$ can be considered as an n-tuple in $(2^{k+1})^n$. According to König's lemma, there exist infinite sequences $x_0, \dots, x_k \in 2^\omega$ such that, for any m, there exists a number $n \ge m$ that satisfies $x_\ell \restriction m = u_\ell^n \restriction m$ for any $\ell \le k$. Hence, $x_0 = x$ and $x_k = y$, and since Q is a tree, $Q(x_\ell \restriction m, x_{\ell+1} \restriction m, \gamma \restriction m)$ holds for any $\ell < k$ and any m. We conclude that $x_\ell \,\mathsf{E}\, x_{\ell+1}$ for any $\ell < k$ according to (2) for Q; this implies that $x \,\mathsf{E}\, y$ since E is an equivalence relation. Finally, Q_{xy} is ill-founded in view of (2) for Q. □

4. EQUIVALENCE OF NORMAL TREES

Recall that an analytic (i.e., $\mathbf{\Sigma}^1_1$) equivalence relation U is *complete* when $\mathsf{F} \le_{\mathrm{B}} \mathsf{U}$ holds for any other analytic equivalence relation F. The elementary construction that results in such a relation is well known.

Example 6 ("universal" complete analytic equivalence relation)**.** We begin with a $\mathbf{\Sigma}^1_1$-set $U \subseteq \mathcal{N}^3$ that is universal in the sense that, for any $\mathbf{\Sigma}^1_1$-set $P \subseteq \mathcal{N}^2$, there exists an $x \in \mathcal{N}$ such that P coincides with the section $U_x = \{\langle y, z\rangle : \langle x, y, z\rangle \in U\}$. (See [11] about the existence of universal sets in Borel and projective classes.) Define a set $P \subseteq \mathcal{N}^3$ so that each section P_x coincides with the closure of the section U_x in the sense of equivalence, i.e., with the least equivalence relation that includes U_x. Formally, $\langle y, z\rangle \in P_x$ when there is a finite sequence $y = y_0, y_1, y_2, \dots, y_n, y_{n+1} = z$ such that, for any $k \le n$, either $\langle y_k, y_{k+1}\rangle$ belongs to U_x, or $\langle y_{k+1}, y_k\rangle$ belongs to U_x, or just $y_k = y_{k+1}$.

It is clear that P is still a $\mathbf{\Sigma}^1_1$-set in $\mathcal{N}^3$ and each section P_x is an (analytic) equivalence relation. Moreover, if U_x itself is an equivalence relation, then simply $P_x = U_x$. Thus, the family of all sections P_x, $x \in \mathcal{N}$, is identical to the family of all analytic equivalence relations on $\mathcal{N}$. We argue that an (analytic) equivalence relation U on $\mathcal{N}^2$ defined so that $\langle x, y\rangle \,\mathsf{U}\, \langle x', y'\rangle$ when $x = x'$ and $\langle y, y'\rangle \in P_x$ is complete. Indeed, take an arbitrary analytic equivalence relation F on $\mathcal{N}$. Then, $\mathsf{F} = P_x$ for an appropriate x according to the aforesaid, and so the map $\vartheta(y) = \langle x, y\rangle$ is a continuous reduction of F to U, which was to be proved. □

However, the nature of the completeness property of the equivalence relations defined by such a general method is not quite convenient for certain applications. This is the reason why one sometimes considers other examples of complete relations in which the completeness is realized in a more special way. To give an important example of such a more special complete analytic equivalence relation, we apply the concept of a normal tree (see Definition 2).

Definition 7 (equivalence of normal trees [13])**.** Suppose that $S, T \in \mathbf{NT}$. By $\textsc{Emb}(S, T)$ we will denote the set of all finite sequences $f \in \mathbb{N}^{<\omega}$ such that $\langle u, s\rangle \in S \Rightarrow \langle u, s +_{\mathtt{cw}} (f \restriction n)\rangle \in T$ for any $n \le \mathtt{lh}\, f$, $u \in 2^n$, and $s \in \mathbb{N}^n$.

It is clear that $\textsc{Emb}(S, T)$ is a tree in $\mathbb{N}^{<\omega}$ that contains Λ.

We assume that $S \le_{\mathrm{NT}} T$ when the tree $\textsc{Emb}(S, T)$ is ill-founded, or, which is equivalent,

$$\exists\, \gamma \in \mathbb{N}^\omega\, \forall\, n\, \forall\, u \in 2^n\, \forall\, s \in \mathbb{N}^n\, \big(\langle u, s\rangle \in S \Rightarrow \langle u, s +_{\mathtt{cw}} \gamma \restriction n\rangle \in T\big).$$

We say that $S \,\mathsf{E}_{\mathrm{NT}}\, T$ when $S \le_{\mathrm{NT}} T$ and $T \le_{\mathrm{NT}} S$.[3] □

Thus, $S \le_{\mathrm{NT}} T$ points to the existence of a certain translation embedding of T in S. The relation $\le_{\mathrm{NT}}$ is a partial order on $\mathbf{NT}$; accordingly, E_{NT} is an equivalence relation on $\mathbf{NT}$. Next, applying

[3] In [17], $\le_{\mathrm{NT}}$ and E_{NT} are denoted by $\le^*_{\max}$ and $E^*_{\max}$, respectively.

componentwise addition to the sequences γ that guarantee $\le_{\rm NT}$, we can easily show that $S \mathsf{E}_{\rm NT} T$ is equivalent to the existence of a $\gamma \in \mathbb{N}^\omega$ such that, for any n and any $u \in 2^n$ and $s \in \mathbb{N}^n$, we have simultaneously the following:

$$\langle u, s\rangle \in S \Rightarrow \langle u, s +_{\tt cw} \gamma \restriction n\rangle \in T \qquad \text{and} \qquad \langle u, s\rangle \in T \Rightarrow \langle u, s +_{\tt cw} \gamma \restriction n\rangle \in S. \tag{4}$$

One may notice that, by definition, any $T \in \mathbf{NT}$ is a subset of the countable set $(2 \times \mathbb{N})^{<\omega}$. Thus, $\mathbf{NT}$ is a subset of the Polish space $\mathcal{P}((2 \times \mathbb{N})^{<\omega})$, which, as usual, can be identified with the product $2^{(2\times\mathbb{N})^{<\omega}}$. (Straightforward calculations show that $\mathbf{NT}$ is a closed set.) Thus, formally, the relations $\le_{\rm NT}$ and $\mathsf{E}_{\rm NT}$ are subsets of $\mathcal{P}((2 \times \mathbb{N})^{<\omega}) \times \mathcal{P}((2 \times \mathbb{N})^{<\omega})$.

Theorem 8 ($\approx$[13, Theorem 5]). $\mathsf{E}_{\rm NT}$ *is a complete analytic equivalence relation on* $\mathbf{NT}$.

Proof. The fact that $\le_{\rm NT}$ and $\mathsf{E}_{\rm NT}$ are $\mathbf{\Sigma}^1_1$-relations can be verified by direct evaluation. (The principal quantifier expresses the existence of $\gamma \in \mathbb{N}^\omega$ with certain properties.) To verify that $\mathsf{E}_{\rm NT}$ is an equivalence relation, it suffices to verify that $\le_{\rm NT}$ is a transitive relation. Suppose that $R \le_{\rm NT} S$ and $S \le_{\rm NT} T$, where R, S, and T are normal trees in $(2 \times \mathbb{N})^{<\omega}$. Then the trees $U = \textsc{Emb}(R, S)$ and $V = \textsc{Emb}(S, T)$ (the trees on $\mathbb{N}^{<\omega}$) are ill-founded; formally, ${\tt rnk}(U) = {\tt rnk}(V) = \infty$. Hence, according to Lemma 4, the tree $W = U +_{\tt cw} V$ satisfies ${\tt rnk}(W) = \infty$. On the other hand, one can easily verify that $W \subseteq \textsc{Emb}(R, T)$. Thus, the tree $\textsc{Emb}(R, T)$ is ill-founded, which was to be proved.

To prove the completeness of $\mathsf{E}_{\rm NT}$, consider an arbitrary analytic equivalence relation E on 2^ω. Then, E is a $\mathbf{\Sigma}^1_1$-subset of $2^\omega \times 2^\omega$; therefore, there exists a closed set $P \subseteq 2^\omega \times 2^\omega \times \mathbb{N}^\omega$ such that $\mathsf{E} = {\tt dom}\, P = \{\langle x, y\rangle : \exists\, z\, P(x, y, z)\}$. Next, there exists a tree $Q \subseteq (2\times2\times\mathbb{N})^{<\omega}$ (a tree on $2\times2\times\mathbb{N}$) such that $P = [Q] = \{\langle x, y, \gamma\rangle : \forall\, n\, Q(x \restriction n, y \restriction n, \gamma \restriction n)\}$. In other words,

$$x \,\mathsf{E}\, y \;\Leftrightarrow\; Q_{xy} \text{ is ill-founded} \;\Leftrightarrow\; {\tt rnk}(Q_{xy}) = \infty \qquad \text{for any } x, y \in 2^\omega. \tag{5}$$

It follows from Theorem 5 that there exists another tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ that satisfies conditions (i)–(v) of Theorem 5. In particular, according to (v), the tree Q_{xy} is ill-founded if and only if R_{xy} is ill-founded. We do not assert that the map

$$x \mapsto \vartheta(x) = \big\{\langle u, s\rangle \in (2 \times \mathbb{N})^{<\omega} :\ R(u, x \restriction {\tt lh}\, u, s)\big\}, \qquad x \in 2^\omega, \tag{6}$$

is a Borel reduction of E to $\mathsf{E}_{\rm NT}$. One can easily verify that ϑ is a Borel, and even continuous, map. It follows immediately from (iii) that $\vartheta(x) \in \mathbf{NT}$. The reduction property is derived from the following lemma.

Lemma 9. *If a tree* $R \subseteq (2\times2\times\mathbb{N})^{<\omega}$ *satisfies conditions* (i)–(iv) *of Theorem* 5 *and* $x, y \in 2^\omega$, *then* $\textsc{Emb}(\vartheta(x), \vartheta(y)) = R_{xy}$.

Proof. Suppose that $f \in \textsc{Emb}(\vartheta(x), \vartheta(y))$ and $m = {\tt lh}\, f$. Then, by definition, we obtain $R(u, x \restriction m, s) \Rightarrow R(u, y \restriction m, s +_{\tt cw} f)$ for any $u \in 2^m$ and $s \in \mathbb{N}^m$. Take $u = x \restriction m$ and $s = 0^m$ (a sequence of m zeros); then, $R(x \restriction m, x \restriction m, 0^n) \Rightarrow R(x \restriction m, y \restriction m, f)$. However, the left-hand side holds according to (ii). Hence, the right-hand side also holds because $f \in R_{xy}$.

To prove the reverse inclusion, we assume that $f \in R_{xy}$, i.e., $R(x \restriction m, y \restriction m, f)$, where $m = {\tt lh}\, f$, and therefore $R(x \restriction n, y \restriction n, f \restriction n)$ holds for any $n \le m$ since R is a tree. Let $n \le m$, $u \in 2^n$, and $s \in \mathbb{N}^n$. We must prove that $R(u, x \restriction n, s) \Rightarrow R(u, y \restriction n, s +_{\tt cw} (f \restriction n))$. Thus, we assume that $R(u, x \restriction n, s)$. On the other hand, as pointed out above, we have $R(x \restriction n, y \restriction n, f \restriction n)$. Now, $R(u, y \restriction n, s +_{\tt cw} (f \restriction n))$ follows from (iv), which was to be proved. $\square$

To complete the proof of Theorem 8, consider arbitrary $x, y \in 2^\omega$. Then $x \,\mathsf{E}\, y$ is equivalent to the fact that the tree Q_{xy} is ill-founded, then to the fact that the tree R_{xy} is ill-founded, then to the fact (by Lemma 9) that $\textsc{Emb}(\vartheta(y), \vartheta(x))$ is ill-founded, and finally, to the fact that $\vartheta(x) \,\mathsf{E}_{\rm NT}\, \vartheta(y)$ (by Definition 7). $\square$

5. BOREL APPROXIMATIONS

According to Theorem 8, E_{NT} is a complete analytic equivalence relation on $\mathbf{NT}$, so that any $\mathbf{\Sigma}^1_1$-equivalence relation E satisfies $\mathsf{E} \le_{\mathrm{B}} \mathsf{E}_{\mathrm{NT}}$. What can be said about Borel equivalence relations, which constitute a proper part of analytic relations? Quite general arguments (see Section 6 below) show that an ω_1-decreasing sequence of canonical upper Borel approximations of any complete analytic equivalence relation contains a cofinal subsequence consisting of Borel equivalence relations and that any such subsequence proves to be $\le_{\mathrm{B}}$-cofinal among all Borel equivalence relations.

In the case of the equivalence relation E_{NT}, it turns out that such an approximating sequence of Borel equivalence relations can be determined directly in a mathematically sensible way.

Definition 10. Suppose that $S, T \in \mathbf{NT}$ and $\xi < \omega_1$.

Set $S \le^{\xi}_{\mathrm{NT}} T$ when the tree $\mathrm{E{\small MB}}(S,T)$ satisfies $\mathtt{rnk}(\mathrm{E{\small MB}}(S,T)) \ge \xi$.

Define $S \,\mathsf{E}^{\xi}_{\mathrm{NT}}\, T$ when $S \le^{\xi}_{\mathrm{NT}} T$ and $T \le^{\xi}_{\mathrm{NT}} S$ hold. $\square$

We stress that the inequality $\mathtt{rnk}(\mathrm{E{\small MB}}(S,T)) \ge \xi$ implies that either $\mathrm{E{\small MB}}(S,T)$ (a tree in $\mathbb{N}^{<\omega}$) is well-founded and its rank is a countable ordinal $\ge \xi$, or $\mathrm{E{\small MB}}(S,T)$ is ill-founded and then, by definition, $\mathtt{rnk}(\mathrm{E{\small MB}}(S,T)) = \infty$ is greater than any ordinal.

Recall that the set $\mathbf{NT}$ of all normal trees is a closed subset of the Polish space $\mathcal{P}((2\times\mathbb{N})^{<\omega})$ (which is identified with the product $2^{(2\times\mathbb{N})^{<\omega}}$ homeomorphic to the Cantor discontinuum), and therefore $\mathbf{NT}$ itself is a Polish space.

Lemma 11. (i) *The relations* $\mathsf{E}^{\xi}_{\mathrm{NT}}$ *are Borel equivalence relations on* $\mathbf{NT}$.

(ii) *More precisely, if* $\nu < \omega_1$, *then the relation* $\mathsf{E}^{\omega\nu}_{\mathrm{NT}}$ *is a* $\mathbf{\Pi}^0_{1+2\nu}$*-set in the space*[4] $\mathcal{P}((2\times\mathbb{N})^{<\omega}) \times \mathcal{P}((2\times\mathbb{N})^{<\omega})$ *and the relation* $\mathsf{E}^{\omega\nu+k}_{\mathrm{NT}}$, $k \ge 1$, *is a* $\mathbf{\Sigma}^0_{1+2\nu+1}$*-set.*

(iii) *Moreover,* $\le_{\mathrm{NT}} = \bigcap_{\xi<\omega_1} \le^{\xi}_{\mathrm{NT}}$ *and* $\mathsf{E}_{\mathrm{NT}} = \bigcap_{\xi<\omega_1} \mathsf{E}^{\xi}_{\mathrm{NT}}$.

Proof. (i) The fact that $\mathsf{E}^{\xi}_{\mathrm{NT}}$ is an equivalence relation can be verified as in the proof of Theorem 8 with reference to Lemma 4.

(iii) The equalities follow from the fact that a tree $R \subseteq \mathbb{N}^{<\omega}$ is ill-founded when $\mathtt{rnk}(R) = \infty$; i.e., $\mathtt{rnk}(R) \ge \xi$ for any $\xi < \omega_1$.

(ii) To estimate the Borel class of $\le^{\xi}_{\mathrm{NT}}$ and $\mathsf{E}^{\xi}_{\mathrm{NT}}$, consider the sets

$$\mathcal{T}^{\xi} = \{T \in \mathcal{T} \colon \mathtt{rnk}(T) \ge \xi\} \qquad \text{and} \qquad \mathcal{T}^{\xi}_s = \{s \in T \in \mathcal{T} \colon \mathtt{rnk}_T(s) \ge \xi\},$$

where $\xi < \omega_1$, $s \in \mathbb{N}^{<\omega}$, and $\mathcal{T}$ is the set of all trees $T \subseteq \mathbb{N}^{<\omega}$. (It is clear that $\mathcal{T}$ is a closed subset of the Polish space $\mathcal{P}(2^{<\omega})$, which can be identified with the product $2^{\mathbb{N}^{<\omega}}$.) We argue that each $\mathcal{T}^{\xi}_s$ is a Borel set; moreover,

($*$) $\mathcal{T}^{\omega\nu}_s$ is a $\mathbf{\Pi}^0_{2\nu}$-subset of $\mathcal{T}$, i.e., the intersection of $\mathcal{T}$ (a closed set) with a $\mathbf{\Pi}^0_{2\nu}$-set, while $\mathcal{T}^{\omega\nu+k}_s$, $k \ge 1$, is a $\mathbf{\Sigma}^0_{2\nu+1}$-subset of $\mathcal{T}$.[5]

This assertion is proved by induction on ξ simultaneously for all s. If $\xi = 0$, then it is obvious that the set $\mathcal{T}^{\xi}_s = \{T \in \mathcal{T} \colon s \in T\}$ is even open–closed in $\mathcal{T}$. The inductive step is easily performed using the equalities $\mathcal{T}^{\xi+1}_s = \bigcup_{k\in\mathbb{N}} \mathcal{T}^{\xi}_{s^\wedge k}$ for every ξ and $\mathcal{T}^{\lambda}_s = \bigcap_{\xi<\lambda} \bigcup_{k\in\mathbb{N}} \mathcal{T}^{\xi}_{s^\wedge k}$ for any limit ordinal $\lambda < \omega_1$.

Hence, each set $\mathcal{T}^{\omega\nu} = \mathcal{T}^{\omega\nu}_{\Lambda}$ belongs to $\mathbf{\Pi}^0_{2\nu}$ as a subset of $\mathcal{T}$; accordingly, each set $\mathcal{T}^{\omega\nu+k}$, $k \ge 1$, belongs to $\mathbf{\Sigma}^0_{2\nu+1}$ as a subset of $\mathcal{T}$.

However, by definition, $\le^{\xi}_{\mathrm{NT}}$ coincides with the h-preimage of the set $\mathcal{T}^{\xi}$, where $h(S,T) = \mathrm{E{\small MB}}(S,T)$ for any $S, T \in \mathbf{NT}$. Elementary calculations show that h, as a function from the closed

[4] Or in the space $\mathbf{NT}\times\mathbf{NT}$, which is equivalent in the present case since $\mathbf{NT}$ is closed.

[5] Recall that the product of ordinals $\alpha\nu$ is equal to the sum of ν copies of the ordinal α, which, generally speaking, is different from $\nu\alpha$. For example, $2\omega = \omega < \omega\cdot 2 = \omega + \omega$.

set $\mathbf{NT}\times\mathbf{NT}$ into the closed set $\mathcal{T}$, is a $\mathbf{\Sigma}^0_2$-measurable function; i.e., the h-preimages of open sets belong to the class $\mathbf{\Sigma}^0_2$. (More precisely, for any $f\in\mathbb{N}^{<\omega}$, the set $\{\langle S,T\rangle\colon s\in \mathrm{E{\small MB}}(S,T)\}$ is closed, while the set $\{\langle S,T\rangle\colon s\notin \mathrm{E{\small MB}}(S,T)\}$ is open in $\mathbf{NT}\times\mathbf{NT}$.) Thus, each relation $\le^{\omega\nu}_{\text{NT}}$ and, hence, each $\mathsf{E}^{\omega\nu}_{\text{NT}}$ is a $\mathbf{\Pi}^0_{2\nu}$-combination of sets from the class $\mathbf{\Sigma}^0_2$, i.e., a set from $\mathbf{\Pi}^0_{1+2\nu}$; an analogous result is derived for the relations $\le^{\omega\nu+k}_{\text{NT}}$ and $\mathsf{E}^{\omega\nu+k}_{\text{NT}}$, $k\ge 1$. □

6. A CANONICAL INTERSECTION PROPERTY

The intersections mentioned in Lemma 11(iii) belong to a remarkable type.

Definition 12. The intersection $Z=\bigcap_{\xi<\omega_1}Z_\xi$ of sets Z_ξ of a fixed Polish space X has a *canonical intersection property* (CIP) when, for any $\mathbf{\Pi}^1_1$-set C in X that satisfies $Z\subseteq C$, there exists an index $\eta<\omega_1$ such that $\bigcap_{\xi<\eta}Z_\xi\subseteq C$ already holds. □

A general method for constructing intersections that satisfy the CIP is known from the classical descriptive set theory. It is based on the following result, which was actually obtained in [16]. We set

$$\mathcal{T}^\infty=\{T\in\mathcal{T}\colon\ T \text{ is an ill-founded tree}\}$$

and recall that $\mathcal{T}$ stands for the set of all trees $T\subseteq\mathbb{N}^{<\omega}$; in addition, we set $\mathcal{T}^\xi=\{T\in\mathcal{T}\colon \mathtt{rnk}(T)\ge\xi\}$ for every $\xi<\omega_1$.

Proposition 13. $\mathcal{T}^\infty=\bigcap_{\xi<\omega_1}\mathcal{T}^\xi$, *and this intersection satisfies the* CIP.

Proof (sketch)**.** The complementary $\mathbf{\Pi}^1_1$-set $\mathcal{T}_\infty=\{T\in\mathcal{T}\colon T \text{ is well-founded}\}$ admits a decomposition $\mathcal{T}_\infty=\bigcup_{\xi<\omega_1}\mathcal{T}_\xi$ into Borel constituents $\mathcal{T}_\xi=\{T\in\mathcal{T}\colon \mathtt{rnk}(T)=\xi\}$. The following index restriction theorem by Luzin and Sierpiński is the key result of [16]: any $\mathbf{\Sigma}^1_1$-set $Y\subseteq\mathcal{T}_\infty$ is covered by a union of a countable number of constituents $\mathcal{T}_\xi$. This immediately implies the CIP for the sequence of sets $\mathcal{T}^\xi$. □

Lemma 14. *The intersections indicated in assertion* (iii) *of Lemma* 11 *satisfy the* CIP.

Proof. For any pair of trees $S,T\in\mathbf{NT}$, we set $\varphi(S,T)=\mathrm{E{\small MB}}(S,T)$. Thus, φ is a Borel map $\mathbf{NT}^2\to\mathcal{T}$, and it follows immediately from the definitions that $S\le_{\text{NT}}T$ is equivalent to $\varphi(S,T)\in\mathcal{T}^\infty$. In other words, $\le_{\text{NT}}$ coincides with the φ-preimage of the set $\mathcal{T}^\infty$, and similarly, each $\le^\xi_{\text{NT}}$ coincides with the φ-preimage of the set $\mathcal{T}^\xi$. This allows us to immediately derive the CIP for the intersection $\le_{\text{NT}}=\bigcap_{\xi<\omega_1}\le^\xi_{\text{NT}}$ from the same property for the intersection $\mathcal{T}^\infty=\bigcap_{\xi<\omega_1}\mathcal{T}^\xi$. □

Corollary 15. *If* E *is a Borel equivalence relations on a Polish space* X, *then there exists an ordinal* $\xi<\omega_1$ *such that* $\mathsf{E}\le_{\text{B}}\mathsf{E}^\xi_{\text{NT}}$.

Proof. According to Theorem 8, we have $\mathsf{E}\le_{\text{B}}\mathsf{E}_{\text{NT}}$. Therefore, there exists a Borel map $\vartheta\colon X\to\mathbf{NT}$ such that $x\,\mathsf{E}\,y\Leftrightarrow\vartheta(x)\,\mathsf{E}_{\text{NT}}\,\vartheta(y)$. Set $\varphi(x,y)=\langle\vartheta(x),\vartheta(y)\rangle$. The map φ is also Borel; moreover, the φ-image $\varphi(P)$ of the set $P=(X\times X)\setminus\mathsf{E}$ is a $\mathbf{\Sigma}^1_1$-set that does not intersect E_{NT}. Hence, according to Lemma 14, there exists an ordinal $\xi<\omega_1$ such that $\varphi(P)$ does not intersect $\mathsf{E}^\xi_{\text{NT}}$. This means that ϑ is a reduction of the relation E not only to E_{NT} but also to the approximating Borel equivalence relation $\mathsf{E}^\xi_{\text{NT}}$. □

For the reader's convenience, we give an elementary proof of the following result, which is "folklore" in descriptive set theory.

Theorem 16. *Suppose that* E *is a* $\mathbf{\Sigma}^1_1$*-equivalence relation on a Polish space* X, $\mathsf{E}=\bigcap_{\xi<\omega_1}E_\xi$, *all sets* E_ξ *are Borel subsets of* $X\times X$, *and the sequence of sets* E_ξ *is* $\subseteq$*-decreasing and satisfies the* CIP. *Then, the set of all ordinals* $\xi<\omega_1$ *such that* E_ξ *is an equivalence relation is cofinal in* ω_1.

If it is known in addition that E *is a complete* $\mathbf{\Sigma}^1_1$*-equivalence relation, then, for any Borel equivalence relation* F, *there exists an ordinal* $\xi<\omega_1$ *such that* E_ξ *is an equivalence relation and* $\mathsf{F}\le_{\text{B}}E_\xi$.

Proof. To prove the first assertion, it suffices to verify that, for any $\xi < \omega_1$, there exists an ordinal ν, $\xi < \nu < \omega_1$, such that E_ν is an equivalence relation.

Step 1. We argue that, for any $\xi < \omega_1$, there exists an ordinal $\zeta = \zeta(\xi)$, $\xi < \zeta < \omega_1$, such that $\langle x, y\rangle \in E_\zeta \Rightarrow \langle y, x\rangle \in E_\xi$. Indeed, the set $P = \{\langle x, y\rangle : \langle y, x\rangle \in E_\xi\}$ is a Borel superset of E in $X \times X$ (since E is a symmetric relation). In this situation, the CIP provides an ordinal ζ, $\xi < \zeta < \omega_1$, such that $E_\zeta \subseteq P$.

Step 2. We argue that, for any $\xi < \omega_1$, there exists an ordinal $\eta = \eta(\xi)$, $\xi < \eta < \omega_1$, such that $\langle x, z\rangle \in E_\xi$ for any x and z such that, for a certain y, the pairs $\langle x, y\rangle$ and $\langle y, z\rangle$ belong to E_η. The argument includes two steps. First, we consider the set

$$P = \bigl\{\langle x, y\rangle : \forall\, \langle y, z\rangle\, (\langle y, z\rangle \in \mathsf{E} \Rightarrow \langle x, z\rangle \in E_\xi)\bigr\}.$$

It is a $\mathbf{\Pi}^1_1$-set in $X \times X$ and a superset of E because E is a transitive relation. Thus, due to the CIP, there exists an ordinal ζ, $\xi < \zeta < \omega_1$, that satisfies $E_\zeta \subseteq P$. This means that we have $\langle x, z\rangle \in E_\xi$ whenever $\langle x, y\rangle \in E_\zeta$ and $\langle y, z\rangle \in \mathsf{E}$. Now, consider the set

$$Q = \bigl\{\langle y, z\rangle : \forall\, \langle x, y\rangle\, (\langle x, y\rangle \in E_\zeta \Rightarrow \langle x, z\rangle \in E_\xi)\bigr\}.$$

According to the aforesaid, it is a $\mathbf{\Pi}^1_1$-set and a superset of E; therefore, there exists an ordinal η, $\zeta < \eta < \omega_1$, such that $E_\eta \subseteq Q$. It is clear that η possesses the required properties.

Final argument. Set $\xi_0 = \xi$ and $\xi_{n+1} = \eta(\zeta(\xi_n))$ for any n. The ordinal $\nu = \mathtt{sup}_n\, \xi_n$ possesses the required properties.

To derive the second assertion of the theorem, we can argue as in the proof of Corollary 15. □

7. REDUCTION TO IDEALS

It is not quite clear from Definition 7 that the equivalence relations E_{NT} and $\mathsf{E}^\xi_{\text{NT}}$ can somehow be reduced to ideals. However, this is actually the case.

The possibility of generating E_{NT} by a $\mathbf{\Sigma}^1_1$-ideal was established in [17]. To expound this result, we denote by $\mathcal{I}$ the ideal on $(2 \times \mathbb{N})^{<\omega}$ that is finitely generated by all sets of the form $S \,\Delta\, T$, where $S, T \subseteq (2 \times \mathbb{N})^{<\omega}$ are normal trees and $S \,\mathsf{E}_{\text{NT}}\, T$. In other words, $\mathcal{I}$ consists of all subsets of $(2 \times \mathbb{N})^{<\omega}$ that admit a covering by unions of a finite number of symmetric differences $S \,\Delta\, T$ of the aforementioned type.

Theorem 17 (proved in [17]). *The ideal $\mathcal{I}$ is a $\mathbf{\Sigma}^1_1$-set in the Polish space $\mathcal{P}((2 \times \mathbb{N})^{<\omega})$. In addition, the equivalence relation E_{NT} coincides with $\mathsf{E}_{\mathcal{I}} \restriction \mathbf{NT}$; this means that $S \,\mathsf{E}_{\text{NT}}\, T$ is equivalent to $S \,\Delta\, T \in \mathcal{I}$ for any $S, T \in \mathbf{NT}$.*

Proof. The fact that $\mathcal{I}$ belongs to the class $\mathbf{\Sigma}^1_1$ is quite clear: the principal quantifier expresses the existence of a finite set of elements in $\mathbf{NT}$ whose properties can also be expressed by a $\mathbf{\Sigma}^1_1$-relation because E_{NT} is a $\mathbf{\Sigma}^1_1$-set.

Assuming that $S \,\Delta\, T \in \mathcal{I}$, let us prove that $S \,\mathsf{E}_{\text{NT}}\, T$ (nontrivial direction). By definition, we have $S \,\Delta\, T \subseteq \bigcup_{i=1}^k (S_i \,\Delta\, T_i)$, where $S_i, T_i \in \mathbf{NT}$ and $S_i \,\mathsf{E}_{\text{NT}}\, T_i$. Then, by definition, the trees $R_i = \text{Emb}(S_i, T_i)$ and $R'_i = \text{Emb}(T_i, S_i)$ are ill-founded. We must derive that the trees $\text{Emb}(S, T)$ and $\text{Emb}(T, S)$ have the same property. To prove the ill-foundedness of $\text{Emb}(S, T)$, note that the tree $R = R_1 +_{\mathtt{cw}} \ldots +_{\mathtt{cw}} R_k$ satisfies $\mathtt{rnk}(R) \geq \min\{\mathtt{rnk}(R_1), \ldots, \mathtt{rnk}(R_k)\}$ according to Lemma 4; therefore, R is ill-founded. Thus, it remains to verify that $R \subseteq \text{Emb}(S, T)$. To this end, consider an arbitrary $r = r_1 +_{\mathtt{cw}} \ldots +_{\mathtt{cw}} r_k \in R$, where all sequences $r_i \in R_i$, $i = 1, \ldots, k$, have the same length, say m.

To prove that $r \in \text{Emb}(S, T)$, suppose the contrary, i.e., $r \notin \text{Emb}(S, T)$; in other words, there exists a pair $\langle u, s\rangle \in S$ that satisfies $\langle u, s +_{\mathtt{cw}} (r \restriction n)\rangle \notin T$, where $n = \mathtt{lh}\, u = \mathtt{lh}\, s$. Under this assumption, $\langle u, s +_{\mathtt{cw}} t\rangle \notin T$ whenever $t \in 2^n$ and $t \le_{\mathtt{cw}} r \restriction n$. In particular, $\langle u, s\rangle \notin T$, so $\langle u, s\rangle \in S \,\Delta\, T$ and thus $\langle u, s\rangle \in S_{i_1} \,\Delta\, T_{i_1}$ for a certain $1 \le i_1 \le k$. However, we know that $r_{i_1} \in R_{i_1} = \mathrm{E{\small MB}}(S_{i_1}, T_{i_1})$. This implies $\langle u, s_1\rangle \in S_{i_1} \cap T_{i_1}$, where $s_1 = s +_{\mathtt{cw}} (r_{i_1} \restriction n)$.

However, according to the aforesaid, $\langle u, s_1\rangle \in S \setminus T$; therefore, repeating the same procedure, we obtain $\langle u, s_1\rangle \in S_{i_2} \,\Delta\, T_{i_2}$ for a certain $1 \le i_2 \le k$. In this case, $i_2 \ne i_1$ because $\langle u, s_1\rangle \in S_{i_1} \cap T_{i_1}$. This implies $\langle u, s_2\rangle \in S_{i_2} \cap T_{i_2}$, where $s_2 = s_1 +_{\mathtt{cw}} (r_{i_2} \restriction n)$, since r_{i_2} belongs to the tree $R_{i_2} = \mathrm{E{\small MB}}(S_{i_2}, T_{i_2})$. Moreover, $\langle u, s_2\rangle$ still belongs to $S_{i_1} \cap T_{i_1}$ because all S_i and T_i are normal trees.

After k steps of this construction, all indices $1 \le i \le k$ will be considered, and the final sequence $s_k = s +_{\mathtt{cw}} (r \restriction n)$ will satisfy $\langle u, s_k\rangle \in S_i \cap T_i$ for any $i = 1, \dots, k$. Hence, $\langle u, s_k\rangle \notin S \,\Delta\, T$. However, $\langle u, s_k\rangle \in S$ since $\langle u, s\rangle \in S$ and S is a normal tree. Thus, $\langle u, s_k\rangle$ belongs to T, which contradicts our assumptions. $\square$

What about the equivalence relations $\mathsf{E}^{\xi}_{\mathrm{NT}}$? If we replace E_{NT} by $\mathsf{E}^{\xi}_{\mathrm{NT}}$ in the definition of $\mathcal{I}$ above, then the theorem remains valid: $\mathsf{E}^{\xi}_{\mathrm{NT}}$ coincides with $\mathcal{I} \restriction \mathbf{NT}$ for the "new" $\mathcal{I}$. However, for such an ideal $\mathcal{I}$, which depends on ξ, it is impossible to establish a class better than $\mathbf{\Sigma}^1_1$, which is of little interest because the relations $\mathsf{E}^{\xi}_{\mathrm{NT}}$ are Borel. However, a more complex variant of this construction gives Borel ideals that generate the relations $\mathsf{E}^{\xi}_{\mathrm{NT}}$.

In the arguments below, the variables d and e denote natural numbers.

Definition 18. For any $d \in \mathbb{N}$, $d \ge 1$, the *d-width* $\mathrm{W{\small ID}}_d(X)$ of a set $X \subseteq (2 \times \mathbb{N})^{<\omega}$ is the set of all finite sequences $f \in \mathbb{N}^{<\omega}$ such that, for any $n \le \mathtt{lh}\, f$ and $u \in 2^n$, one has

$$\forall s_1 \in \mathbb{N}^n \,\exists t_1 \in \mathbb{N}^n \,\forall s_2 \in \mathbb{N}^n \,\exists t_2 \in \mathbb{N}^n \,\dots\, \forall s_d \in \mathbb{N}^n \,\exists t_d \in \mathbb{N}^n$$

$$\Big(\text{if } \langle u, s_1 +_{\mathtt{cw}} t_1 +_{\mathtt{cw}} \dots +_{\mathtt{cw}} s_{k-1} +_{\mathtt{cw}} t_{k-1} +_{\mathtt{cw}} s_k\rangle \in X \text{ for any } 1 \le k \le d, \text{ then}$$

$$t_1 +_{\mathtt{cw}} \dots +_{\mathtt{cw}} t_d \le_{\mathtt{cw}} f \restriction n \text{ and } \tau <_{\mathtt{cw}} s_1 +_{\mathtt{cw}} t_1 +_{\mathtt{cw}} \dots +_{\mathtt{cw}} s_d +_{\mathtt{cw}} t_d \text{ whenever } \langle u, \tau\rangle \in X\Big). \quad \square$$

It is clear that $\mathrm{W{\small ID}}_d(X)$ is a tree in $\mathbb{N}^{<\omega}$ (possibly, an empty tree).

The definition of $\mathrm{W{\small ID}}_d(X)$ can be interpreted in terms of a game $G^d_{fnu}(X)$ of length d, where $n \in \mathbb{N}$ and $u \in 2^n$; player **I** makes moves $s_1, \dots, s_d \in \mathbb{N}^n$, while player **II** makes moves $t_1, \dots, t_d \in \mathbb{N}^n$; **I** must play so that $\langle u, s_1 +_{\mathtt{cw}} t_1 +_{\mathtt{cw}} \dots +_{\mathtt{cw}} s_{k-1} +_{\mathtt{cw}} t_{k-1} +_{\mathtt{cw}} s_k\rangle \in X$ for any k, whereas **II** must play so that $t_1 +_{\mathtt{cw}} \dots +_{\mathtt{cw}} t_d \le_{\mathtt{cw}} f \restriction n$; finally, **II** wins if $\tau <_{\mathtt{cw}} s_1 +_{\mathtt{cw}} t_1 +_{\mathtt{cw}} \dots +_{\mathtt{cw}} s_d +_{\mathtt{cw}} t_d$ for any $\tau \in \mathbb{N}^n$ such that $\langle u, \tau\rangle \in X$. Moreover, **II** also wins when, for a certain $k \le d$, **I** cannot make a move s_k such that $\langle u, s_1 +_{\mathtt{cw}} t_1 +_{\mathtt{cw}} \dots +_{\mathtt{cw}} s_{k-1} +_{\mathtt{cw}} t_{k-1} +_{\mathtt{cw}} s_k\rangle \in X$; accordingly, **I** wins when $t_1 +_{\mathtt{cw}} \dots +_{\mathtt{cw}} t_k \not\le_{\mathtt{cw}} f \restriction n$ for a certain $k \le d$.

In this terminology, $f \in \mathrm{W{\small ID}}_d(X)$ is equivalent to the fact that **II** wins the game $G^d_{fnu}(X)$ (i.e., has a winning strategy in this game) whatever $n \le \mathtt{lh}\, f$ and $u \in 2^n$. Thus, the assumption $f \in \mathrm{W{\small ID}}_d(X)$ can be informally understood as the possibility to leave X in at most d jumps whose total length is no greater than f.

Here, it is relevant to note that $\mathrm{W{\small ID}}_d(X) \subseteq \mathrm{W{\small ID}}_{d+1}(X)$. Indeed, assume that $f \in \mathrm{W{\small ID}}_d(X)$. Then, for any $n \le \mathtt{lh}\, f$ and $u \in 2^n$, a winning strategy for player **II** in $G^{d+1}_{fnu}(X)$ is to simply follow any winning strategy in the game $G^d_{fnu}(X)$. After d moves, **I** has no admissible move s_{d+1}; hence, **II** wins.

Lemma 19. *If* $X, Y \subseteq (2 \times \mathbb{N})^{<\omega}$ *and* $d, e \ge 1$, *then* $\mathrm{W{\small ID}}_d(X) +_{\mathtt{cw}} \mathrm{W{\small ID}}_e(Y) \subseteq \mathrm{W{\small ID}}_{d+e}(X \cup Y)$.

Proof. Consider arbitrary $f \in \mathrm{W{\small ID}}_d(X)$ and $g \in \mathrm{W{\small ID}}_e(Y)$ with $\mathtt{lh}\, f = \mathtt{lh}\, g$ and prove that $h = f +_{\mathtt{cw}} g$ belongs to $\mathrm{W{\small ID}}_{d+e}(X \cup Y)$. Suppose that $n \le \mathtt{lh}\, f$ and $u \in 2^n$. Fix an arbitrary winning strategy for **II** in the game $G^d_{fnu}(X)$ and call it an *X-strategy*. Fix an arbitrary winning strategy for **II** in $G^e_{gnu}(Y)$ and call it a *Y-strategy*. A winning strategy for **II** in $G^{d+e}_{hnu}(X \cup Y)$ can be determined as follows. Let $s_1, t_1, \dots, s_{d+e}, t_{d+e}$ be a complete sequence of moves. By definition,

$S_k = s_1 + t_1 + \ldots + s_{k-1} + t_{k-1} + s_k \in X \cup Y$ for every k. Define $K = \{k\colon S_k \in X\}$ and $K' = \{k\colon S_k \in Y \setminus X\}$. Let $K = \{k_1, \ldots, k_m\}$ and $K' = \{k'_1, \ldots, k'_{m'}\}$ in increasing order. It is obvious that $m + m' = d + e$.

(a) Consider a derived sequence $\sigma_1, \tau_1, \ldots, \sigma_m, \tau_m$ such that $\tau_i = t_{k_i}$ and

$$\sigma_i = s_{k_{i-1}+1} +_{\mathtt{cw}} t_{k_{i-1}+1} +_{\mathtt{cw}} s_{k_{i-1}+2} +_{\mathtt{cw}} t_{k_{i-1}+2} +_{\mathtt{cw}} \cdots +_{\mathtt{cw}} s_{k_i-1} +_{\mathtt{cw}} t_{k_i-1} +_{\mathtt{cw}} s_{k_i}$$

for any $1 \le i \le m$. Assume that **II** plays in $G^{d+e}_{hnu}(X \cup Y)$ so that *each move* $t_{k_i} = \tau_i$ *is made according to the* X*-strategy applied to the subsequence* $\sigma_1, \tau_1, \ldots, \sigma_{i-1}, \tau_{i-1}, \sigma_i$. This determines all the moves t_k, $k \in K$, and, assuming that **II** follows this strategy, we immediately obtain $m \le d$.

(b) Define another derived sequence $\sigma'_1, \tau'_1, \ldots, \sigma'_{m'}, \tau'_{m'}$ so that $\tau'_i = t_{k'_i}$ and

$$\sigma'_i = s_{k'_{i-1}+1} +_{\mathtt{cw}} t_{k'_{i-1}+1} +_{\mathtt{cw}} s_{k'_{i-1}+2} +_{\mathtt{cw}} t_{k'_{i-1}+2} +_{\mathtt{cw}} \cdots +_{\mathtt{cw}} s_{k'_i-1} +_{\mathtt{cw}} t_{k'_i-1} +_{\mathtt{cw}} s_{k'_i}$$

for any $1 \le i \le m'$. Assume that **II** plays in $G^{d+e}_{hnu}(X \cup Y)$ so that *each move* $t_{k'_i} = \tau'_i$ *is made according to the* Y*-strategy applied to the subsequence* $\sigma'_1, \tau'_1, \ldots, \sigma'_{i-1}, \tau'_{i-1}, \sigma'_i$. This determines all the moves $t_{k'}$, $k' \in K'$, and, assuming that **II** follows this strategy, we immediately obtain $m' \le e$.

Items (a) and (b) completely determine the strategy for **II** in the game $G^{d+e}_{hnu}(X \cup Y)$. Let us show that this is a winning strategy. Indeed, it follows from the aforesaid that $m = d$ and $m' = e$. Moreover, we have $\tau <_{\mathtt{cw}} \tau_m$ and $\tau <_{\mathtt{cw}} \tau'_{m'}$ whenever $\langle u, \tau\rangle \in X$ and $\langle u, \tau\rangle \in Y$ (according to the choice of the X- and Y-strategies, respectively). Therefore, t_{d+e} (irrespective of whether this is τ_m or $\tau'_{m'}$) satisfies $\tau <_{\mathtt{cw}} t_{d+e}$ for every τ such that $\langle u, \tau\rangle \in X \cup Y$, which was to be proved. $\square$

Definition 20. Let $\mathcal{I}_{\rm NT}$ be the family of all sets $X \subseteq (2 \times \mathbb{N})^{<\omega}$ such that, for a certain $d \ge 1$ (and hence for any $d' \ge d$), the tree $\mathrm{WID}_d(X)$ is ill-founded.

For any $\xi < \omega_1$, let $\mathcal{I}^{\xi}_{\rm NT}$ be the family of all sets $X \subseteq (2 \times \mathbb{N})^{<\omega}$ such that $\mathtt{rnk}(\mathrm{WID}_d(X)) \ge \xi$ for a certain $d \ge 1$. $\square$

Recall that each ideal $\mathcal{I} \subseteq \mathcal{P}(D)$ on a set D generates an equivalence relation $\mathsf{E}_{\mathcal{I}}$ on $\mathcal{P}(D)$ such that $X \mathrel{\mathsf{E}_{\mathcal{I}}} Y$ when $X \mathbin{\Delta} Y \in \mathcal{I}$. In the following theorem, $D = (2 \times \mathbb{N})^{<\omega}$.

Theorem 21. (i) *The sets* $\mathcal{I}_{\rm NT}$ *and* $\mathcal{I}^{\xi}_{\rm NT}$ *(for any* $\xi < \omega_1$*) are ideals on* $(2 \times \mathbb{N})^{<\omega}$.

(ii) *The equivalence relation* $\mathsf{E}_{\rm NT}$ *coincides with* $\mathsf{E}_{\mathcal{I}_{\rm NT}} \restriction \mathbf{NT}$, *where* $\mathsf{E}_{\mathcal{I}_{\rm NT}}$ *is the equivalence relation on* $\mathcal{P}((2 \times \mathbb{N})^{<\omega})$ *generated by the ideal* $\mathcal{I}_{\rm NT}$.

(iii) *Similarly, for any* ξ, $\mathsf{E}^{\xi}_{\rm NT}$ *coincides with* $\mathsf{E}_{\mathcal{I}^{\xi}_{\rm NT}} \restriction \mathbf{NT}$.

(iv) $\mathcal{I}_{\rm NT}$ *is a* $\mathbf{\Sigma}^1_1$*-set* (as a subset of the Polish space $\mathcal{P}((2 \times \mathbb{N})^{<\omega})$), *and each* $\mathcal{I}^{\omega\nu+k}_{\rm NT}$ *is a* $\mathbf{\Sigma}^0_{\omega+2\nu+1}$*-set; hence,* $\mathcal{I}^{\omega\nu+k}_{\rm NT}$ *is a* $\mathbf{\Sigma}^0_{2\nu+1}$*-set provided that* $\nu \ge \omega$.

Proof. (i) Suppose that sets $X, Y \subseteq (2 \times \mathbb{N})^{<\omega}$ belong to $\mathcal{I}^{\xi}_{\rm NT}$ and, hence, the trees $\mathrm{WID}_d(X)$ and $\mathrm{WID}_e(Y)$ have ranks $\ge \xi$. By Lemma 19, $\mathrm{WID}_d(X) +_{\mathtt{cw}} \mathrm{WID}_e(Y) \subseteq \mathrm{WID}_{d+e}(X \cup Y)$. Then, we obtain $\mathtt{rnk}(\mathrm{WID}_{d+e}(X \cup Y)) \ge \xi$ according to Lemma 4; therefore, $X \cup Y \in \mathsf{E}^{\xi}_{\rm NT}$. Finally, to see that $Z = (2 \times \mathbb{N})^{<\omega}$ does not belong to $\mathcal{I}^{\xi}_{\rm NT}$, we note that $\mathrm{WID}_d(Z) = \varnothing$ for any d by obvious considerations.

(ii) Consider a pair of trees $S, T \in \mathbf{NT}$. We must prove that $S \mathsf{E}_{\rm NT} T$ is equivalent to $S \Delta T \in \mathcal{I}_{\rm NT}$. Suppose that $S \mathsf{E}_{\rm NT} T$. Then the trees $E = \mathrm{EMB}(S, T)$ and $F = \mathrm{EMB}(T, S)$ are ill-founded; therefore, the tree $G = E +_{\mathtt{cw}} F$ is also ill-founded according to Lemma 4. However, it is clear that $G \subseteq E \cap F$. (Indeed, S and T are right $\le_{\mathtt{cw}}$-transitive.) Thus, it suffices to prove that $E \cap F \subseteq \mathrm{WID}_1(S \mathbin{\Delta} T)$. Consider an arbitrary $f \in E \cap F$. By definition, for any pair $\langle u, s\rangle \in S \cup T$, $\mathtt{lh}\, u = \mathtt{lh}\, s = n \le \mathtt{lh}\, f$, we have $\langle u, s +_{\mathtt{cw}} (f \restriction n)\rangle \in S \cap T$. In particular, since $\langle u, s\rangle \in S \mathbin{\Delta} T \Rightarrow \langle u, s +_{\mathtt{cw}} (f \restriction n)\rangle \notin S \mathbin{\Delta} T$, it follows immediately that $f \in \mathrm{WID}_1(S \mathbin{\Delta} T)$.

To prove the converse implication, suppose that $S \Delta T \in \mathcal{I}_{\rm NT}$, i.e., the tree $\text{WID}_d(S \Delta T)$ is ill-founded for a certain $d \geq 1$. It suffices to prove that $\text{WID}_d(S \Delta T) \subseteq \text{EMB}(S,T)$. Suppose the contrary, i.e., $f \in \text{WID}_d(S \Delta T)$, but $f \notin \text{EMB}(S,T)$. The latter means that there exists a pair $\langle u,s\rangle \in S$, $\mathtt{lh}\, u = \mathtt{lh}\, s = n \leq \mathtt{lh}\, f$, such that $\langle u, s +_{\tt cw} (f \restriction n)\rangle \notin T$. Then, $\langle u,s\rangle$ does not belong to T either; therefore, both pairs $\langle u,s\rangle$ and $\langle u, s +_{\tt cw} (f \restriction n)\rangle$ belong to $S \setminus T$. Hence,

($*$) $\langle u, s+g\rangle \in S \setminus T$ for any $g \in \mathbb{N}^n$, $g \leq_{\tt cw} (f \restriction n)$.

Now, consider a play in the game $G^d_{fnu}(S \Delta T)$ in which player **II** follows his winning strategy (which exists because $f \in \text{WID}_d(S \Delta T)$), while **I** plays $s_k = 0^n$ (a sequence of n zeros) at each move k. Let t_k, $1 \leq k \leq d$, be a sequence of moves of player **II**. Since $s_k = 0^n \ \forall k$, we have $t_1 +_{\tt cw} \ldots +_{\tt cw} t_k \leq_{\tt cw} (f \restriction n)$ by definition provided that $\langle u, s +_{\tt cw} t_1 +_{\tt cw} \ldots +_{\tt cw} t_{k-1}\rangle \in S \Delta T$; therefore, according to ($*$), $t_1 +_{\tt cw} \ldots +_{\tt cw} t_k \leq_{\tt cw} (f \restriction n)$ and $\langle u, s +_{\tt cw} t_1 +_{\tt cw} \ldots +_{\tt cw} t_k\rangle \in S \Delta T$ for any k. In particular, $\langle u, s +_{\tt cw} t_1 +_{\tt cw} \ldots +_{\tt cw} t_d\rangle \in S \Delta T$, which contradicts the choice of the strategy.

(iii) The same arguments lead to the required result. Now, the reference to Lemma 4 at the beginning of the proof of (ii) looks as follows: the trees $E = \text{EMB}(S,T)$ and $F = \text{EMB}(T,S)$ have ranks $\geq \xi$; hence, the tree $G = E +_{\tt cw} F$ also has this property.

(iv) For any $d \geq 1$, the map $X \mapsto \text{WID}_d(X)$ is a function of a finite Borel level. Hence, for any $\mathbf{\Sigma}^0_\alpha$-family H of subsets of $(2 \times \mathbb{N})^{<\omega}$, the set

$$\bigl\{X \subseteq (2 \times \mathbb{N})^{<\omega} \colon \exists\, d \geq 1\, (\text{WID}_d(X) \in H)\bigr\}$$

belongs to $\mathbf{\Sigma}^0_{\omega+\alpha}$. It remains to recall that the set $\mathcal{T}^{\omega\nu+k}$ of all trees $T \subseteq \mathbb{N}^{<\omega}$ satisfying $\mathtt{rnk}(T) \geq \omega\nu + k$ belongs to the class $\mathbf{\Sigma}^0_{2\nu+1}$ according to ($*$) in the proof of Lemma 11. $\square$

Rosendal proved in [17] that any Borel equivalence relation is Borel-reducible to an equivalence relation of the form $\mathsf{E}_{\mathcal{J}}$, where $\mathcal{J}$ is a Borel ideal. The ideals $\mathcal{J}$ in this proof are chosen as appropriate upper Borel approximations of a certain $\mathbf{\Sigma}^1_1$-ideal $\mathcal{I}$. (The ideal $\mathcal{I}$ was defined at the beginning of this section.) The following result shows that the Borel ideals in the Rosendal theorem can be chosen by a more direct and efficient method.

Corollary 22. *For any Borel equivalence relation* E *on a Polish space, there exists an ordinal* $\xi < \omega_1$ *such that* $\mathsf{E} \leq_{\rm B} \mathsf{E}_{\mathcal{I}^\xi_{\rm NT}}$.

Proof. According to Corollary 15, there exists an ordinal $\xi < \omega_1$ such that $\mathsf{E} \leq_{\rm B} \mathsf{E}^\xi_{\rm NT}$. On the other hand, Theorem 21(iii) implies that $\mathsf{E}^\xi_{\rm NT} \leq_{\rm B} \mathsf{E}_{\mathcal{I}^\xi_{\rm NT}}$. $\square$

Lemma 11, Corollary 15, and Theorem 21 allow us to rapidly complete the proof of Theorem 1: the sequence of equivalence relations $\mathsf{E}_\nu = \mathsf{E}^{\omega\nu+2}_{\rm NT}$, where $\nu < \omega_1$, possesses the required properties. (Naturally, we must apply an arbitrary homeomorphism from $\mathcal{P}((2 \times \mathbb{N})^{<\omega})$ to the space 2^ω that carries over the domain $\mathbf{NT}$ of all equivalence relations $\mathsf{E}^{\omega\nu+2}_{\rm NT}$ to a certain closed set $D \subseteq 2^\omega$.)

8. ARE APPROXIMATING RELATIONS COMPLETE FOR THEIR BOREL CLASSES?

The equivalence relations $\mathsf{E}^\xi_{\rm NT}$ lead to other questions, the most interesting of which is the question of completeness of these relations in the Borel classes pointed out in Lemma 11(ii) (cf. the completeness of the relations $\mathsf{E}_{\rm NT}$ in Theorem 8). We do not know the answer for the present; however, there is a promising approach.

Definition 23. For any tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$, we define

$$A(R) = \bigl\{\langle x,y\rangle \in 2^\omega \times 2^\omega \colon \mathtt{rnk}(R_{xy}) = \infty\bigr\},$$

$$A^\xi(R) = \bigl\{\langle x,y\rangle \in 2^\omega \times 2^\omega \colon \mathtt{rnk}(R_{xy}) \geq \xi\bigr\},$$

where, according to (1), $R_{xy} = \{s \in \mathbb{N}^{<\omega} \colon R(x \restriction \mathtt{lh}\, s, y \restriction \mathtt{lh}\, s, s)\}$. A tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ is said to be *ξ-bounded*, where $\xi < \omega_1$, if $A(R) = A^\xi(R)$. $\square$

Thus, $A(R)$ is a $\mathbf{\Sigma}^1_1$-set in $2^\omega \times 2^\omega$, any $\mathbf{\Sigma}^1_1$-set $A \subseteq 2^\omega \times 2^\omega$ admits a representation in such a form, and the intersection $A(R) = \bigcap_{\xi<\omega_1} A^\xi(R)$ possesses the CIP. (To derive the CIP, note that the sets $A(R)$ and $A^\xi(R)$ are the preimages of the sets $\mathcal{T}^\infty$ and $\mathcal{T}^\xi$, respectively, in the sense of the continuous map $\langle x, y\rangle \mapsto R_{xy}$. It remains to recall that $\mathcal{T}^\infty = \bigcap_\xi \mathcal{T}^\xi$ satisfies the CIP according to Proposition 13.)

The following lemma represents a classical result. We refer to [15] for very similar results and constructions. We call an arbitrary set $X \subseteq 2^\omega \times 2^\omega$ such that $\langle x, y\rangle \in X \Leftrightarrow \langle x', y'\rangle \in X$ holds whenever $x, x', y, y' \in 2^\omega$ satisfy $x \restriction m = x' \restriction m$ and $y \restriction m = y' \restriction m$ an *m-Lipschitz set*. Obviously, any such set is open–closed in $2^\omega \times 2^\omega$.

Lemma 24. *If $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ is a tree, $\nu < \omega_1$, and $k \ge 1$, then $A^k(R)$ is a k-Lipschitz set, $A^{\omega(1+\nu)}(R)$ belongs to $\mathbf{\Pi}^0_{1+2\nu}$, and $A^{\omega(1+\nu)+k}(R)$ belongs to $\mathbf{\Sigma}^0_{1+2\nu+1}$.*

Conversely, if $\nu < \omega_1$, $k \ge 1$, and $A \subseteq 2^\omega \times 2^\omega$ is a k-Lipschitz set, or a $\mathbf{\Pi}^0_{1+2\nu}$-set, or a $\mathbf{\Sigma}^0_{1+2\nu+1}$-set, then there exists a tree $R \subseteq (2 \times 2 \times \mathbb{N})^{<\omega}$ such that $A = A^k(R) = A(R)$, or $A = A^{\omega(1+\nu)}(R) = A(R)$, or $A = A^{\omega(1+\nu)+k}(R) = A(R)$, respectively.[6]

Proof. The verification of the first assertion is analogous to evaluating the Borel class of the relations $\mathsf{E}^\xi_{\text{NT}}$ in the proof of Lemma 11(ii). Set

$$R^s_{xy} = \{t \in \mathbb{N}^{<\omega} \colon s^\wedge t \in R_{xy}\} \qquad \text{and} \qquad A^\xi_s(R) = \{\langle x, y\rangle \in 2^\omega \times 2^\omega \colon \mathtt{rnk}(R^s_{xy}) \ge \xi\}$$

for any $x, y \in 2^\omega$, $s \in \mathbb{N}^{<\omega}$, and $\xi < \omega_1$. We argue that, under the hypotheses of the lemma, $A^k_s(R)$ is a $(\mathtt{lh}\, s + k)$-Lipschitz set, $A^{\omega(1+\nu)}_s(R)$ belongs to $\mathbf{\Pi}^0_{1+2\nu}$, and $A^{\omega(1+\nu)+k}_s(R)$ belongs to $\mathbf{\Sigma}^0_{1+2\nu+1}$. To verify the first assertion, it suffices to note that $A^k_s(R)$ is equal to the set of all pairs $\langle x, y\rangle$ such that R_{xy} contains at least one sequence of the form $s^\wedge t$, where $t \in \mathbb{N}^k$. The two other assertions are derived by induction by means of the following obvious equalities:

$$A^{\xi+1}_s(R) = \bigcup_{j\in\mathbb{N}} A^\xi_{s^\wedge j}(R) \qquad \text{and} \qquad A^\lambda_S(R) = \bigcap_{\xi<\lambda} \bigcup_{j\in\mathbb{N}} A^\xi_{s^\wedge j}(R) \quad \text{for a limit } \lambda.$$

Finally, note that $A^\xi(R) = A^\xi_\Lambda(R)$.

The converse assertions of the lemma are proved by induction on ν and k.

The case of Lipschitz sets. Consider a k-Lipschitz set $A \subseteq 2^\omega \times 2^\omega$. There exists a set $P \subseteq 2^k \times 2^k$ such that

$$A = \{\langle x, y\rangle \in 2^\omega \times 2^\omega \colon \exists\, \langle u, v\rangle \in P\, (u \subset x \wedge v \subset y)\}.$$

Define R as the tree of all triples $\langle u, v, s\rangle \in 2^{<\omega} \times 2^{<\omega} \times \mathbb{N}^{<\omega}$ such that $\mathtt{lh}\, u = \mathtt{lh}\, v = \mathtt{lh}\, s$ and there exists a pair $\langle u', v'\rangle \in P$ such that u is comparable with u' (i.e., $u \subseteq u'$ or $u' \subseteq u$) and v is comparable with v'. Thus, we have $R_{xy} = \mathbb{N}^{<\omega}$ when $\langle x, y\rangle \in A$; otherwise, R_{xy} does not contain any sequence of length k. Hence, $A = A^k(R) = A^\omega(R) = A(R)$.

The case of $\mathbf{\Sigma}^0_{1+2\nu+1}$. Each $\mathbf{\Sigma}^0_{1+2\nu+1}$-set $A \subseteq 2^\omega \times 2^\omega$ is equal to a union of the form $A = \bigcup_n A_n$, where all A_n are sets from $\mathbf{\Pi}^0_{1+2\nu}$. Then, by the induction hypothesis, for any n, there exists a tree $R^n \subseteq (2\times 2\times\mathbb{N})^{<\omega}$ such that $A_n = A(R^n) = A^{\omega(1+\nu)}(R^n)$. Define a tree $R \subseteq (2\times 2\times\mathbb{N})^{<\omega}$ as follows: $R(u^\wedge i, v^\wedge j, n^\wedge s)$ when $R^n(u, v, s)$ for any $i, j = 0, 1$ and $n \in \mathbb{N}$ (and, separately, $\langle \Lambda, \Lambda, \Lambda\rangle \in R$). Then, $R_{xy} = \{\Lambda\} \cup \{n^\wedge s \colon s \in {R^n}_{xy}\}$ for any $x, y \in 2^\omega$, and, hence, the cross-section tree R_{xy} coincides with $\sum^*_n {R^n}_{xy}$ (see Section 2 for the operation of countable sum). Hence, $\mathtt{rnk}(R_{xy}) =$

[6] Note that $1 + \nu = \nu$ and $1 + 2\nu = 2\nu$ for $\nu \ge \omega$.

$\sup \mathtt{rnk}(R^n{}_{xy})$.[7] In particular, $\mathtt{rnk}(R_{xy}) = \infty$ if and only if $\mathtt{rnk}(R^n{}_{xy}) = \infty$ for a certain n. Hence, $A = A(R)$ according to the choice of the trees R^n. Next, if $\langle x, y\rangle \notin A$, then $\langle x, y\rangle \notin A_n$ for any n, so $\mathtt{rnk}(R^n{}_{xy}) < \omega(1+\nu)$, and this implies $\mathtt{rnk}(R_{xy}) < \omega(1+\nu)+1$. We conclude that $A = A(R) = A^{\omega(1+\nu)+1}(R) = A^{\omega(1+\nu)+k}(R)$ for any $k \geq 1$, which was to be proved.

The case of $\mathbf{\Pi}^0_{1+2\nu}$. Here, $\nu < \omega_1$ may be either a limit or a nonlimit ordinal. Any $\mathbf{\Pi}^0_{2\nu}$-set $A \subseteq 2^\omega \times 2^\omega$ coincides with an intersection of the form $A = \bigcap_n A_n$, where each A_n is a set from the class $\mathbf{\Sigma}^0_{1+2\eta_n+1}$, $\eta_n < \nu$, or simply a k_n-Lipschitz set for a certain k_n in the case when $\nu = 0$ (then $\mathbf{\Pi}^0_1$ are closed sets). By the induction hypothesis, for every n, there exists a tree $R^n \subsetneq (2\times 2\times \mathbb{N})^{<\omega}$ such that $A_n = A(R^n) = A^{\omega(1+\eta_n)+1}(R^n)$ or $A_n = A(R^n) = A^{k_n}(R^n)$ for $\nu = 0$. In this situation, one can easily define a tree $R \subseteq (2\times 2\times \mathbb{N})^{<\omega}$ such that, for any $x, y \in 2^\omega$, R_{xy} is isomorphic to $\prod^*_{n\in\mathbb{N}} R^n{}_{xy}$. (See Section 2 for the operation of countable product. The tree R can actually be defined as the set of all triples of the form $\langle u, v, \sigma\rangle$, where u and v belong to 2^m for a certain m and $\sigma = \langle s_0, \dots, s_m\rangle$, where each s_k belongs to $\mathbb{N}^m$ and $\langle u, v, s_k\rangle \in R^k$, with the ordering $\langle u, v, \sigma\rangle \preccurlyeq \langle u', v', \sigma'\rangle$ for $u \subseteq u'$, $v \subseteq v'$, and $\sigma \preccurlyeq \sigma'$ in the sense pointed out in Section 2.)

According to Lemma 3, $\mathtt{rnk}(R_{xy}) = \infty$ if and only if $\mathtt{rnk}(R^n{}_{xy}) = \infty$ for any n. Hence, $A = A(R)$ due to the choice of the trees R^n. Next, if $\langle x, y\rangle \notin A$, then $\langle x, y\rangle \notin A_n$ and, hence, $\mathtt{rnk}(R^n{}_{xy}) \leq \omega(1+\eta_n) < \omega(1+\nu)$ (or $< k_n < \omega$ for $\nu = 0$) for at least one n. On the other hand, the inequality $\mathtt{rnk}(R_{xy}) \leq \mathtt{min}_n\, \mathtt{rnk}(R^n{}_{xy}) + n$ holds by virtue of Lemma 3, so $\mathtt{rnk}(R_{xy}) \leq \omega(1+\nu)$. Thus, $A = A(R) = A^{\omega(1+\nu)}(R)$, which was to be proved. $\square$

In the light of this lemma, we could try to prove the completeness of, say, the relation $\mathsf{E}^{\omega(1+\nu)+2}_{\mathrm{NT}}$ for the Borel class $\mathbf{\Sigma}^0_{1+2\nu+1}$ in the following way. Consider a certain $\mathbf{\Sigma}^0_{1+2\nu+1}$-equivalence relation E on 2^ω. Thus, E, as a set of pairs, is of class $\mathbf{\Sigma}^0_{1+2\nu+1}$ in $2^\omega \times 2^\omega$. It follows from Lemma 24 that there exists a tree $Q \subsetneq (2\times 2\times \mathbb{N})^{<\omega}$ such that $\mathsf{E} = A^{\omega(1+\nu)+1}(Q) = A(Q)$, or, in other words,

$$x \,\mathsf{E}\, y \iff \mathtt{rnk}(Q_{xy}) \geq \omega(1+\nu)+1 \iff \mathtt{rnk}(Q_{xy}) = \infty$$

for any $x, y \in 2^\omega$. In particular, Q is an $(\omega(1+\nu)+1)$-bounded tree in the sense of Definition 23.

Conjecture 25. In this particular case, there exists a tree $R \subsetneq (2\times 2\times \mathbb{N})^{<\omega}$ that satisfies conditions (i)–(v) of Theorem 5 and the following additional condition: R is an $(\omega(1+\nu)+2)$-bounded tree and, therefore,

$$x \,\mathsf{E}\, y \iff \mathtt{rnk}(R_{xy}) \geq \omega(1+\nu)+2 \iff \mathtt{rnk}(R_{xy}) = \infty$$

for any $x, y \in 2^\omega$. $\square$

Accepting this conjecture, we obtain

$$x \,\mathsf{E}\, y \iff \vartheta(x)\, \mathsf{E}^{\omega(1+\nu)+2}_{\mathrm{NT}}\, \vartheta(y) \iff \vartheta(x)\, \mathsf{E}_{\mathrm{NT}}\, \vartheta(y),$$

where the map ϑ is defined according to (6) in Section 4. (Indeed, due to Lemma 9, the equality $\textsc{Emb}(\vartheta(x), \vartheta(y)) = R_{xy}$ holds for any x and y.) Thus, as in the proof of Theorem 8, the map ϑ is a Borel reduction of the relation E to $\mathsf{E}^{\omega(1+\nu)+2}_{\mathrm{NT}}$. In view of the arbitrary character of E in the class $\mathbf{\Sigma}^0_{1+2\nu+1}$, the relation $\mathsf{E}^{\omega(1+\nu)+2}_{\mathrm{NT}}$ turns out to be complete for the class of all $\mathbf{\Sigma}^0_{1+2\nu+1}$-equivalence relations. Thus, since $\mathsf{E}^{\omega(1+\nu)+2}_{\mathrm{NT}}$ itself belongs to $\mathbf{\Sigma}^0_{1+2(1+\nu)+1}$ according to Lemma 11(ii), we conclude that, for $\nu \geq \omega$ (then $1+\nu = \nu$), the class $\mathbf{\Sigma}^0_{1+2\nu+1}$ contains a complete equivalence relation.

However, this argument is based on Conjecture 25, which, to our regret, we could neither prove nor disprove for the present.

[7] Recall that $\sup X$ denotes the least ordinal that is strictly greater than any ordinal in X, or ∞ if X contains ∞.

9. OPEN QUESTIONS

There are good grounds to assume that $\mathsf{E}^{\eta}_{\rm NT} <_{\rm B} \mathsf{E}^{\omega\nu}_{\rm NT} <_{\rm B} \mathsf{E}^{\omega\nu+n}_{\rm NT}$ for $\eta < \omega\nu$ and $n \ge 1$. Here, the main idea is that there do not exist $\le_{\rm B}$-maximal Borel equivalence relations (as pointed out in [6]); therefore, the sequence of equivalence relations $\mathsf{E}^{\xi}_{\rm NT}$ must have countably many $<_{\rm B}$-increasing indices (in the strict sense). On the other hand, it is likely that $\mathsf{E}^{\omega\nu+n}_{\rm NT} \sim_{\rm B} \mathsf{E}^{\omega\nu+n+1}_{\rm NT}$ for $n \ge 1$.

There are a few more interesting questions.

What Borel classes contain complete equivalence relations?

Here, one can consider another similar problem. Some time ago, it seemed quite possible (see, for example, [9]) that the equivalence relation T, called the *equality of countable sets of real numbers*,[8] is not Borel-reducible to any equivalence relation $\mathsf{E}_{\mathcal{I}}$ generated by a Borel ideal $\mathcal{I} \subseteq \mathcal{P}(\mathbb{N})$. However, Rosendal [17] disproved this conjecture: there exists a Borel ideal of the form $\mathcal{I}^{\xi}_{\rm NT}$ such that $\mathsf{T} \le_{\rm B} \mathsf{E}_{\mathcal{I}^{\xi}_{\rm NT}}$ (see Corollary 22 above). What is the minimal ordinal ξ that guarantees this relation?

In conclusion, note that all estimates for the Borel classes of equivalence relations in this paper concern only real Borel classes in spaces of the Cantor discontinuum type. A somewhat more fundamental concept of "potential" Borel classes of equivalence relations introduced in [6] may require appropriate modification of the proofs.

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[8] T is defined on the set $\mathbb{R}^{\omega}$ of all countable sequences of real numbers so that $x \mathsf{T} y$ holds when the sets $\{x(n) : n \in \omega\}$ and $\{y(n) : n \in \omega\}$ (countable sets of real numbers) are equal to each other.

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